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CLIQUE PARTITIONS OF GLUED GRAPHS

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กราฟปะติด คือ กราฟที่ได้จากการรวมกราฟสองกราฟที่ไม่มีจุดยอดร่วมกัน โดยการปะติดจุดยอด และเส้นเชื่อมของกราฟย่อยเชื่อมโยงที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้นของทั้งสองกราฟนั้น เราเรียกกราฟย่อยที่กล่าวมาว่า *กราฟโคลน* และเรียกกราฟสองกราฟที่ไม่มีจุดยอดร่วมกันว่า *กราฟต้นฉบับ* *ผลแบ่งกั้นกราฟด้วยคลีก* คือ เซตของคลีกของกราฟ ซึ่งเส้นเชื่อมแต่ละเส้นเป็นเส้นเชื่อมของคลีกเหล่านั้นเพียงคลีกเดียวเท่านั้น *จำนวนคลีกแบ่งกั้นกราฟ* คือ จำนวนสมาชิกที่น้อยที่สุดของผลแบ่งกั้นกราฟด้วยคลีก

งานวิจัยนี้เราศึกษาหาขอบเขตของจำนวนคลีกแบ่งกั้นกราฟของกราฟปะติดในพจน์ของจำนวนคลีกแบ่งกั้นกราฟของกราฟต้นฉบับ นอกจากนั้นเราหาค่าหรือขอบเขตของจำนวนคลีกแบ่งกั้นกราฟของกราฟปะติดคงสภาพคลีก และกราฟปะติดที่กำหนดกราฟโคลนของกราฟปะติดเป็นกราฟบริบูรณ์ที่มี 2 จุดยอดและกราฟบริบูรณ์ที่มี 3 จุดยอด

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A *glued graph* results from combining two vertex-disjoint graphs by identifying nontrivial connected isomorphic subgraphs of both graphs. Such subgraphs are referred to as the *clones*. The two vertex-disjoint graphs are referred to as the *original graphs*. A *clique partition* of a graph is a set of its cliques which together contain each edge exactly once. The *clique partition number* of a graph is the smallest cardinality of its clique partitions.

We study bounds of clique partition numbers of glued graphs in terms of clique partition numbers of their original graphs. Also, we investigate values or bounds of clique partition numbers of clique-preserving glued graphs and glued graphs with specified clones such as complete graphs K_2 and K_3.

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CHAPTER I

INTRODUCTION

We separate this chapter into four sections. The first section, we shall give, briefly, the history of a glued graph and a clique partition of a graph. Later we give basic definitions and some poperties of them. Finally, some remarks for cliques of glued graphs are stated in the last section.

1.1 Motivation and outline

A glued graph at H-clone results from combining two vertex-disjoint graphs by identifying a subgraph H of each original graph. The glue operator is a mathematical operator defined by Uiyyasathian [15] since 2003. In 2006, Promsakon [10], who studied some basic properties, vertex-colorbilities and edge-colorbilities of glued graphs. In the next year, Charoenpanitseri [3] studied the total colorings of glued graphs. In 2008, the subject of clique coverings of glued graphs was studied by Pimpasalee [9] of which main results are useful for our study. Later, Saduakdee [13, 14] studied the perfection of glued graphs of perfect original graphs.

A clique partition of a graph G is a collection of complete subgraphs of G that partitions the edge set of G. In this thesis, we study the problem of finding clique partitions with minimum size among all clique partitions of a glued graph. The question of calculating clique partition numbers was raised by Orlin [8] in 1977. Clique partitions of the variety classes of graphs were investigated by many authors, see [5, 6, 16]. Since 1948, DeBruijn and Erdős [1] had already proved that partitioning a complete graph K_n into smaller cliques required at least n cliques.

In 1986, Gregory et al. [6] investigated the lower bound of clique partition numbers of the cocktail party graph, T_v, $cp(T_v) \geq v$ for $v \geq 8$ and gave a characterization of the cases where equality holds. In 1996, Monson [7] listed other results that the effect of the vertex and edge deletion on the clique partition number of a graph. More recently, Cavers [2] collected the clique partition numbers of graphs and introduced some new results in 2005.

Our purpose in this thesis is to study bounds of clique partition numbers of glued graphs in terms of these clique partition numbers of their original graphs. Also, we investigate values or bounds of clique partition numbers of glued graphs without new cliques and particularly with specified clones such as complete graphs K_2 and K_3.

The definitions of glued graphs and clique partitions are located in Section 1.2, along with examples and some basic properties.

In Chapter 2, we give some preliminary results and study a bound of the clique partition number of a glued graph. Also, we investigate clique partition numbers of clique-preserving glued graphs.

In Chapter 3, clique partitions of glued graphs at K_2-clones and K_3-clones are considered.

Finally, conclusions and open problems are in Chapter 4.

Throughout the thesis, we consider only simple graphs. $V(G)$ and $E(G)$ stand for the vertex set and edge set of a graph G, respectively. The number of elements in $E(G)$ is represented by $e(G)$.

1.2 Glued graphs

Let G_1 and G_2 be two nontrivial vertex-disjoint graphs. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph between G_1 and G_2 at H_1 and H_2 with respect to f*, denoted by $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs.* We use $u \equiv v$ to denote the vertex in the glued graph $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ where $u \in V(G_1)$, $v \in V(G_2)$ and $f(u) = v$.

The *glued graph between G_1 and G_2 at H-clone*, written $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2} = G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ and H is the copy of H_1 and H_2 in the resulting graph.

We denote $G_1 \triangleleft\!\triangleright G_2$ an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph with respect to any of their isomorphisms.

The clone of a glued graph is called a *K_n-clone* if it is a complete graph K_n.

The notation $K_n(v_1, v_2, \ldots, v_n)$ denotes a complete graph on vertices $v_1, v_2, \ldots, v_n$.

Example 1.2.1. Let G_1 and G_2 be graphs as shown in Figure 1.2.1

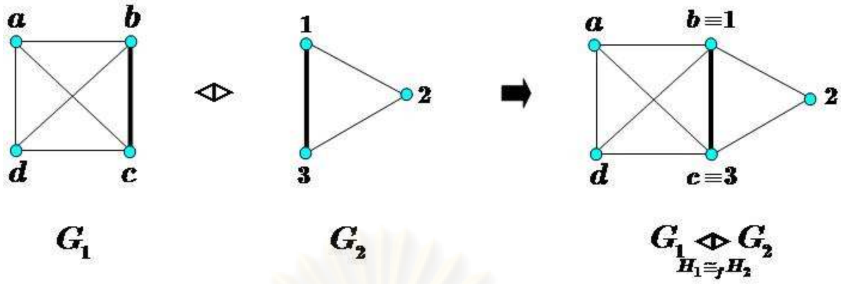

Figure 1.2.1: A glued graph between G_1 and G_2 with respect to f

Let $H_1 = K_2(b, c) \subseteq G_1$ and $H_2 = K_2(1, 3) \subseteq G_2$. We consider an isomorphism f between H_1 and H_2, as follows: $f(b) = 1$ and $f(c) = 3$. We show the glued graph between G_1 and G_2 with respect to f, $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}$, in Figure 1.2.1.

□

The following example shows that different isomorphisms can give the different or the same result.

Example 1.2.2. Let G_1 and G_2 be graphs as shown in Figure 1.2.2.

Let $H_1 = K_3(a, b, c)$ and $H_2 = K_3(1, 3, 4)$. Thus H_1 and H_2 are nontrivial connected subgraphs of G_1 and G_2, respectively. Consider three isomorphisms between H_1 and H_2, namely f, g and h, as follows:

$$f(a) = 1,\ f(b) = 3,\ f(c) = 4,$$
$$g(a) = 3,\ g(b) = 4,\ g(c) = 1,$$
$$h(a) = 4,\ h(b) = 1,\ h(c) = 3.$$

We show glued graphs between G_1 and G_2 with respect to f, g and h in Figure 1.2.2. Moreover, it is easy to see that $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2} \cong \underset{H_1 \cong_g H_2}{G_1 \Diamond G_2}$ but $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2} \ncong \underset{H_1 \cong_h H_2}{G_1 \Diamond G_2}$.

□

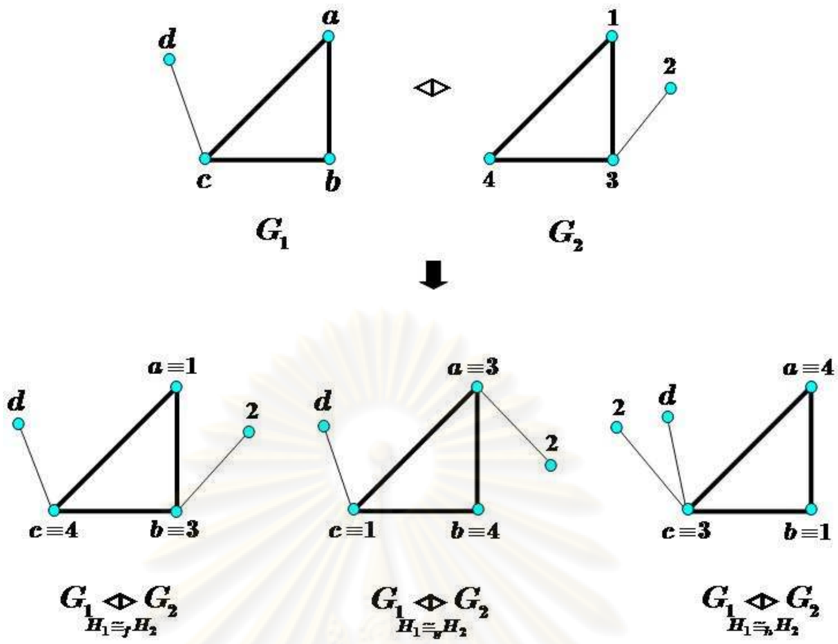

Figure 1.2.2: Glued graphs with different isomorphisms

Note that a glued graph of simple orginal graphs could have multiple edges. This is illustrated in Example 1.2.3.

Example 1.2.3. Let G_1 and G_2 be graphs as shown in Figure 1.2.3.

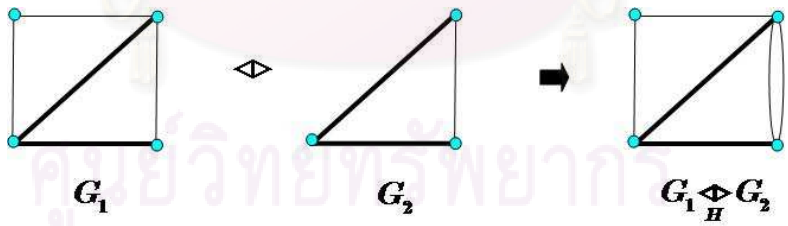

Figure 1.2.3: A glued graph containing multiple edges

□

Promsakon [10] proved that the glued graph of simple original graphs, $G_1 \underset{H}{\diamond} G_2$, is a simple graph if and only if there are no vertices u and v in H such that there are edges $e_1 \in E(G_1) \setminus E(H)$ and $e_2 \in E(G_2) \setminus E(H)$ whose endpoints are u and v.

In this thesis, we consider only simple connected glued graphs. Next, we collect some basic properties of glued graphs in the following remark.

Remark 1.2.4.

1. The original graphs are subgraphs of their glued graph.
2. The graph gluing does not create or destroy any edge.
3. A glued graph between disconnected graphs is also disconnected and a glued graph between connected graphs is also connected.
4. If $u \in V(G_1) \smallsetminus V(H)$ and $v \in V(G_2) \smallsetminus V(H)$ where G_1 and G_2 are graphs and H is a clone of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, then u and v are not adjacent in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$.

More details concerning glued graphs can be explored in Promsakon's thesis [10]. In the next section, we introduce the definition of clique partitions of graphs.

1.3 Clique partitions of graphs

A *clique* of a graph G is a complete subgraph of G. Note that a clique is not necessarily maximal. An *n-clique* or a *clique of order* n is a clique with n vertices. A *clique partition* of a graph G is a set of cliques of G which together contain each edge of G exactly once. A *minimum clique partition* of a graph G is a clique partition of G with the smallest cardinality among all clique partitions of G, and the size of a minimum clique partition of G is called the *clique partition number* of a graph G, denoted by $cp(G)$.

Example 1.3.1. Let G be the graph as shown in Figure 1.3.1.

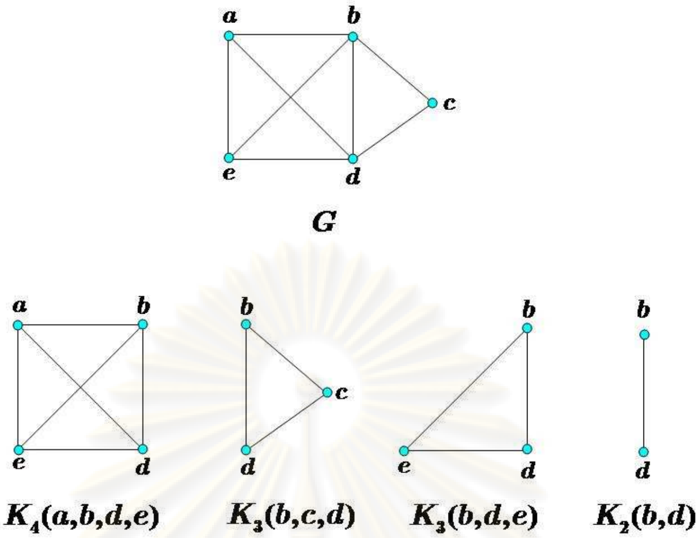

Figure 1.3.1: Some cliques of a graph

Note that $K_4(a,b,d,e)$, $K_3(b,c,d)$, $K_3(b,d,e)$ and $K_2(b,d)$ in Figure 1.3.1 are cliques of G while only $K_4(a,b,d,e)$ and $K_3(b,c,d)$ are maximal cliques. $\square$

Example 1.3.2. Consider the graph G illustrated in Figure 1.3.2.

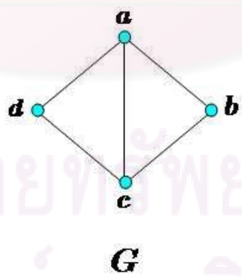

Figure 1.3.2: A clique partition of a graph

Let $\mathscr{P}_1 = \{K_3(a,c,d), K_2(a,b), K_2(b,c)\}$ and $\mathscr{P}_2 = \{K_3(a,b,c), K_2(a,d), K_2(c,d)\}$. Note that $\mathscr{P}_1$ and $\mathscr{P}_2$ are clique partitions of G. Thus $cp(G) \leq 3$. Since G cannot be partitioned by using 2 cliques, $cp(G) = 3$. This implies that $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions of G. Hence, it is possible that a graph has more than one minimum clique partition. $\square$

Remark 1.3.3. If G is a K_3-free graph, then $cp(G) = e(G)$ because all cliques in G have order 2.

