การออกแบบเชิงตัวเลขของระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงด้วยวิธีอสมการ

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NUMERICAL DESIGN OF RETARDED FRACTIONAL DELAY
DIFFERENTIAL SYSTEMS BY THE METHOD OF INEQUALITIES

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A Thesis Submitted in Partial Fulfillment of the Requirements
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วัน กวง เหงียน : การออกแบบเชิงตัวเลขของระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงด้วยวิธีอสมการ (NUMERICAL DESIGN OF RETARDED FRACTIONAL DELAY DIFFERENTIAL SYSTEMS BY THE METHOD OF INEQUALITIES) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.สุชิน อรุณสวัสดิ์วงศ์, 47 หน้า.

คลาสของระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงมีรูปแบบทั่วไปมากและรวมระบบตรรกยะและระบบพลวัตที่มีการประวิงเวลาแบบหน่วงเป็นกรณีพิเศษในทางปฏิบัติ ระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงเกิดขึ้นเมื่อลักษณะของกระบวนการที่ถูกควบคุมอธิบายได้ด้วยสมการอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงและ/หรือตัวควบคุมที่ใช้มีอันดับเศษส่วน วัตถุประสงค์หลักของวิทยานิพนธ์ฉบับนี้คือการพัฒนาขั้นตอนวิธีเชิงคำนวณเพื่อทำให้สามารถออกแบบระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วงด้วยวิธีอสมการได้

ในการออกแบบด้วยวิธีอสมการ โดยปกติกำหนดปัญหาออกแบบในรูปแบบที่เหมาะสมสำหรับการหาคำตอบด้วยวิธีเชิงตัวเลข ดังนั้นขั้นตอนการออกแบบดังกล่าวจึงประกอบด้วยการคำนวณสองขั้นตอน ในขั้นตอนแรกหาจุดเสถียรภาพในปริภูมิของพารามิเตอร์การออกแบบ ในขั้นที่สองค้นหาคำตอบการออกแบบในบริเวณเสถียรภาพโดยเริ่มต้นจากจุดเสถียรภาพที่หาได้ ในวิทยานิพนธ์ฉบับนี้ได้พัฒนาวิธีทดสอบเสถียรภาพเชิงเลขและวิธีสำหรับคำนวณพิกัดเชิงเสถียรภาพของฟังก์ชันลักษณะสมบัติสำหรับระบบอนุพันธ์เศษส่วนที่มีการประวิงเวลาแบบหน่วง นอกจากนั้นได้ออกแบบตัวควบคุมอันดับเศษส่วนสำหรับกระบวนการที่มีหนึ่งอินพุตหนึ่งเอาต์พุตและกระบวนการที่มีหลายอินพุตหลายเอาต์พุต ด้วยวิธีอสมการร่วมกับเกณฑ์การออกแบบของผลตอบสนองต่อสัญญาณขั้นบันไดโดยใช้ขั้นตอนวิธีที่พัฒนาขึ้น ผลการออกแบบแสดงให้เห็นอย่างชัดเจนถึงประสิทธิผลของขั้นตอนการออกแบบที่เป็นระบบและประโยชน์ของเครื่องมือเชิงคำนวณที่พัฒนาขึ้นในวิทยานิพนธ์ฉบับนี้

ภาควิชา วิศวกรรมไฟฟ้า ลายมือชื่อนิสิต Quang
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KEY WORD: FRACTIONAL CONTROL SYSTEMS / FRACTIONAL CONTROLLERS / DELAY SYSTEMS / STABILITY ANALYSIS / STABILIZATION / ABSCISSA OF STABILITY / THE METHOD OF INEQUALITIES / CONTROL DESIGN

VAN QUANG NGUYEN: NUMERICAL DESIGN OF RETARDED FRACTIONAL DELAY DIFFERENTIAL SYSTEMS BY THE METHOD OF INEQUALITIES. THESIS PRINCIPAL ADVISOR: ASST. PROF. SUCHIN ARUNSAWATWONG, Ph.D., 47 pp.

The class of retarded fractional delay differential systems (RFDDS) is very general and includes rational systems and retarded delay differential systems as special cases. In practice, a RFDDS can arise when the controlled plant is characterized by fractional delay differential equations and/or when the controller used is of fractional order. The primary objective of this thesis is to develop computational methods to make possible the design of RFDDSs by the method of inequalities (MoI).

In the design by the MoI, design problems are usually formulated so that they are suitable for solution by numerical methods. Accordingly, the design procedure comprises two phases of computation. First, seek a stability point in design-parameter space. Second, search in the stability region for a design solution by starting at the stability point so obtained. In this thesis, for RFDDSs, a computational stability test and a method for computing the abscissa of stability of the characteristic function are devised. Moreover, fractional controllers are designed for SISO and MIMO plants by the MoI together with conventional step-response criteria, using the developed computational tools. The numerical results evidently show the effectiveness of the systematic design procedure as well as the usefulness of the tools.

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Field of study: Electrical Engineering **Principal Advisor's signature:**
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Regarding the love and support of members in my family for which a word of thanks is by no means enough, I would like to dedicate this work and express my heartfelt appreciation to them.

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Symbols

$\mathrm{Re}(s)$	The real part of s
$\mathrm{Im}(s)$	The imaginary part of s
$H(\rho)$	A right half complex plane with $\mathrm{Re}(s) \geq \rho$
α	The abscissa of stability of a characteristic function
I_{MN}	An integral operator (See the definition in Section 2.2.2)

Acronyms

RFDDS	Retarded Fractional Delay Differential System
RS	Rational System
RDDS	Retarded Delay Differential System
MoI	Method of Inequalities

CHAPTER I

INTRODUCTION

1.1 Introduction

Recently, much research effort has been given to fractional differential systems. This is mainly due to the fact that many physical processes have their mathematical models described by or more adequately characterized by fractional delay differential equations (see, e.g., [1, 2] and the references therein). Moreover, it is demonstrated [2–4] that if appropriately designed, feedback systems with fractional order controllers can yield better performances than those with integer order controllers.

Many investigators have been prompted to develop methods for designing fractional differential systems in order to enhance the system's performances and robustness (see, e.g., [1–3, 5] and also the references therein). However, these methods cannot effectively handle complicated design problems in a systematic way. Evidently, it is the lack of effective computational tools that hinders the advance in the field. Therefore, it is desirable to have a systematic method that can solve the design problems for fractional differential systems efficaciously.

It is readily appreciated that methods based on numerical optimization are useful and effective in designing control systems (see, e.g., [6, 7]). The method of inequalities (MoI) [8–11] is a general multiobjective optimization method that has been successfully applied to many difficult design problems (see, e.g., [12–18] and also the references cited in [11]). The method facilitates a realistic formulation of the design problem so that the constraints and performance specifications can be directly expressed as inequalities, whereas all tedious computations are carried out by a computer using efficient numerical algorithms. So far, it appears that no one has considered the design of fractional differential systems using the MoI.

This work is concerned primarily with the numerical design of retarded fractional delay differential systems (RFDDSs) by the MoI, where the transfer functions of the systems are of the form

$$\frac{q_0(s) + \sum_{k=1}^{n_2} q_k(s)e^{-\beta_k s} + \sum_{k=1}^{\tilde{n}_2} \tilde{q}_k(s)e^{-v_k(s)}}{p_0(s) + \sum_{k=1}^{n_1} p_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s)e^{-u_k(s)}}, \tag{1.1}$$

the delays γ_k and β_k are such that $0 < \gamma_1 < \ldots < \gamma_{n_1}$ and $0 < \beta_1 < \ldots < \beta_{n_2}$, the polynomials $p_k, q_k, \tilde{p}_k$ and $\tilde{q}_k$ are of the form $\sum_{j=0}^{l_k} a_j s^{\alpha_j}$ with all $\alpha_j \geq 0$, $\deg p_0 > \deg p_k$ for $k = 1, 2, \ldots, n_1$, $\deg p_0 \geq \deg q_0$ and $\deg p_0 > \deg q_k$ for $k = 1, 2, \ldots, n_2$, u_k and v_k are polynomials of the form $\sum_{j=1}^{m_k} b_j s^{\delta_j}$ with $0 < \delta_j \leq 1$, and $b_j \geq 0$, and none of u_k and v_k assumes the form αs. Transfer function (1.1) can occur when the plant is described by fractional differential equations and/or when the controller used is of fractional order. Obviously, the class of systems with transfer function (1.1) is very general and includes rational systems (RSs) and retarded delay differential systems (RDDSs) as special cases. This is illustrated in Figure 1.1.

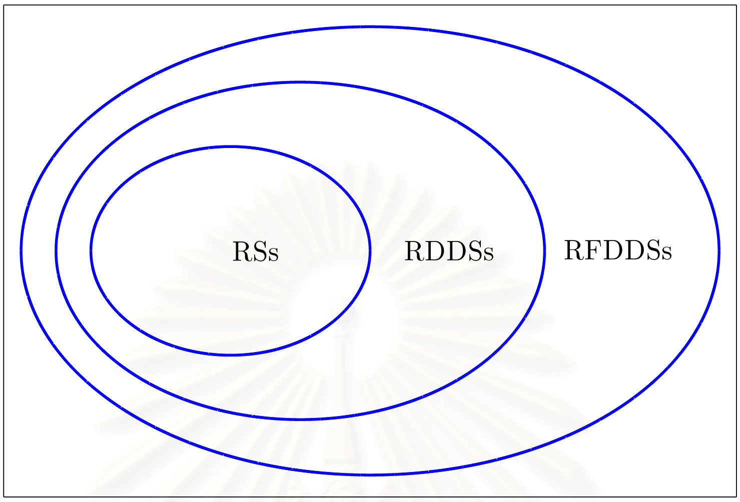

Figure 1.1: RFDDSs include rational systems and retarded delay diferential systems

In the design of RFDDSs by the MoI [8, 9, 11, 19] as well as parameter optimization methods[1] (see, e.g., [6,7] and also [20]), one usually has to cope with stability problems and the problem of computing the systems' time responses efficiently. There are methods for computing the time responses of RFDDSs based on the numerical inversion of Laplace transforms (see, e.g., [1,21,22]). However, the stability problems still remain unresolved, i.e., computational tools are needed for checking the systems' stability and determining a design parameter that stabilizes the systems (see, e.g., [8,9,11,19,23] for details). This is because numerical search algorithms are in general able to seek a design solution only if they start from a *stability point* (i.e., a point for which the system is stable so that all the associated performance measures are finite or defined). Once such a point is obtained, the algorithms search for a design solution inside the space of all stability points.

The principal objective of this thesis is to develop computational tools for resolving the stability problems in the design of RFDDSs by the MoI. Once the computational tools are obtained, we will develop numerical procedures for designing RFDDSs by the MoI and make use of these procedures in the design of fractional controllers for RFDDSs. Accordingly, the thesis is organized as follows. Chapter 2 describes the general principle in the design of dynamical and control systems by the MoI and explains the design formulation for RFDDSs. In Chapter 3, computational tools are developed for resolving the stability problems associated with the design of RFDDSs by the MoI, and also, numerical examples demonstrating the effectiveness of the tools are given. Numerical examples are given in Chapter 4, where fractional controllers are designed for SISO and MIMO systems to show the effectiveness of the design procedure and the usefulness of the developed computational tools. Finally, discussion and conclusions are given in Chapter 5, in which we also suggest some future

[1] that is, methods that search for a design solution in a parameter space using numerical optimization algorithms.

work.

1.2 Literature Review

1.2.1 The Method of Inequalities

Extensively used in the literature, the MoI [8, 11] has proved effective for designing dynamical and control systems (see, e.g., [12–18]). The method allows constraints and design specifications to be expressed directly as inequalities, which are usually solved by numerical methods.

It is noted [8, 11] that in using the MoI, a design problem must be formulated so that it is suitable for solution by numerical methods.

The design formulation proposed by Zakian and Al-Naib [8] in connection with RSs has been used by many researchers (e.g., [12, 13, 24] and also many references cited in [11]). The formulation is then extended by Arunsawatwong [23] to RDDSs. Evidently, numerical procedures have been devised for resolving the associated stability problems of RSs (see, e.g., [25]) and RDDSs (see [26]). The formulation may be further extended to RFDDSs; inevitably, the associated stability problems have to be resolved and these fundamental problems may have hindered the application of the MoI to the design of RFDDSs.

1.2.2 Stability of RFDDSs

Recently, much research effort has been given to control systems of fractional order owing to the following reasons. First, many physical processes are described, or more adequately characterized, by fractional delay differential equations. Second, the use of appropriately designed fractional controllers can bring better performances to feedback control systems. However, advanced mathematical tools are generally required for handling systems of fractional order.

Bonnet and Partington [27] have investigated the stability for a very general class of fractional systems (i.e., RSs and RDDSs also belong to the class) (see also [28, 29]). Based on this stability analysis, Hwang and Cheng [30] developed a numerical procedure for testing the stability of these systems. In conjunction with the MoI, Hwang and Cheng's procedure provides a useful computational tool for checking the stability of RFDDSs. So far, however, no one has considered developing numerical procedures for stabilizing RFDDSs. Once such a procedure is available, RFDDSs can be designed by the MoI in a systematic way using numerical methods.

The abscissa of stability has proved useful in stabilizing RSs and RDDSs. Zakian [25] proposed a bisection algorithm for computing the abscissa of stability of a polynomial which allows one to stabilize RSs by numerical methods. The algorithm makes repeated use of a stability test, which is in this case the well-known Routh test. Later, Arunsawatwong [26] extends Zakian's approach to retarded delay differential systems, whose characteristic functions are quasipolynomials. The extension is possible since there are only a finite number of roots in any given right half of the complex plane. In connection with the stability result achieved by Bonnet and Partington [27], the abscissa of stability can be used for stabilizing RFDDSs by numerical methods provided that a practical method for computing it is available.

1.3 Objectives

The purpose of this thesis is threefold.

1. Develop computational tools which enable one to design retarded fractional delay differential systems (RFDDSs) by the method of inequalities (MoI), namely a stability test and a practical method for computing the abscissa of stability for RFDDSs.

2. Establish a numerical procedure for designing RFDDSs by the MoI using the developed computational tools.

3. Perform numerical designs of fractional systems so as to illustrate the effectiveness of the developed design procedure.

1.4 Scope of Thesis

The scope of this research work is specified as follows. Consider linear RFDDSs whose transfer function is given in (1.1).

1. Develop computational tools including a stability test and a practical method for computing the abscissa of stability of RFDDSs.

