กึ่งกรุปการแปลงเชิงเส้นซึ่งให้โครงสร้างเนียร์ริง

นางสาวเรืองลักษณ์ จงโชตินนท์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2550

LINEAR TRANSFORMATION SEMIGROUPS
ADMITTING NEARRING STRUCTURE

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2007

Thesis Title	LINEAR TRANSFORMATION SEMIGROUPS ADMITTING NEARRING STRUCTURE
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Field of Study	Mathematics
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เรืองลักษณ์ จงโชตินนท์ : กึ่งกรุปการแปลงเชิงเส้นซึ่งให้โครงสร้างเนียร์ริง. (LINEAR TRANSFORMATION SEMIGROUPS ADMITTING NEARRING STRUCTURE)
อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ. ดร. สุรีย์พร ชาวแพรกน้อย, 36 หน้า.

เราเรียกระบบ $(N,+,\cdot)$ ว่าเป็น *เนียร์ริงซ้าย* [ขวา] เมื่อ

(i) $(N,+)$ เป็นกรุป

(ii) $(N,\cdot)$ เป็นกึ่งกรุป

(iii) $z\cdot(x+y)=z\cdot x+z\cdot y$ $[(x+y)\cdot z=x\cdot z+y\cdot z]$ สำหรับทุก $x,y,z\in N$

เรากล่าวว่ากึ่งกรุป S *ให้โครงสร้างของเนียร์ริงซ้าย* [ขวา] เมื่อ

(1) $(S,+,\cdot)$ เป็นเนียร์ริงซ้าย [ขวา] สำหรับบางการดำเนินการ + บน S โดยที่ $\cdot$ เป็นการดำเนินการบน S หรือ

(2) $(S^0,+,\cdot)$ เป็นเนียร์ริงซ้าย [ขวา] สำหรับบางการดำเนินการ + บน S^0 โดยที่ $\cdot$ เป็นการดำเนินการบน S^0

ให้ V เป็นปริภูมิเวกเตอร์บนริงการหาร R และ $L_R(V)$ เป็นกึ่งกรุปภายใต้การประกอบ ที่ประกอบด้วยการแปลงเชิงเส้น $\alpha:V\to V$ ทั้งหมด *กึ่งกรุปการแปลงเชิงเส้น* บน V หมายถึงกึ่งกรุปย่อยของ $L_R(V)$ เราศึกษากึ่งกรุปการแปลงเชิงเส้นหลากหลายชนิด เราศึกษาว่าเมื่อใดกึ่งกรุปเหล่านี้ให้โครงสร้างของเนียร์ริงซ้ายและโครงสร้างของเนียร์ริงขวา

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต....เรืองลักษณ์ จงโชตินนท์....

สาขาวิชา..........คณิตศาสตร์.......... ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์หลัก..สุรีย์พร ชาวแพรกน้อย

ปีการศึกษา.............2550...............

4972460823 : MAJOR MATHEMATICS
KEY WORDS : NEARRINGS / LINEAR TRANSFORMATION SEMIGROUPS

RUANGLAK JONGCHOTINON : LINEAR TRANSFORMATION SEMIGROUPS ADMITTING NEARRING STRUCTURE. THESIS PRINCIPAL ADVISOR : ASST. PROF. SUREEPORN CHAOPRAKNOI, Ph.D. 36 pp.

A system $(N, +, \cdot)$ is called *a left [right] nearring* if

(i) $(N, +)$ is a group,

(ii) $(N, \cdot)$ is a semigroup and

(iii) $z \cdot (x + y) = z \cdot x + z \cdot y$ $[(x + y) \cdot z = x \cdot z + y \cdot z]$ for all $x, y, z \in N$.

A semigroup S is said to *admit a left [right] nearring structure* if

(1) $(S, +, \cdot)$ is a left [right] nearring for some operation $+$ on S where $\cdot$ is the operation on S or

(2) $(S^0, +, \cdot)$ is a left [right] nearring for some operation $+$ on S^0 where $\cdot$ is the operation on S^0.

Let V be a vector space over a division ring R and $L_R(V)$ the semigroup under composition of all linear transformations $\alpha : V \to V$. By a *linear transformation semigroup* on V we mean a subsemigroup of $L_R(V)$. Various types of linear transformation semigroups are studied. We determine when they admit the structure of a left nearring and a right nearring.

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Field of Study : ...Mathematics... Principal Advisor's Signature : Sureeporn Cha
Academic Year : ....2007..........

ACKNOWLEDGEMENTS

I am very grateful to my supervisor, Assistant Professor Dr. Sureeporn Chaopraknoi, for her consistent encouragement, helpfulness and valuable suggestions throughout the preparation and completion of this thesis. Also, I am sincerely grateful to Assistant Professor Dr. Sajee Pianskool, for her willingness to sacrifice her time to suggest and advise me. Moreover, I wish to express my gratitude to my thesis committee and all of my teachers who have taught me for my knowledge and skills.

In particular, I would like to thank the Development and Promotion of Science and Technology Talents Project (DPST) for the financial support.

Finally, I would like to express my sincere gratitude to my beloved family for their love and encouragement throughout my study.

CONTENTS

CHAPTER I
INTRODUCTION

The multiplicative structure of a ring is by definition a semigroup with zero. However, ring theory is a classical subject in mathematics and had been widely studied before semigroup theory was considered important and of interest by its own. Because the multiplicative structure of a ring is a semigroup with zero, it is valid to ask whether a given semigroup S, S^0 isomorphics to the multiplicative structure of some ring. If it does, S is said to *admit a ring structure*. Let $\mathcal{R}$ be the class of all semigroups admitting ring structure. If S is a member of $\mathcal{R}$, then there is an isomorphism φ from S^0 onto $(R, \cdot)$ for some ring $(R, +, \cdot)$. If we define an addition $\oplus$ on S^0 by

$$x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)) \text{ for all } x, y \in S^0,$$

then $(S^0, \oplus, \cdot)$ is a ring which is isomorphic to $(R, +, \cdot)$ through the mapping φ. Hence a semigroup $S \in \mathcal{R}$ if and only if there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is the operation on S^0. Many well-known theorems in ring theory are useful to study whether a semigroup S is a member of $\mathcal{R}$. For examples, Wedderburn's theorem tells us any finite nonabelian group is not a member of $\mathcal{R}$. Because every Boolean ring is a commutative ring, we conclude that any left [right] zero semigroup, that is, a semigroup S in which $xy = x\,[xy = y]$ for all $x, y \in S$, containing more than one element is not a member of $\mathcal{R}$. It is interesting to know that S.R. Kogalovski [8] announced in 1961 that the class $\mathcal{R}$ is not axiomatizable.

In fact, semigroups admitting ring structure have long been studied. In 1970, R.E. Peinado [10] gave a brief survey of semigroups admitting ring structure. D.D. Chu and H.I. Shyr [2] proved a nice result that the multiplicative semigroup $\mathbb{N}$ of natural numbers is a member of $\mathcal{R}$ by showing that $(\mathbb{N}^0, \cdot) \cong (\mathbb{Z}_2[X], \cdot)$. For various studies in this area, see [6], [9], [11], [12], [13], [15] and [7].

We know that nearrings generalize rings and many examples and important results of nearrings can be found in [3], the book written by J.R. Clay. Some important right nearrings which are not rings are $(M(A), +, \circ)$ and $(M(G), +, \circ)$ where A is an abelian group, G is a group, $M(A)$ is the set of all functions $f : A \to A$, $M(G)$ is the set of all functions $f : G \to G$ and $+$ and $\circ$ are usual addition and composition of functions, respectively. Observe that $(M(A), +)$ is an abelian group, so we say that $(M(A), +, \circ)$ is an additively commutative right nearring. If A and G are containing more than one element, then $(M(A), +, \circ)$ and $(M(G), +, \circ)$ have no multiplicative zero. However, every left nearring has a multiplicative right zero and every right nearring has a multiplicative left zero. It follows that if a semigroup S has no right [left] zero, then S is not isomorphic to the multiplicative structure of a left [right] nearring. Then the most reasonable definition of a semigroup admitting a left [right] nearring structure is as follows : A semigroup S is said to *admit a left [right] nearring structure* if S or S^0 is isomorphic to the multiplicative structure of some left [right] nearring, or equivalently

(1) there is an operation $+$ on S such that $(S, +, \cdot)$ is a left [right] nearring where $\cdot$ is the operation on S or

(2) there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a left [right] nearring where $\cdot$ is the operation on S^0.

There were some studies of semigroups admitting left [right] nearring structure. These can be seen from [4] and [5].

In this research, we characterize when linear transformation semigroups of various types admit a left nearring structure and a right nearring structure.

Chapter II contains examples, basic definitions and quoted results which are refered for our research.