Definition 1.3.4. Two edges e and f in a graph G are *clique-independent edges* of G if there is no clique in G containing both e and f. A set of pairwise clique-independent edges is called a *clique-independent set*.

Example 1.3.5. Let G be the graph as shown in Figure 1.3.3.

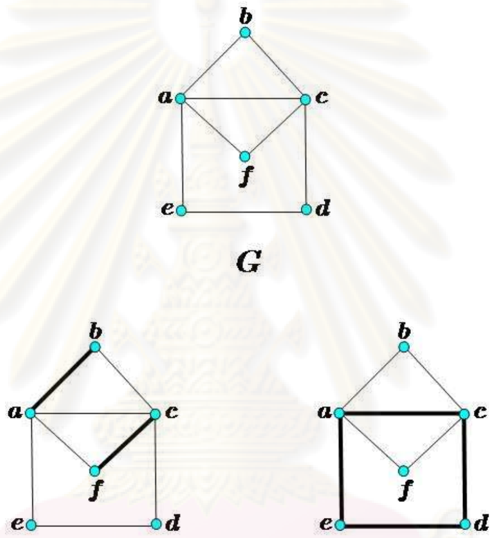

Figure 1.3.3: A pair of clique-independent edges and a clique-independent set of a graph

In Figure 1.3.3, note that bf is not an edge in G, so there is no clique in G containing both ab and cf. Thus, ab and cf are clique-independent edges of G. In contrast, ab and ac are not clique-independent edges of G because they both are in $K_3(a, b, c)$.

Let $I = \{ac, cd, de, ea\}$ be a subset of the edge set of G. Since G does not contain ad and ce, it can be concluded that I is a set of pairwise clique-independent edges, so I is a clique-independent set of G.

□

Example 1.3.5 suggests some properties in the following remark.

Remark 1.3.6.

1. Let e, f be any two edges in a graph G. If there exist two endpoints of e and f that are not adjacent in G, then e and f are clique-independent edges of G.

2. Since different elements in a clique-independent set I of a graph G must be partitioned by different cliques of G, $|I| \leq cp(G)$.

1.4 Remarks for our terminologies

Let G be any graph. For convenience, if an edge e in $E(G)$ is covered by a 2-clique in a clique partition of G, then we will also refer to e as such a clique. Moreover, we will also refer to $E(G)$ as a set of 2-cliques.

Example 1.4.1. Let G_1, G_2 be graphs and $G_1 \underset{H}{\diamond} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 1.4.1.

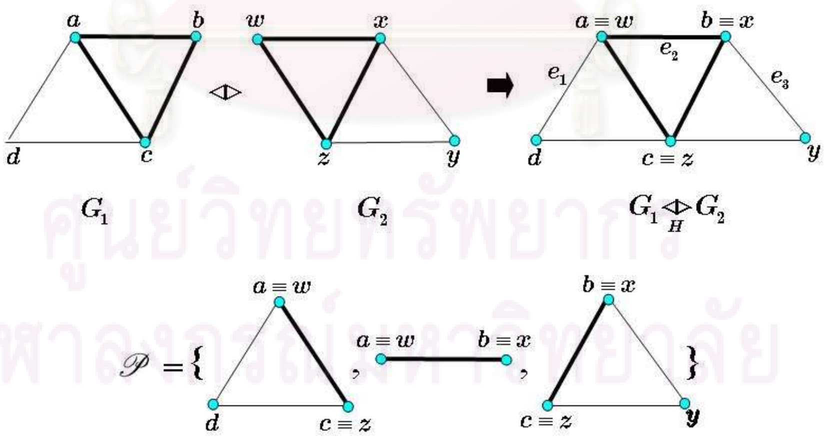

Figure 1.4.1: A minimum clique partition of the glued graph

Let $\mathscr{P} = \{K_3(a \equiv w, c \equiv z, d), K_3(b \equiv x, y, c \equiv z), K_2(a \equiv w, b \equiv x)\}$. Then $\mathscr{P}$ is a clique partition of $G_1 \underset{H}{\diamond} G_2$. Thus $cp(G_1 \underset{H}{\diamond} G_2) \leq 3$. Consider $I = \{e_1, e_2, e_3\}$ in $G_1 \underset{H}{\diamond} G_2$ as in the Figure 1.4.1. Then I is a clique-independent set of $G_1 \underset{H}{\diamond} G_2$.

Thus $cp(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \geq |I| = 3$, so $cp(G) = 3$. This implies that $\mathscr{P}$ is a minimum clique partition of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$.

Note that the vertices a, b and c in G_1 correspond to vertices $a \equiv w$, $b \equiv x$ and $c \equiv z$, respectively, in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. Also, the vertices w, x and z in G_2 correspond to vertices $a \equiv w$, $b \equiv x$ and $c \equiv z$, respectively, in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. Thus corespondence cliques in $\mathscr{P}$ must be cliques of G_1 or G_2, i.e., $K_3(a \equiv w, c \equiv z, d) = K_3(a, b, c)$, $K_2(a, b) = K_2(a \equiv w, b \equiv x) = K_2(w, x)$ and $K_3(b \equiv x, y, c \equiv z) = K_3(x, y, z)$. □

Hence throughout this thesis, we simplify the terminologies by considering subgraphs of original graphs as subgraphs of their glued graphs, and subgraphs in the clone of a glued graph are subgraphs in the corresponding clones of both original graphs. For example,

- if Q is a clique in G_1, then Q is also a clique in $G_1 \triangleleft\!\triangleright G_2$
- if e is an edge in the clone of $G_1 \triangleleft\!\triangleright G_2$, then e is also an edge in G_1 and G_2.

CHAPTER II

BOUNDS OF CLIQUE PARTTION NUMBERS OF GLUED GRAPHS

There are two sections in this chapter. The first one, we give some preliminary results, and show a bound of clique partition numbers of glued graphs along with its sharpness. In the last section, we study clique partitions of clique-preserving glued graphs.

2.1 Preliminaries

Let H be a subgraph of a graph G. We write $G - H$ for the subgraph of G obtained by deleting the set of edges $E(H)$. Note that $G - e$ stands for the subgraph of G resulting from removing edge e out of G for any edge e of G.

Theorems 2.1.1−2.1.5 [7, 8, 12, 2] are known results about the effects of an edge deletion and an n-clique deletion on the clique partition number. These help us to investigate bounds of clique partition numbers of glued graphs at K_2-clones and K_3-clones which will be considered in the next chapter.

Theorem 2.1.1. [7] *Let e be an edge of a graph G and s the order of the smallest clique containing the edge e among all of the minimum clique partitions of the graph G. Then $cp(G) - 1 \leq cp(G - e) \leq cp(G) + s - 2$.*

The next example illustrates the notation s in Theorem 2.1.1.

Example 2.1.2. Let G be the graph with an edge e as shown in Figure 2.1.1

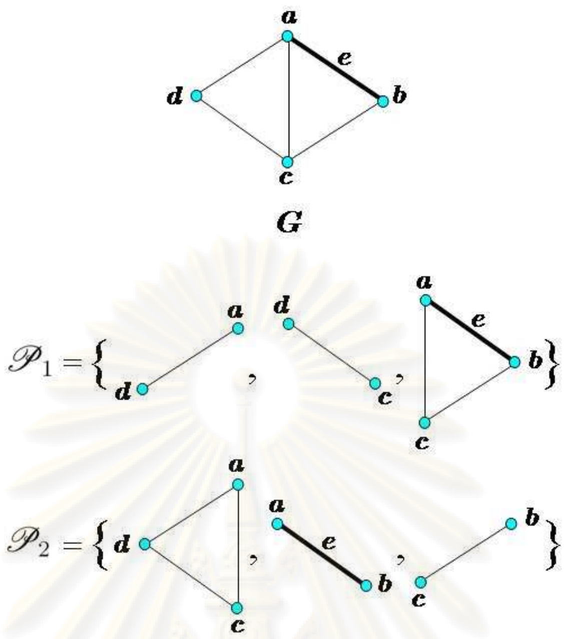

Figure 2.1.1: Two minimum clique partitions of a graph

Let s be the order of the smallest clique containing the edge e among all of the minimum clique partitions of the graph G. It can be easily seen that $\mathscr{P}_1 = \{K_2(a,d), K_2(c,d), K_3(a,b,c)\}$ and $\mathscr{P}_2 = \{K_3(a,c,d), K_2(a,b), K_2(b,c)\}$ are the only two minimum clique partitions of G. Then the smallest clique containing an edge e is an element in $\mathscr{P}_2$, namely $K_2(a,b)$. Thus $s = 2$.

□

Theorem 2.1.3. [8] *For $n \geq 3$, $cp(K_n - e) = n - 1$ where e is any edge of K_n.*

Theorem 2.1.4. [12] *For $n \geq 4$, $n - 1 \leq cp(K_n - K_3) \leq 2n - 5$.*

Theorem 2.1.5. [2] *If an n-vertex graph G is neither the complete graph nor trivial graph, $cp(G) + cp(\overline{G}) \geq n$.*

Propositions 2.1.6 and 2.1.7 are further results of a clique deletion and a path deletion.

Proposition 2.1.6. *Let G be graph contain path P_3. Then $cp(G)-2 \leq cp(G-P_3)$.*

Proof. Let $\mathscr{P}$ and $\mathscr{P}'$ be minimum clique partitions of P_3 and $G-P_3$, respectively. Then $|\mathscr{P}| = 2$ and $\mathscr{P} \cup \mathscr{P}'$ is a clique partition of G. Thus,

$$cp(G) \leq \left|\mathscr{P} \cup \mathscr{P}'\right| = |\mathscr{P}| + \left|\mathscr{P}'\right| = 2 + cp(G - P_3).$$

Hence, $cp(G) - 2 \leq cp(G - P_3)$. □

Proposition 2.1.7. *For any graph G and a clique C of G, $cp(G)-1 \leq cp(G-C)$, and the equality holds if and only if there exists a minimum clique partition of G containing C.*

Proof. Let $\mathscr{P}$ and $\mathscr{P}'$ be minimum clique partitions of G and $G-C$, respectively. Then $\mathscr{P}' \cup \{C\}$ is a clique partition of G, so $cp(G) \leq \left|\mathscr{P}' \cup \{C\}\right|$. Note that $\left|\mathscr{P}' \cup \{C\}\right| = \left|\mathscr{P}'\right| + 1 = cp(G - C) + 1$. Hence $cp(G) - 1 \leq cp(G - C)$.

For necessity, suppose that every minimum clique partition of G does not contain C. Since $\mathscr{P}' \cup \{C\}$ is a partition of G containing C, $\left|\mathscr{P}' \cup \{C\}\right| > |\mathscr{P}|$. Hence,

$$cp(G - C) + 1 = \left|\mathscr{P}'\right| + 1 = \left|\mathscr{P}' \cup \{C\}\right| > |\mathscr{P}| = cp(G).$$

For sufficiency, assume that $C \in \mathscr{P}$. Then $\mathscr{P} \setminus \{C\}$ is a clique partition of $G - C$. Thus $cp(G - C) \leq |\mathscr{P} \setminus \{C\}| = |\mathscr{P}| - 1 = cp(G) - 1$. Since $cp(G) - 1 \leq cp(G - C)$ always, we have $cp(G - C) = cp(G) - 1$. □

A generalized concept of a clique partition is a clique covering which is defined as follows.

A *clique covering* of a graph G is a set of cliques of G which together contain each edge of G at least once. The *clique covering number* of a graph G, denoted by $cc(G)$, is the smallest cardinality of clique coverings of G.

For any glued graph $G_1 \diamond G_2$, Pimpasalee [9] proved that $cc(G_1 \diamond G_2) \leq cc(G_1) + cc(G_2)$. So we investigate whether or not $cp(G_1) + cp(G_2)$ is an upper bound of clique partition numbers of any glued graphs. However, we find that the clique partition number of a glued graph can be more than, less than or equal to $cp(G_1) + cp(G_2)$. Examples 2.1.8–2.1.10 show a glued graph with its clique partition number for each of the possible cases.

It is important to first note here that, thoughout this thesis for convenience, we refer K_n in the glued graph $G_1 \underset{K_n}{\diamond} G_2$ to be only the K_n-clone, not an arbitrary copy of K_n.

Example 2.1.8. *A glued graph $G_1 \diamond G_2$ with $cp(G_1 \diamond G_2) > cp(G_1) + cp(G_2)$.*

For $m, n > 2$, we will show that $cp(K_m \underset{K_2}{\diamond} K_n) = \min\{m, n\}$.
Let $2 < n \leq m$. Since $K_m \underset{K_2}{\diamond} K_n$ can be partitioned into the sets of clique K_m and all $(n-1)$ cliques in a minimum clique partition of $K_n - K_2$, $cp(K_m \underset{K_2}{\diamond} K_n) \leq 1 + (n-1) = n$. Since $n \leq m$ and $cp(K_n - K_2) = n - 1$ by Theorem 2.1.3, we obtain that $cp(K_m \underset{K_2}{\diamond} K_n) > n - 1$. Thus $cp(K_m \underset{K_2}{\diamond} K_n) \geq n$. It follows that $cp(K_m \underset{K_2}{\diamond} K_n) = n$, so $cp(K_m \underset{K_2}{\diamond} K_n) = \min\{m, n\}$. Note that $cp(K_m) = 1 = cp(K_n)$. Hence, $cp(K_m) + cp(K_n) = 2 < \min\{m, n\} = cp(K_m \underset{K_2}{\diamond} K_n)$.

□

Example 2.1.9. *A glued graph $G_1 \diamond G_2$ with $cp(G_1 \diamond G_2) < cp(G_1) + cp(G_2)$.*

Let P_m and P_n be paths with m and n vertices, respectively, where $m, n > 2$. Let $P_m \underset{H}{\diamond} P_n$ be any glued graph at the H-clone. Promsakon [10] proved that $P_m \underset{H}{\diamond} P_n$ is a path. Since a path is a K_3-free graph, $cp(P_m \underset{H}{\diamond} P_n) = e(P_m \underset{H}{\diamond} P_n) = e(P_m) + e(P_n) - e(H) = cp(P_m) + cp(P_n) - e(H)$. This shows that $cp(P_m \underset{H}{\diamond} P_n) < cp(P_m) + cp(P_n)$.

□

Example 2.1.10. *A glued graph $G_1 \diamond G_2$ with $cp(G_1 \diamond G_2) = cp(G_1) + cp(G_2)$.*

Let G_1, G_2 be graphs and $G_1 \underset{H}{\diamond} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 2.1.2.

