ค่าประมาณความผิดพลาดภายหลังสำหรับสมการเชิงอนุพันธ์ย่อยเชิงวงรีแบบกึ่งเชิงเส้น

นายสุทธิศักดิ์ จำปาหวาย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2552

A POSTERIORI ERROR ESTIMATES FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2009

Thesis Title	A POSTERIORI ERROR ESTIMATES FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS
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Field of Study	Mathematics
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ในวิทยานิพนธ์ฉบับนี้เราหาขอบเขตบนและขอบเขตล่างของค่าประมาณความผิดพลาดภายหลังสำหรับวิธีการชิ้นประกอบของสมการเชิงอนุพันธ์ย่อยเชิงวงรีแบบกึ่งเชิงเส้นบนโดเมนรูปหลายเหลี่ยมในปริภูมิสองมิติ โดยที่เราพิจารณาปัญหาแบบ Dirichlet ที่มีเงื่อนไขค่าขอบเป็นศูนย์ การประมาณค่าอยู่บนพื้นฐานของ Lagrange element และอยู่บนสมมติฐานการอินทิเกรตได้อย่างแม่นตรง ซึ่งเราวัดค่าประมาณความผิดพลาดอยู่ในรูปของนร์อมแบบพลังงาน ภายใต้เงื่อนไขของฟังก์ชัน $f(x,u)$ มีอนุพันธ์อันดับหนึ่งเทียบกับตัวแปรที่สอง

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต..............................

สาขาวิชา..........คณิตศาสตร์.......... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..............................

ปีการศึกษา............2552...............

5072519723 : MAJOR MATHEMATICS
KEYWORDS : A POSTRIORI ERROR ESTIMATES / SEMI-LINEAR ELLIPTIC

SUTTISAK JAMPAWAI : A POSTERIORI ERROR ESTIMATES FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS. THESIS ADVISOR : KHAMRON MEKCHAY, Ph.D. 32 pp.

We derive upper and lower bounds for a posteriori error estimates in finite element solutions of semi-linear elliptic partial differential equations (PDEs) over polygonal domains in two space dimensions. We consider the Dirichlet problem for semi-linear PDEs with vanishing boundary. The estimate is based on Lagrange element, and the error estimates are computed in the energy norm with assumption of exact integration. The proof is based on the condition of function $f(x, u)$ which have first derivative in second argument.

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Field of Study : ....Mathematics.... Advisor's Signature : ....

Academic Year : ....2009...........

ACKNOWLEDGEMENTS

First I would like to thank my advisor, Khamron Mekchay, Ph.D., for giving me invaluable opportunity to work on challenging and extremely interesting the thesis over the past couple years. Next, I would like to thank Associate Professor Pornchai Satravaha, Ph.D., Assistant Professor Anusorn Chonwerayuth, Ph.D. and Assistant Professor Settapat Chinviriyasit, Ph.D., my thesis committee, for variable suggestions. Next, the Institute for the Promotion of Teaching Science and Technology (IPST), for the DPST scholarship which provided me the great opportunity to study through out M.Sc. Program. Finally, I would like to thank all of my teachers and lecturers who have taught me for my knowledge and skills.

If I did the wrong thing, I do apologize. It does not happen by intention.

CONTENTS

CHAPTER I

INTRODUCTION

The finite element method is one of the main tools for the numerical treatment of partial differential equations (PDEs). It is based on the variational formulation of the differential equation, it is much more flexible than finite difference methods and can thus be applied to more complicated problems.

Adaptive finite element methods started in the late 70's and now are standard tools in science and engineering. AFEMs are effective tools to obtain good approximate solutions with low computational costs, especially in presence of singularities.

A key of AFEMs is an *a posteriori error estimation*. A posteriori error estimates are computable estimates for the error in suitable norms, typically in energy norm, in term of the approximate solution and data of the problem.

For elliptic PDEs, AFEMs are boil down to iterations of the form

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{REFINE}$$

Given a current mesh and data, SOLVE find the approximate solution; ESTIMATE computes error estimates in suitable norm based on *a posteriori error estimators*; REFINE refines the current mesh to obtain a finer mesh according to the error indicators. The ultimate purpose is to construct a sequence of meshes (approximate solutions) that will eventually reducing error in an efficient way in term of degree of freedom.

For elliptic PDEs, a posteriori error estimation techniques were developed for computing quantities η_T to approximate the error in energy norm or other norms on each finite element T. These formed basis of adaptive mesh procedures designed to control and minimize the error. In the last 30 years, many results for elliptic error estimation techniques were obtained: we refer to Babu$\check{s}$ka and Rheinboldt as representative of the work. Here are list of recent results of AFEM for elliptic-type PDEs.

- W.Dorfler [4] designed steps of AFEM and proved the convergence of algorithm for Poisson equation in two dimensions.

- P. Morin, R.H. Nochetto and K.G. Siebert [7] extended the result of W.Dorfler [4] for linear elliptic PDEs,

$$-\nabla \cdot (A\nabla u) = f \qquad \text{in } \Omega,$$

  where $\Omega \subseteq \mathbb{R}^d (d \geq 1)$, A is a piecewise constant function, and f is a function on Ω. Here, they introduced *oscillation* that is important in proving convergence of AFEM algorithm.

- K. Mekchay and R.H.Nochetto [6], extended the result of [7] and proved convergence of AFEM algorithm for the general linear elliptic PDEs,

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f \qquad \text{in } \Omega,$$

  where A, f, b, c are functions on Ω.

- In the book by M. Ainsworth and J. T. Oden [1], they derived a posteriori error estimates for nonlinear problems in general elliptic PDEs in term of implicit forms.

- Recently in 2006, R.H. Nochetto, A. Schmidt, K.G. Siebert and A. Veeser [8], they computed upper bounds and lower bounds of a posteriori error estimates in the maximum norm for semi-linear Poisson equation,

$$-\Delta u + f(x,u) = 0 \qquad \text{in } \Omega,$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous in $\overline{\Omega} \times \mathbb{R}$ and non-decreasing in the second argument.

In this thesis, we are interested in deriving an explicit a posteriori error estimation in energy norm for semi-linear elliptic PDE,

$$-\nabla \cdot (A\nabla u) = f(x,u) \qquad \text{in } \Omega,$$

where A is a function on Ω and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has first derivative in second argument. We estimated upper and local lower bounds of a posteriori error estimates in the energy norm.

We organized this thesis into three parts. In Chapter II, we gave definitions and theorems that are important in deriving weak and discrete formulations of our model problem. Nearly last, we gave finite element space and the theorems for deriving upper bound. In Chapter III, we formulated the weak form of the model problem including also the discrete problem. In Chapter IV, we derived upper and local lower bounds. Finally, we gave conclusion and some idea for designing AFEM algorithm.

CHAPTER II

PRELIMINARY

2.1 Sobolev Spaces

We first introduced weak derivatives and defined Sobolev spaces, refer to the book of S. C. Brenner and L. R. Scott [3].

We reviewed Lebesgue integrations and restricted our attention for simplicity to a real-valued functions f on a given domain Ω, that are Lebesgue measurable. We denoted the Lebesgue integral of f by

$$\int_{\Omega} f(x)dx.$$

For $1 \leq p < \infty$, let

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p},$$

and for $p = \infty$, set

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}\{|f(x)| : x \in \Omega\}.$$

In either cases, we define the *Lebesgue spaces*

$$L^p(\Omega) := \{f : \|f\|_{L^p(\Omega)} < \infty\}.$$

A multi-index α is an n-tuple of non-negative integers. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. The length of α is given by $|\alpha| := \sum_{i=1}^{n} \alpha_i$. For $\phi \in C^\infty$, denoted by $D^\alpha \phi$ the usual partial derivatives $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi$. Note that the *order* of this derivative is given by $|\alpha|$.