2. Develop procedures for performing numerical design of RFDDSs by the MoI.

3. Design fractional controllers for SISO plants, i.e., a time-delay plant and a heat-conduction process, and a binary distillation column by the MoI using performance specifications based on the systems' step responses.

1.5 Methodology

This thesis extends a known formulation used in connection with the MoI for designing RSs and RDDSs to RFDDSs.

First, by extending Hwang and Cheng's procedure [30], we obtain a new stability test for checking whether a RFDDS has no poles in any given right half of the complex plane. A practical method for computing the abscissa of stability of the system is devised by making repeated use of the new stability test in a bisection scheme based on Zakian's iteration [25].

Next, numerical procedures for designing RFDDSs are developed based on the computational tools obtained. Fractional controllers are then designed for some RFDDSs using step-response performance specifications.

All the numerical procedures involved in this thesis are implemented in FORTRAN programming language.

1.6 Contributions

The outcome of this research work includes

1. A numerical procedure for checking the stability of RFDDSs,
2. A practical method for computing the abscissa of stability of the characteristic function in (3.1), and a procedure for stabilizing systems with transfer function (1.1) by numerical methods.
3. A systematic procedure for designing RFDDSs by the MoI.

CHAPTER II

CONTROL DESIGN BY THE METHOD OF INEQUALITIES

This chapter explains the general principle in designing dynamical and control systems by the MoI (Section 2.1) and describes the formulation used for the the design of RFDDSs by the MoI (Section 2.2).

2.1 The Method of Inequalities

The MoI requires that a design problem be formulated as a set of inequalities

$$\phi_i(p) \le C_i, \quad i = 1, 2, ..., m, \tag{2.1}$$

where $p \in \mathbb{R}^n$ is the design parameter vector, $\phi_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ represent performance measures or physical properties of the system, and the bounds C_i are the supremal values of $\phi_i(p)$ that can be tolerated. Any point p satisfying (2.1) is an acceptable design solution.

In practice, numerical methods are usually employed to compute a design solution, i.e., a solution of (2.1). In order to formulate the design problem so that it is suitable for solution by numerical methods, Zakian [8, 9, 11, 19, 31] advocates that the numerical solution of inequalities (2.1) comprises two phases of computation.

- **Phase I:** Determine a point p such that $\phi_i(p) < \infty \; \forall i$.
- **Phase II:** Determine a solution of inequalities (2.1) by starting at the point obtained in Phase I.

Each phase gives rise to distinct computational problems.

2.1.1 Phase I

Define *a stability point* as a point p satisfying

$$\phi_i(p) < \infty \quad \forall i \tag{2.2}$$

and define *the stability region* Ω as the set containing all stability points. The main problem of Phase I is in the computation of a stability point.

In general, inequalities (2.2) are not soluble by numerical methods. That is, a stability point cannot be generated using only the functions ϕ_i and a descent method. This is because a gradient or similar properties of ϕ_i cannot be defined outside the stability region Ω.

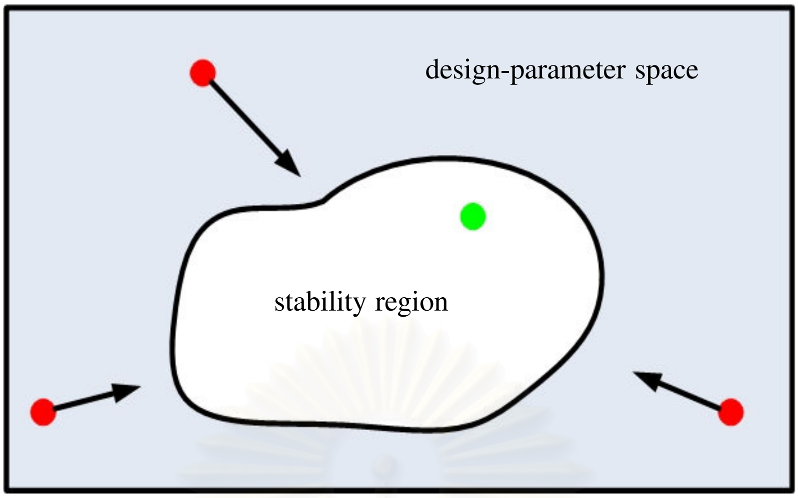

Figure 2.1: Computation in Phase I

It is suggested [8, 11, 19] that a possible method for obtaining a stability point by numerical methods is to replace condition (2.2) by an equivalent inequality

$$\alpha(p) < 0 \tag{2.3}$$

such that (i) $\alpha(p) < \infty$ for all $p \in \mathbb{R}^n$ and (ii) α can be computed economically in practice.

Once there exists such a function α, condition (2.3) becomes soluble by numerical methods. Consequently, iterative numerical methods can be used to locate a stability point by starting from any arbitrary point in $\mathbb{R}^n$.

For RSs, Zakian and Al-Naib [8] chose α to be the abscissa of stability of a characteristic polynomial. In this connection, an economical algorithm for computing the abscissa of stability was given in [25]. This useful approach was extended later to the case of RDDSs by Arunsawatwong [23, 26]. As will be seen later, the approach can be further extended to be applicable to the whole class of RFDDSs.

The computation in Phase I is illustrated in Figure 2.1.

2.1.2 Phase II

By starting from a stability point, a search method locates a solution of (2.1) within the stability region Ω. To avoid stepping outside the stability region, one checks the stability of the system at every point generated by the search method. In doing this, one needs to determine the sign of $\alpha(p)$, which is less demanding than to compute the value of $\alpha(p)$. Therefore, in Phase II, only a stability test is required.

Once the system is found to be stable, the performance $\phi_i(p)$ are to be computed. In contrast to Phase I, the main problem in Phase II is in computing the functions ϕ_i economically for given values of p. Usually, the functions ϕ_i that are the most difficult to compute are those defined in terms of the time responses of the system. Once the time responses are obtained, such functions ϕ_i can be

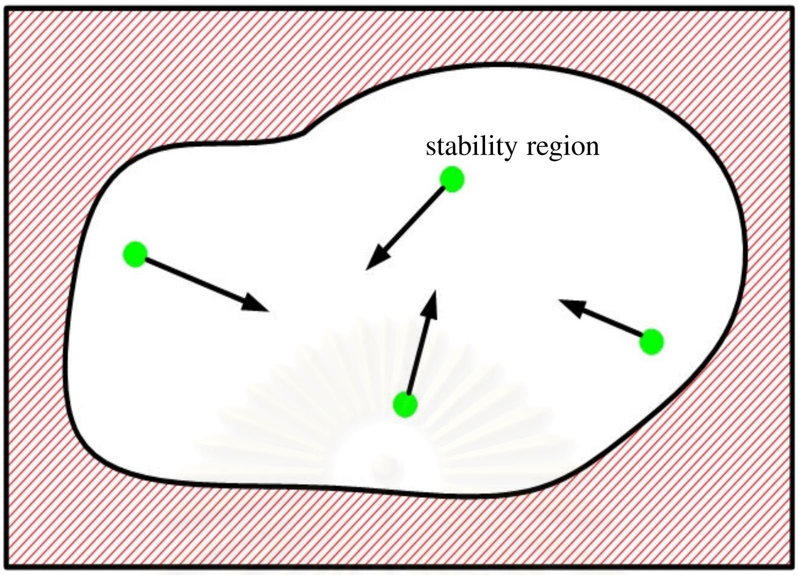

Figure 2.2: Computation in Phase II

computed using known numerical methods (see Chapter 4 for this). Therefore, it is evident that one needs an efficient and reliable algorithm for computing the time responses.

The computation in Phase II is illustrated in Figure 2.2.

2.2 Design formulation for RFDDSs

Following the general principle given in Section 2.1, this section describes a practical formulation for the design of RFDDSs by the MoI, which is suitable for solution by numerical methods.

2.2.1 Stability Test & Stabilization

It is known [27] that a system characterized by the transfer function (1.1) is BIBO stable if and only if the characteristic function

$$f(s) = p_0(s) + \sum_{k=1}^{n_1} p_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s)e^{-u_k(s)} \tag{2.4}$$

has all zeros with negative real parts. Accordingly, a RFDDS is stable if and only if

$$\alpha < 0, \quad \alpha \triangleq \max\{\mathrm{Re}\, s : \ f(s) = 0\}. \tag{2.5}$$

In conjunction with the MoI, inequality (2.5) provides a useful criterion for stabilizing RFDDSs (including RSs and RDDSs) by numerical methods.

Usually, inequality (2.5) is replaced by a practical sufficient condition

$$\alpha(p) \le -\varepsilon \quad (\varepsilon > 0). \tag{2.6}$$

It should be noted that the bound $-\varepsilon$ is introduced in (2.6) so as to ensure that the system is stable as long as the magnitude of error in the computed value of $\alpha(p)$ is less than ε.

For RFDDSs, once the abscissa of stability α can be efficiently computed, a stability point is readily obtainable by simply solving the inequality (2.6) by iterative numerical methods. See Chapter 3 for details.

2.2.2 Computation of Time Responses

Many methods have been developed for the numerical inversion of Laplace transforms (see, e.g., [1, 21, 22] and many references cited in [32, 33]). It is noted, in Taiwo's design of distillation columns by the MoI [14], that Zakian's method [22] is used for computing the systems' step responses. Evidently, the method, which is based on Zakian's I_{MN} approximants [22, 34], is fast and quite simple to implement; see [32, 33] for comparisons of several methods. Moreover, it is demonstrated [32] that the method is suitable for inverting fractional functions in Laplace domain.

Regarding the disadvantage of Zakian's method in inverting oscilatory functions (see, e.g., [32]), it should be noted in designing a system by the MoI that the search for a design solution is done inside the stability region where the system is stable and therefore, its time responses should not be very oscilatory. As the search progresses, trial points generated by the algorithm lie well inside the stability region and thus the accuracy of the computed time responses is improved.

Hence, in this thesis, we compute the time responses of RFDDSs by employing Zakian's I_{MN} approximants [22, 34], which result in a useful formula for numerically inverting Laplace transforms (see also [33, 35]).

Definition of I_{MN} Approximants Zakian [22, 34] defines the I_{MN} approximant of $x(t)$ for $t \geq 0$ by the improper integral

$$I_{MN}(x, t) \triangleq \int_0^\infty x(\lambda t) \sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda} d\lambda, \tag{2.7}$$

where (α_i, K_i) are defined constants, and the nonnegative integers (M, N) are, respectively, the orders of the numerator and the denominator of the Laplace transform of $\sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda}$.

In this work, we restrict our attention only to the *full grade* I_{MN} approximants [34], whose constants (α_i, K_i) are defined by

$$\sum_{i=1}^{N} \frac{K_i}{z + \alpha_i} = \mathrm{e}_{MN}^{-z} \quad \text{and} \quad \mathrm{Re}(\alpha_i) > 0 \;\forall i, \tag{2.8}$$

where e_{MN}^{-z} denotes the $[M/N]$ Padé approximant to e^{-z}. The *full grade* I_{MN} approximants have many remarkably useful properties [34, 35] and have been successfully applied to many practical problems (see, for example, [8, 33, 36]).

Inversion Formula Let $X(s)$ denote the Laplace transform of $x(t)$, evaluated at s. That is,

$$X(s) \triangleq \mathcal{L}[x(t)] = \int_0^\infty x(t) \mathrm{e}^{-st} dt, \tag{2.9}$$

where s is a complex number such that the integral converges to a finite limit. From (2.7), it can be readily verified [22] that

$$I_{MN}(x,t) = \frac{1}{t}\sum_{i=1}^{N} K_i X\Big(\frac{\alpha_i}{t}\Big), \quad t > 0. \tag{2.10}$$

Evidently, (2.10) provides a useful formula for the numerical inversion of Laplace transforms. In this thesis, we normally use $M = 11$ and $N = 18$ with double precision arithmetic operations. However, whenever there is a doubt in the accuracy of the obtained results, we recompute by using $M = 30$ and $N = 40$ with quad-precision arithmetic operations. The details of how to choose appropriate values of M and N for the inversion formula (2.10) can be found in [33].

CHAPTER III

STABILITY AND STABILIZATION OF RFDDSs

In this chapter, a computational stability test and an algorithm for computing the abscissa of stability for RFDDSs are developed. By using the algorithm, one can stabilize RFDDSs by iterative numerical methods. Details of how the stability test is developed are given in Section 3.1. Section 3.2 describes the algorithm for computing the abscissa of stability and explains how to stabilize a RFDDS by numerical methods, using the developed algorithm. Numerical examples are given in Section 3.3 to demonstrate the effectiveness of the developed computational tools.

Consider a linear RFDDS whose characteristic function is described by

$$f(s) \triangleq p_0(s) + \sum_{k=1}^{n} p_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}} \tilde{p}_k(s)e^{-u_k(s)}, \tag{3.1}$$

where $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n$, the polynomials $p_k(s)$ and $\tilde{p}_k(s)$ are of the form $\sum_{j=0}^{l_k} a_j s^{\alpha_j}$ with $\alpha_j \geq 0$ and $\deg p_0 > \deg p_k$ for all k, $u_k(s)$ are of the form $\sum_{j=1}^{m_k} b_j s^{\delta_j}$ with $0 < \delta_j \leq 1$ and $b_j \geq 0$, and none of u_k assumes the form αs. It is readily deduced [27] that the system is BIBO stable if and only if f has all the zeros with negative real parts.

3.1 Stability Test

In this section, we develop a stability test for checking whether characteristic function f has no zeros in a given right half of the complex plane. The test is an extension of the method proposed by Hwang and Cheng [30] for checking the BIBO stability of the system.

Let $H(\rho)$ denote the right half plane given by

$$H(\rho) \triangleq \{s \in \mathbb{C} : \mathrm{Re}(s) \geq \rho\},$$

where $\rho \in \mathbb{R}$ is given. Function f is said to be *stable with respect to* $H(\rho)$ (or $H(\rho)$-stable) if none of its zeros lies in $H(\rho)$. Following this, it is readily appreciated that a linear RFDDS is BIBO stable if and only if f is $H(0)$-stable. Clearly, Hwang and Cheng's procedure [30] is used only for testing the $H(0)$-stability of f, whereas the procedure developed here can be used in testing the $H(\rho)$-stability of f for any given value of ρ.

In developing the stability tests in this work and in [30], the key idea used is the well-known Cauchy's residue theorem (see, e.g., [37]), which states as follows.