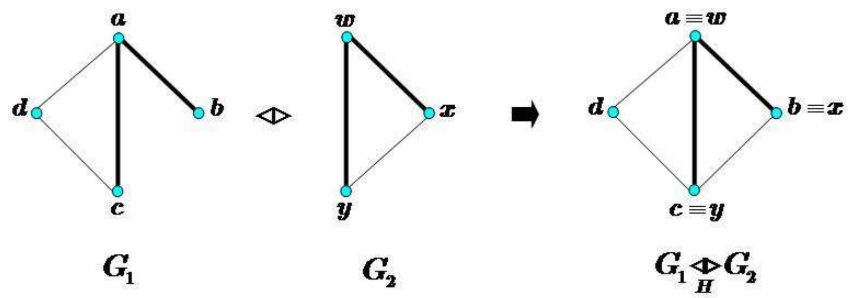

Figure 2.1.2: A glued graph $G_1 \Diamond G_2$ with clique partition number $cp(G_1) + cp(G_2)$

It is obvious that $cp(G_1) = 2$, $cp(G_2) = 1$ and $cp(G_1 \underset{H}{\Diamond} G_2) = 3$. Hence $cp(G_1 \underset{H}{\Diamond} G_2) = 3 = 2 + 1 = cp(G_1) + cp(G_2)$.

$\square$

Although $cp(G_1) + cp(G_2)$ is not a bound of $cp(G_1 \Diamond G_2)$ in general, we still look for another bound of $cp(G_1 \Diamond G_2)$ in terms of $cp(G_1)$ and $cp(G_2)$. Recall that, $E(G)$ is also used as a set of clique partition of G in which all elements have order 2.

Theorem 2.1.11. *Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Diamond} G_2$ be any glued graph at the H-clone . Then*

$$1 \leq cp(G_1 \underset{H}{\Diamond} G_2) \leq \min\{cp(G_1) + cp(G_2 - H), cp(G_2) + cp(G_1 - H)\}.$$

Proof. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be minimum clique partitions of G_1 and G_2, respectively. Let $\mathscr{P}'$ and $\mathscr{P}''$ be minimum clique partitions of $G_1 - H$ and $G_2 - H$, respectively. Both $\mathscr{P}_1 \cup \mathscr{P}''$ and $\mathscr{P}_2 \cup \mathscr{P}'$ are clique partitions of $G_1 \underset{H}{\Diamond} G_2$. Thus, $cp(G_1 \underset{H}{\Diamond} G_2) \leq \left|\mathscr{P}_1 \cup \mathscr{P}''\right|$ and $cp(G_1 \underset{H}{\Diamond} G_2) \leq \left|\mathscr{P}_2 \cup \mathscr{P}'\right|$. Note that $\mathscr{P}_1$ and $\mathscr{P}''$ are disjoint, so are $\mathscr{P}_2$ and $\mathscr{P}'$. Then $cp(G_1 \underset{H}{\Diamond} G_2) \leq \left|\mathscr{P}_1 \cup \mathscr{P}''\right| = |\mathscr{P}_1| + \left|\mathscr{P}''\right| = cp(G_1) + cp(G_2 - H)$. Similarly, $cp(G_1 \underset{H}{\Diamond} G_2) \leq cp(G_2) + cp(G_1 - H)$. Thus,

$$cp(G_1 \underset{H}{\Diamond} G_2) \leq \min\{cp(G_1) + cp(G_2 - H), cp(G_2) + cp(G_1 - H)\}.$$

Hence, $1 \leq cp(G_1 \underset{H}{\Diamond} G_2) \leq \min\{cp(G_1) + cp(G_2 - H), cp(G_2) + cp(G_1 - H)\}$. $\square$

Next, we give examples to show the sharpness of the bound in Theorem 2.1.11.

Example 2.1.12. Let G_1 be a Hamiltonian graph on n vertices with a Hamiltonian path P and $G_2 = \overline{G}_1 \cup P$.

Figure 2.1.3: The sharpness of the lower bound in Theorem 2.1.11

Then the resulting glued graph $G_1 \underset{P}{\diamond} G_2$ is a complete graph as illustrated in Figure 2.1.3, so $cp(G_1 \underset{P}{\diamond} G_2) = 1$. Furthermore, note that neither G_1 nor G_2 is a complete graph, so its clique partition number is more than 1. It is noticeable that the graph gluing of original graphs with any arbitrary large clique partition number could yield a resulting glued graph with clique partition number 1. □

Example 2.1.13. Let $m \geq n \geq 2$. Consider the glued graph $K_m \underset{K_2}{\diamond} K_n$ is shown in Figure 2.1.4.

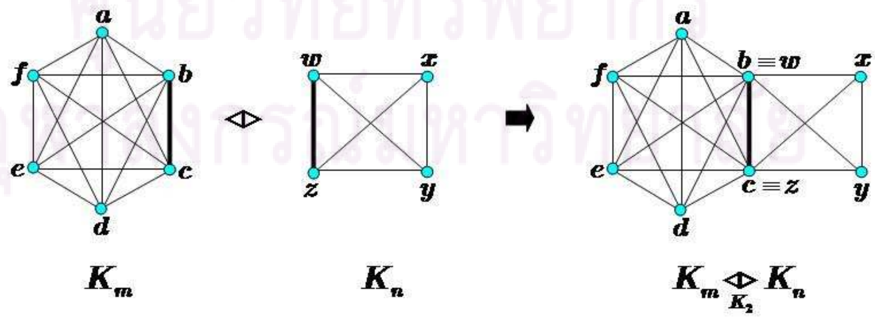

Figure 2.1.4: The sharpness of the upper bound in Theorem 2.1.11

We have that $cp(K_m) = 1 = cp(K_n)$ and by Example 2.1.8, $cp(K_m \underset{K_2}{\diamond} K_n) = \min\{m, n\} = n$. Since $cp(K_m - K_2) = m - 1$ and $cp(K_n - K_2) = n - 1$ by

Theorem 2.1.3, we have that $cp(K_m) + cp(K_n - K_2) = 1 + (n-1) = n$ and $cp(K_n) + cp(K_m - K_2) = 1 + (m-1) = m$. Thus, $\min\{cp(K_m) + cp(K_n - K_2), cp(K_n) + cp(K_m - K_2)\} = n$. Hence, $cp(K_m \underset{K_2}{\Leftrightarrow} K_n) = n = \min\{cp(K_m) + cp(K_n - K_2), cp(K_n) + cp(K_m - K_2)\}$.

□

In next section, we study bounds of clique partition numbers of clique-preserving glued graphs.

2.2 Clique partitions of clique-preserving glued graphs

Our purpose in this section is to study a lower bound of clique partition numbers of clique-preserving glued graphs and gives a characterization when its value is at such lower bound.

Definition 2.2.1. An edge $e = ab$ in any glued graph $G_1 \Leftrightarrow G_2$ is called *a new edge for the original graph* G_i, $i = 1$ or 2 if the corresponding vertices of a and b in G_i are not adjacent. A clique in any glued graph $G_1 \Leftrightarrow G_2$ is called *a new clique for the original graph* G_i, $i = 1$ or 2 if all corresponding vertices of $v_1, ..., v_n$ in G_i do not form a clique in G_i. A *clique-preserving glued graph* is a glued graph which does not have a new clique for any original graphs.

Example 2.2.2. Let G_1, G_2 be graphs and $G_1 \underset{H}{\Leftrightarrow} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 2.2.1.

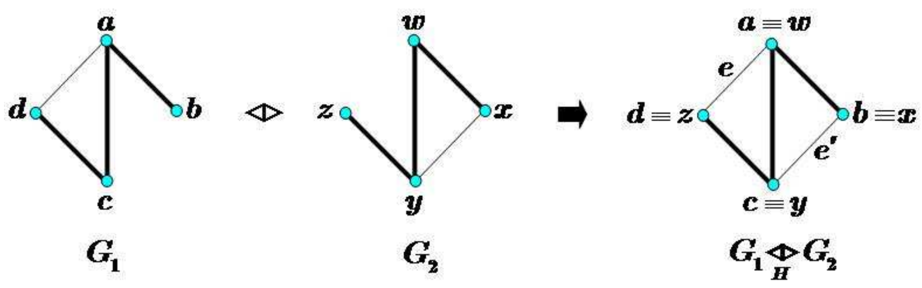

Figure 2.2.1: A glued graph containing new cliques

In Figure 2.2.1, e' is an edge in $G_1 \underset{H}{\Leftrightarrow} G_2$ but b and c are not adjacent in G_1, so e' is a new edge for G_1. Similarly, e is a new edge for G_2. Note that $K_3(a \equiv w, b \equiv x, c \equiv y)$ is a clique in $G_1 \underset{H}{\Leftrightarrow} G_2$ but G_1 does not contain $K_3(a, b, c)$, so $K_3(a \equiv w, b \equiv x, c \equiv y)$ is a new clique for G_1. Similarly, $K_3(a \equiv w, c \equiv y, d \equiv z)$ is a new clique for G_2. Hence, $G_1 \underset{H}{\Leftrightarrow} G_2$ is not a clique-preserving glued graph. □

When a glued graph $G_1 \Leftrightarrow G_2$ is a clique-preserving glued graph, a clique in a minimum clique partition of $G_1 \Leftrightarrow G_2$ must be a clique in G_1 or G_2. Thus, being a clique-preserving glued graph benefits the investigation of its clique partition number of each original graph.

We first observe some basic properties of a new edge and a new clique of a glued graph in the following remark.

Remark 2.2.3.

1. If a glued graph $G_1 \Leftrightarrow G_2$ has a new clique for G_i, $i = 1$ or 2, then $G_1 \Leftrightarrow G_2$ has a new edge for G_i.

2. A new edge of a glued graph cannot be a new edge for both original graphs at the same time.

3. Both endpoints of a new edge of a glued graph must lie in the clone.

Proposition 2.2.4. [9] *If H is an induced subgraph of both G_1 and G_2, then $G_1 \underset{H}{\Leftrightarrow} G_2$ is a clique-preserving glued graph.*

Back to Example 2.1.12, we have a glued graph with the property that $cp(G_1 \underset{P}{\Diamond} G_2) < \max\{cp(G_1), cp(G_2)\}$. We next consider a condition to guarantee that a resulting glued graph needs at least as many cliques to partition as its original graphs do.

Theorem 2.2.5. *If* $G_1 \Diamond G_2$ *is a clique-preserving glued graph, then*

$$cp(G_1 \Diamond G_2) \geq \max\{cp(G_1), cp(G_2)\}.$$

Proof. Assume that $G_1 \Diamond G_2$ is a clique-preserving glued graph. Then $G_1 \Diamond G_2$ does not have a new clique for G_1. Hence at least $cp(G_1)$ cliques are needed to partition the copy of G_1 in $G_1 \Diamond G_2$. Therefore $cp(G_1) \leq cp(G_1 \Diamond G_2)$. Similarly, $cp(G_2) \leq cp(G_1 \Diamond G_2)$. Hence $cp(G_1 \Diamond G_2) \geq \max\{cp(G_1), cp(G_2)\}$. □

The converse of Theorem 2.2.5 does not hold as shown in Example 2.2.6.

Example 2.2.6. Let G_1, G_2 be graphs and $G_1 \underset{H}{\Diamond} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 2.2.2.

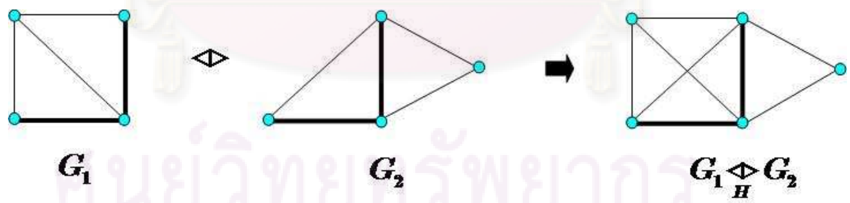

Figure 2.2.2: A glued graph illustrating that the converse of Theorem 2.2.5 does not hold

Since $cp(K_n - e) = n - 1$ where e is an edge in K_n by Theorem 2.1.3, we have that $cp(G_1) = cp(K_4 - e) = 3 = cp(G_2)$. Note that $G_1 \underset{H}{\Diamond} G_2 \cong K_4 \underset{K_2}{\Diamond} K_3$, by Example 2.1.8, $cp(G_1 \underset{H}{\Diamond} G_2) = 3$. We can see that $cp(G_1 \underset{H}{\Diamond} G_2) = 3 = \max\{cp(G_1), cp(G_2)\}$. But 4-clique in $G_1 \underset{H}{\Diamond} G_2$ is a new clique for G_1 and G_2, so $G_1 \underset{H}{\Diamond} G_2$ is not a clique-preserving glued graph. □

Now we consider the set of all cliques in a minimum clique partition of a glued graph which belong to each original graph.

Definition 2.2.7. For a glued graph $G_1 \underset{H}{\Diamond} G_2$, let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{H}{\Diamond} G_2$. We define

$$\mathscr{P}[G_1] = \{C \in \mathscr{P} \mid C \text{ is a clique of } G_1\},$$

$$\mathscr{P}[G_2] = \{C \in \mathscr{P} \mid C \text{ is a clique of } G_2\},$$

$$E_1[\mathscr{P}] = \{e \in E(H) \mid e \text{ is not covered by any clique in } \mathscr{P}[G_1]\} \text{ and}$$

$$E_2[\mathscr{P}] = \{e \in E(H) \mid e \text{ is not covered by any clique in } \mathscr{P}[G_2]\}\,.$$

The following example illustrates the Definition 2.2.7.

Example 2.2.8. Let G_1, G_2 be graphs and $G_1 \underset{H}{\Diamond} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 2.2.3.