Chapter III deals with some linear transformation semigroups with zero. We will show that they do not admit the structure of both a left nearring and a right nearring. Since every ring is a left nearring and a right nearring, semigroups in this chapter do not admit a ring structure. Some results in [1] are useful for this chapter.

In the last chapter, we still consider some linear transformation semigroups with zero. We provide necessary and sufficient conditions for each these linear transformation semigroups to admit a left nearring structure and a right nearring structure. Some techniques in [1] are important for this chapter.

CHAPTER II
PRELIMINARIES

For any set X, the cardinality of X will be denoted by $|X|$.

An element a of a semigroup S is called an *idempotent* if $a^2 = a$. A *left* [*right*] *zero* of a semigroup S is an element $z \in S$ such that $zx = z\,[xz = z]$ for all $x \in S$. An element 0 of S is called a *zero* of S if $0x = x0 = 0$ for all $x \in S$. If S has a left zero z_1 and a right zero z_2, then $z_1 = z_2$ which is the zero of S.

Note that right [left] zeroes of a semigroup S are idempotents of S. The identity of a group G is exactly one idempotent of G.

In a semigroup S, we can adjoin an extra element 0 and define $0x = x0 = 0$ for all $x \in S$. Then $S \cup \{0\}$ becomes a semigroup with zero 0. For a semigroup S, we let

$$S^0 = \begin{cases} S \cup \{0\} & \text{if } |S| = 1 \text{ or } S \text{ has no zero,} \\ S & \text{otherwise.} \end{cases}$$

Observe that if $|S| = 1$, then $S^0 \cong (\mathbb{Z}_2, \cdot)$.

A semigroup S is called a *left* [*right*] *zero semigroup* if $xy = x\,[xy = y]$ for all $x, y \in S$.

A *left* [*right*] *nearring* is a triple $(N, +, \cdot)$ such that

(i) $(N, +)$ is a group,

(ii) $(N, \cdot)$ is a semigroup and

(iii) $z \cdot (x + y) = z \cdot x + z \cdot y\,[(x + y) \cdot z = x \cdot z + y \cdot z]$ for all $x, y, z \in N$.

Throughout, for every $x, y \in N, x \cdot y$ is denoted by xy.

Proposition 2.1. ([3], page 19) *Let* $(N, +, \cdot)$ *be a left* [*right*] *nearring with the additive identity* 0. *Then*

(*i*) $x0 = 0\,[0x = 0]$ *for all* $x \in N$.

(*ii*) $x(-y) = -(xy)\,[(-x)y = -(xy)]$ *for all* $x, y \in N$.

A *zero* of a left [right] nearring $(N, +, \cdot)$ is an element $z \in N$ such that $xz = zx = z$ for all $x \in N$. If z is a zero of a left [right] nearring $(N, +, \cdot)$, then z is a right [left] zero of the semigroup $(N, \cdot)$. From Proposition 1.1 (i), 0 is a right [left] zero of $(N, \cdot)$. Thus $z = 0$. Hence the left [right] nearring $(N, +, \cdot)$ has a zero if and only if $0x = x0 = 0$ for all $x \in N$ where 0 is the identity of $(N, +)$. Such left [right] nearring is called *zero-symmetric*.

Example 2.2. ([3], page 14) Let $(G, +)$ be any group with the identity 0. For each $S \subseteq G \smallsetminus \{0\}$, define $\cdot_S : G \times G \to G$ by

$$x \cdot_S y = \begin{cases} y\,[x] & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(G, +, \cdot_S)$ is a left [right] nearring. Moreover, if $x * y = y\,[x]$ for all $x, y \in G$, then $(G, +, *)$ is also a left [right] nearring. If $|G| \geq 2$, then $(G, +, *)$ is neither a right [left] nearring nor a ring.

An example of a right nearring which is not a left nearring and a ring is shown as follows:

Example 2.3. ([3], pages 7 and 19) Let $(A, +)$ be an abelian group with identity 0, $M(A)$ the set of all mappings $f : A \to A$ and

$$M_0(A) = \{f \in M(A) | f(0) = 0\}.$$

Then $(M(A), +, \circ)$ is a right nearring and $(M_0(A), +, \circ)$ is a zero-symmetric right nearring where $+$ and $\circ$ are the usual addition and composition of functions. Moreover, the zero map θ, that is, $\theta(x) = 0$ for all $x \in A$, is a left zero of $(M(A), \circ)$ which is not a zero if $|A| > 1$ and θ is the zero of $(M_0(A), \circ)$. By Proposition 2.1 (i), $(M(A), +, \circ)$ is neither a left nearring nor a ring where $|A| > 1$.

Hence every left [right] nearring is a genearalization of rings.

A semigroup S is said to *admit a ring structure* if S^0 is isomorphic to a multiplicative structure of some ring, or equivalently, there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is the operation on S^0.

Proposition 2.1 shows that if S has no right [left] zero, then there is no operation $+$ on S such that $(S,+,\cdot)$ is a left [right] nearring. Therefore the definition of semigroups admitting left [right] nearring structure is reasonably given as follows. We say that a semigroup S *admits a left* [*right*] *nearring structure* if S or S^0 is isomorphic to the multiplicative structure of some left [right] nearring, or equivalently,

(1) there is an operation $+$ on S such that $(S,+,\cdot)$ is a left [right] nearring where $\cdot$ is the operation on S or

(2) there is an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is a left [right] nearring where $\cdot$ is the operation on S^0.

Denote by $\mathcal{R}, \mathcal{LNR}$ and $\mathcal{RNR}$ the classes of all semigroups which admit a ring structure, a left nearring structure and a right nearring structure, respectively. Then $\mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

Throughout this research, let V be a vector space over a division ring R and $L_R(V)$ the semigroup under composition of all linear transformations $\alpha : V \to V$. Then $L_R(V)$ admits a ring structure under the usual addition of linear transformations. The image of v under $\alpha \in L_R(V)$ is written by $v\alpha$. For $\alpha \in L_R(V)$, let Ker α and Im α denote the kernel and the image of α, respectively. For $A \subseteq V$, let $\langle A \rangle$ stand for the subspace of V spanned by A. $\dim_R W$ means the dimension of a subspace W of V. For every $\alpha, \beta \in L_R(V)$, $\alpha\beta$ means $\alpha \circ \beta$ where $\circ$ is the composition of function. The following five propositions are simple facts of vector spaces and linear transformations which will be used. The proofs are routine and elementary, so they will be omitted.

Proposition 2.4. *Let B be a basis of V. If $u \in B$ and $v \in \langle B \smallsetminus \{u\} \rangle$, then $(B \smallsetminus \{u\}) \cup \{u+v\}$ is also a basis of V.*

Proposition 2.5. *Let B be a basis of $V, A \subseteq B$ and $\varphi : B \smallsetminus A \to V$ a one-to-one map such that $(B \smallsetminus A)\varphi$ is a linearly independent subset of V. If $\alpha \in L_R(V)$ is defined by*

$$v\alpha = \begin{cases} 0 & \text{if } v \in A, \\ v\varphi & \text{if } v \in B \smallsetminus A, \end{cases}$$

then Ker $\alpha = \langle A \rangle$ and Im $\alpha = \langle B \smallsetminus A \rangle \varphi$.

Proposition 2.6. *Let B be a basis of V and $A \subseteq B$. Then*

(i) $\{v + \langle A \rangle \mid v \in B \setminus A\}$ *is a basis of the quotient space* $V/\langle A \rangle$ *and*

(ii) $\dim_R(V/\langle A \rangle) = |B \setminus A|$.

Proposition 2.7. *Let B be a basis of V. Then for every $v \in V$, there is a unique set of vectors $v_1, \ldots, v_n$ in B, along with a unique set of scalars $a_1, \ldots, a_n$ in R, for which $v = a_1v_1 + \cdots + a_nv_n$.*

Proposition 2.8. *Let B be a basis of V and B_1, B_2 and B_3 are disjoint subsets of B. Then $\langle B_1 \cup B_2 \rangle \cap \langle B_1 \cup B_3 \rangle = \langle B_1 \rangle$.*

Proposition 2.9. *Let B be a basis of V and C a nonempty subset of B. Then* $\bigcap\limits_{v \in C} \langle B \setminus \{v\} \rangle = \langle B \setminus C \rangle$.

Proof. Since $B \setminus C \subseteq B \setminus \{v\}$ for every $v \in C$, $\langle B \setminus C \rangle \subseteq \bigcap\limits_{v \in C} \langle B \setminus \{v\} \rangle$.