Definition 2.1.1. Let Ω be a domain in $\mathbb{R}^n (n \geq 1)$. Defined by $C_0^\infty(\Omega)$ the set of $C^\infty(\Omega)$ functions with compact support in Ω.

Note that a support of a continuous function f is the closure of the open set $\{x : f(x) \neq 0\}$, denoted by $\operatorname{supp}(f)$.

Definition 2.1.2. We say that a given function $f \in L^1(\Omega)$ has a *weak derivative*, $D_w^\alpha f$, provided there is a function $v \in L^1(\Omega)$ such that

$$\int_\Omega v(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \varphi(x) dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

If such a v exists, we define $D_w^\alpha f = v$.

Example 2.1.3. Take $n = 1$, $\Omega = (-1,1)$, and $f(x) = |x|$. We claim that $D_w^1 f$ exists and is given by

$$v(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ -1 & \text{if} \ -1 < x < 0. \end{cases}$$

To see this, we break the interval $(-1,1)$ into the parts in which f is smooth, and we integrate by parts. Let $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} \int_{-1}^1 f(x)\varphi'(x)dx &= \int_{-1}^0 -x\varphi'(x)dx + \int_0^1 x\varphi'(x)dx \\ &= \int_{-1}^0 \varphi(x)dx + \int_0^1 -\varphi(x)dx \\ &= -\int_{-1}^1 v(x)\varphi(x)dx \end{aligned}$$

One may check that $D_w^i f$ does not exist for $i > 1$. $\square$

Definition 2.1.4. Let Ω be a domain in $\mathbb{R}^n$, k be a non-negative integer, and $f \in L^1(\Omega)$. Suppose that the weak derivatives $D_w^\alpha f$ exist for all $|\alpha| \leq k$. Define the *Sobolev norms*

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p} \tag{2.1}$$

and the *semi-norms*

$$|f|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|=k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p} \tag{2.2}$$

in the case $1 \le p < \infty$, and in the case $p = \infty$

$$\|f\|_{W^{k,\infty}(\Omega)} := \max\{\|D_w^\alpha f\|_{L^\infty(\Omega)} : |\alpha| \le k\}.$$

In either case, we define the *Sobolev spaces* via

$$W^{k,p}(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{W^{k,p}(\Omega)} < \infty \right\}. \tag{2.3}$$

For $k = 1$ and $p = 2$, the Sobolev space $W^{1,2}(\Omega)$ is often denoted by $H^1(\Omega)$.

Theorem 2.1.5. *The Sobolev spaces $W^{k,p}(\Omega)$ are Banach spaces.*

Proof. See Theorem 1.3.2 in Brenner and Scott [3]. □

Theorem 2.1.6. *Let Ω be any open set. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. See Theorem 1.3.4 in Brenner and Scott [3]. □

Then, the closure of $C_0^\infty(\Omega) \cap W^{1,p}(\Omega)$ is denoted by $W_0^{1,p}(\Omega)$. Thus, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Here are list of bilinear forms (linear products) for spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H_0^1(\Omega)$.

1. For $u, v \in L^2(\Omega)$, $\langle u, v \rangle := \int_\Omega uv \ dx$ and $\|u\|_0 := \|u\|_{L^2(\Omega)} = \sqrt{\langle u, u \rangle}$.

2. For $u, v \in H^1(\Omega)$, $(u, v)_1 := \langle \nabla u, \nabla v \rangle + \langle u, v \rangle$.

3. For $u, v \in H_0^1(\Omega)$, $(u, v)_1 = \langle \nabla u, \nabla v \rangle$ and $|u|_{H^1(\Omega)} = \sqrt{\langle \nabla u, \nabla u \rangle} = \|\nabla u\|_0$.

4. For $u \in H^1(\Omega)$, $|u|_{H^1(\Omega)} \le \|u\|_{H^1(\Omega)}$.

Note that the Sobolev space $H^1(\Omega)$ is a Hilbert space (See Example 2.2.2 in Brenner and Scott [3]).

Theorem 2.1.7 (Poincar*é* inequality). *Suppose Ω, subset of $\mathbb{R}^n$, is an open bounded domain. Then*

$$\|u\|_0 \leq C_P \|\nabla u\|_0 \qquad \forall u \in H_0^1(\Omega), \tag{2.4}$$

where C_P is a constant that depends only on the domain Ω.

Proof. See p.30 in Braess [2]. □

Corollary 2.1.8. *The semi-norm $|\cdot|_{H^1(\Omega)}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $H_0^1(\Omega)$.*

Proof. Let $f \in H_0^1(\Omega)$. Then

$$\begin{aligned}
|f|^2_{H^1(\Omega)} = \int_\Omega |\nabla f|^2 dx &\leq \int_\Omega |f|^2 dx + \int_\Omega |\nabla f|^2 dx \\
&= \|f\|^2_{H^1(\Omega)} \\
&= \int_\Omega |f|^2 dx + \int_\Omega |\nabla f|^2 dx \\
&\leq C_P^2 \int_\Omega |\nabla f|^2 dx + \int_\Omega |\nabla f|^2 dx \qquad \text{(by Theorem 2.1.7)} \\
&= (1 + C_P^2) \int_\Omega |\nabla f|^2 dx \\
&= (1 + C_P^2) |f|^2_{H^1(\Omega)}
\end{aligned}$$

Therefore, $|\cdot|_{H^1(\Omega)}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$. □

2.2 Finite Element Spaces

In this section, we defined some continuous piecewise spaces of polynomial function that are subspaces of $H^1(\Omega)$.

Definition 2.2.1. Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal domain. A *triangulation* $\mathcal{T}$ of Ω is a collection $\{T\}$ of triangles such that:

1. $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}$;

2. for $T, T' \in \mathcal{T}$ and $T \neq T'$ the set $\overline{T} \cap \overline{T'}$ is empty or consists of a vertex or a common side.

Let $\omega \subseteq \Omega$. We define

- $\mathbb{P}_p(\omega) :=$ the set of all polynomials on ω in two variables of degree less than or equal to p;

- $H_T := \operatorname{diam} T =$ the diameter of triangle T;

- $\rho_T :=$ the diameter of the largest circle inscribed in T. Regularity constant is denoted by

$$\kappa_T := \frac{H_T}{\rho_T}. \tag{2.5}$$

Definition 2.2.2. A family of triangulations $\{\mathcal{T}_H\}$ of Ω is said to be *shape-regular* if there exists a constant K such that $\kappa_T \leq K$ for all $T \in \mathcal{T}_H$ and for all triangulations $\mathcal{T}_H$.

Definition 2.2.3. Let $\mathcal{T}$ be a conforming triangulation. Then finite element subspace of order $p \in \mathbb{N}$ associated with $\mathcal{T}$ is defined by

$$\mathbb{V}^p := \{v \in C(\bar{\Omega}) : \forall T \in \mathcal{T}, v|_T \in \mathbb{P}_p(T)\}. \tag{2.6}$$

If there is no ambiguity, we will use $\mathbb{V}$ for simplicity.

Refinement.

Let $\mathcal{T}_0$ be an initial triangulation of Ω. If we decompose a subset of triangles of $\mathcal{T}_0$ into subtriangles such that the resulting set of triangles is again a triangulation of Ω, we call this a *refinement* of $\mathcal{T}_0$. We may denote this triangulation by $\mathcal{T}_1$. In this way we can construct a sequence of triangulations $\{\mathcal{T}_k\}$ such that $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$.

We used notations as follows:

1. $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$;

2. $\mathbb{V}_H := \{v \in C(\bar{\Omega}) : \forall T \in \mathcal{T}_H, v|_T \in \mathbb{P}_p\}$;

3. $\mathbb{V}_H^\circ := \{v \in \mathbb{V}_H : v(x) = 0, x \in \partial\Omega\}$.