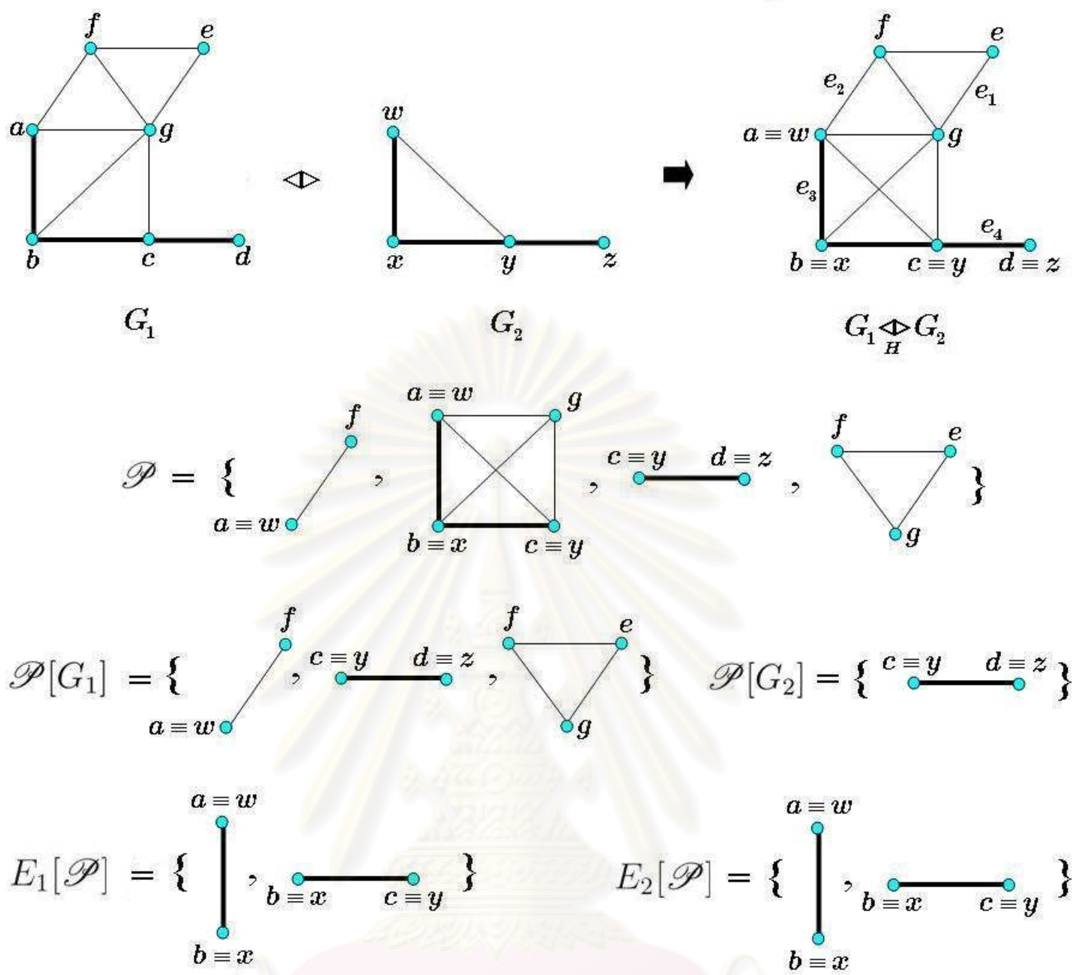

Figure 2.2.3: $\mathscr{P}[G_1]$, $\mathscr{P}[G_2]$, $E_1[\mathscr{P}]$ and $E_2[\mathscr{P}]$ of a glued graph

Let $\mathscr{P} = \{K_2(f, a \equiv w), K_2(c \equiv y, d \equiv z), K_3(f, e, g), K_4(a \equiv w, b \equiv x, c \equiv y, g)\}$. Then $\mathscr{P}$ is a clique partition of $G_1 \underset{H}{\Diamond} G_2$. Thus, $cp(G_1 \underset{H}{\Diamond} G_2) \leq 4$. Consider $I = \{e_1, e_2, e_3, e_4\}$ in $G_1 \underset{H}{\Diamond} G_2$ as in the Figure 2.2.3. Then I is a clique-independent set of $G_1 \underset{H}{\Diamond} G_2$. Thus $cp(G_1 \underset{H}{\Diamond} G_2) \geq |I| = 4$. Therefore, $cp(G_1 \underset{H}{\Diamond} G_2) = 4$. Since $|\mathscr{P}| = 4$, $\mathscr{P}$ is a minimum clique partition of $G_1 \underset{H}{\Diamond} G_2$. It is easy to see that $\mathscr{P}[G_1] = \{K_2(f, a \equiv w), K_2(c \equiv y, d \equiv z), K_3(f, e, g)\}$, $\mathscr{P}[G_2] = \{K_2(c \equiv y, d \equiv z)\}$ and $E_1[\mathscr{P}] = \{K_2(a \equiv w, b \equiv x), K_2(b \equiv x, c \equiv y)\} = E_2[\mathscr{P}]$.

□

Remark 2.2.9. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{H}{\Diamond} G_2$. Then

1. $\mathscr{P}[G_1] \cup \mathscr{P}[G_2] \subseteq \mathscr{P}$.

2. $E_1[\mathscr{P}] \cup E_2[\mathscr{P}] \subseteq E(H)$.

3. $(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])$ is a clique partition of H, moreover $|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| \leq e(H)$.

4. If $\mathscr{P}[G_1] \cap \mathscr{P}[G_2]$ contains any clique of order more than two, then $|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| < e(H)$.

Proposition 2.2.10. *For a minimum clique partition $\mathscr{P}$ of a clique-preserving glued graph $G_1 \Diamond G_2$, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$ and $E_1[\mathscr{P}] \cap E_2[\mathscr{P}] = \varnothing$.*

Proof. Let $G_1 \Diamond G_2$ be a clique-preserving glued graph. Assume that $\mathscr{P}$ is a minimum clique partition of $G_1 \Diamond G_2$. By Remark 2.2.9, we have that $\mathscr{P}[G_1] \cup \mathscr{P}[G_2] \subseteq \mathscr{P}$. Since $G_1 \Diamond G_2$ is a clique-preserving glued graph, every clique in the glued graph must be a copy of cliques in G_1 or G_2. Thus, $\mathscr{P} \subseteq \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Hence, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$.

Suppose that $e \in E_1[\mathscr{P}] \cap E_2[\mathscr{P}]$. Then e is not covered by any clique in $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. This implies that $\mathscr{P}[G_1] \cup \mathscr{P}[G_2]$ is not a clique partition of $G_1 \underset{H}{\Diamond} G_2$, which is a contradiction. Hence, $E_1[\mathscr{P}] \cap E_2[\mathscr{P}] = \varnothing$. □

The following remark helps us to determine the clique partition number of a clique-preserving glued graph.

Remark 2.2.11. Let $\mathscr{P}$ be a minimum clique partition of a clique-preserving glued graph $G_1 \underset{H}{\Diamond} G_2$. For $i = 1, 2$, $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a clique partition of G_i.

Theorem 2.2.12. *For any clique-preserving glued graph* $G_1 \underset{H}{\Diamond} G_2$,

$$cp(G_1 \underset{H}{\Diamond} G_2) \geq cp(G_1) + cp(G_2) - e(H).$$

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{H}{\Diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Note that $\mathscr{P}[G_i]$ and $E_i[\mathscr{P}]$ are disjoint for all $i = 1, 2$, $E_1[\mathscr{P}] \cup E_2[\mathscr{P}]$ and $\mathscr{P}[G_1] \cap \mathscr{P}[G_2]$ are also disjoint. Then,

$$\begin{aligned}
|\mathscr{P}| &= |\mathscr{P}[G_1] \cup \mathscr{P}[G_2]| \\
&= |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| - |\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| \\
&= |\mathscr{P}[G_1] \cup E_1[\mathscr{P}]| + |\mathscr{P}[G_2] \cup E_2[\mathscr{P}]| - |\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| - |E_1[\mathscr{P}] \cup E_2[\mathscr{P}]| \\
&= |\mathscr{P}[G_1] \cup E_1[\mathscr{P}]| + |\mathscr{P}[G_2] \cup E_2[\mathscr{P}]| - |(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])|.
\end{aligned}$$

Note further that $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a clique partitions of G_i for all $i = 1, 2$, we have $|\mathscr{P}[G_i] \cup E_i[\mathscr{P}]| \geq cp(G_i)$ for all $i = 1, 2$ and
$|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| \leq e(H)$. Thus,

$$cp(G_1 \underset{H}{\Diamond} G_2) = |\mathscr{P}| \geq cp(G_1) + cp(G_2) - e(H).$$

□

Example 2.2.13. *The sharpness of the lower bound in Theorem 2.2.12.*

Let G_1, G_2 be graphs and $G_1 \underset{H}{\Diamond} G_2$ be the glued graph at H-clone where all clones are shown as bold edges in Figure 2.2.4.

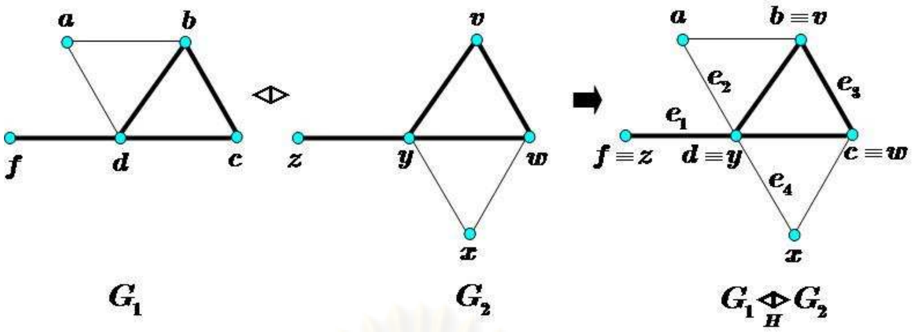

Figure 2.2.4: The sharpness of the lower bound in Theorem 2.2.12

Observe that $cp(G_1) = 4 = cp(G_2)$ and $e(H) = 4$. Let $\mathscr{P} = \{K_2(f \equiv z, d \equiv y), K_3(d \equiv y, a, b \equiv v), K_2(b \equiv v, c \equiv w), K_3(c \equiv w, x, d \equiv y)\}$. Then $\mathscr{P}$ is a clique partition of $G_1 \underset{H}{\Diamond} G_2$, so $cp(G_1 \underset{H}{\Diamond} G_2) \leq |\mathscr{P}| = 4$. Let $I = \{e_1, e_2, e_3, e_4\}$. Then I is a clique-independent set of $G_1 \underset{H}{\Diamond} G_2$. Thus $cp(G_1 \underset{H}{\Diamond} G_2) \geq |I| = 4$. Hence, $cp(G_1 \underset{H}{\Diamond} G_2) = 4 = 4 + 4 - 4 = cp(G_1) + cp(G_2) - e(H)$. □

Theorem 2.2.14. *Let $G_1 \underset{H}{\Diamond} G_2$ be any clique-preserving glued graph. Then $cp(G_1 \underset{H}{\Diamond} G_2) = cp(G_1) + cp(G_2) - e(H)$ if and only if there are minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of G_1 and G_2, respectively, such that for each edge $e \in E(H)$, e must be covered by a 2-clique in $\mathscr{P}_1$ or $\mathscr{P}_2$.*

Proof. For necessity, assume that $cp(G_1 \underset{H}{\Diamond} G_2) = cp(G_1) + cp(G_2) - e(H)$.
Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{H}{\Diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Note that

$$|\mathscr{P}| = |\mathscr{P}[G_1] \cup E_1[\mathscr{P}]| + |\mathscr{P}[G_2] \cup E_2[\mathscr{P}]| - |(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])|\,.$$

Since $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a clique partition of G_i for all $i = 1, 2$, $|\mathscr{P}[G_i] \cup E_i[\mathscr{P}]| \geq cp(G_i)$. Besides, $|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| \leq e(H)$. Together with $|\mathscr{P}| = cp(G_1) + cp(G_2) - e(H)$, we can conclude that $|\mathscr{P}[G_i] \cup E_i[\mathscr{P}]| = cp(G_i)$ for all $i = 1, 2$ and $|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| = e(H)$. Hence, $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a minimum clique partition of G_i for all $i = 1, 2$.

Let e be an edge in the H-clone of $G_1 \underset{H}{\Diamond} G_2$. If $e \in \mathscr{P}[G_1] \cap \mathscr{P}[G_2]$, then e is covered by a 2-clique in $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. Thus $e \in \mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ for all $i = 1, 2$. Suppose that $e \notin \mathscr{P}[G_1] \cap \mathscr{P}[G_2]$. Then there exists a clique C of order more than two in $\mathscr{P}$ covering e. Without loss of generality, assume that $C \in \mathscr{P}[G_1]$. Then $e \in E_2[\mathscr{P}]$, so $e \in \mathscr{P}[G_2] \cup E_2[\mathscr{P}]$.

For sufficiency, assume that G_1 and G_2 have minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively such that satisfy the condition in the right hand side of the statement. Let $A = \{e \in E(H) \mid e \in \mathscr{P}_1\}$ and $B = \{f \in E(H) \mid f \in \mathscr{P}_2\}$.
Note that $|A| + |B| - |A \cap B| = e(H)$ and $(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)$ is a clique partition of $G_1 \underset{H}{\Diamond} G_2$.
Thus, $|(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)| \geq cp(G_1 \underset{H}{\Diamond} G_2)$. Hence,

$$\begin{aligned} cp(G_1) + cp(G_2) - e(H) &= |\mathscr{P}_1| + |\mathscr{P}_2| - |A| - |B| + |A \cap B| \\ &= |(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)| \\ &\geq cp(G_1 \underset{H}{\Diamond} G_2). \end{aligned}$$

By Theorem 2.2.12, $cp(G_1 \underset{H}{\Diamond} G_2) = cp(G_1) + cp(G_2) - e(H)$. $\square$

CHAPTER III

MAIN RESULTS

In this chapter we focus on a glued graph at K_n-clone because the clone is always an induced subgraph of both original graphs, so the resulting glued graph is a clique-preserving glued graph. We first study properties of clique partitions of glued graphs at K_n-clones in Section3.1. Later, we investigate bounds of the clique partition numbers of glued graphs at K_2-clones and K_3-clones in Section3.2 and Section3.3, respectively.

Recall that we refer K_n in the glued graph $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$ to be only the K_n-clone, not an arbitrary copy of K_n.