Conversely, let $w \in V$ be such that $w \notin \langle B \setminus C \rangle$. Then $w = a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_mv_m$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in R$, $u_1, \ldots, u_n \in B \setminus C$ and $v_1, \ldots, v_m \in C$. Since $w \notin \langle B \setminus C \rangle$, there exists $i \in \{1, \ldots, m\}$ such that $b_i \neq 0$. Without loss of generality, suppose that $b_1 \neq 0$. By Proposition 2.7, we have that $w \notin \langle B \setminus \{v_1\} \rangle$, so $w \notin \bigcap\limits_{v \in C} \langle B \setminus \{v\} \rangle$. Hence $\bigcap\limits_{v \in C} \langle B \setminus \{v\} \rangle \subseteq \langle B \setminus C \rangle$. □

Let

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $G_R(V)$ is the unit group of the semigroup $L_R(V)$ or the group of all units of $L_R(V)$. The following known result will be referred.

Proposition 2.10. ([14]) *$G_R(V)$ admits a ring structure if and only if* $\dim_R V \leq 1$.

Next, let

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \text{Ker } \alpha \text{ is infinite}\},$$

$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/\text{Im } \alpha) \text{ is infinite}\}.$$

If $\dim_R V$ is infinite, then 0 belongs to both $OM_R(V)$ and $OE_R(V)$. Since Ker $\alpha\beta \supseteq$ Ker α and Im $\alpha\beta \subseteq$ Im β, for all $\alpha, \beta \in L_R(V)$, it follows that $OM_R(V)$ and $OE_R(V)$ are both subsemigroups of $L_R(V)$ containing 0 if $\dim_R V$ is infinite.

For this case, the semigroups $OM_R(V)$ and $OE_R(V)$ may be referred to respetively as the *opposite semigroup* of $M_R(V)$ and the *opposite semigroup of* $E_R(V)$.

For any cardinal number k with $k \leq \dim_R V$, let

$$K_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \mathrm{Ker}\ \alpha \geq k\},$$
$$CI_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R(V/\mathrm{Im}\ \alpha) \geq k\},$$
$$I_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \mathrm{Im}\ \alpha \leq k\}.$$

Then the zero map 0 on V belongs to all of the above three subsets of $L_R(V)$. Since for $\alpha, \beta \in L_R(V), \mathrm{Ker}\ \alpha\beta \supseteq \mathrm{Ker}\ \alpha$ and $\mathrm{Im}\ \alpha\beta \subseteq \mathrm{Im}\ \beta$, we conclude that all of $K_R(V,k), CI_R(V,k)$ and $I_R(V,k)$ are subsemigroups of $L_R(V)$. Observe that if $\dim_R V$ is infinite, the notations $OM_R(V)$ and $OE_R(V)$ defined previously denote $K_R(V,\aleph_0)$ and $CI_R(V,\aleph_0)$, respectively, that is,

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \mathrm{Ker}\ \alpha \geq \aleph_0\},$$
$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/\mathrm{Im}\ \alpha) \geq \aleph_0\}.$$

We know that if $\dim_R V$ is finite, then for $\alpha \in L_R(V), \dim_R \mathrm{Ker}\ \alpha = \dim_R(V/\mathrm{Im}\ \alpha)$ $= \dim_R V - \dim_R \mathrm{Im}\ \alpha$ since $\dim_R V = \dim_R \mathrm{Ker}\ \alpha + \dim_R \mathrm{Im}\ \alpha$ and $\dim_R V = \dim_R(V/\mathrm{Im}\ \alpha) + \dim_R \mathrm{Im}\ \alpha$. Hence we have

Proposition 2.11. *If* $\dim_R V < \infty$, *then* $K_R(V,k) = CI_R(V,k) = I_R(V, \dim_R V - k)$ *for every cardinal number* $k \leq \dim_R V$.

However, these are not generally true if $\dim_R V$ is infinite. This is shown by the following proposition. This proposition also shows that the semigroups $K_R(V,k), CI_R(V,k)$ and $I_R(V,k)$ should be considered independently if $\dim_R V$ is infinite.

Proposition 2.12. ([1], page 12) *Let* V *be an infinite dimensional vector space and a nonzero cardinal number* $k \leq \dim_R V$. *Then the following statements hold.*

(i) $CI_R(V,k) \neq K_R(V,l)$ *for every cardinal number* $l \leq \dim_R V$.

(ii) *If* $k < \dim_R V$, *then* $I_R(V,k) \neq K_R(V,l)$ *and* $I_R(V,k) \neq CI_R(V,l)$ *for every cardinal number* $l \leq \dim_R V$.

Next, we define $K'_R(V,k)$, $CI'_R(V,k)$ and $I'_R(V,k)$ which are subsets of $K_R(V,k)$, $CI_R(V,k)$ and $I_R(V,k)$ respectively as follows :

$$K'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha > k\} \text{ where } k < \dim_R V,$$
$$CI'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im} \alpha) > k\} \text{ where } k < \dim_R V,$$
$$I'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha < k\} \text{ where } 0 < k \leq \dim_R V.$$

Then 0 belongs to all $K'_R(V,k), CI'_R(V,k)$ and $I'_R(V,k)$, moreover, they are respectively subsemigroups of $K_R(V,k), CI_R(V,k)$ and $I_R(V,k)$. Observe that if $k < \dim_R V$, then $K'_R(V,k) = K_R(V,k')$ and $CI'_R(V,k) = CI_R(V,k')$ where k' is the successor of k. Also, if $0 < k \leq \dim_R V, k$ is a finite cardinal number and $\tilde{k}$ is the predecessor of k, then $I'_R(V,k) = I_R(V,\tilde{k})$.

For $\alpha \in L_R(V)$, let

$$F(\alpha) = \{v \in V \mid v\alpha = v\}.$$

Then for $\alpha \in L_R(V), F(\alpha)$ is a subspace of V and α is called an *almost identical linear transformation* of V if $\dim_R(V/F(\alpha))$ is finite. The set of all almost identical linear transformations of V will be denoted by $AI_R(V)$, that is,

$$AI_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/F(\alpha)) < \infty\}.$$

Observe that 1_V, the identity map on V, belongs to $AI_R(V)$.

Proposition 2.13. ([1], page 14) *$AI_R(V)$ is a subsemigroup of $L_R(V)$.*

Notice that if $\dim_R V < \infty$, then $AI_R(V) = L_R(V)$ which admits a ring structure. Moreover, the semigroup $AI_R(V)$ does not contain 0, the zero map on V, if $\dim_R V$ is infinite.

Since every linear transformation from a vector space V into a vector space W can be defined on a basis of V, for convenience, we may write $\alpha \in L_R(V,W)$ by using a bracket notation. For examples,

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \setminus B_1}$$

means that B is a basis of $V, B_1 \subseteq B$ and

$$v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ v & \text{if } v \in B \setminus B_1 \end{cases}$$

and

$$\beta = \begin{pmatrix} u & w & v \\ w & 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}$$

means that B is a basis of $V, u, w \in B, u \neq w$ and

$$v\beta = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v = w, \\ v & \text{if } v \in B \smallsetminus \{u, w\}. \end{cases}$$

Chapter III deals with linear transformation semigroups on V with zero. The purpose of this chapter is to show that if $\dim_R V$ is infinite, $OM_R(V), OE_R(V)$ and some linear transformation semigroups containing $OM_R(V)$ and $OE_R(V)$ do not admit the structure of both a left nearring and a right nearring.

The semigroups $OM_R(V)$ and $OE_R(V)$ are generalized to be the semigroups $K_R(V,k)$ and $CI_R(V,k)$, respectively. We also determine in the last chapter when the semigroups $K_R(V,k)$ and $CI_R(V,k)$ admit a left nearring and a right nearring structure. Moreover, the semigroups $I_R(V,k), K'_R(V,k), CI'_R(V,k)$ and $I'_R(V,k)$ are also studied in the same matter.

CHAPTER III
SEMIGROUPS WHICH DO NOT ADMIT THE STRUCTURE OF A LEFT [RIGHT] NEARRING

In this chapter, we deal with linear transformation semigroups on V where $\dim_R V$ is infinite. The following linear transformation semigroups on V given in Chapter I are recalled as follows:

$$\begin{aligned}
L_R(V) &= \{\alpha : V \to V \mid \alpha \text{ is a linear transformation}\},\\
G_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\},\\
OM_R(V) &= \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \text{ is infinite}\},\\
OE_R(V) &= \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im} \alpha) \text{ is infinite}\},\\
AI_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is almost identical}\}\\
&= \{\alpha \in L_R(V) \mid \dim_R(V/F(\alpha)) < \infty\}\\
&\quad \text{where } F(\alpha) = \{v \in V \mid v\alpha = v\}.
\end{aligned}$$

3.1 The Semigroups $OM_R(V)$ and $OE_R(V)$

Recall that 0, the zero map on V belongs to both $OM_R(V)$ and $OE_R(V)$ and note that $1_V \notin OM_R(V)$ and $1_V \notin OE_R(V)$.