Remark 2.2.4. $\mathbb{V}_H^\circ \subset H_0^1(\Omega)$

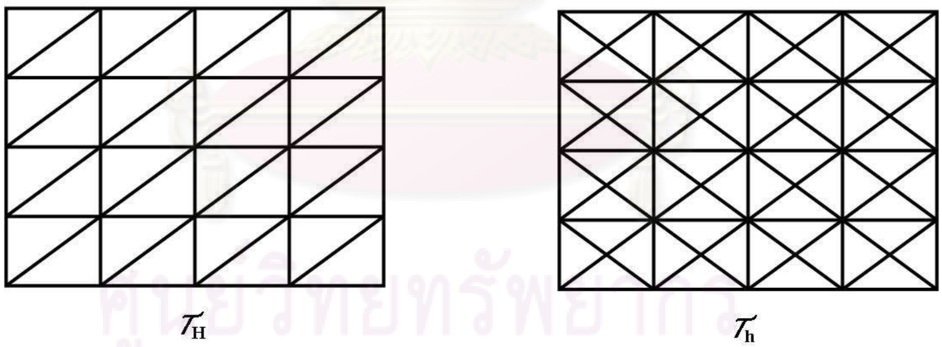

Figure 2.2.1: An example of refinement $\mathcal{T}_h$ of $\mathcal{T}_H$.

Definition 2.2.5. For $T \in \mathcal{T}_H$, we define the *patch element* of T to be

$$\omega_T := \bigcup\{T' \in \mathcal{T} : \overline{T} \cap \overline{T'} \neq \phi\}.$$

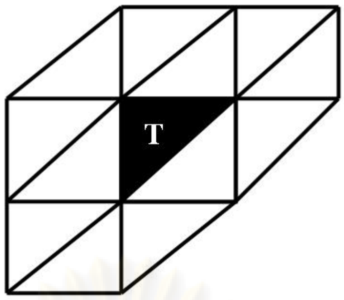

Figure 2.2.2: An example of a patch with respect to the element T.

Theorem 2.2.6 (The Cl$\acute{e}$ment Interpolation). *There is a linear interpolation operator* $\mathcal{I}_H : H_0^1(\Omega) \to \mathbb{V}_H^\circ$ *such that for* $T \in \mathcal{T}_H$ *and* $S \in \partial T$ *we have*

$$\|\varphi - \mathcal{I}_H \varphi\|_{L^2(T)} \leq C H_T \|\nabla \varphi\|_{L^2(\omega_T)} \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.7}$$

$$\|\varphi - \mathcal{I}_H \varphi\|_{L^2(S)} \leq C H_S^{1/2} \|\nabla \varphi\|_{L^2(\omega_T)} \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.8}$$

where H_S *is the diameter of the side* S *and* C *is a constant depending only on the shape regularity.*

Proof. See p.84 in Braess [2]. $\square$

CHAPTER III

THE MODEL PROBLEM

3.1 The Model Problem

First, we introduced the definition of strict monotonicity.

Definition 3.1.1. Let A be an operator. If A satisfies

$$(A(p) - A(q)) \cdot (p - q) \geq \theta |p - q|^2 \tag{3.1}$$

for all $p, q \in \mathbb{R}^n$ and some constant $\theta > 0$. Then A is *strict monotonicity.*

Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a convex polyhedral domain, $f \in C(\bar{\Omega} \times \mathbb{R})$ and has first derivative in second argument and $A : \mathbb{R}^n \to \mathbb{R}^n$ be strict monotonicity.
As the model problem, we consider the semi-linear elliptic PDE with vanishing Dirichlet boundary condition,

$$-\nabla \cdot (A(x) \nabla u) = f(x, u), \qquad \text{in } \Omega, \tag{3.2}$$

$$u = 0, \qquad \text{on } \partial\Omega. \tag{3.3}$$

The *weak formulation* of this problem reads as follows: find $u \in H_0^1(\Omega)$ satisfying

$$\mathcal{B}(u, \varphi) = \mathcal{L}(u; \varphi), \qquad \forall \varphi \in H_0^1(\Omega), \tag{3.4}$$

where

$$\mathcal{B}(u, \varphi) = \int_\Omega A(x) \nabla u \cdot \nabla \varphi dx, \tag{3.5}$$

and

$$\mathcal{L}(u;\varphi) = \int_\Omega f(x,u)\varphi dx. \tag{3.6}$$

The corresponding *discrete* problem then reads as follows: find $u_H \in \mathbb{V}_H^\circ(\Omega)$ such that

$$\mathcal{B}(u_H,\phi) = \mathcal{L}(u_H;\phi), \qquad \forall \phi \in \mathbb{V}_H^\circ(\Omega). \tag{3.7}$$

Remark 3.1.2.

1. If u satisfies equation (3.4), then u is called a *weak solution* of (3.2) and (3.3) by [3].

2. If u_H satisfies equation (3.7), then u_H is called a *finite element solution* and is unique by [7].

Let $\|\cdot\|$ denotes the *energy norm* defined by $\|v\| = \sqrt{\mathcal{B}(v,v)}$ for $v \in H_0^1(\Omega)$. The error $\mathcal{E}_H := u - u_H$ belongs to the space $H_0^1(\Omega)$ and satisfies

$$\mathcal{B}(\mathcal{E}_H,\varphi) = \mathcal{B}(u,\varphi) - \mathcal{B}(u_H,\varphi) = \mathcal{L}(u;\varphi) - \mathcal{B}(u_H,\varphi) \qquad \forall \varphi \in H_0^1(\Omega). \tag{3.8}$$

For convenience, real values $f(x,u)$ and $f(x,u_H)$ are denoted by f and f_H, respectively.

Lemma 3.1.3. $\mathcal{B}(\mathcal{E}_H,\phi) = \langle f - f_H, \phi\rangle, \qquad \forall \phi \in \mathbb{V}_H^\circ(\Omega).$

Proof. Let $\phi \in \mathbb{V}_H^\circ(\Omega)$. Then

$$\begin{aligned}
\mathcal{B}(\mathcal{E}_H,\phi) &= \mathcal{L}(u;\phi) - \mathcal{B}(u_H,\phi), && \text{by equation (3.8)}\\
&= \mathcal{L}(u;\phi) - \mathcal{L}(u_H;\phi), && \text{by equation (3.7)}\\
&= \int_\Omega f(x,u)\phi dx - \int_\Omega f(x,u_H)\phi dx, && \text{by equation (3.6)}\\
&= \int_\Omega [f(x,u) - f(x,u_H)]\phi dx,\\
&= \langle f - f_H, \phi\rangle.
\end{aligned}$$

□

3.2 Coercivity and Continuity

Definition 3.2.1. A bilinear form $\mathcal{B}(\cdot,\cdot)$ on a norm linear space H is said to be *bounded* (or *continuous*) if $\exists\ c_1 < \infty$ such that

$$\mathcal{B}(v,w) \le c_1\|v\|_H\|w\|_H, \qquad \forall v,w \in H,$$

and *coercive* on $V \subset H$ if $\exists\ c_2 < \infty$

$$\mathcal{B}(v,v) \ge c_2\|v\|_H^2, \qquad \forall v \in V.$$

Lemma 3.2.2. *The bilinear form $\mathcal{B}(\cdot,\cdot)$ in equation (3.5) is coercive on $H_0^1(\Omega)$ and bounded on $H^1(\Omega)$.*

Proof. We will first show that $\mathcal{B}(\cdot,\cdot)$ is coercive. Let $v \in H_0^1(\Omega)$. Since A is strict monotonicity, we can choose $p = \nabla v$ and $q = \mathbf{0}$. We get $A\nabla v\cdot\nabla v \ge \theta|\nabla v|^2$. Take integral over Ω,

$$\mathcal{B}(v,v) = \int_\Omega (A\nabla v\cdot\nabla v)dx \ge \int_\Omega \theta|\nabla v|^2dx = \theta\|\nabla v\|_0^2 = \theta|v|_{H^1(\Omega)}^2.$$

Since two norms are equivalent on $H_0^1(\Omega)$, $\mathcal{B}(v,v) \ge \theta\|v\|_{H^1(\Omega)}^2$.