3.1 Some properties of clique partitions of glued graphs at K_n-clones

Theorem 3.1.1. *For $m \geq n > r \geq 2$, $cp(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) \leq (r-1)(n-r)+2$.*

Proof. Let $m \geq n > r \geq 2$. Since $m \geq n$, we can use the m-clique, all cliques in a minimum clique partition of $K_{n-r+2} - e$ and $(r-2)(n-r)$ copies of 2-clique to partition $K_m \underset{K_r}{\triangleleft\!\triangleright} K_n$, where e is an edge of the clone. Note that by Theorem 2.1.3, $cp(K_{n-r+2} - e) = (n-r+2) - 1 = n-r+1$. Hence,

$$cp(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) \leq 1 + (n-r+1) + (r-2)(n-r) = (r-1)(n-r)+2.$$

□

Lemma 3.1.2. *If $G_1 \underset{K_n}{\Diamond} G_2$ has a minimum clique partition containing the K_n-clone, then G_1 or G_2 has a minimum clique partition containing the K_n-clone.*

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_n}{\Diamond} G_2$ containing the K_n-clone. Then $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively. Suppose that all minimum clique partitions of G_1 and G_2 do not contain the K_n-clone. Let $\mathscr{P}_1$ be a minimum clique partition of G_1. Note that

$$\mathscr{P} = (\mathscr{P}[G_1] \smallsetminus \{K_n\}) \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\}) \cup \{K_n\}.$$

Since $\mathscr{P}_1$ does not contain the K_n-clone and $\mathscr{P}[G_1]$ is a clique partition of G_1 containing the K_n-clone, $|\mathscr{P}[G_1]| > |\mathscr{P}_1| = cp(G_1)$, consequently, $|\mathscr{P}[G_1] \smallsetminus \{K_n\}| \geq cp(G_1)$. Thus,

$$\begin{aligned} |\mathscr{P}| &= |\mathscr{P}[G_1] \smallsetminus \{K_n\}| + |\mathscr{P}[G_2] \smallsetminus \{K_n\}| + 1 \\ &\geq cp(G_1) + |\mathscr{P}[G_2] \smallsetminus \{K_n\}| + 1. \end{aligned}$$

Observe that $\mathscr{P}_1 \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\})$ is also a clique partition of $G_1 \underset{K_n}{\Diamond} G_2$ and $|\mathscr{P}_1 \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\})| = cp(G_1) + |\mathscr{P}[G_2] \smallsetminus \{K_n\}|$, this contradicts the minimality of $\mathscr{P}$. Thus, G_1 or G_2 has a minimum clique partition containing the K_n-clone. □

The converse of Lemma 3.1.2 does not hold as shown in Example 3.1.3.

Example 3.1.3. Let G_1, G_2 be graphs and $G_1 \underset{K_3}{\Diamond} G_2$ be the glued graph at K_3-clone where all clones are shown as bold edges in Figure 3.1.1.

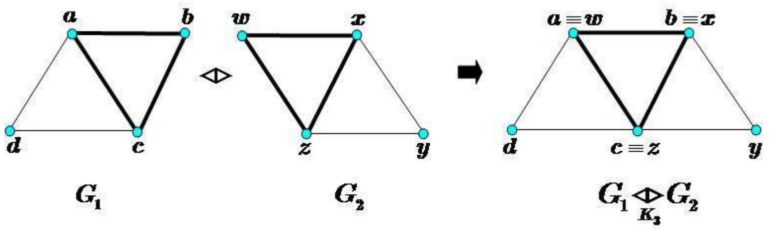

Figure 3.1.1: A glued graph illustrating that the converse of Lemma 3.1.2 does not hold

From Example 1.3.2, $cp(G_1) = 3 = cp(G_2)$. Then $\mathscr{P}_1 = \{K_3(a, b, c), K_2(a, d), K_2(c, d)\}$ and $\mathscr{P}_2 = \{K_3(w, x, z), K_2(x, y), K_2(y, z)\}$ are minimum clique partitions of G_1 and G_2, respectively, containing the K_3-clone. From Example 1.4.1, $\mathscr{P} = \{K_3(a \equiv w, c \equiv z, d), K_3(b \equiv x, y, c \equiv z), K_2(a \equiv w, b \equiv x)\}$ is a minimum clique partition of $G_1 \underset{K_3}{\lhd\!\rhd} G_2$. Note that $\mathscr{P}$ does not contain the K_3-clone and $\mathscr{P}$ is the only minimum clique partition of $G_1 \underset{K_3}{\lhd\!\rhd} G_2$. Hence, $G_1 \underset{K_3}{\lhd\!\rhd} G_2$ does not have a minimum clique partition containing the K_3-clone.

□

Theorem 3.1.4. *If $G_1 \underset{K_n}{\lhd\!\rhd} G_2$ has a minimum clique partition containing the K_n-clone, then $cp(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cp(G_1) + cp(G_2) - 1$*

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_n}{\lhd\!\rhd} G_2$ containing the K_n-clone. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $\mathscr{P}$ contains the K_n-clone, $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_n\}$. Note that $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively, so $|\mathscr{P}[G_1]| \geq cp(G_1)$ and $|\mathscr{P}[G_2]| \geq cp(G_2)$. Thus,

$$\begin{aligned} cp(G_1 \underset{K_n}{\lhd\!\rhd} G_2) &= |\mathscr{P}| = |\mathscr{P}[G_1] \cup \mathscr{P}[G_2]| \\ &= |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| - |\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| \\ &\geq cp(G_1) + cp(G_2) - 1. \end{aligned}$$

By Lemma 3.1.2, there exists a minimum clique partition of G_1 or G_2 containing the K_n-clone. Without loss of generality, assume that G_1 has a minimum clique partition, $\mathscr{P}_1$, containing the K_n-clone. Let $\mathscr{P}_2$ be a minimum clique partition of G_2. Then $(\mathscr{P}_1 \smallsetminus \{K_n\}) \cup \mathscr{P}_2$ is a clique partition of $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$. Note that $\mathscr{P}_1 \smallsetminus \{K_n\}$ and $\mathscr{P}_2$ are disjoint. Therefore,

$$cp(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) \leq |(\mathscr{P}_1 \smallsetminus \{K_n\}) \cup \mathscr{P}_2| = |(\mathscr{P}_1 \smallsetminus \{K_n\})| + |\mathscr{P}_2| = cp(G_1) + cp(G_2) - 1.$$

Hence, $cp(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) = cp(G_1) + cp(G_2) - 1$.

□

Theorem 3.1.5. *Let G_1 and G_2 be graphs containing K_n as a subgraph. If G_1 or G_2 has a minimum clique partition containing the K_n-clone, then*

$$cp(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) \leq cp(G_1) + cp(G_2) - 1.$$

Proof. Assume that G_1 has a minimum clique partition containing the K_n-clone. By Theorem 2.1.11, $cp(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) \leq cp(G_1 - K_n) + cp(G_2)$. Since G_1 has a minimum clique partition containing the K_n-clone, by Proposition 2.1.7, $cp(G_1) - 1 = cp(G_1 - K_n)$. Thus, $cp(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) \leq cp(G_1 - K_n) + cp(G_2) = cp(G_1) + cp(G_2) - 1$. □

3.2 Clique partitions of glued graphs at K_2-clones

In this section, we show bounds of the clique partition numbers of $G_1 \underset{K_2}{\triangleleft\!\triangleright} G_2$. Recall that, we refer K_2 in the glued graph $G_1 \underset{K_2}{\triangleleft\!\triangleright} G_2$ to be only the K_2-clone, not an arbitrary copy of K_2 in our graphs.

Remark 3.2.1. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\triangleleft\!\triangleright} G_2$.

1. $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] \subseteq \{K_2\}$.

2. If the K_2-clone is contained in $\mathscr{P}$, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_2\}$, and, $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively.

3. If the K_2-clone is not contained in $\mathscr{P}$, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \varnothing$, furthermore, if the K_2-clone is contained in $\mathscr{P}[G_i] \smallsetminus \mathscr{P}[G_j]$ for some $i, j \in \{1, 2\}$ and $i \neq j$, then $\mathscr{P}[G_i]$ and $\mathscr{P}[G_j]$ are clique partitions of G_i and $G_j - K_2$, respectively.

Theorem 3.2.2. *For any nontrivial graphs G_1 and G_2,*

$$cp(G_1) + cp(G_2) - 1 \leq cp(G_1 \underset{K_2}{\Diamond} G_2) \leq cp(G_1) + cp(G_2) + s - 2 \tag{3.2.1}$$

where s is the order of the smallest clique containing the K_2-clone among all of the minimum clique partitions of G_1 and G_2.

Proof. To prove the upper bound, without loss of generality, assume that G_2 has a minimum clique partition containing a clique of order s which contains the K_2-clone. By Theorems 2.1.11 and 2.1.1, we have

$$cp(G_1 \underset{K_2}{\Diamond} G_2) \leq cp(G_1) + cp(G_2 - K_2) = cp(G_1) + cp(G_2) + s - 2.$$

To show the lower bound, let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\Diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. If $K_2 \in \mathscr{P}$, then $cp(G_1 \underset{K_2}{\Diamond} G_2) = cp(G_1) + cp(G_2) - 1$ by Theorem 3.1.4. Suppose that $K_2 \notin \mathscr{P}$. By Remark 3.2.1, either $\mathscr{P}[G_1]$ or $\mathscr{P}[G_2]$ is a clique partition of G_1 or G_2, respectively. Without loss of generality, let $\mathscr{P}[G_1]$ be a clique partition of G_1. Then $\mathscr{P}[G_2]$ is a clique partition of $G_2 - K_2$ and $\mathscr{P}[G_2] \cup \{K_2\}$ forms a clique partition of G_2. Hence, $cp(G_1 \underset{K_2}{\Diamond} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1] \cup \mathscr{P}[G_2]| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2] \cup \{K_2\}| - 1 \geq cp(G_1) + cp(G_2) - 1.$ □

Theorem 3.2.3. *Let G_1 and G_2 be any nontrivial graphs. The following statements are equivalent:*

(i) $cp(G_1 \underset{K_2}{\Diamond} G_2) = cp(G_1) + cp(G_2) - 1,$

(ii) *G_1 or G_2 has a minimum clique partition containing the K_2-clone, and*

(iii) *$cp(G_1 - K_2) = cp(G_1) - 1$ or $cp(G_2 - K_2) = cp(G_2) - 1$.*

Proof. $(ii) \Rightarrow (i)$ Assume that G_1 or G_2 has a minimum clique partition containing the K_2-clone. By Theorem 3.1.5, $cp(G_1 \underset{K_2}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 1$. By Theorem 3.2.2, $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$.

$(i) \Rightarrow (iii)$ Assume that $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\oplus} G_2$. Proposition 2.2.10 says $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since only the K_2-clone can possibly belong to $\mathscr{P}[G_1] \cap \mathscr{P}[G_2]$, we consider two cases.

Case 1. $K_2 \in \mathscr{P}$. By Lemma 3.1.2, G_1 or G_2 has a minimum clique partition containing the K_2-clone. Apply Proposition 2.1.7, we have $cp(G_1 - K_2) = cp(G_1) - 1$ or $cp(G_2 - K_2) = cp(G_2) - 1$.

Case 2. $K_2 \notin \mathscr{P}$. Then $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]|$. By Remark 3.2.1, either $\mathscr{P}[G_1]$ or $\mathscr{P}[G_2]$ is a clique partition of G_1 or G_2, respectively. Without loss of generality, let $\mathscr{P}[G_2]$ be a clique partition of G_2. Then $\mathscr{P}[G_1]$ is a clique partition of $G_1 - K_2$. Thus $cp(G_1) + cp(G_2) - 1 = cp(G_1 \underset{K_2}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1 - K_2) + cp(G_2)$, so $cp(G_1) - 1 \geq cp(G_1 - K_2)$. Again, Theorem 2.1.1, $cp(G_1) - 1 = cp(G_1 - K_2)$.

$(iii) \Rightarrow (ii)$ Assume that $cp(G_1) - 1 = cp(G_1 - K_2)$. Let $\mathscr{P}_1$ and $\mathscr{P}'$ be minimum clique partitions of G_1 and $G_1 - K_2$, respectively. Then $|\mathscr{P}_1| - 1 = |\mathscr{P}'|$. Note that $\mathscr{P}' \cup \{K_2\}$ is a clique partition of G_1 and $cp(G_1) = |\mathscr{P}_1| = |\mathscr{P}'| + 1 = |\mathscr{P}' \cup \{K_2\}|$. Hence, $\mathscr{P}' \cup \{K_2\}$ is a minimum clique partition of G_1. □

Corollary 3.2.4. *If $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2)$, then $cp(G_1) \leq cp(G_1 - K_2)$ and $cp(G_2) \leq cp(G_2 - K_2)$.*

Proof. It follows directly from Theorems 2.1.1 and 3.2.3. □

Corollary 3.2.5 follows immediately from Theorem 3.2.3 and Lemma 3.1.2.

Corollary 3.2.5. *If there exists a minimum clique partition of* $G_1 \underset{K_2}{\Diamond} G_2$ *containing the* K_2*-clone, then*

(i) $cp(G_1 \underset{K_2}{\Diamond} G_2) = cp(G_1) + cp(G_2) - 1$, *or*

(ii) $cp(G_1 - K_2) = cp(G_1) - 1$ *or* $cp(G_2 - K_2) = cp(G_2) - 1$.

For any graph G with an edge e, the statement $cp(G - e) \geq cp(G) - 1$ can be rewritten by $cp(G - e) = cp(G) + t$ where $t \geq -1$. Consider a glued graph $G_1 \underset{K_2}{\Diamond} G_2$, if $cp(G_i - K_2) = cp(G_i) + t_i$ where $i = 1, 2$, then its special case, namely $t_i = -1$ for some $i = 1, 2$, is examined in Theorem 3.2.3. Now we study in general.

Theorem 3.2.6. *Let* $G_1 \underset{K_2}{\Diamond} G_2$ *be any glued graph at* K_2*-clone. If* $cp(G_1 - K_2) = cp(G_1) + t_1$ *and* $cp(G_2 - K_2) = cp(G_2) + t_2$ *for some integers* t_1, t_2*, then* $cp(G_1 \underset{K_2}{\Diamond} G_2) = cp(G_1) + cp(G_2) + t$ *where* $t = \min\{t_1, t_2\}$.

Proof. Assume that $cp(G_1 - K_2) = cp(G_1) + t_1$ and $cp(G_2 - K_2) = cp(G_2) + t_2$ for some integers t_1, t_2. First note by Theorem 2.1.1 that $t_1, t_2 \geq -1$.
If $t_i = -1$ for some $i = 1, 2$, then the statement is hold by Theorem 3.2.3. Otherwise, assume that $0 \leq t_1 \leq t_2$. Since a union of a minimum clique partition of $G_1 - K_2$ and a minimum clique partition of G_2 is a clique partition of $G_1 \underset{K_2}{\Diamond} G_2$, $cp(G_1 \underset{K_2}{\Diamond} G_2) \leq cp(G_1) + cp(G_2) + t_1$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\Diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $t_1, t_2 \geq 0$ and by Corollary 3.2.5, the K_2-clone is not in $\mathscr{P}$ and then $\mathscr{P}$ is partitioned into $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. We consider two cases.

Case 1. $\mathscr{P}[G_1]$ is a clique partition of G_1 and $\mathscr{P}[G_2]$ is a clique partition of $G_2 - K_2$. Then $|\mathscr{P}[G_1]| \geq cp(G_1)$ and $|\mathscr{P}[G_2]| \geq cp(G_2 - K_2)$. Thus, $cp(G_1 \underset{K_2}{\Diamond} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2) + t_2 \geq cp(G_1) + cp(G_2) + t_1$. Hence, $cp(G_1 \underset{K_2}{\Diamond} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2) + t_1$.