Theorem 1 ***[Cauchy's Residue]*** *Let $F(s)$ be analytic within and on a simple closed contour Γ except for finitely many points $s_1, s_2, \ldots, s_n$ lying in the interior of Γ. Then*

$$\int_{\Gamma} F(s)\, ds = \mathbf{i}2\pi \sum_{i=1}^{n} \mathrm{Res}(s_i),$$

where the integral is taken in the positive direction and $\mathrm{Res}(s_i)$ *denotes the residue of* $F(s)$ *at the points* s_i.

By defining

$$F(s) \triangleq 1/f(s), \tag{3.2}$$

we can see that f, in general, has no zeros within Γ if and only if the integral $\int_\Gamma F(s)\, ds = 0$. However, it may happen that the sum of the residues of F at all poles is equal to zero. For this reason, Hwang and Cheng [30] suggest replacing the contour integral $\int_\Gamma F(s)\, ds$ with the following integral:

$$J \triangleq \int_\Gamma \frac{F(s)}{(s + h_1 + \mathbf{i}h_2)^k\, F(\mathbf{i}h_2)} ds \tag{3.3}$$

where $k \geq 1$ is a specified integer, and h_1 and h_2 are randomly chosen parameters so that the point $(-h_1 - \mathbf{i}h_2)$ must lie outside Γ. Accordingly, we can see that f has no zeros within Γ if and only if $J = 0$ for any $(-h_1 - \mathbf{i}h_2)$ outside Γ.

The term $(s + h_1 + \mathbf{i}h_2)^k$ in integral (3.3) prevents the sum of the residues from being zero when the function f has zeros inside Γ. Note in passing that the idea of using the term $(s + h_1 + \mathbf{i}h_2)^k$ is due to Tuan and Duc [38].

Since the branch cut of the function f in (3.1) consists of the negative real axis including the origin, we divide the stability test into two cases: (i) $\rho \leq 0$ and (ii) $\rho > 0$. Each case uses a different contour so that the integration path does not cross the branch cut.

Now we are ready to consider the stability test.

Case I: $\rho \leq 0$. For $\rho \leq 0$, we define the integration contour Γ_ρ as follows.

$$\begin{aligned}
\Gamma_\rho &\triangleq \Gamma_{I\rho} \cup \Gamma_{R\rho} \cup \Gamma_{1\rho} \cup \Gamma_{2\rho} \cup \Gamma_r, \\
\Gamma_{I\rho} &= \{s = \rho + \mathbf{i}\,\omega : \omega \in [-R, -a] \cup [a, R]\}, \\
\Gamma_{R\rho} &= \left\{s = \rho + R\,e^{\mathbf{i}\theta} : -\pi/2 \leq \theta \leq \pi/2\right\}, \\
\Gamma_{1\rho} &= \left\{s = x - \mathbf{i}a : \rho \leq x \leq -\sqrt{r^2 - a^2}\right\}, \\
\Gamma_{2\rho} &= \left\{s = x + \mathbf{i}a : \rho \leq x \leq -\sqrt{r^2 - a^2}\right\}, \\
\Gamma_r &= \left\{s = r\,e^{\mathbf{i}\theta} : 0 \leq |\theta| \leq \pi - \arcsin(a/r)\right\},
\end{aligned} \tag{3.4}$$

where $R \to \infty$, $r > a > 0$ and $r \to 0$. See Fig. 3.1.

Next define the contour integral

$$J(\rho) = \int_{\Gamma_\rho} \frac{F(s)}{(s + h_1 + \mathbf{i}h_2)^k\, F(\rho + \mathbf{i}h_2)} ds. \tag{3.5}$$

We can easily deduce that, for $\rho < 0$, f is $H(\rho)$-stable if and only if f has no zero in the real interval $[\rho, 0]$ and $J(\rho) = 0$ for any $(-h_1 - \mathbf{i}h_2)$ outside Γ.

The procedure for implementing the stability test is as follows. First, check whether f has no zero in $[\rho, 0]$, which can be done conveniently and speedily by efficient numerical search algorithms.

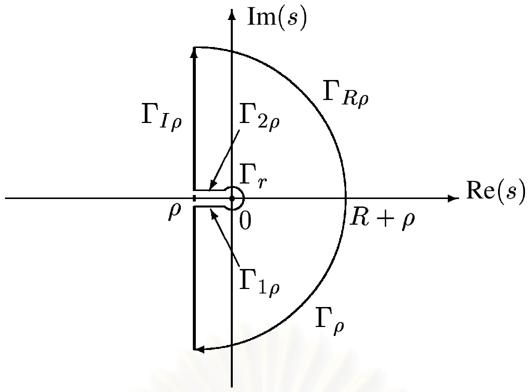

Figure 3.1: The contour Γ_ρ for $\rho \leq 0$.

If no zero is found, we then compute $J(\rho)$ defined in (3.5). If $J(\rho)$ is sufficiently close to zero, we conclude that f is $H(\rho)$-stable.

Now it remains only to explain how to compute the integral $J(\rho)$.

Proposition 2 *Consider $F(s)$ defined in (3.2) and the contour $\Gamma_{R\rho}$ in (3.4).*

$$\lim_{R\to\infty}\int_{\Gamma_{R\rho}} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds = 0. \tag{3.6}$$

Proof: Let $J_{R\rho}$ denote the integral along $\Gamma_{R\rho}$. Recalling that $F(s) = 1/f(s)$, we obtain

$$J_{R\rho} = \int_{\Gamma_{R\rho}} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds = \int_{\Gamma_{R\rho}} \frac{f(\rho+\mathbf{i}h_2)}{(s+h_1+\mathbf{i}h_2)^k\,f(s)}ds.$$

Let s be an arbitrary point on $\Gamma_{R\rho}$. Note that

$$f(s) = p_0(s) + \sum_{k=1}^{n} p_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}} \tilde{p}_k(s)e^{-u_k(s)},$$

where $\deg p_0(s) > \deg p_k(s)$ and all $u_k(s) = \sum_{j=1}^{m_k} b_j s^{\delta_j}$ are not of the form αs. Then it readily follows that

$$\lim_{R\to\infty} |p_0(s) + \sum_{k=1}^{n} p_k(s)e^{-\gamma_k s}| = +\infty,$$

and

$$\lim_{R\to\infty} |\sum_{k=1}^{\tilde{n}} \tilde{p}_k(s)e^{-u_k(s)}| = 0.$$

Therefore,

$$\lim_{R\to\infty} |f(s)| = +\infty.$$

This implies that, for each sufficiently large value of R, there exists a positive constant M_R such that $|f(s)| > M_R$ for all $s \in \Gamma_{R\rho}$ and $M_R \to \infty$ as $R \to \infty$. Besides, when R is sufficiently large,

$$|s + h_1 + \mathbf{i}h_2| > R/2.$$

By changing the variable $s = \rho + R\,e^{\mathbf{i}\theta}$, we can rewrite $J_{R\rho}$ as

$$J_{R\rho} = \int_{-\pi/2}^{\pi/2} \frac{f(\rho + \mathbf{i}h_2)\,\mathbf{i}R\,e^{\mathbf{i}\theta}d\theta}{(\rho + R\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)^k\, f(\rho + R\,e^{\mathbf{i}\theta})}.$$

Therefore, it is deduced that

$$\begin{aligned} |J_{R\rho}| &\leq \int_{-\pi/2}^{\pi/2} \frac{|f(\rho + \mathbf{i}h_2)\,\mathbf{i}R\,e^{\mathbf{i}\theta}|d\theta}{|\,(\rho + R\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)^k\, f(\rho + R\,e^{\mathbf{i}\theta})|} \\ &= \int_{-\pi/2}^{\pi/2} \frac{|f(\rho + \mathbf{i}h_2)|Rd\theta}{|\,(\rho + R\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)\,|^k|f(\rho + R\,e^{\mathbf{i}\theta})|} \\ &< \int_{-\pi/2}^{\pi/2} \frac{|f(\rho + \mathbf{i}h_2)|Rd\theta}{(R/2)^k M_R} = \frac{2^k|f(\rho + \mathbf{i}h_2)|\pi}{R^{k-1}M_R} \end{aligned}$$

Obviously,

$$\lim_{R\to\infty} \frac{2^k|f(\rho + \mathbf{i}h_2)|\pi}{R^{k-1}M_R} = 0,$$

and we obtain $\lim_{R\to\infty} J_{R\rho} = 0$ for every $k \geq 1$. Q.E.D.

Proposition 3 *Consider $F(s)$ defined in (3.2) and the contour Γ_r in (3.4).*

$$\lim_{r\to 0} \int_{\Gamma_r} \frac{F(s)}{(s + h_1 + \mathbf{i}h_2)^k\, F(\rho + \mathbf{i}h_2)} ds = 0. \tag{3.7}$$

Proof: Let J_r denote the integral along Γ_r. By changing the variable $s = r\,e^{\mathbf{i}\theta}$, we obtain

$$J_r = \int_{-\pi+\arcsin(a/r)}^{\pi-\arcsin(a/r)} \frac{f(\rho + \mathbf{i}h_2)\,\mathbf{i}r\,e^{\mathbf{i}\theta}d\theta}{(r\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)^k\, f(re^{\mathbf{i}\theta})}$$

Since $f(s)$ has no zero at the origin (otherwise, the zero would have been detected before), we can specify a number $\epsilon > 0$ such that $|f(s)| \geq \epsilon$ for any $|s| \leq r$ when r is sufficiently small. In addition, when r is sufficiently small, $|r\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2| > h = |h_1 + \mathbf{i}h_2|/2 > 0$. Consequently,

$$\begin{aligned} |J_r| &\leq \int_{-\pi+\arcsin(a/r)}^{\pi-\arcsin(a/r)} \frac{|f(\rho + \mathbf{i}h_2)\,\mathbf{i}r\,e^{\mathbf{i}\theta}|d\theta}{|\,(r\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)^k\, f(re^{\mathbf{i}\theta})|} \\ &\leq \int_{-\pi}^{\pi} \frac{|f(\rho + \mathbf{i}h_2)\,\mathbf{i}r\,e^{\mathbf{i}\theta}|d\theta}{|\,(r\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)^k\, f(re^{\mathbf{i}\theta})|} = \int_{-\pi}^{\pi} \frac{|f(\rho + \mathbf{i}h_2)|\,rd\theta}{|\,(r\,e^{\mathbf{i}\theta} + h_1 + \mathbf{i}h_2)\,|^k|f(re^{\mathbf{i}\theta})|} \\ &< \int_{-\pi}^{\pi} \frac{|f(\rho + \mathbf{i}h_2)|\,rd\theta}{h^k\epsilon} = \frac{2\pi|f(\rho + \mathbf{i}h_2)|\,r}{h^k\epsilon}. \end{aligned}$$

Obviously,

$$\lim_{r\to 0} \frac{2\pi|f(\rho + \mathbf{i}h_2)|\,r}{h^k\epsilon} = 0.$$

Therefore, $\lim_{r\to 0} J_r = 0$. Q.E.D.

Hence, it follows from Propositions 2 and 3 and from (3.5) that

$$J(\rho) = J_{I\rho} + J_{1\rho} + J_{2\rho} \tag{3.8}$$

where $J_{I\rho}$, $J_{1\rho}$ and $J_{2\rho}$ denote the contour integrals over the paths $\Gamma_{I\rho}$, $\Gamma_{1\rho}$ and $\Gamma_{2\rho}$, respectively.

Now consider $J_{I\rho}$. As $r \to 0$ and $R \to \infty$, we obtain

$$J_{I\rho} = \int_{\rho-\mathbf{i}\infty}^{\rho+\mathbf{i}\infty} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}h_2)} ds. \tag{3.9}$$

Because $J_{I\rho}$ in (3.9) is an improper integral, after the change of variable $s = \rho + \mathbf{i}\,\omega = \rho + \mathbf{i}\tan\frac{x}{2}$, it follows that

$$J_{I\rho} = \int_{-\pi}^{\pi} \frac{F(\rho+\mathbf{i}\tan\frac{x}{2})}{\left[h_1+\rho+\mathbf{i}\left(h_2+\tan\frac{x}{2}\right)\right]^k F(\rho+\mathbf{i}\,h_2)} \frac{\mathbf{i}\,dx}{(2\cos^2\frac{x}{2})}. \tag{3.10}$$

By defining $G_\rho(x)$ as the integrand in (3.10), we arrive at the following initial value problem

$$\begin{cases} \dfrac{dy_{r\rho}(x)}{dx} = \mathrm{Re}\left\{G_\rho(x)\right\}, \quad y_{r\rho}(-\pi) = 0, \\ \dfrac{dy_{i\rho}(x)}{dx} = \mathrm{Im}\left\{G_\rho(x)\right\}, \quad y_{i\rho}(-\pi) = 0, \end{cases} \tag{3.11}$$

with $J_{I\rho} = y_{r\rho}(\pi) + \mathbf{i}\,y_{i\rho}(\pi)$.

The remaining task is to compute $J_{1\rho}$ and $J_{2\rho}$. As $r \to 0$, we obtain

$$J_{1\rho} = \int_{\rho}^{0} \frac{F(-x\,e^{-\mathbf{i}\pi})\,dx}{(-x\,e^{-\mathbf{i}\pi}+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}\,h_2)}, \tag{3.12}$$

$$J_{2\rho} = \int_{\rho}^{0} \frac{-F(-x\,e^{\mathbf{i}\pi})\,dx}{(-x\,e^{\mathbf{i}\pi}+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}\,h_2)}. \tag{3.13}$$

Following the approach used in computing $J_{I\rho}$, we easily deduce that $J_{1\rho}$ and $J_{2\rho}$ can be obtained by solving the following initial value problems.

$$\begin{cases} \dfrac{dy_{rj}(x)}{dx} = \mathrm{Re}\left\{G_j(x)\right\}, \quad y_{rj}(\rho) = 0, \\ \dfrac{dy_{ij}(x)}{dx} = \mathrm{Im}\left\{G_j(x)\right\}, \quad y_{ij}(\rho) = 0, \end{cases} \tag{3.14}$$

and

$$J_{j\rho} = y_{rj}(0) + \mathbf{i}\,y_{ij}(0), \quad \text{for } j = 1, 2, \tag{3.15}$$

where $G_j(x)$ denote the integrands in (3.12) and (3.13).