Theorem 3.1.1. *If $S(V)$ is $OM_R(V)$ or $OE_R(V)$, then the following statements hold.*

(i) $S(V)$ does not admit a left nearring structure.

(ii) $S(V)$ does not admit a right nearring structure.

Proof. Let B be a basis of V. Then B is infinite, so there are subsets B_1, B_2 of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B| = |B_1| = |B_2|$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1} \quad \text{and } \beta = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2}. \tag{1}$$

So Ker $\alpha = \langle B_2 \rangle, \dim_R(V/\text{Im } \alpha) = \dim_R(V/\langle B_1 \rangle) = |B_2|, \text{Ker } \beta = \langle B_1 \rangle$ and $\dim_R(V/\text{Im } \beta) = \dim_R(V/\langle B_2 \rangle) = |B_1|$. Then α and β are elements of $OM_R(V)$ and $OE_R(V)$. Obviously,

$$\alpha^2 = \alpha,\ \beta^2 = \beta,\ \alpha\beta = \beta\alpha = 0.$$

(i) Suppose that $S(V) \in \mathcal{LNR}$. Then there is an operation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \circ)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in S(V)$. So

$$\alpha\lambda = \alpha(\alpha \oplus \beta) = \alpha^2 \oplus \alpha\beta = \alpha \oplus 0 = \alpha,$$
$$\beta\lambda = \beta(\alpha \oplus \beta) = \beta\alpha \oplus \beta^2 = 0 \oplus \beta = \beta.$$

We therefore deduce from these equalities and (1) that

$$\text{for every } v \in B_1,\ v\lambda = v\alpha\lambda = v\alpha = v,$$
$$\text{for every } v \in B_2,\ v\lambda = v\beta\lambda = v\beta = v.$$

Consequently, $v\lambda = v$ for every $v \in B$. Since B is a basis of $V, \lambda = 1_V$. This is a contradiction because $1_V \notin OM_R(V)$ and $1_V \notin OE_R(V)$. Hence $S(V) \notin \mathcal{LNR}$.

(ii) Suppose that $S(V) \in \mathcal{RNR}$. Then there is an operation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \circ)$ is a right nearring. Then $\lambda = \alpha \oplus \beta \in S(V)$. Consequently,

$$\lambda\alpha = (\alpha \oplus \beta)\alpha = \alpha^2 \oplus \beta\alpha = \alpha \oplus 0 = \alpha,$$
$$\lambda\beta = (\alpha \oplus \beta)\beta = \alpha\beta \oplus \beta^2 = 0 \oplus \beta = \beta.$$

We therefore deduce from these equalities and (1) that

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda\alpha = v\alpha = v, \\ &\text{for every } v \in B_2,\ v\lambda\beta = v\beta = v. \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda\beta = v\beta = 0, \\ &\text{for every } v \in B_2,\ v\lambda\alpha = v\alpha = 0. \end{aligned} \tag{3}$$

By (1) and (3), we have that

$$\text{for every } v \in B_1,\ v\lambda \in \text{Ker } \beta = \langle B_1 \rangle,$$
$$\text{for every } v \in B_2,\ v\lambda \in \text{Ker } \alpha = \langle B_2 \rangle.$$

Then

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda\alpha = v\lambda, \\ &\text{for every } v \in B_2,\ v\lambda\beta = v\lambda. \end{aligned} \tag{4}$$

By (2) and (4), we have $v\lambda = v$ for every $v \in B$. Since B is a basis of V, $\lambda = 1_V$. This is a contradiction because $1_V \notin OM_R(V)$ and $1_V \notin OE_R(V)$. Hence $S(V) \notin \mathcal{RNR}$.

Therefore the theorem is proved. $\square$

Since every ring is both a left nearring and a right nearring, we have

Corollary 3.1.2. *The semigroups $OM_R(V)$ and $OE_R(V)$ do not admit a ring structure.*

3.2 Semigroups Containing $OM_R(V)$ and Semigroups Containing $OE_R(V)$

Also, $\dim_R V$ is assumed to be infinite in this section. The following proposition is needed for our study.

Proposition 3.2.1. ([1], pages 22, 23 and 25) *The following statements hold.*

(i) $OM_R(V) \cup H$ *is a subsemigroup of* $L_R(V)$ *where* H *is a subsemigroup of* $G_R(V)$.

(ii) $OE_R(V) \cup H$ *is a subsemigroup of* $L_R(V)$ *where* H *is a subsemigroup of* $G_R(V)$.

(iii) $OM_R(V) \cup T$ *is a subsemigroup of* $L_R(V)$ *where* T *is a subsemigroup of* $AI_R(V)$.

(iv) $OE_R(V) \cup T$ *is a subsemigroup of* $L_R(V)$ *where* T *is a subsemigroup of* $AI_R(V)$.

It is shown in this section that any linear transformation semigroup on V of Proposition 3.2.1 does not admit the structure of a left nearring and a right nearring.

Theorem 3.2.2. *If H is a subsemigroup of $G_R(V)$ and $S(V)$ is the semigroup $OM_R(V) \cup H$ or the semigroup $OE_R(V) \cup H$, then the following statements hold.*

(i) $S(V)$ does not admit a left nearring structure.

(ii) $S(V)$ does not admit a right nearring structure.

Proof. Let B be a basis of V and $u \in B$ a fixed element. Since B is infinite, $B \setminus \{u\}$ has subsets B_1, B_2 such that $B \setminus \{u\} = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and

$|B_1| = |B_2| = |B \setminus \{u\}| (= |B|)$. Then $B = B_1 \cup B_2 \cup \{u\}$ and these three sets are pairwise disjoint. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \cup \{u\} \\ v & 0 \end{pmatrix}_{v \in B_1}, \ \beta = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v \end{pmatrix}_{v \in B_2}, \ \gamma = \begin{pmatrix} u & B_1 \cup B_2 \\ u & 0 \end{pmatrix}. \quad (1)$$

So Ker $\alpha = \langle B_2 \cup \{u\} \rangle$, $\dim_R(V/\text{Im } \alpha) = \dim_R(V/\langle B_1 \rangle) = |B_2 \cup \{u\}|$, Ker $\beta = \langle B_1 \cup \{u\} \rangle$, $\dim_R(V/\text{Im } \beta) = \dim_R(V/\langle B_2 \rangle) = |B_1 \cup \{u\}|$, Ker $\gamma = \langle B_1 \cup B_2 \rangle$ and $\dim_R(V/\text{Im } \gamma) = \dim_R(V/\langle u \rangle) = |B_1 \cup B_2|$. Then $\alpha, \beta, \gamma \in S(V)$. Obviously,

$$\alpha^2 = \alpha, \ \beta^2 = \beta, \ \alpha\beta = \beta\alpha = \gamma\alpha = \alpha\gamma = \gamma\beta = \beta\gamma = 0.$$

(i) Suppose that $S(V) \in \mathcal{LNR}$. Then there is an operation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \circ)$ is a left nearring. So $\lambda = \alpha \oplus \beta \in S(V)$. It is obtained that

$$\alpha\lambda = \alpha(\alpha \oplus \beta) = \alpha^2 \oplus \alpha\beta = \alpha \oplus 0 = \alpha,$$
$$\beta\lambda = \beta(\alpha \oplus \beta) = \beta\alpha \oplus \beta^2 = 0 \oplus \beta = \beta,$$
$$\gamma\lambda = \gamma(\alpha \oplus \beta) = \gamma\alpha \oplus \gamma\beta = 0 \oplus 0 = 0.$$

We therefore deduce from these equalities and (1) that

$$\text{for every } v \in B_1, \ v\lambda = v\alpha\lambda = v\alpha = v,$$
$$\text{for every } v \in B_2, \ v\lambda = v\beta\lambda = v\beta = v,$$
$$u\lambda = u\gamma\lambda = 0,$$

that is,

$$\lambda = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B_1 \cup B_2}.$$

Thus Ker $\lambda = \langle u \rangle$ and $\dim_R(V/\text{Im } \lambda) = \dim_R(V/\langle B_1 \cup B_2 \rangle) = |\{u\}| = 1$. Hence $\lambda \notin S(V)$ which is contrary to that $\lambda = \alpha \oplus \beta \in S(V)$. Therefore $S(V) \notin \mathcal{LNR}$.