Case 2. $\mathscr{P}[G_2]$ is a clique partition of G_2 and $\mathscr{P}[G_1]$ is a clique partition of $G_1 - K_2$. Then $|\mathscr{P}[G_2]| \geq cp(G_2)$ and $|\mathscr{P}[G_1]| \geq cp(G_1 - K_2)$. Thus $cp(G_1 \underset{K_2}{\Diamond} G_2) =$

$|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2) + t_1$. Hence $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + t$ where $t = \min\{t_1, t_2\}$. □

Theorem 3.2.7. *For $m \geq n \geq 3$, let G_1 and G_2 be nontrivial graphs of order m and n, respectively. Let s be the order of the smallest clique containing the K_2-clone among all minimum clique partitions of G_1 and G_2. Then*

(i) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$ *if and only if* $s = 2$, *and*

(ii) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + n - 2$ *if and only if* $s = n$.

Proof. (i) By the Definition of s, $s = 2$ if and only if G_i has a minimum clique partition containing the K_2-clone for some $i = 1, 2$. Then by Theorem 3.2.3, $s = 2$ if and only if $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$.

(ii) Assume that $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + n - 2$. Since s is the order of the smallest clique containing the K_2-clone, $s \leq n, m$. Without loss of generality, we may assume that G_1 has a minimum clique partition $\mathscr{P}_1$ containing clique C of order s which contains the K_2-clone. Let $\mathscr{P}_2$ be a minimum clique partition of G_2. Then the union of $\mathscr{P}_2$, $\mathscr{P}_1 \setminus C$ and a minimum clique partition, say $\mathscr{C}$, of $C - K_2$ is a clique partition of $G_1 \underset{K_2}{\diamond} G_2$. Note that $cp(C - K_2) = s - 1$ by Theorem 2.1.3. Thus, $cp(G_1) + cp(G_2) + n - 2 = cp(G_1 \underset{K_2}{\diamond} G_2) \leq |\mathscr{P}_2 \cup (\mathscr{P}_1 \setminus C) \cup \mathscr{C}| \leq cp(G_1) + cp(G_2) - 1 + cp(\mathscr{C} - K_2) = cp(G_1) + cp(G_2) + s - 2$, which implies that $n \leq s$. Hence, $s = n$.

In the other direction, if $s = n$, then $G_2 = K_n$. Thus $cp(G_2) = 1$. Note that by Theorem 2.1.11, $cp(G_1 \underset{K_2}{\diamond} G_2) \leq cp(G_1) + cp(G_2 - K_2)$ and by Theorem 2.1.3, $cp(G_2 - K_2) = n - 1$. Thus, $cp(G_1 \underset{K_2}{\diamond} G_2) \leq cp(G_1) + cp(G_2 - K_2) = cp(G_1) + n - 1 = cp(G_1) + cp(G_2) + n - 2$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since by definition of s, $s = n$ and $G_2 = K_n$, we have that $K_n \notin \mathscr{P}$. Then $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and $G_2 - K_2$, respectively, consequently, $|\mathscr{P}[G_1]| \geq cp(G_1)$ and $|\mathscr{P}[G_2]| \geq cp(G_2 -$

K_2). Note that $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]|$. Then $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2 - K_2)$. Again Theorem 2.1.3, $cp(G_2 - K_2) = cp(K_n - K_2) = n - 1$. Thus, $cp(G_1 \underset{K_2}{\diamond} G_2) = |\mathscr{P}| \geq cp(G_1) + n - 1 = cp(G_1) + cp(G_2) + n - 2$. Hence, $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + n - 2$. □

Now a characterization of $G_1 \underset{K_2}{\diamond} G_2$ when its value is the upper bound in equation (3.2.1) namely $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + s - 2$ where $s \geq 3$ is obtained in the following theorem.

Theorem 3.2.8. *Let $G_1 \underset{K_2}{\diamond} G_2$ be any glued graph of G_1 and G_2 at K_2-clone, and s the order of the smallest clique containing the K_2-clone among all of the minimum clique partitions of G_1 and G_2 where $s \geq 3$. Then $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + s - 2$ if and only if, for each $i = 1, 2$, $cp(G_i - K_2) \geq cp(G_i) + s - 2$.*

Proof. Assume that $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + s - 2$. We have that $cp(G_1 \underset{K_2}{\diamond} G_2) \leq cp(G_j) + cp(G_i - K_2)$ for all $i, j \in \{1, 2\}$ and $i \neq j$. It follows that $cp(G_i - K_2) \geq cp(G_i) + s - 2$ for all $i = 1, 2$.

Conversely, assume that $cp(G_i - K_2) \geq cp(G_i) + s - 2$ for all $i = 1, 2$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\diamond} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $s \geq 3$ and by Corollary 3.2.5, the K_2-clone is not in $\mathscr{P}$. Then $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]|$. Without loss of generality, let $\mathscr{P}[G_1]$ is a clique partition of G_1, then $\mathscr{P}[G_2]$ is a clique partition of $G_2 - K_2$. Thus,

$$\begin{aligned} cp(G_1 \underset{K_2}{\diamond} G_2) &= |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \\ &\geq cp(G_1) + cp(G_2 - K_2) \\ &\geq cp(G_1) + cp(G_2) + s - 2. \end{aligned}$$

By Theorem 3.2.2, $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + s - 2$. □

3.3 Clique partitions of glued graphs at K_3-clones

In this section, we now focus on clique partitions of glued graphs at K_3-clones. Recall that, we refer K_3 in the glued graph $G_1 \underset{K_3}{\diamond} G_2$ to be only the K_3-clone, not an arbitrary copy of K_3.

Definition 3.3.1. Let G be a graph containing a 3-clique T and $\mathscr{P}$ a clique partition of G. Then we say that

1. $\mathscr{P}$ is *type 1 with respect to* T, if $\mathscr{P}$ contains T.
2. $\mathscr{P}$ is *type 2 with respect to* T, if $\mathscr{P}$ contains a clique of order at least 4 covering T.
3. Otherwise, $\mathscr{P}$ is *type 3 with respect to* T, that is, each edge of T is covered by different cliques in $\mathscr{P}$.

Example 3.3.2. Let G be a graph containing a 3-clique $T = K_3(a, b, c)$ shown in Figure 3.3.1.

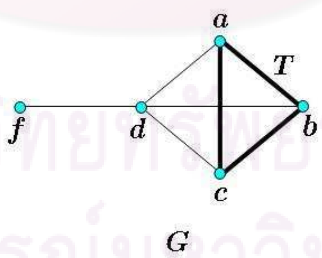

Figure 3.3.1: A graph illustrating types of its clique partitions

Let $\mathscr{P}_1 = \{K_2(a, d), K_2(c, d), K_2(d, f), K_2(b, d), K_3(a, b, c)\}$, $\mathscr{P}_2 = \{K_2(d, f), K_4(a, b, c, d)\}$ and $\mathscr{P}_3 = \{K_2(d, f), K_2(b, d), K_2(a, b), K_2(b, c), K_3(a, c, d)\}$.

It is clear that $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$ are clique partitions of G. Since $\mathscr{P}_1$ contains T, $\mathscr{P}_1$ is type 1 with respect to T. Moreover, $K_4(a, b, c, d)$ in $\mathscr{P}_2$ covers T, so $\mathscr{P}_2$ is type 2 with respect to T. The edges of T are covered by different cliques in $\mathscr{P}_3$, so $\mathscr{P}_3$ is type 3 with respect to T. $\square$

Remark 3.3.3. Let $G_1 \underset{K_3}{\diamond} G_2$ be any glued graph at K_3-clone and $\mathscr{P}$ a minimum clique partition of $G_1 \underset{K_3}{\diamond} G_2$.

1. If $\mathscr{P}$ is type 1 with respect to the K_3-clone, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_3\}$, and hence, $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively.

2. If $\mathscr{P}$ is type 2 with respect to the K_3-clone, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \varnothing$, furthermore, $\mathscr{P}[G_i]$ and $\mathscr{P}[G_j]$ are clique partitions of G_i and $G_j - K_3$, respectively, for some $i, j \in \{1, 2\}$ and $i \neq j$.

3. If $\mathscr{P}$ is type 3 with respect to the K_3-clone, then each element in $\mathscr{P}[G_1] \cap \mathscr{P}[G_2]$ is a proper subset of $E(K_3)$, consequently $|\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| = 0, 1$ or 2.

Theorem 3.3.4. *Let G_1 and G_2 be graphs containing K_3 as a subgraph. Then*

$$cp(G_1 \underset{K_3}{\diamond} G_2) \geq cp(G_1) + cp(G_2) - 3. \tag{3.3.1}$$

Proof. It follows immediately from Theorem 2.2.12. $\square$

Theorem 3.3.5. *Let $G_1 \underset{K_3}{\diamond} G_2$ be any glued graph. Then $cp(G_1 \underset{K_3}{\diamond} G_2) = cp(G_1) + cp(G_2) - 3$ if and only if there exist minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of G_1 and G_2, respectively, such that for each edge $e \in E(K_3)$, e must be covered by a 2-clique in $\mathscr{P}_1$ or $\mathscr{P}_2$.*

Proof. It follows immediately from Theorem 2.2.14. $\square$

Theorem 3.3.6. *Let* $G_1 \underset{K_3}{\Leftrightarrow} G_2$ *be any glued graph at* K_3*-clone and* $\mathscr{P}$ *a minimum clique partition of* $G_1 \underset{K_3}{\Leftrightarrow} G_2$*. If* $\mathscr{P}$ *is type 1 or type 2 with respect to the* K_3*-clone, then* $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) \geq cp(G_1) + cp(G_2) - 1$*.*

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_3}{\Leftrightarrow} G_2$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$.

Case 1. $\mathscr{P}$ is type 1 with respect to the K_3-clone.

Then $\mathscr{P}$ contains the K_3-clone. By Theorem 3.1.4, $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) = cp(G_1) + cp(G_2) - 1$.

Case 2. $\mathscr{P}$ is type 2 with respect to the K_3-clone.

Then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \varnothing$. Note that there is a clique Q of order at least 4 covering the K_3-clone in $\mathscr{P}$. Without loss of generality, let $Q \in \mathscr{P}[G_1]$. Then $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2] \cup \{K_3\}$ are clique partitions of G_1 and G_2, respectively. This implies that $|\mathscr{P}[G_1]| \geq cp(G_1)$ and $|\mathscr{P}[G_2] \cup \{K_3\}| \geq cp(G_2)$. Thus, $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2] \cup \{K_3\}| - 1 \geq cp(G_1) + cp(G_2) - 1$. □

The converse of Theorem 3.3.6 does not hold as shown in Example 3.3.7.

Example 3.3.7. Let G_1, G_2 be graphs and $G_1 \underset{H}{\Leftrightarrow} G_2$ be the glued graphs whose the clone is H as shown as bold edges in Figure 3.3.2.

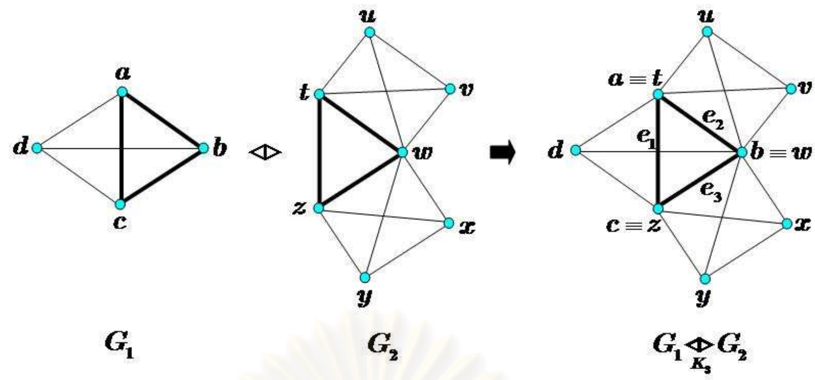

Figure 3.3.2: A glued graph illustrating that the converse of Theorem 3.3.6 does not hold

In Figure 3.3.2, we have that $cp(G_1) = 1$, $cp(G_2) = 3$ and $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) = 4$. Note that $cp(G_1) + cp(G_2) - 1 = 1 + 3 - 1 = 3 < 4 = cp(G_1 \underset{K_3}{\Leftrightarrow} G_2)$. Let e_1, e_2 and e_3 be edges of the K_3-clone. Let $\mathscr{P} = \{K_2(d, b \equiv w), K_3(d, a \equiv t, c \equiv z), K_4(u, v, b \equiv w, a \equiv t), K_4(b \equiv w, x, y, c \equiv z)\}$. Then $\mathscr{P}$ is a minimum clique partition of $G_1 \underset{K_3}{\Leftrightarrow} G_2$. Note that e_1, e_2 and e_3 are contained in different cliques in $\mathscr{P}$. Hence, $\mathscr{P}$ is type 3 with respect to the K_3-clone. $\square$

A characterization of $G_1 \underset{K_3}{\Leftrightarrow} G_2$ when its clique partition number is at the lower bound in equation (3.3.1) is provided in Theorem 3.3.5.

The following theorem shows all possible values of $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2)$ when G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone.

Theorem 3.3.8. *Let $G_1 \underset{K_3}{\Leftrightarrow} G_2$ be any glued graph at the K_3-clone. If G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone, then*

$$cp(G_1) + cp(G_2) - 3 \leq cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) \leq cp(G_1) + cp(G_2) - 1.$$

Proof. Assume that G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone. Then there exists a minimum clique partition of G_1 or G_2 containing the K_3-clone. By Theorem 3.1.5, $cp(G_1 \underset{K_3}{\Leftrightarrow} G_2) \leq cp(G_1) + cp(G_2) - 1$.

By Theorem 3.3.4, $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2) \geq cp(G_1) + cp(G_2) - 3$. Hence,

$$cp(G_1) + cp(G_2) - 3 \leq cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2) \leq cp(G_1) + cp(G_2) - 1.$$

□

We will study bounds of $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2)$ when neither G_1 nor G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone. Lemmas 3.3.9, 3.3.11 and 3.3.13 provide our desired upper bounds of $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2)$.