Case II: $\rho > 0$. For $\rho > 0$, we define the contour as follows.

$$\begin{aligned} \Gamma_\rho &\triangleq \Gamma_{I\rho} \cup \Gamma_{R\rho}, \\ \Gamma_{I\rho} &= \left\{s = \rho + \mathbf{i}\,\omega : \omega \in [-R, R]\right\}, \\ \Gamma_{R\rho} &= \left\{s = \rho + R\,e^{\mathbf{i}\theta} : -\pi/2 \le \theta \le \pi/2\right\}, \end{aligned} \tag{3.16}$$

where $R \to \infty$. See Fig. 3.2. Define

$$J(\rho) \triangleq \int_{\Gamma_\rho} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}h_2)} ds. \tag{3.17}$$

Then, we can easily deduce that f is $H(\rho)$-stable if and only if $J(\rho)=0$ for any $(-h_1-\mathbf{i}h_2)$ outside Γ.

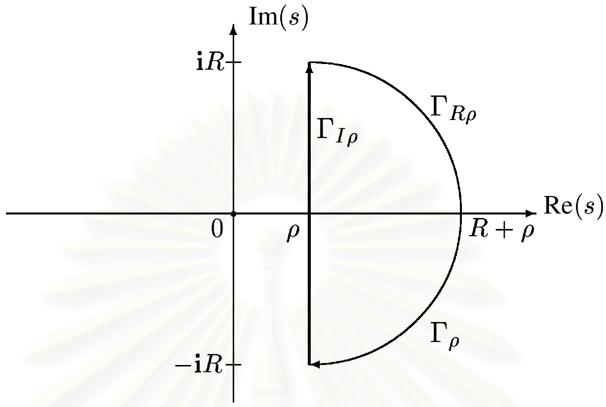

Figure 3.2: The contour Γ_ρ for $\rho > 0$.

In the following, how to compute $J(\rho)$ is described. It is easy to see from Proposition 2 that as $R \to \infty$, the integral along the contour $\Gamma_{R\rho}$ vanishes. Hence,

$$J(\rho) = \int_{\Gamma_{I_\rho}} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}h_2)} ds = \int_{\rho-\mathbf{i}\infty}^{\rho+\mathbf{i}\infty} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\, F(\rho+\mathbf{i}h_2)} ds. \tag{3.18}$$

Again, by the change of variable $s = \rho + \mathbf{i}\,\omega = \rho + \mathbf{i}\tan\frac{x}{2}$, we obtain

$$J(\rho) = \int_{-\pi}^{\pi} \frac{F(\rho+\mathbf{i}\tan\frac{x}{2})}{\left[h_1+\rho+\mathbf{i}\left(h_2+\tan\frac{x}{2}\right)\right]^k F(\rho+\mathbf{i}\,h_2)} \frac{\mathbf{i}\,dx}{(2\,\cos^2\frac{x}{2})}. \tag{3.19}$$

By defining $G_\rho(x)$ as the integrand in (3.19), we obtain the following initial value problem.

$$\begin{cases} \dfrac{dy_{r\rho}(x)}{dx} = \mathrm{Re}\,\{G_\rho(x)\}, \quad y_{r\rho}(-\pi)=0, \\ \dfrac{dy_{i\rho}(x)}{dx} = \mathrm{Im}\,\{G_\rho(x)\}, \quad y_{i\rho}(-\pi)=0, \end{cases} \tag{3.20}$$

and $J(\rho) = y_{r\rho}(\pi) + \mathbf{i}\,y_{i\rho}(\pi)$.

It is found that sometimes initial value problems (3.11) and (3.20) can be stiff, especially when the path $\Gamma_{I\rho}$ gets very close to the right-most pole of F, whose real part is the abscissa of stability. This frequently happens when the stability test is used repeatedly in computing the abscissa of stability (see Section 3.2). Therefore, it is advisable to solve the initial value problems (3.11) and (3.20) using stiff ODE solvers, for example, Radau5 code [39].

3.2 Stabilization of RFDDSs

It has been demonstrated [25, 26] that the abscissa of stability is very useful in stabilizing RSs as well as RDDSs by numerical methods. This idea is still useful in the numerical stabilization of RFDDSs.

3.2.1 Numerical Stabilization Using the Abscissa of Stability

Let α denote the abscissa of stability of the characteristic function f of a RFDDS, which is defined by

$$\alpha \triangleq \max\{\mathrm{Re}(s) : f(s) = 0\}. \tag{3.21}$$

It is readily deduced [27] that the system is BIBO stable if and only if

$$\alpha < 0. \tag{3.22}$$

Evidently, inequality (3.22) provides a practical way of stabilizing the system by numerical methods once the abscissa of stability can be computed economically. In practice, we use a practical sufficient criterion for the stability of the system,

$$\alpha(p) < -\varepsilon \quad (\varepsilon > 0), \tag{3.23}$$

where p is the design parameter.

3.2.2 Computation of the Abscissa of Stability for RFDDSs

In this section, the method for computing the abscissa of stability of f in (3.1) is described. It is based on Zakian's iteration [25], which makes repeated use of a stability test. Here, the Routh test of the shifted polynomial in [25] is replaced by the $H(\rho)$-stability test of f developed in Section 3.1.

The essentials of the iteration are as follows. First, determine an interval (a_0, b_0) that contains α; i.e., f is $H(b_0)$-stable but $H(a_0)$-unstable. Then, let $x_1 = (a_0 + b_0)/2$ and determine whether f is $H(x_1)$-stable. If so, then $\alpha \in (a_0, x_1)$; otherwise, $\alpha \in (x_1, b_0)$. The procedure is repeated until α is located within a sufficiently small interval.

More explicitly, the iteration is expressed as follows. Let $n = 0, 1, 2, \ldots$ and let $\{x_n\}$ be a sequence of real numbers generated by

$$\begin{aligned} x_{n+1} &= x_n + h_{n+1}, \\ h_{n+1} &= \begin{cases} |h_n|/2 & \text{if } f \text{ is } H(x_n)\text{-unstable} \\ -|h_n|/2 & \text{if } f \text{ is } H(x_n)\text{-stable} \end{cases} \end{aligned} \tag{3.24}$$

where $h_0 = b_0 - a_0$ and $x_0 = a_0$.

It can be shown [25] that the infinite sequence $\{x_n\}$ in (3.24) converges to the abscissa of stability, α, with the property

$$|\alpha - x_{n+1}| \leq \frac{|h_0|}{2^{n+1}}. \tag{3.25}$$

Iteration (3.24) has the following noteworthy feature [25]. It always converges with the rate of convergence given by (3.25). The convergence is disrupted, but not catastrophically, if failure to obtain the correct result of the $H(x_n)$-stability test of f occurs in the iteration. The error $|\alpha - x_n|$ achieves its minimal value just prior to the first failure to obtain the correct result in the $H(x_n)$-stability test, and thereafter the error always remains less than twice the minimal value, irrespective of further miscalculation.

It now remains to explain how to determine a_0 and b_0. Given $a < 0$ and $b > 0$, set $a_0 = a$ and $b_0 = b$. There are only three possibilities.

1. If f is $H(a_0)$-unstable and is $H(b_0)$-stable, then $a_0 = a$ and $b_0 = b$.
2. If f is $H(a_0)$-stable, we keep setting $a_0 = 2a_0$ until f is $H(a_0)$-unstable and then set $b_0 = a_0/2$.
3. If f is $H(b_0)$-unstable, we keep setting $b_0 = 2b_0$ until f is $H(b_0)$-stable and then set $a_0 = b_0/2$.

Following is the pseudo-code of the algorithm.

input: permissible error $\epsilon > 0$, $a < 0$, $b > 0$,
output: the abscissa of stability α

begin
  $n = 0$; $a_n = a$; $b_n = b$;
  % First, find an initial interval containing α
  if f is $H(a_n)$-stable,
    while f is $H(a_n)$-stable,
      $a_n = 2 * a_n$;
    end
    $b_n = a_n/2$;
  elseif f is $H(b_n)$-unstable,
    while f is $H(b_n)$-unstable,
      $b_n = 2 * b_n$;
    end
    $a_n = b_n/2$;
  end
  $c = (a_n + b_n)/2$;
  % Now, start doing bisection
  while $|a_n - b_n| > \epsilon$,
    $n = n + 1$;
    if f is $H(c)$-stable,
      $a_n = a_{n-1}$; $b_n = c$;
    else
      $a_n = c$; $b_n = b_{n-1}$;
    end
    $c = (a_n + b_n)/2$;
  end
  $\alpha = c$;
end

3.3 Numerical Examples

In this section, numerical examples are given to show the effectiveness of the stability test and to demonstrate how to stabilize a RFDDS using the method for computing the abscissa of stability. In solving the initial value problems, we use the FORTRAN code Radau5 [39].

3.3.1 Example 1

Consider the following characteristic function [30, 40].

$$f(s) = (\sqrt{s})^3 - 1.5(\sqrt{s})^2 - 1.5(\sqrt{s})^2 e^{-\tau s} + 4\sqrt{s} + 8. \tag{3.26}$$

It is shown [40] that for $\tau \in (0.99830, 1.57079)$, the system is BIBO stable. Therefore, f is $H(0)$-unstable for $\tau = 0.99$ and $H(0)$-stable for $\tau = 1$.

Table 3.1: Stability test of f in (3.26) for $\tau = 0.99,\ 1$ with parameters k, h_1 and h_2, where N_c is the number of function calls.

k	h_1	h_2	J	N_c
$\tau\ =$	0.99			
1	3.9204572	−2.1184570	$0.1425918 + \mathbf{i}\,0.7859342$	21374
2	2.2235993	1.0918359	$-0.0405419 + \mathbf{i}\,0.1515234$	7472
3	5.4785590	−8.8245291	$-0.0531885 + \mathbf{i}\,0.0381619$	4107
k	h_1	h_2	$J \times 10^8$	N_c
$\tau\ =$	1.00			
1	3.0051836	−2.4528789	$-2.1067 - \mathbf{i}\,3.6143$	23891
2	0.4782589	1.5748952	$-0.3593 - \mathbf{i}\,0.2598$	9039
3	1.2774739	4.7541852	$-0.4067 + \mathbf{i}\,3.0108$	6186

We perform the $H(0)$-stability test of f in (3.26) for $\tau = 0.99$ and $\tau = 1$ using different values of k, h_1 and h_2; the results are given in Table 3.1. Furthermore, we verify the stability result by computing the abscissa of stability of f for the various values of τ with the permissible error $\epsilon = 10^{-8}$. The computed abscissae of stability (α_{comp}), which are close to zero, are shown in Table 3.2. This demonstrates the effectiveness of the proposed method.

Table 3.2: The computed α for Example 1.

τ	0.99830	0.99840	1.57078	1.57080
α_{comp}	0.74×10^{-5}	-0.14×10^{-4}	-0.17×10^{-5}	0.38×10^{-6}

3.3.2 Example 2

The characteristic function of the feedback control system of a heat-conduction process, denoted by f_1, is given by

$$f_1(s) = \sigma A\sqrt{\lambda s}(1 - e^{-2L\sqrt{\lambda s}}) + 2p\, e^{-L\sqrt{\lambda s}}, \tag{3.27}$$

where p is the adjustable gain of the proportional controller used and

$$A = 2, \quad L = 1, \quad \sigma = 0.5, \quad \lambda = 1.$$

For a description of the heat-conduction process, see Section 4.1.3 in Chapter 4.

By using the graphical stability test proposed by Callier and Desoer [41], Lertsatienchai [42] shows that the system is stable for $p < 17.7985$ and unstable for $p > 17.7985$. Hence, $p = 17.7985$ is the critical gain of the closed loop system.

We perform the $H(0)$-stability test of $f_1(s)$ in (3.27) for various values of the gain p using the method in Section 3.1. The values of $J(0)$ are shown in Table 3.3. The results show that for all stable cases $J(0) \approx 0$, and that the critical gain must be in the interval $(17.798, 17.799)$. This agrees well with Lertsatienchai's result [42].

Table 3.3: Stability test results with $k = 2$.

p	h_1	h_2	$J(0)$
15	6.0818	−0.9552	$(-0.04 - \mathbf{i}\,0.54) \times 10^{-8}$
	4.2558	−2.3985	$(0.04 + \mathbf{i}\,3.41) \times 10^{-9}$
17.798	10.9483	7.3350	$(-0.16 - \mathbf{i}\,0.27) \times 10^{-8}$
	5.8581	8.0661	$(0.20 - \mathbf{i}\,0.37) \times 10^{-8}$
17.799	1.2031	2.7357	$1.7967 + \mathbf{i}\,0.5787$
	10.4406	−9.1844	$0.1107 + \mathbf{i}\,0.0043$
18.5	2.7785	3.8160	$1.3455 - \mathbf{i}\,0.3424$
	3.6394	−0.8452	$-1.3771 + \mathbf{i}\,1.3702$

3.3.3 Example 3

Consider the functions f_2 and f_3 given by

$$\begin{aligned} f_2(s) &\triangleq (s+1)^n f_1(s), \\ f_3(s) &\triangleq (s^2 - 2s + 5)^n f_1(s), \end{aligned} \tag{3.28}$$

where f_1 is given by (3.27) with $p = 10$ and $n \geq 1$ is an integer. It readily follows from (3.28) that

$$\begin{aligned} f_2(s) &= \sqrt{s}(s+1)^n + 20(s+1)^n e^{-\sqrt{s}} - \sqrt{s}(s+1)^n e^{-2\sqrt{s}}, \\ f_3(s) &= \sqrt{s}(s^2 - 2s + 5)^n + 20(s^2 - 2s + 5)^n e^{-\sqrt{s}} - \sqrt{s}(s^2 - 2s + 5)^n e^{-2\sqrt{s}}. \end{aligned} \tag{3.29}$$

It should be noted that the abscissa of stability of f_1 with $p = 10$ is equal to -1.61. Hence, from the definition of f_2 and f_3 in (3.28), it is clear that the abscissae of stability of f_2 and f_3 are respectively equal to -1 and 1 for all $n \geq 1$. Notice that when $n > 1$, the right-most roots are repeated and thus we expect severe effect of roundoff errors during computation. Besides, the accuracy in computing the abscissa of stability of f_2 also depends on the line search algorithm implemented on its branch cut. In this thesis, quadratic interpolation is used for the line search algorithm.