Finally, we will show that $\mathcal{B}(\cdot,\cdot)$ is bounded. Let $v,w \in H^1(\Omega)$. By the Cauchy-Schwarz inequality,

$$\mathcal{B}(v,w) = \int_\Omega (A\nabla v\cdot\nabla w)dx \le \|A\nabla v\|_0\|\nabla w\|_0.$$

Since A is smooth, A is bounded on Ω. Let $C(A) = \|A\|_{L^\infty(\Omega)}$. Then

$$\begin{aligned}\mathcal{B}(v,w) &\le \|A\|_{L^\infty(\Omega)}\|\nabla v\|_0\|\nabla w\|_0 = C(A)|v|_{H^1(\Omega)}|w|_{H^1(\Omega)},\\ &\le C(A)\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}.\end{aligned}$$

□

Theorem 3.2.3. *The norm $\|\cdot\|_{H^1(\Omega)}$ is equivalent to the norm $|\!|\!|\cdot|\!|\!|$ in $H_0^1(\Omega)$.*

Proof. By the coercivity and continuous of the bilinear form $\mathcal{B}$ (Lemma 3.2.2), we get Theorem 3.2.3. □

3.3 L^2-Estimates

Under the assumptions and notations of the model problem we can estimate the L^2-error as follows.

Theorem 3.3.1 (Duality). *Let $L^2(\Omega)$ be a space with the norm $\|\cdot\|_0$ and the scalar product $\langle\cdot,\cdot\rangle$. Let $H_0^1(\Omega)$ be a subspace which is also a Hilbert space under another norm $\|\cdot\|$. Then the finite element solution u_H of equation (3.7) in $\mathbb{V}_H^\circ \subset H_0^1(\Omega)$ satisfies*

$$\|u-u_H\|_0 \le C\left(\|u-u_H\| \sup_{g\in L^2(\Omega),\|g\|_0\le 1} \inf_{v\in\mathbb{V}_H^\circ} \|\varphi_g - v\| + \|f-f_H\|_0\right). \qquad (3.9)$$

Here, for $g \in L^2(\Omega)$ we denote $\varphi_g \in H_0^1(\Omega)$ the corresponding unique solution of the (linear) dual equation

$$\mathcal{B}(w,\varphi_g) = \langle g, w\rangle\,, \qquad \text{for all } w \in H_0^1(\Omega). \qquad (3.10)$$

Proof. By considering w as a function on $L^2(\Omega)$, $w \in (L^2(\Omega))^*$, the dual space of $L^2(\Omega)$. We can compute the dual norm

$$\|w\|_0 = \sup_{g\in L^2(\Omega),\|g\|_0\le 1} \langle g, w\rangle\,. \qquad (3.11)$$

Here and in (3.9), the supremum is taken only over those g with $\|g\|_0 \le 1$. We recall that u and u_H are given by

$$\mathcal{B}(u,v) = \langle f, v\rangle\,, \qquad \text{for all } v \in H_0^1(\Omega),$$

$$\mathcal{B}(u_H,v) = \langle f_H, v\rangle\,, \qquad \text{for all } v \in \mathbb{V}_H^\circ.$$

By Lemma 3.1.3, $\mathcal{B}(u-u_H,v) = \langle f-f_H, v\rangle$ for all $v \in \mathbb{V}_H^\circ$. Moreover, if we insert $w := u-u_H \in H_0^1(\Omega)$ in (3.10), for any $v \in \mathbb{V}_H^\circ$ and $g \in L^2(\Omega)$, by continuity of the bilinear $\mathcal{B}$ and Cauchy-Schwarz inequality we get

$$\begin{aligned}\langle g, u-u_H\rangle &= \mathcal{B}(u-u_H,\varphi_g),\\ &= \mathcal{B}(u-u_H,\varphi_g-v)+\mathcal{B}(u-u_H,v),\\ &\le C\|u-u_H\|\cdot\|\varphi_g-v\|+\langle f-f_H,v\rangle\,,\\ &\le C\|u-u_H\|\cdot\|\varphi_g-v\|+\|f-f_H\|_0\|v\|_0.\end{aligned}$$

Let $\varphi_{g,H}\in\mathbb{V}_H^\circ$ be a finite element solution of φ_g. By Cé́a's Lemma [p.55, 2]

$$\|\varphi_g-\varphi_{g,H}\|\le C\inf_{v\in\mathbb{V}_H^\circ}\|\varphi_g-v\|.$$

By taking $v=\varphi_{g,H}\in\mathbb{V}_H^\circ$, it follows that

$$\begin{aligned}\langle g,u-u_H\rangle &\le C\|u-u_H\|\cdot\|\varphi_g-\varphi_{g,H}\|+\|f-f_H\|_0\|\varphi_{g,H}\|_0,\\ &\le C\|u-u_H\|\inf_{v\in\mathbb{V}_H^\circ}\|\varphi_g-v\|+\|f-f_H\|_0\|\varphi_{g,H}\|_0.\end{aligned}$$

Since Ω is the convex polyhedral domain, the solution of equation (3.10) has H^2-regular (Regularity theorem [p.89, 2]). Therefore, there is constant c depending only on Ω, $\mathcal{B}$ and $H^2(\Omega)$ such that $\varphi_{g,H}\in H^1(\Omega)$ satisfying

$$\|\varphi_{g,H}\|_0\le\|\varphi_{g,H}\|_{H^2(\Omega)}\le c\|g\|_0.$$

Then φ_g is bounded on its domain. Thus φ_g is also bounded. Now the duality argument (3.11) leads to the conclusion,

$$\begin{aligned}\|u-u_H\|_0 &= \sup_{g\in L^2(\Omega),\|g\|_0\le 1}\langle g,u-u_H\rangle\\ &\le C\left(\|u-u_H\|\sup_{g\in L^2(\Omega),\|g\|_0\le 1}\inf_{v\in\mathbb{V}_H}\|\varphi_g-v\|+\|f-f_H\|_0\right).\end{aligned}$$

□

Corollary 3.3.2. *Under the hypotheses of Theorem 3.3.1 and f has first derivative in second argument with $\|f_u\|_{L^\infty(\Omega)} \leq \rho < 1$ for some positive ρ. Then*

$$\|u - u_H\|_0 \leq C_f H|u - u_H|_{H^1(\Omega)}$$

where H is the maximum of H_T for $T \in \mathcal{T}_H$ and a constant C_f depends only on ρ, the shape regularity and the data.

Proof. By its assumption and theorem 7.3 in the book of Braess [p.90, 2], the right bracket of (3.9) of Theorem 3.3.1 becomes

$$\|u - u_H\|_0 \leq CH|u - u_H|_{H^1(\Omega)} + \|f - f_H\|_0. \tag{3.12}$$

Apply the mean value theorem to f, we get $|f(\cdot, u) - f(\cdot, u_H)| = |f_u(\cdot, u^*)||u - u_H|$ for some u^*. Take L^2-norm to both sides and estimate $\|f_u(\cdot, u^*)\|_0 \leq \|f_u\|_{L^\infty(\Omega)} \leq \rho$, equation (3.12) lead to

$$\|u - u_H\|_0 \leq CH|u - u_H|_{H^1(\Omega)} + \rho\|u - u_H\|_0.$$

Since $\rho < 1$, we can absorb the second term on right-hand side with the term on the left-hand side:

$$\|u - u_H\|_0 \leq \left(\frac{C}{1 - \rho}\right) H|u - u_H|_{H^1(\Omega)}$$

where a constant C depends on Ω, the shape regularity, and the data. $\square$

CHAPTER IV

A POSTERIORI ERROR ESTIMATES

In the first step, we decomposed the residual equation in Lemma 3.1.3 for the true error into local contributions from each element.