Lemma 3.3.9. *Let $G_1 \underset{K_3}{\lhd\!\!\rhd} G_2$ be any glued graph at K_3-clone. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 2 with respect to the K_3-clone of G_1 and G_2, respectively, then $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2) \leq cp(G_1) + cp(G_2) + s - 6$ where $s = \min\{2r_1, 2r_2\}$, r_1 and r_2 are the orders of the cliques containing the K_3-clone in $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively.*

Proof. Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 2 with respect to the K_3-clone of G_1 and G_2, respectively. Then there exists a clique Q_1 of order $r_1 \geq 4$ in $\mathscr{P}_1$ containing the K_3-clone. Similarly, there exists a clique Q_2 of order $r_2 \geq 4$ in $\mathscr{P}_2$ containing the K_3-clone. Let $s = \min\{2r_1, 2r_2\}$. Suppose that $s = 2r_1$. Then $G_1 - K_3$ can be partitioned by the union of $\mathscr{P}_1 \setminus \{Q_1\}$ and a minimum clique partition of $Q_1 - K_3$. By Theorem 2.1.4, $cp(Q_1 - K_3) \leq 2r_1 - 5$, so,

$$cp(G_1 - K_3) \leq |\mathscr{P}_1| - 1 + 2r_1 - 5 = cp(G_1) + 2r_1 - 6.$$

By Theorem 2.1.11, we have that $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2) \leq cp(G_1 - K_3) + cp(G_2)$.
Hence, $cp(G_1 \underset{K_3}{\lhd\!\!\rhd} G_2) \leq cp(G_1) + cp(G_2) + s - 6$ where $s = \min\{2r_1, 2r_2\}$. □

Example 3.3.10. *The sharpness of the upper bound in Lemma 3.3.9.*

Let $m \geq 4$. Consider the glued graph $K_4 \underset{K_3}{\lhd\!\!\rhd} K_m$. K_4 and K_m have minimum clique partitions which are type 2 with respect to the K_3-clone, say $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, such that $r_1 = 4$ and $r_2 = m$, where r_1 and r_2 are order of the cliques containing the K_3-clone in $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Note that $cp(K_4) = 1 = cp(K_m)$ and $\left|V(K_4 \underset{K_3}{\oplus} K_m)\right| = m+1$. Since $\overline{K_4 \underset{K_3}{\oplus} K_m} \cong K_{1,m-3}$, we have that $cp(\overline{K_4 \underset{K_3}{\oplus} K_m}) = m-3$. By Theorem 2.1.5, $cp(K_4 \underset{K_3}{\oplus} K_m) + cp(\overline{K_4 \underset{K_3}{\oplus} K_m}) \geq \left|V(K_4 \underset{K_3}{\oplus} K_m)\right|$. Thus, $cp(K_4 \underset{K_3}{\oplus} K_m) + (m-3) \geq m+1$, so $cp(K_4 \underset{K_3}{\oplus} K_m) \geq 4$. Since $E(K_4 - K_3) \cup \{K_m\}$ is a clique partition of $K_4 \underset{K_3}{\oplus} K_m$, $cp(K_4 \underset{K_3}{\oplus} K_m) \leq |E(K_4 - K_3) \cup \{K_m\}| = 3 + 1 = 4$. Therefore, $cp(K_4 \underset{K_3}{\oplus} K_m) = 4$. Hence, $cp(K_4 \underset{K_3}{\oplus} K_m) = 4 = 1 + 1 + 8 - 6 = cp(K_4) + cp(K_m) + 2r_1 - 6$. □

Lemma 3.3.11. *Let $G_1 \underset{K_3}{\oplus} G_2$ be any glued graph at K_3-clone. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 3 with respect to the K_3-clone of G_1 and G_2, respectively, then $cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) + s - 6$ where $s = \min\{s_1, s_2\}$, s_i is the sum of orders of all cliques in $\mathscr{P}_i$ containing edges of the K_3-clone for all $i = 1, 2$.*

Proof. Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 3 with respect to the K_3-clone of G_1 and G_2, respectively. Then each edge of the K_3-clone is contained in different cliques in $\mathscr{P}_1$, say Q_1, Q_2 and Q_3 of orders q_1, q_2 and q_3, respectively. Similarly there are cliques R_1, R_2 and R_3 of orders r_1, r_2 and r_3, respectively, in $\mathscr{P}_2$. Let $s = \min\{q_1 + q_2 + q_3, r_1 + r_2 + r_3\}$. Suppose that $s = q_1 + q_2 + q_3$. Then $G_1 - K_3$ can be partitioned by the union of $\mathscr{P}_1 \smallsetminus \{Q_1, Q_2, Q_3\}$ and a minimum clique partition of Q_i deleted an edge of the K_3-clone for all $i = 1, 2, 3$. By Theorem 2.1.3, we have that $cp(Q_i - e_i) = q_i - 1$ where Q_i covers an edge e_i of the K_3-clone for all $i = 1, 2, 3$. Thus,

$$cp(G_1 - K_3) \leq |\mathscr{P}_1| - 3 + (q_1 - 1) + (q_2 - 1) + (q_3 - 1) = cp(G_1) + q_1 + q_2 + q_3 - 6.$$

By Theorem 2.1.11, we have that $cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1 - K_3) + cp(G_2)$. Hence,

$$cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) + s - 6 \text{ where } s = \min\{q_1 + q_2 + q_3, r_1 + r_2 + r_3\}.$$

□

Example 3.3.12. *The sharpness of the upper bound in Lemma 3.3.11.*

Let $p, q, r \geq 3$. Let G_1 be the graph obtained from $K_3(a, b, c)$, K_p, K_q and K_r by identifying each of three edges in $K_3(a, b, c)$ with an edge in K_p, K_q, and K_r, respectively, as shown in Figure 3.3.3. In the same way, G_2 is the graph obtained from $K_3(u, v, w)$, K_l, K_m and K_n by identifying each of three edges in $K_3(u, v, w)$ with an edge in K_l, K_m, and K_n, respectively, where $l, m, n \geq \max\{p, q, r\}$. Consider the glued graph of G_1 and G_2 at $K_3(a, b, c)$ and $K_3(u, v, w)$, denoted by $G_1 \underset{K_3}{\diamond} G_2$ which is shown in Figure 3.3.3

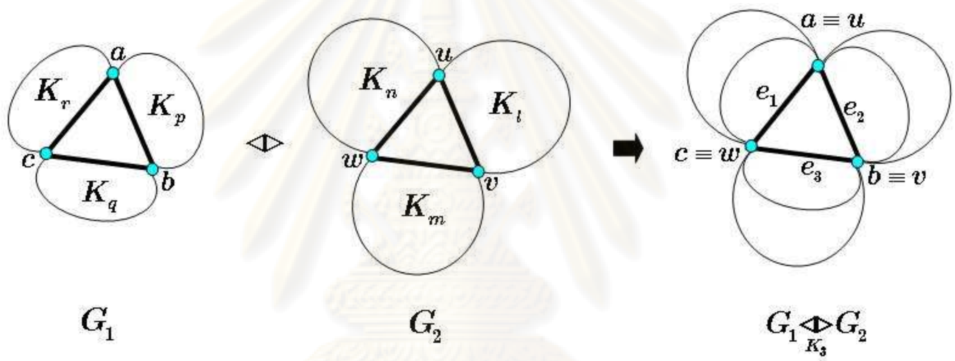

Figure 3.3.3: The sharpness of the upper bound in Lemma 3.3.11

It is easily seen that both G_1 and G_2 have minimum clique partitions which are type 3 with respect to the K_3- clone, say $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Moreover, $cp(G_1) = 3 = cp(G_2)$. Let s_i be the sum of orders of all cliques in $\mathscr{P}_i$ containing edges of the K_3-clone for all $i = 1, 2$. Then $s_1 = p + q + r$ and $s_2 = l + m + n$. Since $l, m, n \geq \max\{p, q, r\}$, $s_2 \geq s_1$. By Lemma 3.3.11, we have that

$$cp(G_1 \underset{K_3}{\diamond} G_2) \leq cp(G_1) + cp(G_2) + s_1 - 6 = 3 + 3 + (p + q + r) - 6 = p + q + r. \quad (3.3.2)$$

Let e_1, e_2 and e_3 be edges of the clone of $G_1 \underset{K_3}{\diamond} G_2$, and $\mathscr{P}$ a minimum clique partition of $G_1 \underset{K_3}{\diamond} G_2$. If e_1, e_2 and e_3 are covered by a 3-clique in $\mathscr{P}$, then $cp(G_1 \underset{K_3}{\diamond} G_2) = |\mathscr{P}| \geq 1 + (p-1) + (q-1) + (r-1) + (l-1) + (m-1) + (n-1) + 3 > p + q + r$. Thus by equation (3.3.2), $G_1 \underset{K_3}{\diamond} G_2$ does not have a minimum clique partition which is type 1 with respect to the K_3-clone. This implies that each e_i must be covered by distinct cliques in any minimum clique partition of $G_1 \underset{K_3}{\oplus} G_2$ for all $i \in \{1,2,3\}$. Note that $G_1 \underset{K_3}{\oplus} G_2 \cong (K_p \underset{K_2}{\oplus} K_l) \cup (K_q \underset{K_2}{\oplus} K_m) \cup (K_r \underset{K_2}{\oplus} K_n)$. By Example 2.1.8, we have that $cp(K_p \underset{K_2}{\oplus} K_l) = p$, $cp(K_q \underset{K_2}{\oplus} K_m) = q$ and $cp(K_r \underset{K_2}{\oplus} K_n) = r$. Thus, $cp(G_1 \underset{K_3}{\oplus} G_2) \geq cp(K_p \underset{K_2}{\oplus} K_l) + cp(K_q \underset{K_2}{\oplus} K_m) + cp(K_r \underset{K_2}{\oplus} K_n) = p+q+r$. Hence, $cp(G_1 \underset{K_3}{\oplus} G_2) = p+q+r = 3+3+(p+q+r)-6 = cp(G_1)+cp(G_2)+s-6$ where $s = \min\{s_1, s_2\}$.

□

Lemma 3.3.13. *Let $G_1 \underset{K_3}{\oplus} G_2$ be any glued graph at K_3-clone. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 2 and 3 with respect to the K_3-clone of G_1 and G_2, respectively, then $cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) + s - 6$ where $s = \min\{2r, t\}$, r is the order of a clique containing the K_3-clone in $\mathscr{P}_1$ and t is the sum of orders of all cliques in $\mathscr{P}_2$ containing edges of the K_3-clone.*

Proof. Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions which are type 2 and 3 with respect to the K_3-clone of G_1 and G_2, respectively. Since $\mathscr{P}_1$ is type 2 with respect to the K_3-clone, there exists a clique R of order $r \geq 4$ in $\mathscr{P}_1$ containing the K_3-clone. Then $G_1 - K_3$ can be partitioned by the union of $\mathscr{P}_1 \setminus \{R\}$ and a minimum clique partition of $R - K_3$. By Theorem 2.1.4, $cp(R - K_3) \leq 2r - 5$. Thus $cp(G_1 - K_3) \leq |\mathscr{P}_1| - 1 + 2r - 5 = cp(G_1) + 2r - 6$. Since $\mathscr{P}_2$ is type 3 with respect to the K_3-clone, there exists three cliques in $\mathscr{P}_2$ such that each one covers different edge in the K_3-clone, say Q_1, Q_2, Q_3 of orders q_1, q_2 and q_3, respectively. Then $G_2 - K_3$ can be partitioned by the union of $\mathscr{P}_2 \setminus \{Q_1, Q_2, Q_3\}$ and a minimum clique partition of Q_i deleted an edge of the K_3-clone for all $i = 1, 2, 3$. By Theorem 2.1.3, $cp(Q_i - e_i) = q_i - 1$ where Q_i covers an edge e_i in the K_3-clone for all $i = 1, 2, 3$. Thus,

$$cp(G_2 - K_3) \leq |\mathscr{P}_2| - 3 + (q_1 - 1) + (q_2 - 1) + (q_3 - 1) = cp(G_2) + q_1 + q_2 + q_3 - 6.$$

By Theorem 2.1.11, we have that $cp(G_1 \underset{K_3}{\diamond} G_2) \leq cp(G_1) + cp(G_2) + s - 6$ where $s = \min\{2r, q_1 + q_2 + q_3\}$.

□

Example 3.3.14. *The sharpness of the upper bound in Lemma 3.3.13.*

Let $l, m, n \geq 3$. Let G be the graph obtained from $K_3(u, v, w)$, K_l, K_m and K_n by identifying each of three edges in $K_3(u, v, w)$ with an edge in K_l, K_m, and K_n, respectively. Consider the glued graph of G and K_4 at $K_3(u, v, w)$ and 3-clique in K_4, denoted by $G \underset{K_3}{\diamond} K_4$ which is shown in Figure 3.3.4.

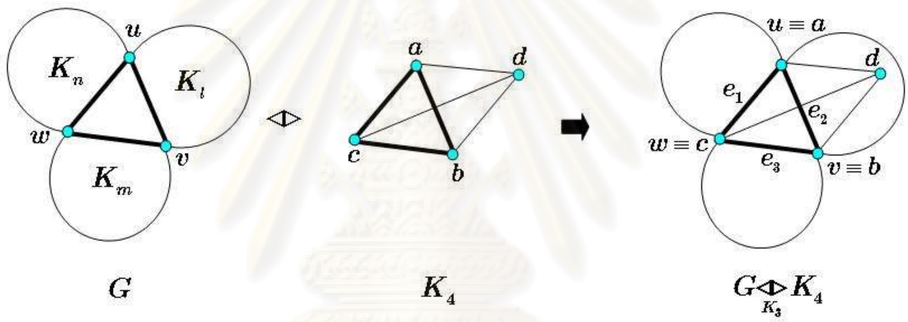

Figure 3.3.4: The sharpness of the upper bound in Lemma 3.3.13

It is easily seen that G and K_4 have minimum clique partitions which are type 2 and type 3 with respect to the K_3-clone, say $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Note that $cp(K_4) = 1$ and $cp(G) = 3$. Let r be the order of the clique containing the K_3-clone in $\mathscr{P}_1$ and t the sum of orders of all cliques in $\mathscr{P}_2$ containing edges of the K_3-clone. Then $r = 4$ and $t = l + m + n$. Since $l + m + n \geq 3$, $t > 2r$. By Lemma 3.3.13, we have that

$$cp(G \underset{K_3}{\diamond} K_4) \leq cp(G_1) + cp(G_2) + 2r - 6 = 1 + 3 + 8 - 6 = 6. \qquad (3.3.3)$$

Let e_1, e_2 and e_3 be edges of the clone of $G \underset{K_3}{\diamond} K_4$, and $\mathscr{P}$ a minimum clique partition of $G \underset{K_3}{\diamond} K_4$. If e_1, e_2 and e_3 are covered by a 3-clique in $\mathscr{P}$, then $cp(G \underset{K_3}{\diamond} K_4) = |\mathscr{P}| \geq 1+(l-1)+(m-1)+(n-1)+3 \geq 1+3+3+3 = 10$. If e_1, e_2 and e_3 are covered by a 4-clique in $\mathscr{P}$, then $cp(G\underset{K_3}{\oplus}K_4) = |\mathscr{P}| \geq 1+(l-1)+(m-1)+(n-1) \geq 7$. Thus by the equation (3.3.3), $G\underset{K_3}{\oplus}K_4$ does not have a minimum clique partition which is type 1 or type 2 with respect to the K_3-clone. This implies that e_1, e_2 and e_3 must be covered by an n-clique, l-clique and m-clique, respectively, in $\mathscr{P}$ of $G\underset{K_3}{\oplus}K_4$. Since $\mathscr{P}_2 \subseteq \mathscr{P}$ and $cp(K_4 - K_3) = 3$, $cp(G\underset{K_3}{\oplus}K_4) \geq cp(G) + K_4 - K_3 = 3 + 3 = 6$. Hence $cp(G\underset{K_3}{\oplus}K_4) = 6 = cp(G) + cp(K_4) + s - 6$ where $s = \min\{2r, t\}$. □

Theorem 3.3.15 follows immediately from Lemmas 3.3.9, 3.3.11 and 3.3.13

Theorem 3.3.15. *Let $G_1\underset{K_3}{\oplus}G_2$ be any graph at the K_3-clone and $\mathscr{P}_i$ a minimum clique partition of G_i for all $i = 1, 2$. Then*

$$cp(G_1\underset{K_3}{\oplus}G_2) \leq cp(G_1) + cp(G_2) - 6 + \min\{\sigma_1, \sigma_2\}$$

where for each $i = 1, 2$ $\sigma_i = \begin{cases} 2s_i & \text{if } \mathscr{P}_i \text{ is type 2 with respect to the } K_3\text{-clone,} \\ s_i & \text{if } \mathscr{P}_i \text{ is type 3 with respect to the } K_3\text{-clone} \end{cases}$

and s_i is the sum of orders of all cliques in $\mathscr{P}_i$ containing edges of the K_3-clone.