Table 3.4: The abscissae of stability of test function f_2.

n	permissible error	actual error	α_{comp}
1	1×10^{-4}	0.42×10^{-4}	-0.9999581333
	1×10^{-5}	0.25×10^{-5}	-0.9999975049
	1×10^{-6}	0.34×10^{-6}	-1.0000003377
2	1×10^{-4}	0.21×10^{-3}	-0.9997943714
	1×10^{-5}	0.13×10^{-3}	-0.9998723965
	1×10^{-6}	0.35×10^{-3}	-0.9996510458
3	1×10^{-4}	0.11×10^{-1}	-0.9891804825
	1×10^{-5}	0.16×10^{-1}	-0.9844916687
	1×10^{-6}	0.15×10^{-1}	-0.9847371004
4	1×10^{-4}	0.67×10^{-1}	-0.9328956603
	1×10^{-5}	0.42×10^{-1}	-0.9584787572
	1×10^{-6}	0.69×10^{-1}	-0.9308129399

Table 3.5: The abscissae of stability of test function f_3.

n	permissible error	actual error	α_{comp}
1	1×10^{-4}	0.24×10^{-4}	1.0000242478
	1×10^{-5}	0.29×10^{-5}	1.0000028711
	1×10^{-6}	0.31×10^{-6}	1.0000003056
2	1×10^{-4}	0.16×10^{-4}	1.0000159501
	1×10^{-5}	0.45×10^{-4}	1.0000449319
	1×10^{-6}	0.37×10^{-4}	1.0000372677
3	1×10^{-4}	0.12×10^{-2}	1.0011814091
	1×10^{-5}	0.16×10^{-2}	1.0015600809
	1×10^{-6}	0.14×10^{-2}	1.0014076181
4	1×10^{-4}	0.30×10^{-2}	1.0029858523
	1×10^{-5}	0.62×10^{-2}	1.0061657232
	1×10^{-6}	0.47×10^{-2}	1.0046522891

The abscissae of stability of f_2 and f_3 expressed in (3.29) with $n = 1, 2, 3, 4$ are computed and the computational results are shown in Tables 3.4 and 3.5, where

$$\text{actual error} = \begin{cases} |\alpha_{\text{comp}} + 1| & \text{for } f_2 \\ |\alpha_{\text{comp}} - 1| & \text{for } f_3 \end{cases}.$$

From Tables 3.4 and 3.5, we can see that the actual errors are greater than the permissible errors in most of the cases where $n > 1$. This indicates that, for those difficult cases, iteration (3.24) fails to converge to the correct value of α owing to the wrong determination of $H(\rho)$-stability effected by the roundoff errors. However, by noting that a closed-loop system seldom has repeated roots, one can see that such cases hardly happen in practice. As shown in this example, even if they happen, iteration (3.24) is still able to produce results with reasonable accuracy of 2–3 decimal digits. Note that in practice, this level of accuracy is adequate for many applications.

3.3.4 Example 4

Consider a unity feedback control system shown in Fig. 3.3, where the plant transfer function is given by

$$G(s) = \frac{e^{-\sqrt{s}}}{s(s-1)}. \tag{3.30}$$

Obviously, the plant is unstable.

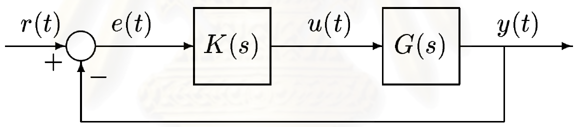

Figure 3.3: A unity feedback control system.

The problem is to stabilize the control system using a PD controller

$$K(s) = p_1 + p_2 s \quad \text{where } p_1, p_2 > 0.$$

The characteristic function of the closed loop system is

$$f(s) = s(s-1) + (p_1 + p_2 s)e^{-\sqrt{s}}. \tag{3.31}$$

Table 3.6: Stabilization of an unstable RFDDS.

p_0	$(3,\ 2)$	$(1,\ 4)$	$(1.5,\ 20)$
α_{comp}	0.5657	0.0709	0.3602
p	$(0.7162, 4.3345)$	$(0.6850, 4.3220)$	$(0.8760, 7.0325)$
α_{comp}	-0.0119	-0.0172	-0.0612

In conjunction with the method of inequalities, a stabilizing controller is obtained by solving the inequality

$$\alpha(p) \leq \varepsilon, \quad \varepsilon = -0.001. \tag{3.32}$$

For the details on this, see [26]. By using the MBP, the solutions of inequality (3.32) are located. The stabilizing controllers are obtained from three different starting points. The results are shown in Table 3.6.

This example demonstrates that the stabilization problem for RFDDSs can be solved easily by iterative numerical methods once a practical method of computing the abscissa of stability is available.

CHAPTER IV

CONTROLLER DESIGN

In this chaper, fractional controllers are designed by the MoI for SISO and MIMO plants. Section 4.1.2 presents the design of a fractional PI controller for a time-delay plant. In Section 4.1.3, a fractional phase-lead controller is obtained for a heat-conduction process. Two fractional PI controllers, a decentralized fractional PI and a full fractional PI controller, are achieved for a binary distillation column in Section 4.2. Throughout this chapter, the unity feedback control configuration is used. See Fig. 4.1.

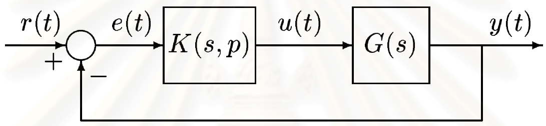

Figure 4.1: A unity feedback control system

By using step-response criteria, the control design objectives are threefold: to achieve good step responses (i.e., small maximum overshoot, settling time, and rise time), to avoid saturation of control signals, and, for MIMO systems, to reduce loop interaction.

In this work, the design inequalities (2.1) are solved by using a numerical search algorithm called *the moving boundaries process* (MBP). See [8] for the details of the algorithm. For the efficiency in the design process, tasks such as the $H(\rho)$-stability test, the computation of $\alpha(p)$ and the step responses are implemented in FORTRAN programming language. The FORTRAN code Radau5 [39] is employed for solving the initial value problems in the $H(\rho)$-stability test. FORTRAN compiler Salford FTN77PE is used. All the computations are carried out by a computer with Intel 1.66 GHz microprocessor, RAM 512 MB, using OS Window XP SP2.

4.1 Controller Design for SISO Systems

4.1.1 Design Formulation & Specifications

Consider an SISO control system shown in Fig. 4.1, where $G(s)$ is the plant transfer function, $K(s,p)$ is the controller transfer function with design parameter p. Suppose that the reference r is a unit step function.

The design parameter p is to be determined so that it satisfies the following design specifications.

$$\phi_i(p) \le C_i, \quad i = 1, 2, 3, 4, \tag{4.1}$$

where

$$\left.\begin{aligned} \phi_1 &\triangleq \sup\{[y(t) - y_\infty]/y_\infty : t \geq 0\} \\ \phi_2 &\triangleq \min\{t : y(t) = 0.9\, y_\infty\} \\ \phi_3 &\triangleq \min\{\tau : |y(t) - y_\infty| \leq 0.02\, y_\infty \;\forall\, t \geq \tau\} \\ \phi_4 &\triangleq \sup\{|u(t)| : t \geq 0\} \end{aligned}\right\}, \tag{4.2}$$

and y_∞ denotes the steady state value of the step response y, that is,

$$y_\infty \triangleq \lim_{t\to\infty} y(t).$$

In many practical applications, the actuator has a saturation characteristic. To avoid the saturation during the operation of the system, the design requirement $\phi_4 \leq C_4$ in (4.1) needs to be taken into consideration. This fundamental requirement makes the design problem become difficult to solve even for an SISO system. However, it should be noted that the MoI can solve such a design problem effectively in a systematic manner (see, for example, [12, 13]).

In addition to the design specifications in (4.1), if there are constraints on the design parameter p (or permissible ranges of p), they can be incorporated into the design inequalities (4.1) very easily. This still results in the inequalities of the form (2.1).

4.1.2 Controller Design for a Time-Delay Plant

In this Section, we consider a typical second-order plant with time-delay whose transfer function is given by

$$G(s) = \frac{2e^{-2s}}{(s+1)(s+2)}. \tag{4.3}$$

Assume that a fractional PI controller is used and the controller transfer function $K(s,p)$ is

$$K(s,p) = \frac{p_1 + p_2 s^{p_3}}{s^{p_3}}, \tag{4.4}$$

where $p = [p_1,\, p_2,\, p_3]^T$ is the vector of design parameters that are constrained to be positive.

It is easy to verify that the transfer function of the closed-loop system takes the form in (1.1) and is given by

$$H(s) = \frac{2(p_1 + p_2 s^{p_3})e^{-2s}}{s^{p_3}(s+1)(s+2) + 2(p_1 + p_2 s^{p_3})e^{-2s}}. \tag{4.5}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\; C_2 = 5.7,\; C_3 = 6.5,\; C_4 = 1.1. \tag{4.6}$$

First, a stability point is obtained by solving the inequality

$$\alpha(p) \leq -0.1.$$

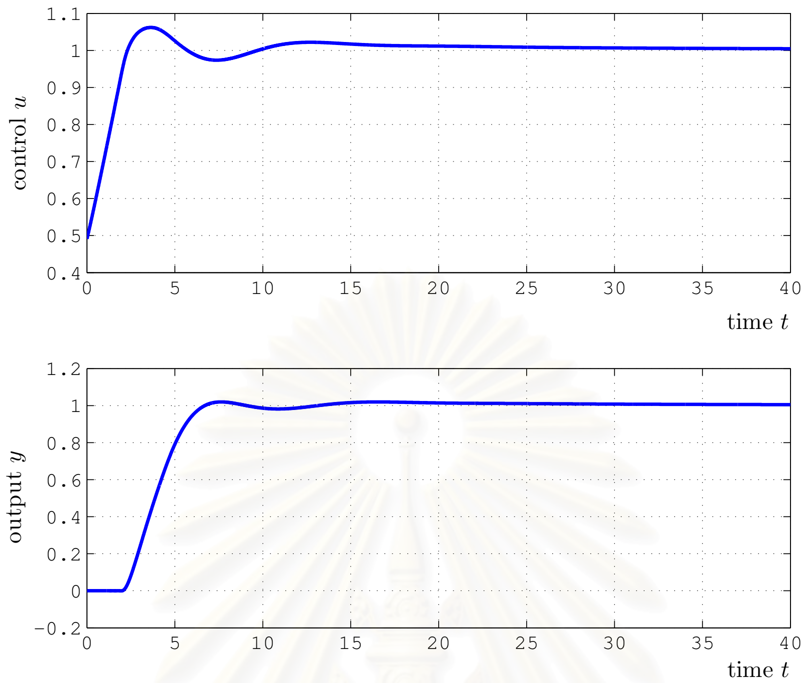

Figure 4.2: Step responses of time-delay system

Once the stability point is obtained, the $H(-0.1)$-stability test is used repeatedly so as to maintain the closed-loop stability during the search. After a number of iterations, the MBP algorithm locates a design solution,

$$p = [0.225,\ 0.491,\ 1.043]^T, \tag{4.7}$$

where $\alpha(p) = -0.2714$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 5.62,\ \phi_3 = 6.37,\ \phi_4 = 1.06.$$

The control and output signals of the system with the controller parameter p in (4.7) are shown in Fig. 4.2.

4.1.3 Controller Design for a Heat-Conduction Process

Consider the heat-conduction process in a metallic rod; see Fig. 4.3. The rod has length L, cross-sectional area A and is made of a material with density ρ, heat capacity C and thermal conductivity σ.

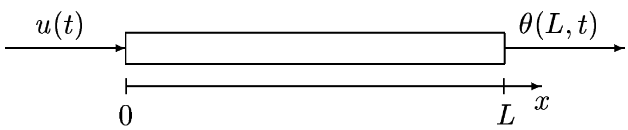

Figure 4.3: The heating metallic rod.

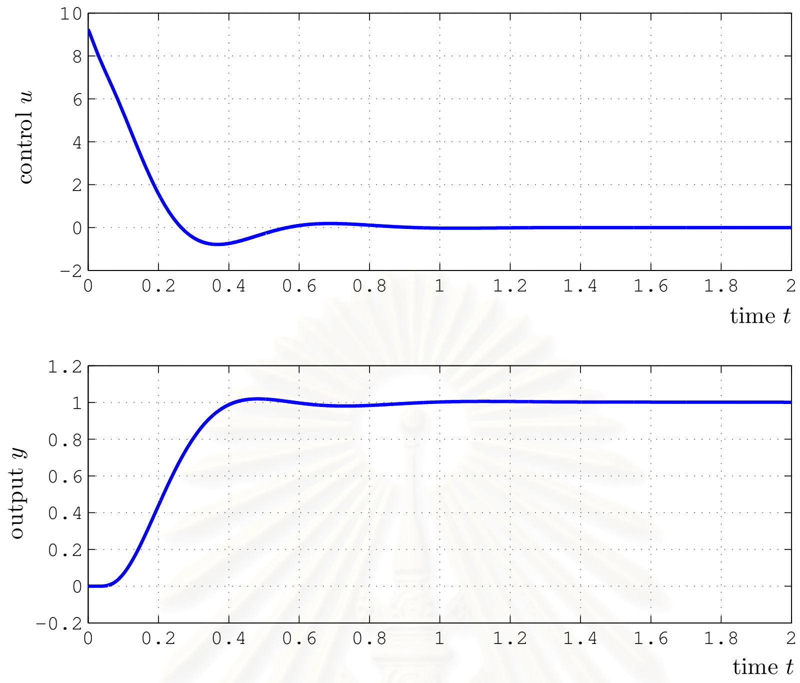

Figure 4.4: Step responses of heat-conduction system

The control signal $u(t)$ is the heat flow injected at $x = 0$. The output $y(t) = \theta(L,t)$ is the temperature measured at $x = L$. In using the feedback configuration in Fig. 4.1, the control objective is to keep the temperature $y(t)$ as close as possible to a reference signal.

Schwarz and Friedland [43] show that with appropriate boundary conditions, the transfer function of the process is

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{1}{\sigma A\sqrt{\lambda s}.\sinh(\sqrt{\lambda s}\,L)}, \tag{4.8}$$

where the parameter $\lambda \triangleq C\rho/\sigma$. Setting $A = 2$, $L = 1$, $\sigma = 0.5$, and $\lambda = 1$, we obtain

$$G(s) = \frac{1}{\sqrt{s}\,\sinh(\sqrt{s}\,)}. \tag{4.9}$$

In general, the nonrational transfer function in (4.9) occurs when the plant is governed by a heat-conduction or diffusion equation (see, for example, [36, 43]).

Assume that a fractional phase-lead controller is used and its transfer function is

$$K(s,p) = \frac{p_1(s^{p_4} + p_2)}{(s^{p_4} + p_3)}, \tag{4.10}$$

where $p_3 > p_2 > 0$ and $p_4 > 0$. It is worth noting [44] that the controller in (4.10) can be realized in practice.