Let $\varphi \in H_0^1(\Omega)$ be chosen arbitrarily. Then, by writing the single integral over the whole domain as a sum of integrals over the individual elements gives

$$\mathcal{B}(\mathcal{E}_H, \varphi) = \sum_{T \in \mathcal{T}_H} \left\{ \int_T f \varphi dx - \int_T A \nabla u_H \cdot \nabla \varphi dx \right\}.$$

Applying Green's theorem to each of the terms in the second term and rearranging terms leads to

$$\begin{aligned}
\mathcal{B}(\mathcal{E}_H, \varphi) &= \sum_{T \in \mathcal{T}_H} \left\{ \int_T f \varphi dx + \int_T \nabla \cdot (A \nabla u_H) \varphi dx - \int_{\partial T} \frac{\partial (A \nabla u_H)}{\partial n_T} \varphi ds \right\}, \\
&= \sum_{T \in \mathcal{T}_H} \left\{ \int_T (f - f_H) \varphi dx + \int_T (f_H + \nabla \cdot (A \nabla u_H)) \varphi dx - \int_{\partial T} \frac{\partial (A \nabla u_H)}{\partial n_T} \varphi ds \right\}, \\
&= \sum_{T \in \mathcal{T}_H} \left\{ \int_T (f - f_H) \varphi dx + \int_T \mathcal{R}_T(u_H) \varphi dx - \int_{\partial T} \frac{\partial (A \nabla u_H)}{\partial n_T} \varphi ds \right\}.
\end{aligned} \tag{4.1}$$

Here $\mathcal{R}_T(u_H)$ is the interior residual

$$\mathcal{R}_T(u_H) := f_H + \nabla \cdot (A \nabla u_H) \qquad \text{in } T,$$

and n_T is the unit outward normal vector to ∂T. Each of these quantities is well-defined thanks to the smoothness of the data and regularity of the approximation u_H when restricted to a single element.

The contribution from the final term in equation (4.1) can be rewritten by observing that the trace of the function φ matches along an edge shared by two elements, giving

$$\mathcal{B}(\mathcal{E}_H,\varphi)=\sum_{T\in\mathcal{T}_H}\left\{\int_T(f-f_H)\varphi dx+\int_T\mathcal{R}_T\varphi dx\right\}-\sum_{S\in\partial\mathcal{T}_H}\int_S\left[\!\left[\frac{\partial(A\nabla u_H)}{\partial n}\right]\!\right]\varphi ds, \tag{4.2}$$

where $\partial\mathcal{T}_H$ is the set of inter-element sides (edges and faces) of $\mathcal{T}_H$ and the final summation is over the set $\partial\mathcal{T}_H$ consisting of the inter-element sides S on the interior of the mesh. The quantity

$$\left[\!\left[\frac{\partial(A\nabla u_H)}{\partial n}\right]\!\right]=(A\nabla u_H)_T\cdot n_T-(A\nabla u_H)_{T'}\cdot n_{T'}$$

defined on the side S separating elements T and T' represents the *jump discontinuity* in the approximation to the normal flux on the interface. Then, they will be denoted by

$$J_S(u_H):=-\left[\!\left[\frac{\partial(A\nabla u_H)}{\partial n}\right]\!\right].$$

Thus equation (4.2) then becomes

$$\mathcal{B}(\mathcal{E}_H,\varphi)=\sum_{T\in\mathcal{T}_H}\left\{\int_T(f-f_H)\varphi dx+\int_T\mathcal{R}_T\varphi dx\right\}+\sum_{S\in\partial\mathcal{T}_H}\int_S J_S\varphi ds. \tag{4.3}$$

The equation (4.3) is written again to be

$$\mathcal{B}(\mathcal{E}_H,\varphi)=\int_\Omega(f-f_H)\varphi dx+\sum_{T\in\mathcal{T}_H}\int_T\mathcal{R}_T\varphi dx+\sum_{S\in\partial\mathcal{T}_H}\int_S J_S\varphi ds. \tag{4.4}$$

Finally, we defined local indicators and error estimators for finding upper bounds and local lower bounds.

Definition 4.0.3. For $T\in\mathcal{T}_H$ and $S\in\partial\mathcal{T}_H$ an inter-element side, we define the *local error indicator* $\eta_H(T)$ by

$$\eta_H^2(T):=H_T^2\|\mathcal{R}_T\|_{L^2(T)}^2+\sum_{S\subset\partial T}H_S\|J_S\|_{L^2(S)}^2,$$

and the *error estimator* $\eta_H(\omega)$ for $\omega \subseteq \Omega$ by

$$\eta_H^2(\omega) := \sum_{T\in\mathcal{T}_H, T\subseteq\omega} \eta_H^2(T).$$

4.1 Upper bounds

Let $\mathcal{I}_H$ be the Cl$\acute{e}$ment interpolation operator. For a given $\varphi \in H_0^1(\Omega)$, where $\mathcal{I}_H\varphi \in \mathbb{V}_H^\circ$, and by the Lemma 3.1.3 and the identity (4.4), we get

$$0 = \sum_{T\in\mathcal{T}_H} \int_T \mathcal{R}_T \mathcal{I}_H \varphi dx + \sum_{S\in\partial\mathcal{T}_H} \int_S J_S \mathcal{I}_H \varphi ds$$

and then subtracting from identity (4.4) gives

$$\begin{aligned} \mathcal{B}(\mathcal{E}_H, \varphi) = & \int_\Omega (f - f_H)\varphi dx + \sum_{T\in\mathcal{T}_H} \int_T \mathcal{R}_T(\varphi - \mathcal{I}_H\varphi) dx \\ & + \sum_{S\in\partial\mathcal{T}_H} \int_S J_S(\varphi - \mathcal{I}_H\varphi) ds, \qquad \forall \varphi \in H_0^1(\Omega). \end{aligned} \tag{4.5}$$

Applying the Cauchy-Schwarz inequality gives

$$\begin{aligned} \mathcal{B}(\mathcal{E}_H, \varphi) \le & \|f - f_H\|_0 \|\varphi\|_0 + \sum_{T\in\mathcal{T}_H} \|\mathcal{R}_T\|_{L^2(T)} \|\varphi - \mathcal{I}_H\varphi\|_{L^2(T)} \\ & + \sum_{S\in\partial\mathcal{T}_H} \|J_S\|_{L^2(S)} \|\varphi - \mathcal{I}_H\varphi\|_{L^2(S)}. \end{aligned}$$

By the Cl$\acute{e}$ment interpolation (Theorem 2.2.6), we get

$$\begin{aligned} \mathcal{B}(\mathcal{E}_H, \varphi) \le & \|f - f_H\|_0 \|\varphi\|_0 + \sum_{T\in\mathcal{T}_H} C H_T \|\mathcal{R}_T\|_{L^2(T)} \|\nabla\varphi\|_{L^2(\omega_T)} \\ & + \sum_{S\in\partial\mathcal{T}_H} C H_S^{1/2} \|J_S\|_{L^2(S)} \|\nabla\varphi\|_{L^2(\omega_T)}. \end{aligned}$$

Applying Cauchy-Schwarz inequality leads to

$$\begin{aligned} \mathcal{B}(\mathcal{E}_H, \varphi) \le & \|f - f_H\|_0 \|\varphi\|_0 \\ & + C\|\nabla\varphi\|_0 \left\{ \sum_{T\in\mathcal{T}_H} H_T^2 \|\mathcal{R}_T\|_{L^2(T)}^2 + \sum_{S\in\partial\mathcal{T}_H} H_S \|J_S\|_{L^2(S)}^2 \right\}^{1/2}. \end{aligned} \tag{4.6}$$