Corollary 3.3.16. *For $m, n > 3$, $cp(K_m\underset{K_3}{\oplus}K_n) \leq \min\{2m, 2n\} - 4$.*

Lemma 3.3.17. [11] *For $2 \leq m < n - 1$, $cp(K_n - K_m) \leq cp(K_{n+1} - K_{m+1})$.*

Lemma 3.3.18. *For $m, n \geq 4$, all minimum clique partitions of $K_m\underset{K_3}{\oplus}K_n$ are type 2 with respect to the K_3-clone.*

Proof. Let $m \geq n$ and $\mathscr{P}$ be a minimum clique partition of $K_m\underset{K_3}{\oplus}K_n$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[K_m] \cup \mathscr{P}[K_n]$. If $\mathscr{P}$ is type 1 with respect to the K_3-clone, then by Theorem 3.1.4, $cp(K_m\underset{K_3}{\oplus}K_n) = cp(K_m) + cp(K_n) - 1 = 1$, it is impossible because $K_m\underset{K_3}{\oplus}K_n$ is not a complete graph. Suppose that $\mathscr{P}$ is type 3 with respect to the K_3-clone. By Remark 3.3.3, we have 3 cases.

Case 1. $|\mathscr{P}[K_m] \cap \mathscr{P}[K_n]| = 2$. Since the original graphs are complete graphs, this caes does not occur. Otherwise, $\mathscr{P}$ is not a minimum clique partition of $K_m\underset{K_3}{\oplus}K_n$.

Case 2. $|\mathscr{P}[K_m] \cap \mathscr{P}[K_n]| = 1$. Then $\mathscr{P}[K_m]$ and $\mathscr{P}[K_n]$ are clique partitions of $K_m - e$ and $K_n - e$, respectively, where e is an edge of the K_3-clone and $e \in \mathscr{P}[K_m] \cap \mathscr{P}[K_n]$. By Theorem 2.1.3, $|\mathscr{P}[K_m]| \geq cp(K_m - e) = m - 1$ and $|\mathscr{P}[K_n]| \geq cp(K_n - e) = n - 1$. Thus, $cp(K_m \underset{K_3}{\oplus} K_n) = |\mathscr{P}| = |\mathscr{P}[K_m]| + |\mathscr{P}[K_n]| - 1 \geq (m-1) + (n-1) - 1 = m + n - 3 \geq 2n - 3$, which contradicts the upper bound of $cp(K_m \underset{K_3}{\oplus} K_n)$ (see Corollary 3.3.16).

Case 3. $|\mathscr{P}[K_m] \cap \mathscr{P}[K_n]| = 0$. Then $|\mathscr{P}| = |\mathscr{P}[K_m]| + |\mathscr{P}[K_n]|$.
Let e_1, e_2 and e_3 be edges of the K_3-clone. Without loss of generality, we may assume that $\mathscr{P}[K_m]$ and $\mathscr{P}[K_n]$ are clique partitions of $K_m - e_1$ and $K_n - P_3$ where $E(P_3) = \{e_2, e_3\}$. By Theorems 2.1.3 and Proposition 2.1.6 , $|\mathscr{P}[K_m]| \geq cp(K_m - e_1) = m - 1$ and $|\mathscr{P}[K_n]| \geq cp(K_n - P_3) = n - 2$. Thus $cp(K_m \underset{K_3}{\oplus} K_n) = |\mathscr{P}| \geq (m-1) + (n-2) \geq 2n - 3$, which contradicts the upper bound of $cp(K_m \underset{K_3}{\oplus} K_n)$. Hence, $\mathscr{P}$ is type 2 with respect to the K_3-clone. $\square$

Theorem 3.3.19. *For* $4 \leq n \leq m - 2$, $n - 1 \leq cp(K_m \underset{K_3}{\oplus} K_n) \leq 2n - 4$.

Proof. The upper bound, follows immediately from Corollary 3.3.16. Let $\mathscr{P}$ be a minimum clique partition of $K_m \underset{K_3}{\oplus} K_n$. By Proposition 2.2.10, $\mathscr{P} = \mathscr{P}[K_m] \cup \mathscr{P}[K_n]$. By Lemma 3.3.18, we have that $\mathscr{P}$ is type 2 with respect to the K_3-clone, so $|\mathscr{P}| = |\mathscr{P}[K_m]| + |\mathscr{P}[K_n]|$. Since $m > n$ and $\mathscr{P}$ is type 2 with respect to the K_3-clone, $\mathscr{P}$ contains an m-clique, so $|\mathscr{P}[K_m]| = cp(K_m) = 1$ and $\mathscr{P}[K_n]$ is a clique partition of $K_n - K_3$. By Lemma 3.3.17 and Theorem 2.1.3, $cp(K_n - K_3) \geq cp(K_{n-1} - K_2) = (n-1) - 1 = n - 2$. Thus $cp(K_m \underset{K_3}{\oplus} K_n) = |\mathscr{P}| = |\mathscr{P}[K_m]| + |\mathscr{P}[K_n]| \geq cp(K_m) + cp(K_n - K_3) \geq 1 + (n-2) = n - 1$. $\square$

CHAPTER IV

CONCLUSIONS AND OPEN PROBLEMS

4.1 Conclusions

In Chapter 2, we have found bounds of the clique partition number of a glued graph at arbitrary clone. We have focused on clique partition numbers of clique-preserving glued graphs in Section 2.2. A characterization for lower bounds of clique partition numbers of clique-preserving glued graphs is obtained. Some properties of clique partition numbers of glued graphs at K_n-clones are studied in Chapter 3. Furthermore, we have investigated clique partition numbers of glued graphs at K_2-clones and K_3-clones. The results are as follows:

A bound of clique partition numbers of glued graphs:

For any graphs G_1 and G_2 containing H as a subgraph,

$$1 \leq cp(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \min\{cp(G_1) + cp(G_2 - H), cp(G_2) + cp(G_1 - H)\}.$$

Clique partition numbers of clique-preserving glued graphs:

1. If $G_1 \lhd\!\rhd G_2$ is a clique-preserving glued graph, then

$$cp(G_1 \lhd\!\rhd G_2) \geq \max\{cp(G_1), cp(G_2)\}.$$

2. If $G_1 \underset{H}{\lhd\!\rhd} G_2$ is a clique-preserving glued graph, then $cp(G_1 \underset{H}{\lhd\!\rhd} G_2) \geq cp(G_1) + cp(G_2) - e(H)$. Moreover, $cp(G_1 \underset{H}{\lhd\!\rhd} G_2) = cp(G_1) + cp(G_2) - e(H)$ if and only if there are minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of G_1 and G_2, respectively,

such that for each edge $e \in E(H)$, e must be covered by a 2-clique in $\mathscr{P}_1$ or $\mathscr{P}_2$.

Some properties of clique partitions of glued graphs at K_n-clones

1. For $m \geq n > r \geq 2$, $cp(K_m \underset{K_r}{\oplus} K_n) \leq (r-1)(n-r)+2$.

2. If $G_1 \underset{K_n}{\oplus} G_2$ has a minimum clique partition containing the K_n-clone, then G_1 or G_2 has a minimum clique partition containing the K_n-clone.

3. If $G_1 \underset{K_n}{\oplus} G_2$ has a minimum clique partition containing the K_n-clone, then $cp(G_1 \underset{K_n}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$

4. If G_1 or G_2 has a minimum clique partition containing the K_n-clone, then $cp(G_1 \underset{K_n}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 1$

Clique partitions of glued graphs at K_2-clones:

For any graphs G_1 and G_2 containing K_2 as a subgraph:

1. $cp(G_1) + cp(G_2) - 1 \leq cp(G_1 \underset{K_2}{\oplus} G_2) \leq cp(G_1) + cp(G_2) + s - 2$

   where s is the order of the smallest clique containing the clone K_2 among all of the minimum clique partitions of G_1 and G_2.

2. The following statements are equivalent:

   (i) $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$.

   (ii) G_1 or G_2 has a minimum clique partition containing the K_2-clone .

   (iii) $cp(G_1 - K_2) = cp(G_1) - 1$ or $cp(G_2 - K_2) = cp(G_2) - 1$.

3. If $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2)$, then $cp(G_1) \leq cp(G_1 - K_2)$ and $cp(G_2) \leq cp(G_2 - K_2)$.

4. If $G_1 \underset{K_2}{\diamond} G_2$ has a minimum clique partition containing the K_2-clone, then

   (i) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$, or

   (ii) $cp(G_1 - K_2) = cp(G_1) - 1$ and $cp(G_2 - K_2) = cp(G_2) - 1$.

5. If $cp(G_1 - K_2) = cp(G_1) + t_1$ and $cp(G_2 - K_2) = cp(G_2) + t_2$ for some integers t_1 and t_2, then $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + t$ where $t = min\{t_1, t_2\}$.

6. For $m \geq n \geq 3$, let G_1 and G_2 be nontrivial graphs of orders m and n, respectively. Let s be the order of the smallest clique containing the K_2-clone among all minimum clique partitions of G_1 and G_2. Then

   (i) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$ if and only if $s = 2$, and

   (ii) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + n - 2$ if and only if $s = n$.

7. $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) + s - 2$ if and only if, for each $i = 1, 2$, $cp(G_i - K_2) \geq cp(G_i) + s - 2$ where s is the order of the smallest clique containing the clone K_2 among all of the minimum clique partitions of G_1 and G_2.

Clique partitions of glued graphs at K_3-clones

For any graphs G_1 and G_2 containing K_3 as a subgraph. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be minimum clique partitions of G_1 and G_2, respectively:

1. $cp(G_1 \underset{K_3}{\diamond} G_2) \geq cp(G_1) + cp(G_2) - 3$.

2. $cp(G_1 \underset{K_3}{\diamond} G_2) = cp(G_1) + cp(G_2) - 3$ if and only if there are minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of G_1 and G_2, respectively, such that for each edge $e \in E(H)$, e must be covered by a 2-clique in $\mathscr{P}_1$ or $\mathscr{P}_2$.

3. If $\mathscr{P}$ is type 1 or type 2 with respect to the K_3-clone, then $cp(G_1 \underset{K_3}{\diamond} G_2) \geq cp(G_1) + cp(G_2) - 1$ where $\mathscr{P}$ is a minimum clique partition of $G_1 \underset{K_3}{\diamond} G_2$.

4. If G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone, then $cp(G_1)+cp(G_2)-3 \leq cp(G_1 \underset{K_3}{\triangleleft\!\triangleright} G_2) \leq cp(G_1)+cp(G_2)-1$.

5. $cp(G_1 \underset{K_3}{\triangleleft\!\triangleright} G_2) \leq cp(G_1) + cp(G_2) - 6 + \min\{\sigma_1, \sigma_2\}$ where for each $i = 1, 2$
$$\sigma_i = \begin{cases} 2s_i & \text{if } \mathscr{P}_i \text{ is type 2 with respect to the } K_3\text{-clone,} \\ s_i & \text{if } \mathscr{P}_i \text{ is type 3 with respect to the } K_3\text{-clone} \end{cases}$$
and s_i is the sum of orders of all cliques in $\mathscr{P}_i$ containing edges of the K_3-clone.

6. For $m, n \geq 4$, all minimum clique partitions of $K_m \underset{K_3}{\triangleleft\!\triangleright} K_n$ are type 2 with respect to the K_3-clone.

7. For $4 \leq n \leq m-2$, $n-1 \leq cp(K_m \underset{K_3}{\triangleleft\!\triangleright} K_n) \leq 2n-4$.

4.2 Open problems

We have some open problems for future work as follows:

1. We see in Chapter 1 that a glued geaph can have a new clique. An open problem is to find values or improve bounds of the clique partition numbers of a glued graphs with a new clique.

2. In Section 2.2.1, an open problem is find an upper bound of a clique partition number of a clique-preserving glued graph. Moreover, finding another lower bound can be further investigated.

3. An open problem is to investigate bounds of the clique partition numbers of glued graphs at K_n-clone where $n \geq 4$.

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APPENDIX

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessary to be distinct) called its *endpoints*. The *order* of a graph G, written $n(G)$, is the number of vertices in G. The number of edges in G is represented by $e(G)$.

A *loop* is an edge whose endpoint are equal. An *multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops and no multiple edges.

A graph is *trivial* if it has no edge; otherwise it *nontrivial*.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say G *isomorphic to* H, written $G \cong H$, if there is an isomorphism from G to H.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A subgraph H of G is an *induced subgraph*, denoted by $G[V(H)]$, if vertices of $V(H)$ are adjacent in $G[V(H)]$ whenever they are adjacent in G.

A graph G is H-*free* if G does not contain H as a subgraph.

The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A *complete graph* is a graph in which each pair of vertices is joined by an edge. The complete graph with n vertices is denoted by K_n.

A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possible empty) independent sets called *partite set* of G A *complete bipartite* or *biclique* is a simple bipartite graph such that two vertices are adjacent if and only if they are different partite set. When the sets have orders r and s, the (unlabeled) biclique is denoted $K_{r,s}$.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

A *spanning subgraph* of G is a subgraph with vertex set $V(G)$.

A *Hamiltonian path* is a spanning subgraph that is a path.

The *union* of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $V(G_1) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup \cdots \cup E(G_k)$.

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