It can be readily verified that the closed-loop transfer function takes the form in (1.1) and is given by

$$H(s) = \frac{2p_1(s^{p_4} + p_2)e^{-\sqrt{s}}}{\sqrt{s}(s^{p_4} + p_3)(1 - e^{-2\sqrt{s}}) + 2p_1(s^{p_4} + p_2)e^{-\sqrt{s}}}. \tag{4.11}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\ C_2 = 0.35,\ C_3 = 0.4,\ C_4 = 10.0. \tag{4.12}$$

In order to obtain a stability point, we solve the inequality

$$\alpha(p) \leq -0.1.$$

Then, the $H(-0.1)$-stability test is used repeatedly to prevent the search from stepping out of the stability region. The MBP algorithm locates a design solution,

$$p = [9.240,\ 7.513,\ 15.204,\ 1.101]^T, \tag{4.13}$$

where $\alpha(p) = -4.4383$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 0.34,\ \phi_3 = 0.39,\ \phi_4 = 9.24.$$

The control and output signals of the system with the controller parameter p in (4.13) are shown in Fig. 4.4.

4.2 Controller Design for a Binary Distillation Column

Distillation columns have been widely used in the chemical and petroleum industries. Known as complicated dynamical systems, distillation columns present interesting and challenging control design problems [45,46]. The controller design for such process systems has long been recognized as a complex problem. Attempts have been made by many researchers in designing controllers for distillation columns (see, e.g., [14, 16, 47]). However, it appears that so far no one has considered designing a fractional controller for a binary distillation column. Designing a fractional controller for systems with time-delays seems to be a difficult task.

The objective of this section is to design fractional PI controllers for a binary distillation column by using the MoI. Here, we are concerned primarily with the pilot scale binary distillation column investigated by Wood and Berry [46]. The design specifications are based on the conventional step-response criteria. It is important to note that a fractional PI controller can be realized in practice (see, e.g., [44]). Besides, the use of a fractional PI controller for the distillation column gives rise to a RFDDS.

4.2.1 Description of Distillation Column and Control System

The column is operated in order to separate a two-component mixture of water and methanol. Two variables, the reflux flow rate R and the steam flow rate S fed into the reboiler, are manipulated in the LV structure, which is commonly used in distillation engineering. The controlled variables x_D and x_B are the mole fractions of methanol of the top and bottom products, respectively. See Figure 4.5.

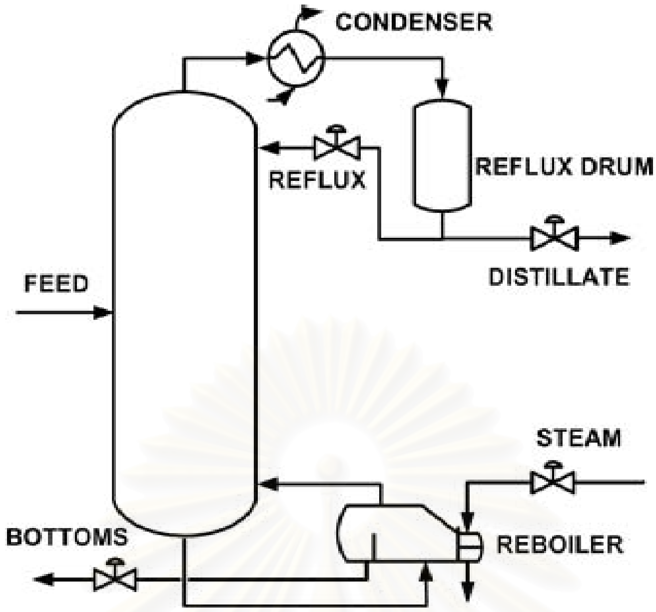

Figure 4.5: Diagram of a binary distillation column

The mathematical model of the column developed by Wood and Berry [46] is a linearization around the steady state operating condition and hence, a relation between incremental variables (see [46] for the details). The transfer matrix from the input $u = [\,\delta R \quad \delta S\,]^T$ to the output $y = [\,\delta x_D \quad \delta x_B\,]^T$ is given by

$$G(s) = \begin{bmatrix} \dfrac{12.8\,e^{-s}}{16.7\,s+1} & \dfrac{-18.9\,e^{-3\,s}}{21\,s+1} \\ \dfrac{6.6\,e^{-7\,s}}{10.9\,s+1} & \dfrac{-19.4\,e^{-3\,s}}{14.4\,s+1} \end{bmatrix}. \tag{4.14}$$

As expected from the LV structure, the reflux flow rate R is used to regulate the composition x_D of top product and the steam flow rate S to regulate the composition x_B of bottom product. However, the interaction between the two control loops is quite obvious, as can be seen from the mathematical model.

4.2.2 Problem Formulation

Consider the unity feedback configuration shown in Fig. 4.1, where $G(s)$ is the transfer matrix given in (4.14).

The performance measures of the system are defined based on step responses as follows. Let $r = [r_1\ r_2]^T$ denote the reference inputs, $u = [u_1\ u_2]^T$ the control signals and $y = [y_1\ y_2]^T$ the outputs. Suppose that r_1 is a unit step function and r_2 is zero. Let $y_{1\infty}$ denote the steady state value of y_1. Define ϕ_i ($i = 1, \ldots, 6$) as follows.

$$
\begin{aligned}
&\phi_1 \triangleq \sup\{[y_1(t) - y_{1\infty}]/y_{1\infty} :\ t \geq 0\}, && \phi_2 \triangleq \min\{t :\ y_1(t) = 0.9\, y_{1\infty}\}, \\
&\phi_3 \triangleq \min\{\tau : |y_1(t) - y_{1\infty}| \leq 0.02\, y_{1\infty}\ \forall\, t \geq \tau\}, && \phi_4 \triangleq \sup\{|y_2(t)| :\ t \geq 0\}, \\
&\phi_5 \triangleq \sup\{|u_1(t)| :\ t \geq 0\}, && \phi_6 \triangleq \sup\{|u_2(t)| :\ t \geq 0\}.
\end{aligned} \tag{4.15}
$$

Now suppose that r_2 is a unit step function and r_1 is zero. Let $y_{2\infty}$ denote the steady state value of y_2. Define ϕ_i ($i = 7, \ldots, 12$) as follows.

$$
\begin{aligned}
&\phi_7 \triangleq \sup\{[y_2(t) - y_{2\infty}]/y_{2\infty} :\ t \geq 0\}, && \phi_8 \triangleq \min\{t :\ y_2(t) = 0.9\, y_{2\infty}\}, \\
&\phi_9 \triangleq \min\{\tau : |y_2(t) - y_{2\infty}| \leq 0.02\, y_{2\infty}\ \forall\, t \geq \tau\}, && \phi_{10} \triangleq \sup\{|y_1(t)| :\ t \geq 0\}, \\
&\phi_{11} \triangleq \sup\{|u_1(t)| :\ t \geq 0\}, && \phi_{12} \triangleq \sup\{|u_2(t)| :\ t \geq 0\}.
\end{aligned} \tag{4.16}
$$

The controller $K(s, p)$ with design parameter p is to be specified so as to fulfill the performance criteria described by the inequalities

$$
\phi_i(p) \leq C_i, \quad i = 1, 2, \ldots, 12, \tag{4.17}
$$

where the performance measures ϕ_i ($i = 1, \ldots, 12$) are defined in (4.15) and (4.16).

To avoid saturation of the actuators during the operation of the system, the design requirements $\phi_i \leq C_i$ ($i = 5, 6, 11, 12$) in (4.17) need to be satisfied. These fundamental requirements make the design problem not easy to solve.

In addition to the design specifications given by inequalities (4.17), if there are constraints on the design parameter p (or permissible ranges of p), they can be incorporated into the design very easily. This results in a set of inequalities of the form (2.1).

4.2.3 Decentralized Fractional PI Controller

Following the design formulation described in Section 3.1, we first attempt using a simple fractional controller. To this end, a decentralized fractional PI controller is considered.

Let $K(s, p)$ be a decentralized controller whose transfer matrix is described by

$$
K(s, p) = \begin{bmatrix} \dfrac{p_1 + p_2\, s^{p_3}}{s^{p_3}} & 0 \\ 0 & \dfrac{p_4 + p_5\, s^{p_6}}{s^{p_6}} \end{bmatrix}, \tag{4.18}
$$

where the design parameter $p \in \mathbb{R}^6$ is to be determined.

Suppose that the bounds C_i in inequalities (2.1) are specified as follows.

$$
\begin{aligned}
&C_1 = 0.05, \quad C_2 = 11, \quad C_3 = 37, \quad C_4 = 0.45, \quad C_5 = 0.2, \quad C_6 = 0.2, \\
&C_7 = 0.05, \quad C_8 = 11, \quad C_9 = 39, \quad C_{10} = 0.45, \quad C_{11} = 0.2, \quad C_{12} = 0.2.
\end{aligned} \tag{4.19}
$$

In the design, we obtain a stability point by solving the inequality

$$
\alpha(p) \leq -0.1.
$$

Then, the $H(-0.1)$-stability test is used repeatedly to ensure that the search is carried out within the stability region. By working interactively with computer, the MBP algorithm finally locates a design solution,

$$p = [0.02,\, 0.15,\, 1.01,\, -0.011,\, -0.09,\, 1.01]^T, \tag{4.20}$$

for which the performance measures $\phi_i(p)$ are

$$\begin{array}{llllll}\phi_1 = 0.0, & \phi_2 = 10.85, & \phi_3 = 36.13, & \phi_4 = 0.43, & \phi_5 = 0.17, & \phi_6 = 0.07, \\ \phi_7 = 0.038, & \phi_8 = 10.58, & \phi_9 = 38.946, & \phi_{10} = 0.42, & \phi_{11} = 0.15, & \phi_{12} = 0.12.\end{array}$$

The step responses of the system with the controller parameter given in (4.20) are shown in Fig. 4.6 and the corresponding control signals are shown in Fig. 4.7. Obviously, the closed-loop system performs satisfactorily with the decentralized controller obtained by the design, i.e., the simultaneous control of both product compositions is possible.

Further improvement of the system performances can be made by using a more complex controller, e.g., a decentralized controller of higher order or a full fractional PI controller, which generally requires more computation effort in the design process. It should be noted, however, that a decentralized controller may be preferable because of the simplicity in its implementation.

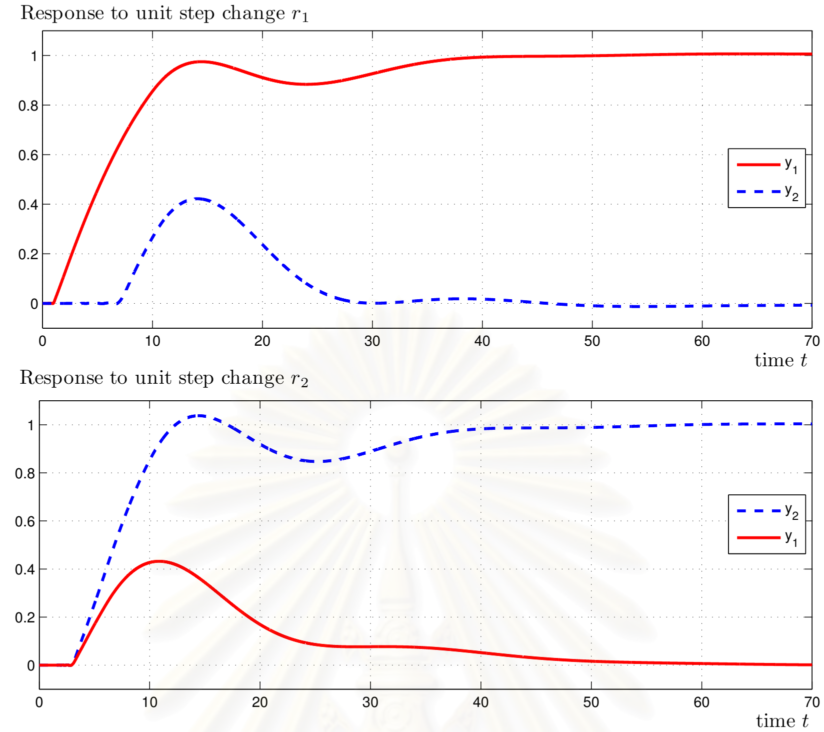

Figure 4.6: Output signals with decentralized fractional PI controller

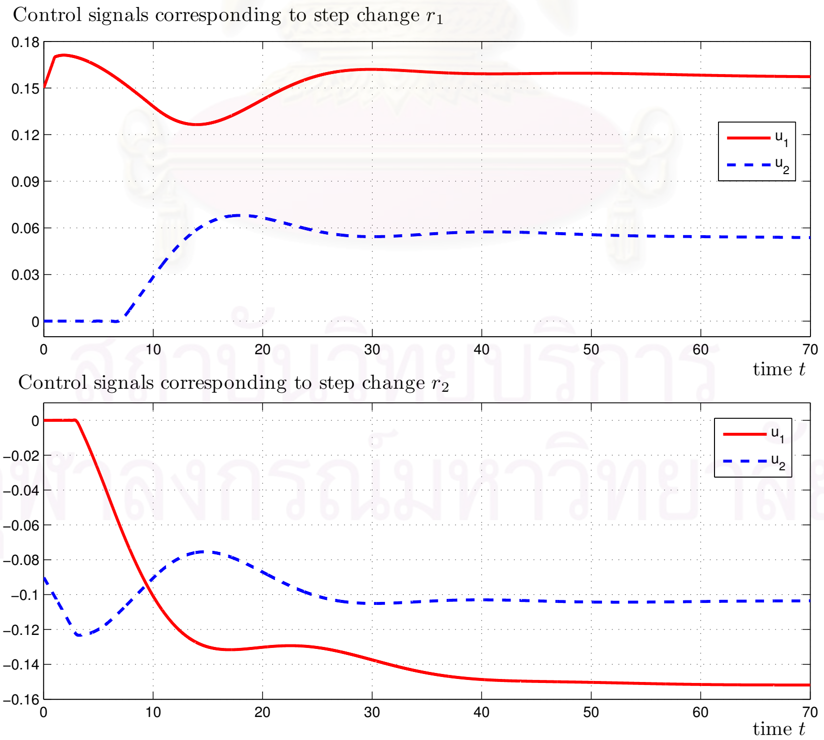

Figure 4.7: Control signals with decentralized fractional PI controller

4.2.4 Full Fractional PI Controller

Evidently, the interaction between control loops is particularly strong and cannot be attenuated by a decentralized controller. Next, a full fractional PI controller is used so as to improve the system performance.