Therefore,

$$\mathcal{B}(\mathcal{E}_H, \varphi) \leq \|f - f_H\|_0 \|\varphi\|_0 + C\eta_H(\Omega)\|\nabla\varphi\|_0 \qquad \forall\varphi \in H_0^1(\Omega).$$

So, substituting $u - u_H \in H_0^1(\Omega)$ in place of φ results in the estimate

$$|||u - u_H|||^2 \leq \|f - f_H\|_0 \|u - u_H\|_0 + C\eta_H(\Omega)\|\nabla(u - u_H)\|_0. \tag{4.7}$$

By Corollary 3.3.2 and its assumptions, we get

$$|||u - u_H|||^2 \leq C_f H \|f - f_H\|_0 |u - u_H|_{H^1(\Omega)} + C\eta_H(\Omega)\|\nabla(u - u_H)\|_0.$$

By equivalence of $|\cdot|_{H^1(\Omega)}$ and $|||\cdot|||$ on $H_0^1(\Omega)$, we get

$$|||u - u_H||| \leq C_1 H \|f - f_H\|_0 + C_2\eta_H(\Omega). \tag{4.8}$$

Theorem 4.1.1 (Upper bound).

$$|||u - u_H||| \leq C_1\eta_H(\Omega) + C_2 H \|f - f_H\|_0$$

where the constants C_1 depends only on the shape regularity, a coercivity constant, the domain Ω, and the data of the problem and C_2 also depends on ρ in Corollary 3.3.2, the shape regularity, and the data of the problem.

Proof. Follows at once from previous arguments. □

4.2 Lower bounds

A key role for estimating the local lower bounds will be played by certain locally supported, nonnegative functions that are commonly referred to as *bubble functions*. The two types of bubble functions are *interior* bubble functions, supported a single element, and *edge* bubble functions, supported on a pair of elements.

Let $\psi_T \in \mathbb{P}_3(T)$ be an interior bubble function with $\text{supp}(\psi_T) = T$ and $0 \le \psi_T \le 1$ and $\max\limits_{x\in T} \psi_T(x) = 1$.

Theorem 4.2.1. *There is a positive constant C such that for all v in a finite-dimensional space $\mathcal{P}(T)$*

$$C^{-1}\|v\|^2_{L^2(T)} \le \int_T \psi_T v^2 dx \le C\|v\|_{L^2(T)},$$

and

$$C^{-1}\|v\|_{L^2(T)} \le \|\psi_T v\|_{L^2(T)} + H_T|\psi_T v|_{H^1(T)} \le C\|v\|_{L^2(T)},$$

where the constant C is independent of v and H_T.

Proof. See Theorem 2.2 in Ainsworth and Oden [1]. □

Let $T_1, T_2 \in \mathcal{T}_H$ be the pair of elements sharing the interior side S. Denote $\omega_S := T_1 \cup T_2$ and let $\psi_S \in \mathbb{P}_2(\omega_S)$ be an edge bubble function with $\text{supp}(\psi_S) = \omega_S$ and $0 \le \psi_S \le 1$ and $\max\limits_{x\in \omega_S} \psi_S(x) = 1$.

Theorem 4.2.2. *Let $S \in \partial T$ be an edge and let ψ_S be the corresponding edge bubble function. Let $\mathcal{P}(S)$ be the finite-dimensional space of functions defined on S. Then for $v \in \mathcal{P}(S)$, there exists a positive constant such that*

$$C^{-1}\|v\|^2_{L^2(S)} \le \int_S \psi_S v^2 ds \le C\|v\|^2_{L^2(S)},$$

and

$$H_T^{-1/2}\|\psi_S v\|_{L^2(T)} + H_T^{1/2}|\psi_S v|_{H^1(T)} \leq C\|v\|^2_{L^2(S)},$$

where the constant C is independent of v and H_T.

Proof. See Theorem 2.4 in Ainsworth and Oden [1]. □

Applying the first part of Theorem 4.2.1, we have

$$\|\overline{\mathcal{R}_T}\|^2_{L^2(T)} \leq C\int_T \psi_T \overline{\mathcal{R}_T}^2 dx, \tag{4.9}$$

where $\overline{\mathcal{R}_T}$ be the L^2-projection of $\mathcal{R}_T$ onto the space of polynomials $\mathbb{P}_p$ over the element $T \in \mathcal{T}_H$. The function $\varphi = \overline{\mathcal{R}_T}\psi_T$ vanishes on the boundary of element T, therefore, $\overline{\mathcal{R}_T}\psi_T$ can be extended to the rest of the domain as a continuous function by defining its values outside the element to be zero. Thus, inserting $\overline{\mathcal{R}_T}\psi_T$ into the equation (4.3) yields

$$\mathcal{B}(\mathcal{E}_H, \overline{\mathcal{R}_T}\psi_T) = \int_T (f - f_H)\overline{\mathcal{R}_T}\psi_T dx + \int_T \mathcal{R}_T\overline{\mathcal{R}_T}\psi_T dx,$$

and therefore

$$\int_T \psi_T\overline{\mathcal{R}_T}^2 dx = \int_T \psi_T\overline{\mathcal{R}_T}(\overline{\mathcal{R}_T} - \mathcal{R}_T)dx - \int_T (f - f_H)\overline{\mathcal{R}_T}\psi_T dx + \mathcal{B}(\mathcal{E}_H, \overline{\mathcal{R}_T}\psi_T). \tag{4.10}$$

Applying the second part of properties of bubble functions, we obtain

$$\|\psi_T\overline{\mathcal{R}_T}\|_{L^2(T)} \leq C\|\overline{\mathcal{R}_T}\|_{L^2(T)}. \tag{4.11}$$

Applying Cauchy-Schwarz inequality to the first term of (4.10) leads to

$$\int_T \psi_T\overline{\mathcal{R}_T}(\overline{\mathcal{R}_T} - \mathcal{R}_T)dx \leq \|\psi_T\overline{\mathcal{R}_T}\|_{L^2(T)}\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)}.$$

By equation (4.11),

$$\int_T \psi_T\overline{\mathcal{R}_T}(\overline{\mathcal{R}_T} - \mathcal{R}_T)dx \leq C\|\overline{\mathcal{R}_T}\|_{L^2(T)}\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)}. \tag{4.12}$$

Similarly, for the second term of (4.10) we obtain

$$\int_T (f - f_H)\overline{\mathcal{R}_T}\psi_T dx \leq C\|f - f_H\|_{L^2(T)}\|\overline{\mathcal{R}_T}\|_{L^2(T)}. \tag{4.13}$$

Since the bilinear form $\mathcal{B}$ is bounded and $\mathrm{supp}(\overline{\mathcal{R}_T}\psi_T) = T$,

$$\mathcal{B}(\mathcal{E}_H, \overline{\mathcal{R}_T}\psi_T) \leq C\|\mathcal{E}_H\|_{H^1(T)}\|\psi_T\overline{\mathcal{R}_T}\|_{H^1(T)},$$

and by Theorem 4.2.1,

$$\|\psi_T\overline{\mathcal{R}_T}\|_{H^1(T)} \leq CH_T^{-1}\|\overline{\mathcal{R}_T}\|_{L^2(T)},$$

the estimation of the last term becomes

$$\mathcal{B}(\mathcal{E}_H, \overline{\mathcal{R}_T}\psi_T) \leq CH_T^{-1}\|\mathcal{E}_H\|_{H^1(T)}\|\overline{\mathcal{R}_T}\|_{L^2(T)}. \tag{4.14}$$

Inserting these estimates into equation (4.10) gives

$$\int_T \psi_T\overline{\mathcal{R}_T}^2 dx \leq C\|\overline{\mathcal{R}_T}\|_{L^2(T)}\left\{\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)} + \|f - f_H\|_{L^2(T)} + H_T^{-1}\|\mathcal{E}_H\|_{H^1(T)}\right\},$$

and rescaling (4.9) leads to the bound

$$\|\overline{\mathcal{R}_T}\|_{L^2(T)} \leq C\left\{\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)} + \|f - f_H\|_{L^2(T)} + H_T^{-1}\|\mathcal{E}_H\|_{H^1(T)}\right\}. \tag{4.15}$$