(ii) Suppose that $S(V) \in \mathcal{RNR}$. Then there is an operation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \circ)$ is a right nearring. Then $\lambda = \alpha \oplus \beta \in S(V)$. Hence

$$\lambda\alpha = (\alpha \oplus \beta)\alpha = \alpha^2 \oplus \beta\alpha = \alpha \oplus 0 = \alpha,$$
$$\lambda\beta = (\alpha \oplus \beta)\beta = \alpha\beta \oplus \beta^2 = 0 \oplus \beta = \beta,$$
$$\lambda\gamma = (\alpha \oplus \beta)\gamma = \alpha\gamma \oplus \beta\gamma = 0 \oplus 0 = 0.$$

We therefore deduce from these equalities and (1) that

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda\alpha = v\alpha = v,\\ &\text{for every } v \in B_2,\ v\lambda\beta = v\beta = v,\\ &\text{for every } v \in V, \quad v\lambda\gamma = 0\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda\beta = v\beta = 0,\\ &\text{for every } v \in B_2,\ v\lambda\alpha = v\alpha = 0.\end{aligned} \tag{3}$$

By (1) and (3), we have that

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda \in \text{Ker } \beta = \langle B_1 \cup \{u\}\rangle,\\ &\text{for every } v \in B_2,\ v\lambda \in \text{Ker } \alpha = \langle B_2 \cup \{u\}\rangle.\end{aligned}$$

Claim that for every $v \in B_1$, $v\lambda \in \langle B_1\rangle$ and for every $v \in B_2$, $v\lambda \in \langle B_2\rangle$. Let $v \in B_1$. Then $v\lambda = a_1v_1 + a_2v_2 + \cdots a_nv_n + au$ for some $a_1, a_2, \ldots, a_n, a \in R$ and $v_1, v_2, \ldots, v_n \in B_1$. Since $0 = v\lambda\gamma = (a_1v_1 + a_2v_2 + \cdots a_nv_n + au)\gamma = au$, $v\lambda \in \langle B_1\rangle$. Similarly, we have that for every $v \in B_2$, $v\lambda \in \langle B_2\rangle$. It is obtained from (1) that

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda\alpha = v\lambda,\\ &\text{for every } v \in B_2,\ v\lambda\beta = v\lambda.\end{aligned} \tag{4}$$

Hence by (2) and (4), we have that for all $v \in \langle B_1 \cup B_2\rangle, v\lambda = v$. Since $u\lambda\gamma = 0$, by the definition of γ, $u\lambda \in \text{Ker } \gamma = \langle B_1 \cup B_2\rangle$, so $(u - u\lambda)\lambda = u\lambda - (u\lambda)\lambda = u\lambda - u\lambda = 0$. Since $B_1 \cup B_2 \cup \{u\}$ is a basis of V and $u\lambda \in \langle B_1 \cup B_2\rangle$, by Proposition 2.4, $B_1 \cup B_2 \cup \{u - u\lambda\}$ is a basis of V. Hence

$$\lambda = \begin{pmatrix} u - u\lambda & v \\ 0 & v \end{pmatrix}_{v \in B_1 \cup B_2}.$$

Thus Ker $\lambda = \langle u - u\lambda\rangle$ and $\dim_R(V/\text{Im } \lambda) = \dim_R(V/\langle B_1 \cup B_2\rangle) = |\{u - u\lambda\}| = 1$. Then $\lambda \notin S(V)$. It is contrary to that $\lambda = \alpha \oplus \beta \in S(V)$. Hence $S(V) \notin \mathcal{RNR}$.

Therefore the proof is complete. □

The following corollary is a direct consequence of the above theorem.

Corollary 3.2.3. *If H is a subsemigroup of $G_R(V)$, then the semigroups $OM_R(V) \cup H$ and $OE_R(V) \cup H$ do not admit a ring structure.*

Remark 3.2.4. Let B be a basis of V and for distinct $u, w \in B$, let $\alpha_{u,w} \in G_R(V)$ be defined by

$$\alpha_{u,w} = \begin{pmatrix} u & w & v \\ w & u & v \end{pmatrix}_{v \in B \setminus \{u,w\}}.$$

Then $H_{u,w} = \{1_V, \alpha_{u,w}\}$ is a subgroup of $G_R(V)$ for all distinct $u, w \in B$, and $H_{u,w} \neq H_{u',w'}$ if u, w, u', w' are elements of B such that $(u, w) \neq (u', w')$. This fact and Theorem 3.2.2 show that if $\dim_R V$ is infinite, there are infinitely many subsemigroups of $L_R(V)$ containing $OM_R(V)$ and infinitely many subsemigroups of $L_R(V)$ containing $OE_R(V)$ which do not admit the structure of a left nearring and a right nearring.

Theorem 3.2.5. *If T is a subsemigroup of $AI_R(V)$ and $S(V)$ is the semigroup $OM_R(V) \cup T$ or the semigroup $OE_R(V) \cup T$, then the following statements hold.*

(i) $S(V)$ does not admit a left nearring structure.

(ii) $S(V)$ does not admit a right nearring structure.

Proof. Let B be a basis of V and let $B_1, B_2 \subseteq B$ be such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Then there is a bijection $\varphi : B_1 \to B_2$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ v\varphi & 0 \end{pmatrix}_{v \in B_1} \quad \text{and } \beta = \begin{pmatrix} B_1 & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in B_2}. \tag{1}$$

So $\text{Ker } \alpha = \langle B_2 \rangle, \dim_R(V/\text{Im } \alpha) = \dim_R(V/\langle B_2 \rangle) = |B_1|, \text{Ker } \beta = \langle B_1 \rangle$ and $\dim_R(V/\text{Im } \beta) = \dim_R(V/\langle B_1 \rangle) = |B_2|$. Thus $\alpha, \beta \in OM_R(V) \cap OE_R(V)$, and so $\alpha, \beta \in S(V)$. Obviously,

$$\alpha^2 = \beta^2 = 0,$$
$$\alpha\beta = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1} \text{ and } \beta\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2}. \tag{2}$$

(i) Suppose that $S(V) \in \mathcal{LNR}$. Then $(S(V), \oplus, \circ)$ is a left nearring for some operation $\oplus$ on $S(V)$. Let $\lambda = \alpha \oplus \beta \in S(V)$. It follows from (2) that

$$\alpha\lambda = \alpha(\alpha \oplus \beta) = \alpha^2 \oplus \alpha\beta = 0 \oplus \alpha\beta = \alpha\beta,$$
$$\beta\lambda = \beta(\alpha \oplus \beta) = \beta\alpha \oplus \beta^2 = \beta\alpha \oplus 0 = \beta\alpha.$$

We therefore from these facts, (1) and (2), we have

$$\begin{aligned}&\text{for every } v \in B_1,\ (v\varphi)\lambda = v\alpha\lambda = v\alpha\beta = v = (v\varphi)\varphi^{-1},\\ &\text{for every } v \in B_2,\ (v\varphi^{-1})\lambda = v\beta\lambda = v\beta\alpha = v = (v\varphi^{-1})\varphi.\end{aligned} \tag{3}$$

We can see from (3) that

$$\begin{aligned}&\lambda|_{B_2} = \varphi^{-1} : B_2 \to B_1 \text{ is a bijection,}\\ &\lambda|_{B_1} = \varphi : B_1 \to B_2 \text{ is a bijection.}\end{aligned} \tag{4}$$

Since $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$, $\lambda|_B : B \to B$ is a bijection. So $\lambda \in G_R(V)$. Hence $\lambda \notin OM_R(V)$ and $\lambda \notin OE_R(V)$. Claim that $\lambda \notin AI_R(V)$. By (4), $B_1 \cap F(\lambda) = \varnothing$, that is $v + F(\lambda) \neq F(\lambda)$ for every $v \in B_1$. Let $v_1, v_2, \ldots, v_n \in B_1$ be distinct and $a_1, a_2 \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\lambda)) = F(\lambda)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\lambda)$, so

$$\begin{aligned}\sum_{i=1}^{n} a_i v_i &= (\sum_{i=1}^{n} a_i v_i)\lambda\\ &= \sum_{i=1}^{n} a_i(v_i\lambda) \in \langle B_2\rangle \qquad \text{from (4).}\end{aligned}$$

Thus $\sum_{i=1}^{n} a_i v_i \in \langle B_1\rangle \cap \langle B_2\rangle = \{0\}$. Then $a_i = 0$ for all $i \in \{1, \ldots, n\}$. So $\{v + F(\lambda) \mid v \in B_1\}$ is a linearly independent subset of $V/F(\lambda)$ and $v + F(\lambda) \neq w + F(\lambda)$ for all distinct $v, w \in B_1$. Hence $\dim_R(V/F(\lambda)) \geq |B_1|$. Since B_1 is infinite, $\lambda \notin AI_R(V)$. Thus $\lambda \notin S(V)$ which is contrary to that $\lambda \in S(V)$. Therefore $S(V) \notin \mathcal{LNR}$.