Let $K(s,p)$ be described by the transfer matrix

$$K(s,p) = \begin{bmatrix} \dfrac{p_1 + p_2\, s^{p_3}}{s^{p_3}} & \dfrac{p_{10} + p_{11}\, s^{p_{12}}}{s^{p_{12}}} \\ \dfrac{p_7 + p_8\, s^{p_9}}{s^{p_9}} & \dfrac{p_4 + p_5\, s^{p_6}}{s^{p_6}} \end{bmatrix}, \tag{4.21}$$

where the design parameter $p \in \mathbb{R}^{12}$ is to be determined.

Let the bounds C_i in inequalities (2.1) be specified as follows.

$$\begin{array}{llllll} C_1 = 0.05, & C_2 = 11, & C_3 = 26, & C_4 = 0.25, & C_5 = 0.2, & C_6 = 0.2, \\ C_7 = 0.05, & C_8 = 11, & C_9 = 20, & C_{10} = 0.25, & C_{11} = 0.2, & C_{12} = 0.2. \end{array} \tag{4.22}$$

Note that the significant reduction of C_3, C_4, C_9 and C_{10} implies a requirement that is far more stringent than that of the case in Section 4.2.3.

However, with the design setting in (4.22), the MBP does not locate any solution. From the computation, it is suggested that the bounds C_5 and C_8 should be slightly relaxed so as to make the design numerically feasible. It should be noted that ϕ_5 and ϕ_{11} are the peaks of control signal u_1 that represent the capability of an actuator. Therefore, we set $C_5 = C_{11} = 0.25$ and $C_8 = 11.2$.

First, we solve the following inequality for a stability point.

$$\alpha(p) \leq -0.03.$$

Then, the $H(-0.03)$-stability test is employed to maintain the closed-loop stability during the search. After a number of iterations, a design solution is located by the MBP algorithm,

$$\begin{array}{lllllll} p = [& 0.04383, & 0.14716, & 1.00999, & -0.01345, & -0.10275, & 1.00210, \\ & 0.02296, & 0.00685, & 0.99819, & -0.01692, & -0.04603, & 1.01996 \quad]^T, \end{array} \tag{4.23}$$

with the performance measures $\phi_i(p)$ given below

$$\begin{array}{llllll} \phi_1 = 0.039, & \phi_2 = 10.71, & \phi_3 = 25.91, & \phi_4 = 0.24, & \phi_5 = 0.24, & \phi_6 = 0.10, \\ \phi_7 = 0.020, & \phi_8 = 11.15, & \phi_9 = 18.42, & \phi_{10} = 0.13, & \phi_{11} = 0.17, & \phi_{12} = 0.14. \end{array}$$

The step responses of the system with the controller parameter given in (4.23) are shown in Fig. 4.8 and the corresponding control signals are shown in Fig. 4.9. Obviously, the closed-loop system performs much better with the full fractional PI controller. The result is as expected because in using decentralized controller, no attempt is made to counteract the strong interaction between the two control loops.

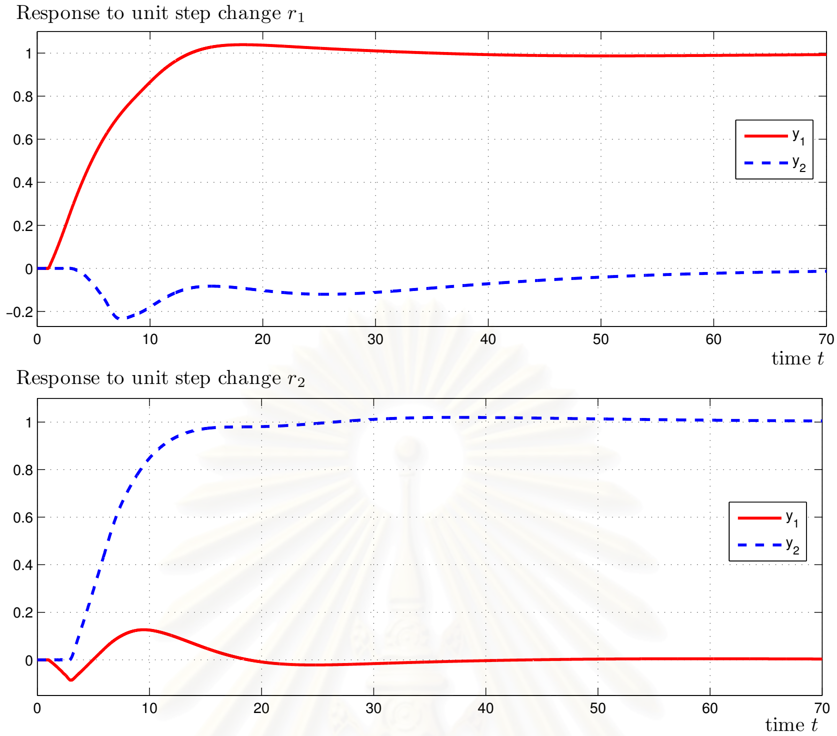

Figure 4.8: Output signals with full fractional PI controller

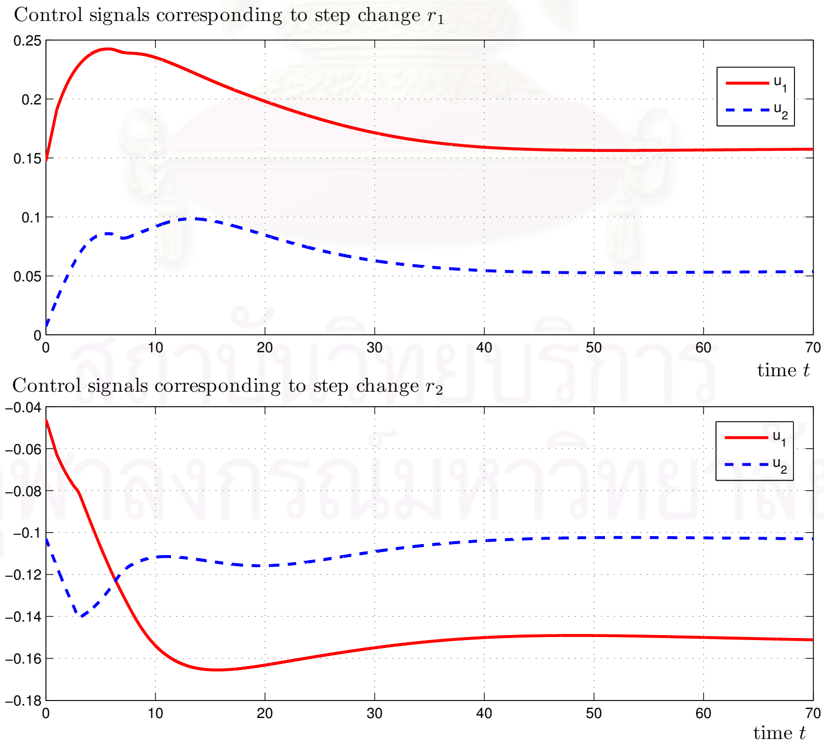

Figure 4.9: Control signals with full fractional PI controller

4.2.5 Comparison with Taiwo's results

In this section, the fractional PI controllers obtained in Sections 4.2.3 and 4.2.4 are compared with the conventional PI controllers given previously by Taiwo [11, 14]. It should be noted that Taiwo also employed the MoI in the design of PI controllers for Wood and Berry's distillation column.

Taiwo's Decentralized PI Controller: The decentralized PI controller obtained by Taiwo for Wood and Berry's distillation column is given in (4.24).

$$K_1(s) = \begin{bmatrix} \dfrac{0.0179 + 0.1644\, s}{s} & 0 \\ 0 & \dfrac{-0.0093 - 0.0581\, s}{s} \end{bmatrix}, \tag{4.24}$$

Accordingly, the step responses of the closed-loop system are shown in Fig. 4.10 and the corresponding control signals in Fig. 4.11.

One can see from Figs. 4.6–4.7, 4.10–4.11 that the step responses of the system obtained by the two decentralized controllers are comparable. Nevertheless, the decentralized fractional PI controller gives responses with smaller settling times. Besides, it reduces the rise time of y_2 when r_2 is a unit step function in a trade-off with the slight oscillation of y_2 and the larger control signal u_2 during the transient period.

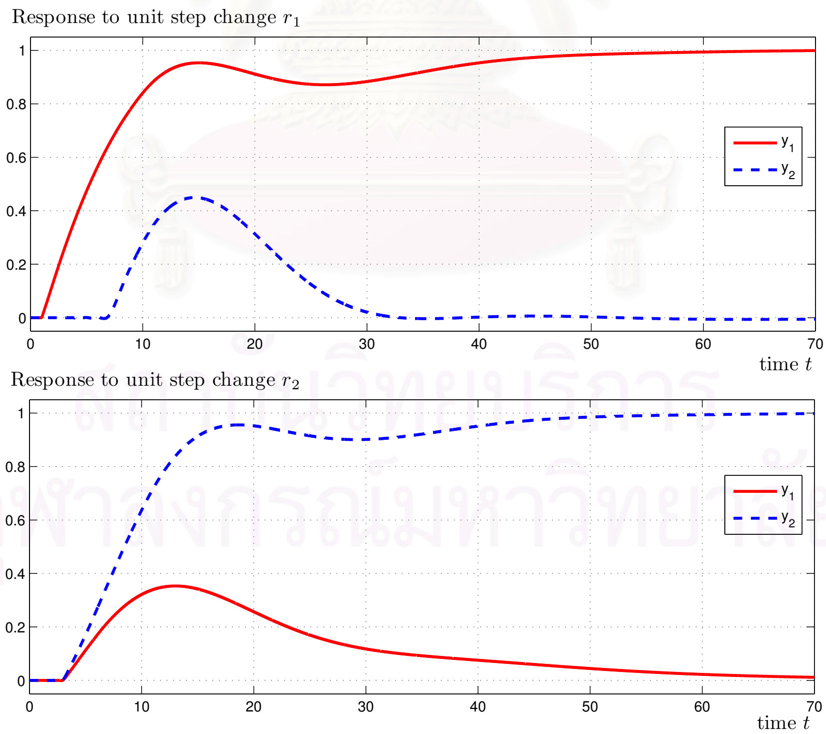

Figure 4.10: Output signals with Taiwo's decentralized PI controller

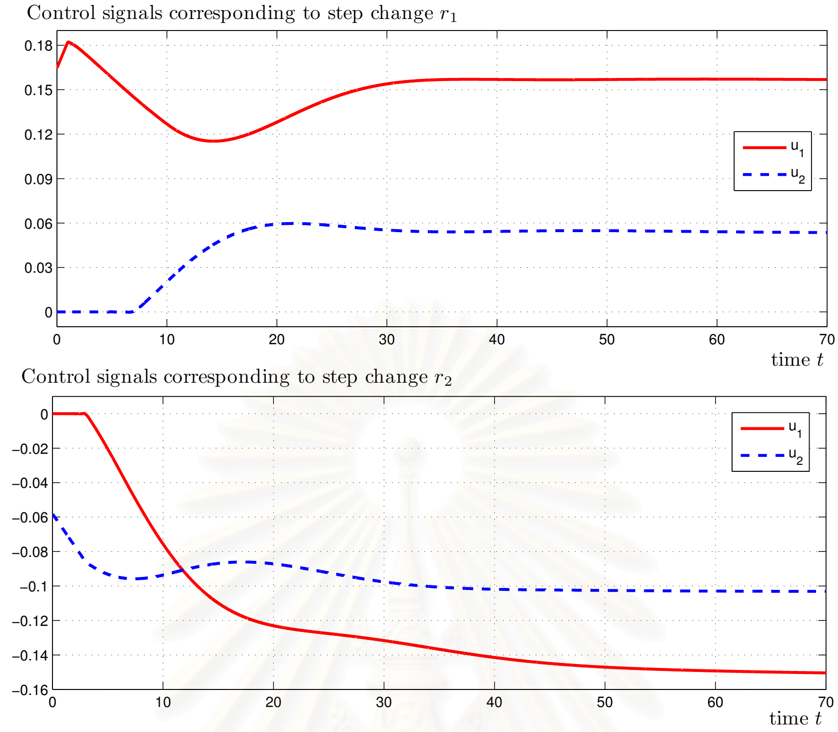

Figure 4.11: Control signals with Taiwo's decentralized PI controller

Taiwo's Full PI controller: The full PI controller obtained by Taiwo for Wood and Berry's distillation column is given in (4.25).

$$K_2(s) = \begin{bmatrix} \dfrac{0.0835 + 0.408\,s}{s} & \dfrac{-0.0176 - 0.05557\,s}{s} \\ \dfrac{0.0334 + 0.00326\,s}{s} & \dfrac{-0.00967 - 0.023\,s}{s} \end{bmatrix}, \tag{4.25}$$

The step responses of the closed-loop system are shown in Fig. 4.12 and the corresponding control signals in Fig. 4.13.

While the system performance cannot be significantly improved by the decentralized fractional PI controller, the full fractional PI controller does give us much better performances than the full PI controller does, as can be seen from Figs. 4.8–4.9, 4.12–4.13. Let $x_i(r_j)$ $(i,j = 1,2,\ x \in \{u, y\})$ denote the signal x_i of the system obtained when r_j is a unit step function. Although the full PI controller gives a very good performance with respect to the output signal $y_1(r_1)$, the output signal $y_2(r_2)$ is much more sluggish and the control signal $u_1(r_1)$ is significantly larger than those obtained by the full fractional PI controller. It has been shown that the fractional PI controllers give better performances than the conventional PI controllers do. This is due to the fact that with a fractional PI controller we have more parameters to adjust than those with a conventional PI controller of the same structure.