By triangle inequality,

$$\|\mathcal{R}_T\|_{L^2(T)} \leq \|\overline{\mathcal{R}_T}\|_{L^2(T)} + \|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)}. \tag{4.16}$$

Hence, the desired bound on the actual residual follows from (4.15) and (4.16),

$$\|\mathcal{R}_T\|_{L^2(T)} \leq C\left\{\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)} + \|f - f_H\|_{L^2(T)} + H_T^{-1}\|\mathcal{E}_H\|_{H^1(T)}\right\}. \tag{4.17}$$

By applying the first part of Theorem 4.2.2,

$$\|\overline{J_S}\|_{L^2(S)}^2 \leq C\int_S \psi_S\overline{J_S}^2 ds \tag{4.18}$$

where $\overline{J_S}$ is the best L^2-projection of J_S onto $\mathbb{P}_p(S)$. We extend $\overline{J_S}$ constantly along the normal such that it is defined on ω_S. The function $\varphi = \overline{J_S}\psi_S$ vanishes on the boundary of the subdomain ω_S. Extending φ by zero outside ω_S to the whole of the domain Ω gives a function $\varphi \in H_0^1(\Omega)$. The residual equation (4.3), with this choice of φ, yields

$$\mathcal{B}(\mathcal{E}_H, \overline{J_S}\psi_S) = \int_{\omega_S}(f - f_H)\overline{J_S}\psi_S dx + \int_{\omega_S}\mathcal{R}_T\overline{J_S}\psi_S dx + \int_S J_S\overline{J_S}\psi_S ds,$$

and thus

$$\begin{aligned}\int_S \psi_S\overline{J_S}^2 ds = \int_S \psi_S\overline{J_S}(\overline{J_S} - J_S)ds + \mathcal{B}(\mathcal{E}_H, \overline{J_S}\psi_S) - \int_{\omega_S}\psi_S\mathcal{R}_T\overline{J_S}dx \\ - \int_{\omega_S}(f - f_H)\overline{J_S}\psi_S dx. \end{aligned} \tag{4.19}$$

Each of these terms can be estimated by using Theorem 4.2.2 and Cauchy-Schwarz inequality. The first term of (4.19) leads to

$$\begin{aligned}\int_S \psi_S\overline{J_S}(\overline{J_S} - J_S)ds &\le \|\psi_S\overline{J_S}\|_{L^2(S)}\|\overline{J_S} - J_S\|_{L^2(S)}, \\ &\le C\|\overline{J_S}\|_{L^2(S)}\|\overline{J_S} - J_S\|_{L^2(S)}. \end{aligned} \tag{4.20}$$

The second term is estimated by the continuity of $\mathcal{B}$,

$$\begin{aligned}\mathcal{B}(\mathcal{E}_H, \overline{J_S}\psi_S) &\le C\|\mathcal{E}_H\|_{H^1(\omega_S)}\|\psi_S\overline{J_S}\|_{H^1(\omega_S)}, \\ &\le CH_S^{-1/2}\|\mathcal{E}_H\|_{H^1(\omega_S)}\|\overline{J_S}\|_{L^2(S)}. \end{aligned} \tag{4.21}$$

The third term is bounded by

$$\begin{aligned}\int_{\omega_S}\psi_S\mathcal{R}_T\overline{J_S}dx &\le \|\mathcal{R}_T\|_{L^2(\omega_S)}\|\psi_S\overline{J_S}\|_{L^2(\omega_S)}, \\ &\le CH_S^{1/2}\|\mathcal{R}_T\|_{L^2(\omega_S)}\|\overline{J_S}\|_{L^2(S)}. \end{aligned} \tag{4.22}$$

Finally the estimation of the last term is

$$\begin{aligned}\int_{\omega_S}(f - f_H)\overline{J_S}\psi_S dx &\le \|f - f_H\|_{L^2(\omega_S)}\|\overline{J_S}\psi_S\|_{L^2(\omega_S)}, \\ &\le CH_S^{1/2}\|f - f_H\|_{L^2(\omega_S)}\|\overline{J_S}\|_{L^2(S)}. \end{aligned} \tag{4.23}$$

As a consequence of these estimates and the bound (4.18), we conclude that

$$\begin{aligned}\|\overline{J_S}\|_{L^2(S)} \le C\{&\|\overline{J_S} - J_S\|_{L^2(S)} + H_S^{1/2}\|\mathcal{R}_T\|_{L^2(\omega_S)} + H_S^{-1/2}\|\mathcal{E}_H\|_{H^1(\omega_S)} \\ &+ H_S^{1/2}\|f - f_H\|_{L^2(\omega_S)}\}.\end{aligned}$$

By triangle inequality similar to (4.16), we obtain

$$\begin{aligned}\|J_S\|_{L^2(S)} \le C\{&\|\overline{J_S} - J_S\|_{L^2(S)} + H_S^{1/2}\|\mathcal{R}_T\|_{L^2(\omega_S)} + H_S^{-1/2}\|\mathcal{E}_H\|_{H^1(\omega_S)} \\ &+ H_S^{1/2}\|f - f_H\|_{L^2(\omega_S)}\}.\end{aligned} \tag{4.24}$$

Applying the estimate (4.17) for interior residual in terms of the true error, giving

$$\begin{aligned}\|J_S\|_{L^2(S)} \le C\{&\|\overline{J_S} - J_S\|_{L^2(S)} + H_S^{1/2}\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(\omega_S)} + H_S^{-1/2}\|\mathcal{E}_H\|_{H^1(\omega_S)} \\ &+ H_S^{1/2}\|f - f_H\|_{L^2(\omega_S)}\}.\end{aligned} \tag{4.25}$$

Theorem 4.2.3. *Let $\mathcal{R}_T$ and J_S denote the interior and boundary residuals associated with the finite element approximation constructed from the subspace $\mathbb{V}_H^\circ$. Suppose that $\overline{\mathcal{R}_T}$ and $\overline{J_S}$ are polynomial approximations to the interior and boundary residuals constructed from finite-dimensional subspace. Then,*

$$\|\mathcal{R}_T\|_{L^2(T)} \le C\left\{\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(T)} + \|f - f_H\|_{L^2(T)} + H_T^{-1}\|\mathcal{E}_H\|_{H^1(T)}\right\} \tag{4.26}$$

and

$$\begin{aligned}\|J_S\|_{L^2(S)} \le C\{&\|\overline{J_S} - J_S\|_{L^2(S)} + H_S^{1/2}\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|_{L^2(\omega_S)} + H_S^{-1/2}\|\mathcal{E}_H\|_{H^1(\omega_S)} \\ &+ H_S^{1/2}\|f - f_H\|_{L^2(\omega_S)}\},\end{aligned} \tag{4.27}$$

where C is a positive constant depending only on the shape regularity of elements and the selection of the finite-dimensional subspace used to approximate the interior and boundary residuals.