(ii) Suppose that $S(V) \in \mathcal{RNR}$. Then $(S(V), \oplus, \circ)$ is a right nearring for some operation $\oplus$ on $S(V)$. Let $\lambda = \alpha \oplus \beta \in S(V)$. It follows from (2) that

$$\begin{aligned}\lambda\alpha &= (\alpha \oplus \beta)\alpha = \alpha^2 \oplus \beta\alpha = 0 \oplus \beta\alpha = \beta\alpha,\\ \lambda\beta &= (\alpha \oplus \beta)\beta = \alpha\beta \oplus \beta^2 = \alpha\beta \oplus 0 = \alpha\beta.\end{aligned}$$

We therefore from these facts, (1) and (2), we have

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda\beta = v\alpha\beta = v,\\ &\text{for every } v \in B_2,\ v\lambda\alpha = v\beta\alpha = v,\end{aligned} \tag{5}$$

and

$$\text{for every } v \in B_1,\ v\lambda\alpha\beta = v\beta\alpha\beta = 0\alpha\beta = 0,$$
$$\text{for every } v \in B_2,\ v\lambda\beta\alpha = v\alpha\beta\alpha = 0\beta\alpha = 0.$$

By these facts and (2), we have that

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda \in \text{Ker } \alpha\beta = \langle B_2 \rangle\,, \\ &\text{for every } v \in B_2,\ v\lambda \in \text{Ker } \beta\alpha = \langle B_1 \rangle\,. \end{aligned} \tag{6}$$

It is obtained from (1) and (5) that

$$\text{for every } v \in B_1,\ (v\lambda)\beta = v = (v\varphi)\varphi^{-1} = (v\varphi)\beta,$$
$$\text{for every } v \in B_2,\ (v\lambda)\alpha = v = (v\varphi^{-1})\varphi = (v\varphi^{-1})\alpha.$$

We can see from (6) that $B_1\lambda, B_1\varphi \subseteq \langle B_2 \rangle$ and $B_2\lambda, B_2\varphi^{-1} \subseteq \langle B_1 \rangle$. Since $\beta|_{\langle B_2 \rangle}$ and $\alpha|_{\langle B_1 \rangle}$ are monomorphisms, we have

$$\text{for every } v \in B_1,\ v\lambda = v\varphi,$$
$$\text{for every } v \in B_2,\ v\lambda = v\varphi^{-1}.$$

Hence

$$\lambda|_{B_1} = \varphi : B_1 \to B_2 \text{ is a bijection,}$$
$$\lambda|_{B_2} = \varphi^{-1} : B_2 \to B_1 \text{ is a bijection.}$$

Since $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$, $\lambda|_B : B \to B$ is a bijection. So $\lambda \in G_R(V)$. Hence $\lambda \notin OM_R(V)$ and $\lambda \notin OE_R(V)$. Similarly (i), we have that $\lambda \notin AI_R(V)$. Thus $\lambda \notin S(V)$ which is contrary to that $\lambda \in S(V)$. Therefore $S(V) \notin \mathcal{RNR}$.

Hence the theorem is proved. □

Also, we have a corollary of Theorem 3.2.5 as follows:

Corollary 3.2.6. *If T is a subsemigroup of $AI_R(V)$, then the semigroups $OM_R(V) \cup T$ and $OE_R(V) \cup T$ do not admit a ring structure.*

Remark 3.2.7. Let B be a basis of V and for each $u \in B$, define $\alpha_u \in L_R(V)$ by

$$\alpha_u = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

Then $F(\alpha_u) = \langle B \smallsetminus \{u\} \rangle$, and hence by Proposition 2.6 (ii), $\dim_R(V/F(\alpha_u)) = |\{u\}|$ for every $u \in B$. Clearly, $\alpha_u \neq \alpha_w$ if u and w are distinct elements of B

and for each $u \in B, \{\alpha_u\}$ is a subsemigroup of $AI_R(V)$ since $\alpha_u^2 = \alpha_u$. This fact and Theorem 3.2.5 show that there are infinitely many subsemigroups of $L_R(V)$ containing $OE_R(V)$ which do not admit the structure of a left nearring and a right nearring.

CHAPTER IV
SEMIGROUPS ADMITTING THE STRUCTURE OF A LEFT [RIGHT] NEARRING

First, let us recall the following linear transformation semigroups on V.

$$\begin{aligned}
K_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R \text{Ker } \alpha \geq k\} \\
&\quad \text{where } k \leq \dim_R V, \\
K'_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R \text{Ker } \alpha > k\} \\
&\quad \text{where } k < \dim_R V, \\
CI_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R (V/\text{Im } \alpha) \geq k\} \\
&\quad \text{where } k \leq \dim_R V, \\
CI'_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R (V/\text{Im } \alpha) > k\} \\
&\quad \text{where } k < \dim_R V, \\
I_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R \text{Im } \alpha \leq k\} \\
&\quad \text{where } k \leq \dim_R V, \\
I'_R(V,k) &= \{\alpha \in L_R(V) \mid \dim_R \text{Im } \alpha < k\} \\
&\quad \text{where } 0 < k \leq \dim_R V.
\end{aligned}$$

Note that these semigroups contain 0, the zero map on V.

4.1 The Semigroups $K_R(V,k)$ and $K'_R(V,k)$

We shall characterize when $K_R(V,k)$ admits the structure of a left nearring or a right nearring. The characterization will generalize Theorem 3.1.1 for the case of $OM_R(V)$ since $OM_R(V) = K_R(V,\aleph_0)$ if $\dim_R V$ is infinite. Since $K'_R(V,k) = K_R(V,k')$ if k' is the successor of k, by the characterization of $K_R(V,k)$ admitting this structure, necessary and sufficient conditions for $K'_R(V,k)$ to admit such a structure are also obtained.

Theorem 4.1.1. *Let k be a cardinal number with $k \leq dim_R V$.*

(1) $K_R(V,k)$ admits the structure of a left nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

(2) $K_R(V,k)$ admits the structure of a right nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

Proof. To prove sufficiency, assume that (i) or (ii) holds. If $k = 0$, then $K_R(V,k) = K_R(V,0) = L_R(V)$. So $K_R(V,k) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$. If $\dim_R V < \infty$ and $k = \dim_R V$, then for every $\alpha \in K_R(V,k)$, $\dim_R \operatorname{Ker} \alpha = \dim_R V < \infty$ implies that $\operatorname{Ker} \alpha = V$ and $\alpha = 0$, respectively. Hence $K_R(V,k) = \{0\} \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $(K_R(V,k), \circ)$ admits the structure of a left nearring or a right nearring. To prove that (i) or (ii) must hold, suppose not. Then either ($k > 0$ and $\dim_R V$ is infinite) or $0 < k < \dim_R V < \infty$.

Case 1 : $k > 0$ and $\dim_R V$ is infinite. Let B be a basis of V. Then B is infinite. So there exist subsets B_1 and B_2 of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\alpha, \beta \in L_R(V)$ be defined by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2} \quad \text{and} \quad \beta = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1}. \tag{1}$$

Then $\operatorname{Ker} \alpha = \langle B_1 \rangle$ and $\operatorname{Ker} \beta = \langle B_2 \rangle$, so $\dim_R \operatorname{Ker} \alpha = |B_1| = |B|$ and $\dim_R \operatorname{Ker} \beta = |B_2| = |B|$. Since $k \leq \dim_R V = |B|$, we have $\alpha, \beta \in K_R(V,k)$. Obviously,

$$\alpha^2 = \alpha,\ \beta^2 = \beta \ \text{ and } \ \alpha\beta = \beta\alpha = 0.$$

If $(K_R(V,k), \oplus, \circ)$ is a left nearring for some operation $\oplus$ on $K_R(V,k)$, then

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha^2 \oplus \alpha\beta = \alpha \oplus 0 = \alpha \ \text{ and} \\ \beta(\alpha \oplus \beta) &= \beta\alpha \oplus \beta^2 = 0 \oplus \beta = \beta. \end{aligned} \tag{2}$$

Let $\lambda = \alpha \oplus \beta \in K_R(V,k)$. Then from (2), $\alpha\lambda = \alpha$ and $\beta\lambda = \beta$. Consequently,

$$\text{for every } v \in B_1,\ v\lambda = v\beta\lambda = v\beta = v,$$
$$\text{for every } v \in B_2,\ v\lambda = v\alpha\lambda = v\alpha = v.$$

Since $B = B_1 \cup B_2, \lambda = 1_V$, the identity map on V. Since $k > 0$, we have $\dim_R \text{Ker } \lambda = 0 < k$. Hence $\lambda \notin K_R(V,k)$, a contradiction.