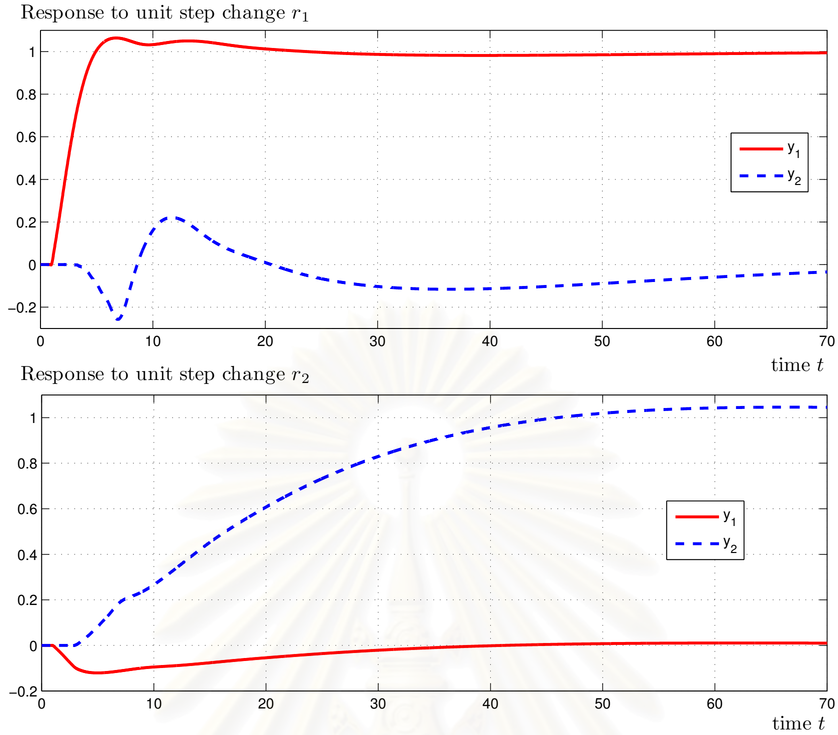

Figure 4.12: Output signals with Taiwo's full PI controller

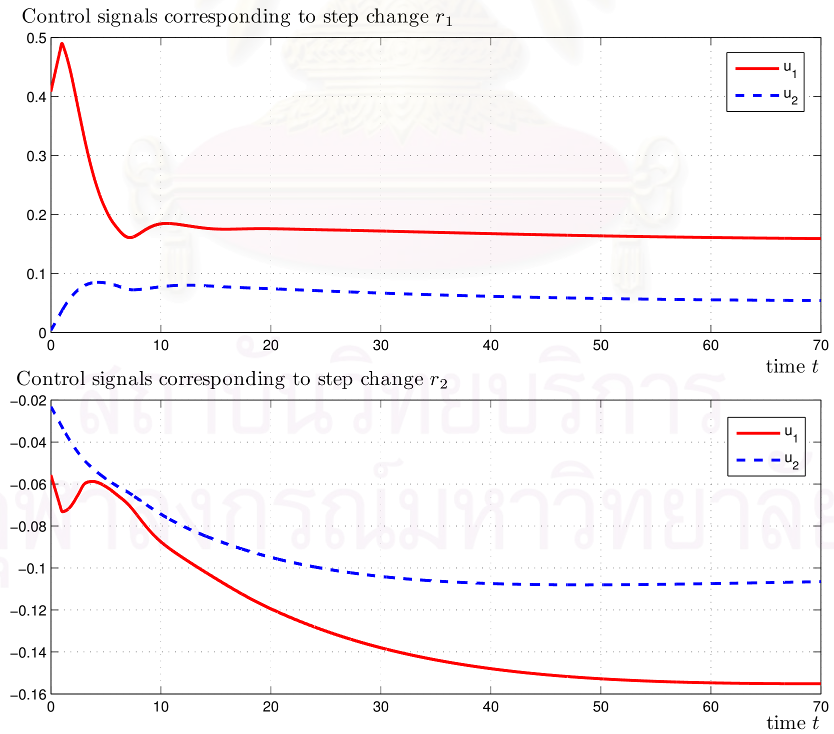

Figure 4.13: Control signals with Taiwo's full PI controller

CHAPTER V

DISCUSSION AND CONCLUSIONS

5.1 Discussion

In this thesis, the design of RFDDSs by the MoI is carried out by using developed computational tools (a stability test and a practical method for computing the abscissa of stability) and an available tool (the formula for the numerical inversion of Laplace transforms based on Zakian's I_{MN} approximants) for computing the time responses of the systems. This section discusses some computational aspects of those tools as such a discussion is necessary for using the tools effectively and may help improve them later.

The abscissa of stability is computed for RFDDSs by using a bisection scheme that makes repeated use of the stability test. Accordingly, the accuracy of the computed abscissa of stability depends largely on the test, that is to say the accuracy in determining the $H(\rho)$-stability for RFDDSs. The $H(\rho)$-stability is checked by comparing a contour integral with zero (assuming that there is no characteristic root on the real interval $[\rho, 0]$ when $\rho \leq 0$); therefore, the accuracy in computing the integral determines the accuracy of the test. (If there are characteristic roots on the real interval $[\rho, 0]$ when $\rho \leq 0$, the accuracy of the test is determined by the efficiency of the line search algorithm employed.) In other words, low levels of accuracy in computing the integral can cause wrong decisions on whether the integral is sufficiently close to zero, which result in the disruption of the convergence of the iteration for computing the abscissa of stability. If the disruption occurs before the stopping criterion of the iteration is satisfied, the abscissa of stability will not be computed to the desired degree of accuracy. However, it should be noted, in most severe cases considered in the numerical examples given in Chapter 3, that the developed method is still able to produce result with reasonable accuracy that is sufficient for many applications. In addition, it has been demonstrated that the method can be used satisfactorily in the design of RFDDSs by the MoI.

The numerical comparison of a contour integral with zero gives rise to the problem of choosing an appropriately small constant according to the level of accuracy required in computing the integral. Once the constant has been chosen, the integral is considered as zero if the magnitude of its computed value is smaller than the constant. In Hwang and Cheng's test [30], they suggest that the constant should be 10^3 times larger than the required level of accuracy. The suggestion still applies to the $H(\rho)$-stability test.

The contour integral used in the $H(\rho)$-stability test has several parameters and terms whose roles are worth considering here. It is noted that many of the computational properties remain the same between the $H(\rho)$-stability test and the test due to Hwang and Cheng (that is an $H(0)$-stability test; see [30] for discussion on computational aspects of the test). The normalizing term $F(\rho + \mathrm{i}h_2)$

in the integrand allows us to compare the integral with zero with reduced risk of failure. The term $(s + h_1 + \mathbf{i}h_2)^k$ reduces, or even eliminates, the risk of the sum of the residues being zero when there are some characteristic roots of the system in the half plane $H(\rho)$. Although the point $(-h_1 - \mathbf{i}h_2)$ is only required to be outside $H(\rho)$, h_1 should not be close to ρ to avoid the stiffness of the initial-value problem obtained from the integral along vertical line $\Gamma_{I\rho}$. In practice, only one or two sets of (h_1, h_2) are sufficient for checking the $H(\rho)$-stability for a RFDDS. With the same level of accuracy required in computing the integral, a larger value of k in the integrand can help reduce the effort in computing the integral significantly. However, when k is too large (e.g., $k > 3$), the integral can be very small and therefore is difficult for numerically comparing with zero.

The Laplace transform inversion method based on Zakian's I_{MN} approximants with $M = 11$ and $N = 18$, i.e., with double precision arithmetic operations, have been used satisfactorily for computing the step responses in the design of RFDDSs by the MoI. To avoid the weakness of the method in computing oscillatory time responses, the computation is performed only at points that are well inside the stability region Ω. This can be assured by stabilizing the systems using criterion $\alpha(p) \le -\varepsilon$ with $\varepsilon > 0$ and performing the $H(-\varepsilon)$-stability test at each trial point generated by the search algorithm. Alternatively, I_{MN} approximants of higher order (e.g., $M = 30$ and $N = 40$) can be used for greater accuracy. In this thesis, I_{MN} approximants with $M = 11$ and $N = 18$ are employed and if there is doubt about the accuracy of the computed results, recomputation with quad-precision can help verify the obtained results. Since the Laplace transform inversion method discussed here is not the main subject to be studied in this thesis, a thorough investigation of the method can be found, for example, in [33] (see also [32]).

It should be noted that the design of control systems with fixed-order controllers by using parameter optimization approaches are generally nonconvex problems (see, e.g., [48]). Therefore, as in solving any nonconvex optimization problems, one must work interactively with the computer. The MBP is capable of locating a design solution only if a good starting point is obtained, which may require several trial steps. If the design procedure cannot obtain a solution, the designer should either try a different controller (usually of higher order or more complexity) or relax the design specifications. Good insight about the system and experience of control design are very useful for seeking a solution to the design problem. In-depth discussion on the design of control systems by the MoI is provided by Zakian and Al-Naib [8] and Zakian [10, 11].

5.2 Conclusions

In the thesis, a systematic computational procedure is developed for designing RFDDSs by the MoI. Accordingly, a design problem must be formulated so that it is suitable for solution by numerical methods. This is an extension of the design formulation which was first used by Zakian and Al-Naib [8] for rational systems and subsequently by Arunsawatwong [23, 26] for retarded delay differential systems. Essentially, this thesis has developed necessary computational tools, namely a stability test and a practical method for computing the abscissa of stability for RFDDSs, to make possible the design of RFDDSs by the MoI.

Extended from the procedure in [30], the $H(\rho)$-stability test has proved effective and provides a useful computational tool for computing the abscissa of stability for RFDDSs. It has been demonstrated that the abscissa of stability is particularly useful in stabilizing RFDDSs. By modifying Zakian's technique [25], a practical method for computing the abscissa of stability is established. The numerical results show that the method developed here is effective, even when the iteration is disrupted by serious round-off error during computation. Also, the numerical example shows that the stabilization problem of RFDDS can be easily solved by numerical methods once a method of computing the abscissa of stability is available. The stability test and the method for computing the abscissa of stability evidently play crucial roles in the design of RFDDSs by the MoI.

In designing RFDDSs by the MoI, we desire to have an efficient and reliable algorithm for computing the time responses. To this end, the Laplace transform inversion formula based on Zakian's I_{MN} approximants can be used satisfactorily in the design procedure. Once the time responses are obtained, the performances defined in terms of the responses (which are usually the most difficult ones to compute) are easily obtainable by simple numerical algorithms. Moreover, it is worth noting that the accuracy in computing time responses of RFDDSs can be significantly improved by using I_{MN} approximants of higher order. In this connection, the rapid advance in the computation power of computers will help.

The MoI facilitates a realistic formulation of the design problem by expressing the physical constraints and performance specifications directly in terms of inequalities. The numerical results evidently show that by using the MoI, one can design RFDDSs effectively in a systematic way. Consequently, one can deal more easily with a sophisticated design problem and, provided that appropriate design criteria are used, can arrive at an accurate and realistic formulation of the design problem. Note that the designer is released from the computational burdens to focus on setting the formulation up. The system is assured to perform satisfactorily provided that the design problem is well formulated and a solution is found.

The design formulation proposed in this thesis for RFDDSs is not only useful for the MoI but also for other numerical optimization methods that search for a solution in a design-parameter space. Accordingly, the computational tools developed in this thesis for resolving stability problems associated with the design of RFDDS by using the MoI remain useful for other numerical optimization approaches.

Despite having been extensively studied by many researchers, fractional controllers are fairly new in comparison with conventional controllers, such as the PID controller. The implementation of fractional controllers is usually more complicated than that of conventional ones. Therefore, fractional controllers are prefered only if they can offer much better performances to the systems. In this connection, the computational tools developed in this thesis allow us to investigate further the advantages of fractional controllers over conventional controllers of integer order.

5.3 Suggestion for Future Work

It should be noted that this thesis is primarily concerned with developing computational tools for resolving the stability problems associated with the design of RFDDSs by the MoI. Therefore, we suggest that further developments should be made to enlarge the class of systems to which the design method is applicable, to arrive at a practical formulation of design problems (i.e., practical performance measures are used), and to improve the accuracy in computing the time responses. The suggested future works are briefly explained as follows.

5.3.1 Neutral Fractional Delay Differential Systems

Transfer function (1.1) represents a neutral fractional delay differential system (NFDDS) if $\deg p_0 = \deg p_k$ for some $k > 0$. The design method developed in this thesis can be applied to NFDDSs once the associated stability problems are resolved. In this connection, the sufficient condition for the stability of a NFDDS obtained by Bonnet and Partington [27] might be useful. Accordingly, a NFDDS is BIBO stable if

$$\alpha(p) < -\varepsilon \quad (\varepsilon > 0),$$

where the abscissa of stability

$$\alpha \triangleq \sup\{\mathrm{Re}(s) : f(s) = 0\},$$

and f is the characteristic function of the system. However, the algorithm developed in thesis for computing the abscissa of stability for RFDDSs is not readily applicable owing to the fact that the $H(\rho)$-stability test cannot be used for NFDDSs.

5.3.2 Practical Design by the Principle of Matching

The formulation described in Section 2.1 gives rise to the need of practical performance measures in order to obtain a realistic design. In this connection, the principle of matching (PoM) [10, 11] allows one to take into consideration practical aspects of control system design.

In a design by the PoM, we are concerned with two connected entities, the system and its environment. The environment affects the system via some signals, i.e., input signals of the system. As far as the performance of the designed system is concerned, the practical model of the environment is characterized by the set of input signals that may occur in practice, called the possible set (denoted by P). A match between the two entities is obtained if any signals in the possible set can be tolerated by the system with respect to some specified well-defined performance criteria. Moreover, if any signals outside the possible set will cause the system to operate unsatisfactorily, the system and its environment are said to be well matched. Let the set of input signals that can be tolerated by the system be defined as the tolerable set (denoted by T). Obviously, a match is obtained if $P \subseteq T$ whereas a good match requires that $P \equiv T$.

A practical design formulation requires a practical way of comparing the possible set P with the tolerable set T in order to judge whether a match (or even a good match) is obtained. Criteria for a match vary significantly depending on the design problem at hand. See [10, 11] for the details.

Once practical matching criteria (in the form of inequalities) are obtained, the design problem is solved by following the formulation presented in Section 2.1.

The application of the PoM to a RFDDS requires a practical method for computing the peak output of the system with respect to a specified possible set.

5.3.3 Development of I_{MN} Recursions for Computing Time Responses

The accuracy in computing oscillatory functions by using I_{MN} approximants in global manner may not be satisfactory. Apparently, no one has developed I_{MN} recursions for solving fractional delay differential equations. In connection with the design method developed in this thesis, the use of I_{MN} recursions for computing time responses of RFDDSs will facilitate higher levels of accuracy.

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Biography

Van Quang Nguyen was born in Hanoi, Vietnam, in 1983. He received his Bachelor's degree in electrical engineering from Hanoi University of Technology, Vietnam, in 2006. He has been granted a scholarship by the AUN/SEED-Net (www.seed-net.org) to pursue his Master's degree in electrical engineering at Chulalongkorn University, Thailand, since 2006. He conducted his graduate study with the Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. His research interest includes the computer aided design of control systems, delay systems and fractional control systems.