Proof. Follows at once from previous arguments. $\square$

Finally, by definition of the indicator and Theorem 4.2.3,

$$\begin{aligned}\eta_H(T)^2 &= H_T^2\|\mathcal{R}_T\|^2_{L^2(T)} + \sum_{S\subset\partial T} H_S\|J_S\|^2_{L^2(S)}\\ &\le CH_T^2\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|^2_{L^2(T)} + CH_T^2\|f - f_H\|^2_{L^2(T)} + C\|\mathcal{E}_H\|^2_{H^1(T)} +\\ &\quad C\sum_{S\subset\partial T}\{H_S\|\overline{J_S} - J_S\|^2_{L^2(S)} + H_S^2\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|^2_{L^2(\omega_S)} + \|\mathcal{E}_H\|_{H^1(\omega_S)} +\\ &\quad H_S^2\|f - f_H\|_{L^2(\omega_S)}\}.\end{aligned}$$

For $\tilde{\omega}_T := \bigcup_{S\subseteq\partial T}\omega_S$, we have

$$\eta_H^2(T) \le C_1\|\mathcal{E}_H\|^2_{H^1(\tilde{\omega}_T)} + C_2\left\{H_T^2\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|^2_{L^2(\tilde{\omega}_T)} + \sum_{S\subset\partial T} H_S\|\overline{J_S} - J_S\|^2_{L^2(S)}\right\} + C_3H_T^2\|f - f_H\|^2_{L^2(\tilde{\omega}_T)}, \tag{4.28}$$

where the constant C_1, C_2 and C_3 depend only on the shape regularity, and the data of the problem. We define the *oscillation* on the element T by

$$osc_H^2(T) = H_T^2\|\overline{\mathcal{R}_T} - \mathcal{R}_T\|^2_{L^2(\tilde{\omega}_T)} + \sum_{S\in\partial T} H_S\|\overline{J_S} - J_S\|^2_{L^2(S)},$$

and for $\omega \subseteq \Omega$, we define

$$osc_H^2(\omega) = \sum_{T\in\mathcal{T}_H, T\subseteq\omega} osc_H^2(T).$$

Therefore, the equation (4.28) becomes

$$\eta_H^2(T) \le C_1\|\mathcal{E}_H\|^2_{H^1(\tilde{\omega}_T)} + C_2 osc_H^2(\tilde{\omega}_T) + C_3H_T^2\|f - f_H\|^2_{L^2(\tilde{\omega}_T)}. \tag{4.29}$$

Theorem 4.2.4 (Local lower bound).

$$\eta_H^2(T) \le C_1\|\mathcal{E}_H\|^2_{H^1(\tilde{\omega}_T)} + C_2 osc_H^2(\tilde{\omega}_T) + C_3H_T^2\|f - f_H\|^2_{L^2(\tilde{\omega}_T)}$$

where the constant C_1, C_2 and C_3 depend only on the shape regularity, and the data of the problem.

Proof. Follows at once from previous arguments. $\square$

4.3 Conclusions

In previous section, we derived the upper and local lower bounds for a posteriori error estimates. The Theorem 4.1.1 gives the upper bound,

$$|||u - u_H||| \leq C_1\eta_H(\Omega) + C_2 H\|f - f_H\|_0,$$

and the Theorem 4.2.4 gives the local lower bounds,

$$\eta_H^2(T) \leq C_1\|u - u_H\|^2_{H^1(\tilde{\omega}_T)} + C_2 osc_H^2(\tilde{\omega}_T) + C_3 H_T^2\|f - f_H\|^2_{L^2(\tilde{\omega}_T)}.$$

Note that the upper bound we have the term $\|f - f_H\|_0$ coming from the nonlinearity of the function $f(x,u)$ which does not appear in the case of linear problems. Similarly, we also have the term $\|f - f_H\|_{L^2(\omega_T)}$ in the local lower bounds.

It is known from [4,6,7] for linear cases the convergence of AFEM relies on the control of error indicators $\eta_H(T)$ and oscillation $osc_H(T)$, based on the assumption that the error $|||u - u_H|||$ reduces if we can control $\eta_H(T)$ and $osc_H(T)$. For our result, in order to design a computable algorithm of AFEM we need to control the term $\|f - f_H\|_{L^2(\omega_T)}$ that appears on the error bounds. Since it is not computable in term of given data and known information like $\eta_H(T)$ or $osc_H(T)$, due to the knowledge of exact solution. We may control this term with the following two ideas. First, if f has first derivative in the second argument and $\|f_u\|_\infty < \rho < 1$ for some constant ρ as in Corollary 3.3.2, then we may absorb the terms $\|f - f_H\|_0$ in the error term $|||u - u_H|||$, namely

$$\|f(\cdot, u) - f(\cdot, u_H)\|_0 \leq \|f_u\|_{L^\infty(\Omega)}\|u - u_H\|_0 \leq \rho|||u - u_H|||.$$

With this we obtain the Corollary 4.3.1.

Corollary 4.3.1. *Upper bounds:* $|||u - u_H||| \leq C\eta_H(\Omega)$.

Local lower bounds: $C_1\eta_H^2(T) \leq C_2 osc_H^2(\tilde{\omega}_T) + \|u - u_H\|^2_{H^1(\tilde{\omega}_T)}$.

In this case, we obtain the same error estimates as for linear cases. Thus the algorithm of AFEM can be designed similarly. Second, we may try approximate $\|f - f_H\|_{L^2(\omega_T)}$ by something that can be computed and use this also as an indicator similar to the role of $\eta_H(T)$ and $osc_H(T)$ in the AFEM algorithm. This may require a further analysis to obtain such the approximation. With the given a posteriori error estimates, one can design the AFEM algorithm as follows.

The Adaptive Finite Element Method (AFEM) consists of loops of the form

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

The procedure SOLVE solves (3.7) for the discrete solution u_H. Note that they requires methods for solving non-linear system like the Newton's method. The procedure ESTIMATE determines the element indicators $\eta_H(T)$, oscillation $osc_H(T)$ and approximation of $\|f - f_H\|_{L^2(T)}$ that are computable for each element. Depending on their relative sizes, these quantities are later used by the procedure MARK to mark element T, and thereby create a subset of $\mathcal{T}_H$ of elements to be refine. Finally, procedure REFINE partitions those elements in the subset to maintain mesh conformity.

NOTATIONS

$\|\cdot\|_V$	The norm on the space V, p.5-6
$\|\cdot\|_W$	The semi-norm on the space W, p.6
$\|\cdot\|_0$	The norm on $L^2(\Omega)$, p.6
$\vert\vert\vert\cdot\vert\vert\vert$	The energy norm, p.12
$\mathcal{B}(\cdot,\cdot)$	The bilinear form , p.11
$\langle\cdot,\cdot\rangle$	The inner product on $L^2(\Omega)$, p.6
$H^1(\Omega)$	The Sobolev spaces of functions in $L^2(\Omega)$ whose first derivatives are also in $L^2(\Omega)$, p.6
$H_0^1(\Omega)$	The space $H^1(\Omega)$ with vanishing on boundary, p.6
$\mathbb{V}_H$	The finite element space, p.9
$\mathbb{V}_H^\circ$	the space $\mathbb{V}_H$ with vanishing on boundary, p.9
$\mathbb{P}_p(\omega)$	The set of all polynomials on ω in two variables of degree less than or equal to p, p.8
$\mathcal{T}_H$	The triangulation of Ω, p.8
∂T	The boundary of the element T, p.10
$\partial\mathcal{T}_H$	The set inter-element sides, p.18
H_T	The diameter of on the element T, p.8
H_S	The diameter of on the side $S \subseteq T$, p.10
H	The maximum of H_T for $T \in \mathcal{T}_H$, p.15

ρ_T	The diameter of the largest circle inscribed in T, p.8
κ_T	The regularity constant on the element T, p.8
ω_S	The union of the pair elements sharing the interior side S, p.21
ω_T	The patch element of the element T, p.9
$\tilde{\omega}_T$	The union of ω_S for $S \subseteq \partial T$, p.26
$\mathcal{I}_H$	The Cl*é*ment Interpolation operator, p.10
$\mathcal{E}_H$	The different between u and u_H, p.12
$\mathcal{R}_T(u_H)$	The interior residual of u_H on the element T, p.17
$J_S(u_H)$	The jump discontinuity of u_H on the element T, p.18
$\eta_H(T)$	The local indicator on the element T, p.18
$osc_H(T)$	The oscillation on the element T, p.26
ψ_T	The interior bubble on the element T, p.21
ψ_S	The edge bubble function on the interior side S, p.21

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