If $(K_R(V,k), \oplus, \circ)$ is a right nearring for some operation $\oplus$ on $K_R(V,k)$, then

$$\begin{aligned}(\alpha \oplus \beta)\alpha &= \alpha^2 \oplus \beta\alpha = \alpha \oplus 0 = \alpha \text{ and}\\ (\alpha \oplus \beta)\beta &= \alpha\beta \oplus \beta^2 = 0 \oplus \beta = \beta.\end{aligned} \tag{3}$$

Let $\lambda = \alpha \oplus \beta \in K_R(V,k)$. Then from (3), $\lambda\alpha = \alpha$ and $\lambda\beta = \beta$. Consequently,

$$\begin{aligned}&\text{for every } v \in B_1,\ v\lambda\beta = v\beta = v,\\ &\text{for every } v \in B_2,\ v\lambda\alpha = v\alpha = v,\end{aligned} \tag{4}$$

and

$$\text{for every } v \in B_1,\ v\lambda\alpha = v\alpha = 0,$$
$$\text{for every } v \in B_2,\ v\lambda\beta = v\beta = 0.$$

It is obtained from (1) that

$$\text{for every } v \in B_1,\ v\lambda \in \text{Ker } \alpha = \langle B_1 \rangle\,,$$
$$\text{for every } v \in B_2,\ v\lambda \in \text{Ker } \beta = \langle B_2 \rangle\,.$$

By (1), we have

$$\text{for every } v \in B_1,\ v\lambda\beta = v\lambda,$$
$$\text{for every } v \in B_2,\ v\lambda\alpha = v\lambda.$$

By (4) and these facts, we have that $v\lambda = v$ for every $v \in B_1 \cup B_2 = B$. Since B is a basis of V, $\lambda = 1_V$. Then we have $\dim_R \text{Ker } \lambda = 0 < k$. Hence $\lambda \notin K_R(V,k)$, a contradiction.

Case 2 : $0 < k < \dim_R V < \infty$. Let B be a basis of V. Since $0 < k < \dim_R V$, there exists $\varnothing \neq B_1 \subseteq B$ such that $|B_1| = k$. Let $u \in B_1$ be fixed. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1} \quad \text{and} \quad \beta = \begin{pmatrix} u & B \smallsetminus \{u\} \\ u & 0 \end{pmatrix}. \tag{5}$$

Then Ker $\alpha = \langle B_1 \rangle$ and Ker $\beta = \langle B \smallsetminus \{u\} \rangle$. So $\dim_R \text{Ker } \alpha = k$ and $\dim_R \text{Ker } \beta = \dim_R V - 1 \geq k$. Thus $\alpha, \beta \in K_R(V, k)$. Obviously,

$$\alpha^2 = \alpha,\ \beta^2 = \beta \ \text{ and } \ \alpha\beta = \beta\alpha = 0.$$

If $(K_R(V, k), \oplus, \circ)$ is a left nearring for some operation $\oplus$ on $K_R(V, k)$, then

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha^2 \oplus \alpha\beta = \alpha \oplus 0 = \alpha \quad \text{and} \\ \beta(\alpha \oplus \beta) &= \beta\alpha \oplus \beta^2 = 0 \oplus \beta = \beta. \end{aligned}$$

Let $\lambda = \alpha \oplus \beta \in K_R(V, k)$. Then $\alpha\lambda = \alpha$ and $\beta\lambda = \beta$. Hence

$$\text{Im } \alpha = \text{Im } \alpha\lambda \subseteq \text{Im } \lambda \ \text{ and } \ \text{Im } \beta = \text{Im } \beta\lambda \subseteq \text{Im } \lambda.$$

It then follows from (5) that

$$B \smallsetminus (B_1 \smallsetminus \{u\}) = (B \smallsetminus B_1) \cup \{u\} \subseteq \text{Im } \alpha \cup \text{Im } \beta \subseteq \text{Im } \lambda.$$

This implies that

$$\dim_R \text{Im } \lambda \geq |B \smallsetminus (B_1 \smallsetminus \{u\})| = \dim_R V - (k - 1). \tag{6}$$

Since $\dim_R \text{Ker } \lambda + \dim_R \text{Im } \lambda = \dim_R V < \infty$ and k is finite, we have that

$$\begin{aligned} \dim_R \text{Ker } \lambda &= \dim_R V - \dim_R \text{Im } \lambda \\ &\leq \dim_R V - (\dim_R V - (k - 1)) \ \text{ from (6)} \\ &= k - 1 < k. \end{aligned}$$

So $\lambda \notin K_R(V, k)$ which is contrary to that $\lambda \in K_R(V, k)$.

Next, suppose that $(K_R(V, k), \oplus, \circ)$ is a right nearring for some operation $\oplus$ on $K_R(V, k)$. Then

$$\begin{aligned} (\alpha \oplus \beta)\alpha &= \alpha^2 \oplus \beta\alpha = \alpha \oplus 0 = \alpha, \\ (\alpha \oplus \beta)\beta &= \alpha\beta \oplus \beta^2 = 0 \oplus \beta = \beta. \end{aligned}$$

Let $\lambda = \alpha \oplus \beta \in K_R(V, k)$. Then $\lambda\alpha = \alpha$ and $\lambda\beta = \beta$. Consequently,

$$\begin{aligned} \text{for every } v \in B \smallsetminus B_1, \quad & v\lambda\beta = v\beta = 0 \ \text{ and} \\ & u\lambda\alpha = u\alpha = 0. \end{aligned} \tag{7}$$

Claim that for every $v \in B \setminus B_1$, $v\lambda \in \langle B \setminus B_1 \rangle$ and $u\lambda \in \langle u \rangle$. If $B_1 = \{u\}$, then by (5) and (7) we have that

$$\text{for every } v \in B \setminus B_1, v\lambda \in \text{Ker } \beta = \langle B \setminus \{u\} \rangle = \langle B \setminus B_1 \rangle \quad \text{and}$$
$$u\lambda \in \text{Ker } \alpha = \langle B_1 \rangle = \langle u \rangle \,.$$

If $B_1 \setminus \{u\} \neq \varnothing$, then for each $w \in B_1 \setminus \{u\}$, define $\beta_w \in L_R(V)$ by

$$\beta_w = \begin{pmatrix} w & B \setminus \{w\} \\ w & 0 \end{pmatrix}. \tag{8}$$

Then for every $w \in B_1 \setminus \{u\}$, Ker $\beta_w = \langle B \setminus \{w\} \rangle$, so $\dim_R \text{Ker } \beta_w = |B \setminus \{w\}| = \dim_R V - 1 \geq k$. Thus $\beta_w \in K_R(V,k)$ for every $w \in B_1 \setminus \{u\}$. Obviously,

$$\text{for every } w \in B_1 \setminus \{u\}, \ \alpha\beta_w = \beta\beta_w = 0.$$

Since $(K_R(V,k), \oplus, \circ)$ is a right nearring, for every $w \in B_1 \setminus \{u\}$,

$$\lambda\beta_w = (\alpha \oplus \beta)\beta_w = \alpha\beta_w \oplus \beta\beta_w = 0 \oplus 0 = 0.$$

Thus if $w \in B_1 \setminus \{u\}$, then for every $v \in B$, $v\lambda\beta_w = 0$. From (5), (7), (8) and this fact, we have that for every $v \in B \setminus B_1$,

$$\begin{aligned} v\lambda &\in \text{Ker } \beta \bigcap \left(\bigcap_{w \in B_1 \setminus \{u\}} \text{Ker } \beta_w \right) \\ &= \langle B \setminus \{u\} \rangle \bigcap \left(\bigcap_{w \in B_1 \setminus \{u\}} \langle B \setminus \{w\} \rangle \right) \\ &= \bigcap_{w \in B_1} \langle B \setminus \{w\} \rangle \qquad\qquad \text{and} \\ u\lambda &\in \text{Ker } \alpha \bigcap \left(\bigcap_{w \in B_1 \setminus \{u\}} \text{Ker } \beta_w \right) \\ &= \langle B_1 \rangle \bigcap \left(\bigcap_{w \in B_1 \setminus \{u\}} \langle B \setminus \{w\} \rangle \right). \end{aligned}$$

Since $|B_1| = k < \infty$, by Proposition 2.8 and Proposition 2.9, we have that

$$\text{for every } v \in B \setminus B_1, v\lambda \in \langle B \setminus B_1 \rangle \quad \text{and} \quad u\lambda \in \langle u \rangle \,.$$

Hence the claim is proved. Since $\lambda\alpha = \alpha$ and $\lambda\beta = \beta$, by the claim and (5),

$$\text{for every } \; v \in B \setminus B_1,\; v\lambda = v\lambda\alpha = v\alpha = v,$$
$$u\lambda = u\lambda\beta = u\beta = u.$$

Then $\langle (B \setminus B_1) \cup \{u\} \rangle \subseteq \text{Im } \lambda$, so

$$\dim_R \text{Im } \lambda \geq |(B \setminus B_1) \cup \{u\}| = (\dim_R V - k) + 1. \tag{9}$$

Since $\dim_R V = \dim_R \text{Ker } \lambda + \dim_R \text{Im } \lambda$ and $\dim_R V$ is finite,

$$\begin{aligned} \dim_R \text{Ker } \lambda &= \dim_R V - \dim_R \text{Im } \lambda \\ &\leq \dim_R V - ((\dim_R V - k) + 1) \;\; \text{from (9)} \\ &= k - 1 < k. \end{aligned}$$

So $\lambda \notin K_R(V, k)$ which is contrary to that $\lambda \in K_R(V, k)$.

Therefore the proof is complete. □

We give a remark here that from Theorem 4.1.1, we conclude that Theorem 3.1.1 for that case of $OM_R(V)$ is a consequence of Theorem 4.1.1.

Corollary 4.1.2. *Let k be a cardinal number with $k < dim_R V$. Then*

(1) $K'_R(V, k)$ admits the structure of a left nearring if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.

(2) $K'_R(V, k)$ admits the structure of a right nearring if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.

Proof. Let k' be the successor of k. Then $k' > 0$ and $K'_R(V, k) = K_R(V, k')$. Suppose that $K'_R(V, k) \in \mathcal{LNR} \cup \mathcal{RNR}$. By Theorem 4.1.1, $\dim_R V < \infty$ and $k' = \dim_R V$. So $k = \dim_R V - 1$.

Conversely, assume that $\dim_R V < \infty$ and $k = \dim_R V - 1$. Then $k' = \dim_R V$. Since $K'_R(V, k) = K_R(V, k')$, by Theorem 4.1.1, $K'_R(V, k) \in \mathcal{LNR} \cap \mathcal{RNR}$. □

We can see from the proofs of Theorem 4.1.1 and Corollary 4.1.2 that $K_R(V, k) = L_R(V)$ or $\{0\}$ and $K'_R(V, k) = \{0\}$ are neccessary conditions of Theorem 4.1.1 and Corollary 4.1.2, respectively. Hence we have

Corollary 4.1.3. *For a cardinal number k with $k \leq \dim_R V, K_R(V,k)$ admits a ring structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $\dim_R V < \infty$ *and* $k = \dim_R V$.

Corollary 4.1.4. *For a cardinal number k with $k < \dim_R V, K'_R(V,k)$ admits a ring structure if and only if* $\dim_R V < \infty$ *and* $k = \dim_R V - 1$.

Remark 4.1.5. If k_1 and k_2 are cardinal numbers such that $k_1 < k_2 \leq \dim_R V$, then $K_R(V,k_1) \supsetneq K_R(V,k_2)$. To see this, let B be a basis of V. Then $k_1 < k_2 \leq |B|$, so there is a subset B_1 of B such that $|B_1| = k_1$. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\dim_R \operatorname{Ker} \alpha = |B_1| = k_1 < k_2$. Thus $\alpha \in K_R(V,k_1) \smallsetminus K_R(V,k_2)$. It then follows that if $\dim_R V$ is infinite, then

$$K_R(V,1) = K'_R(V,0) \supsetneq K_R(V,2) = K'_R(V,1) \supsetneq K_R(V,3) = K'_R(V,2) \supsetneq \ldots$$

and by Theorem 4.1.1, none of these subsemigroups of $L_R(V)$ admits the structure of a left nearring and a right nearring.

4.2 The Semigroups $CI_R(V,k)$ and $CI'_R(V,k)$

We shall characterize when $CI_R(V,k)$ admits the structure of a left nearring or a right nearring. The characterization will generalize Theorem 3.1.1 for the case of $OE_R(V)$ since $OE_R(V) = CI_R(V,\aleph_0)$ if $\dim_R V$ is infinite. Since $CI'_R(V,k) = CI_R(V,k')$ if k' is the successor of k, by the characterization of $CI_R(V,k)$ admitting this structure, necessary and sufficient conditions for $CI'_R(V,k)$ to admit such a structure are also obtained.

From Proposition 2.11, if $\dim_R V < \infty$, then $K_R(V,k) = CI_R(V,k)$ for every cardinal number k with $k \leq \dim_R V$. However, from Proposition 2.12 (i) that if $\dim_R V$ is infinite, then $CI_R(V,k) \neq K_R(V,l)$ for all cardinal numbers k,l with $0 < k \leq \dim_R V$ and $l \leq \dim_R V$. Then characterizing when $CI_R(V,k)$ admits the structure of a left nearring or a right nearring should be also considered.

Theorem 4.2.1. *Let k be a cardinal number with $k \leq dim_R V$.*

(1) $CI_R(V,k)$ admits the structure of a left nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

(2) $CI_R(V,k)$ admits the structure of a right nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

Proof. To prove sufficiency, assume that (i) or (ii) holds. If $k = 0$, then $CI_R(V,k) = CI_R(V,0) = L_R(V) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$. If $\dim_R V < \infty$, then $CI_R(V,k) = K_R(V,k)$. By Theorem 4.1.1, $CI_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $(CI_R(V,k), \circ)$ admits the structure of a left nearring. To prove that (i) or (ii) must hold, suppose not. Then either $0 < k < \dim_R V < \infty$ or ($k > 0$ and $\dim_R V$ is infinite).

Case 1 : $0 < k < \dim_R V < \infty$. Since $\dim_R V < \infty, K_R(V,k) = CI_R(V,k)$. By Theorem 4.1.1, $CI_R(V,k) \notin \mathcal{LNR}$, a contradiction.

Case 2 : $k > 0$ and $\dim_R V$ is infinite. Let B be a basis of V and $B_1, B_2 \subseteq B$ such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\alpha, \beta \in L_R(V)$ be defined by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2} \quad \text{and } \beta = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1}.$$

Then $\dim_R(V/\text{Im } \alpha) = \dim_R(V/\langle B_2 \rangle) = |B_1| = |B| = \dim_R V \geq k$ and $\dim_R(V/\text{Im } \beta) = \dim_R(V/\langle B_1 \rangle) = |B_2| = |B| = \dim_R V \geq k$. So $\alpha, \beta \in CI_R(V,k)$. Suppose that there is an operation $\oplus$ on $CI_R(V,k)$ such that $(CI_R(V,k), \oplus, \circ)$ is a left nearring. Thus $\alpha \oplus \beta \in CI_R(V,k)$. As shown in the proof of Case 1 of Theorem 4.1.1 that $\alpha \oplus \beta = 1_V$. Since $\dim_R(V/\text{Im } 1_V) = 0 < k$, $1_V \notin CI_R(V,k)$. So this is contrary to that $\alpha \oplus \beta \in CI_R(V,k)$.

Similarly, if $(CI_R(V,k), \circ)$ admits a right nearring structure, then (i) or (ii) must hold. $\square$

Corollary 4.2.2. *Let k be a cardinal number with $k < dim_R V$. Then*

(1) $CI'_R(V,k)$ admits the structure of a left nearring if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.

(2) $CI'_R(V,k)$ admits the structure of a right nearring if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.

Proof. Let k' be the successor of k. Then $k' > 0$ and $CI'_R(V,k) = CI_R(V,k')$. Suppose that $\dim_R V < \infty$ and $k = \dim_R V - 1$. Then $k' = \dim_R V$. Since $CI'_R(V,k) = CI_R(V,k')$, by Theorem 4.2.1 $CI'_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $CI'_R(V,k) \in \mathcal{LNR} \cup \mathcal{RNR}$. Since $CI'_R(V,k) = CI_R(V,k')$, by Theorem 4.2.1 $\dim_R V < \infty$ and $k' = \dim_R V$. Hence $\dim_R V < \infty$ and $k = \dim_R V - 1$. □

Notice from the proofs of Theorem 4.1.1, Theorem 4.2.1, Corollary 4.1.2 and Corollary 4.2.2 that necessary conditions of Theorem 4.2.1 and Corollary 4.2.2 are $CI_R(V,k) = L_R(V)$ or $\{0\}$ and $CI'_R(V,k) = \{0\}$, respectively. Hence the following corollaries are obtained directly.

Corollary 4.2.3. *For a cardinal number k with $k \leq \dim_R V, CI_R(V,k)$ admits a ring structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $\dim_R V < \infty$ *and* $k = \dim_R V$.

Corollary 4.2.4. *For a cardinal number k with $k < \dim_R V, CI'_R(V,k)$ admits a ring structure if and only if $\dim_R V < \infty$ and $k = \dim_R V - 1$.*

Remark 4.2.5. Let k_1 and k_2 be cardinal numbers and B a basis of V. If $k_1 < k_2 \leq \dim_R V$, let B_1 be a subset of B such that $|B_1| = k_1$. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\dim_R(V/\mathrm{Im}\ \alpha) = \dim_R(V/\langle B \smallsetminus B_1 \rangle) = |B_1| = k_1 < k_2$, so $\alpha \in CI_R(V,k_1) \smallsetminus CI_R(V,k_2)$. It then follows that if $\dim_R V$ is infinite, then

$$CI_R(V,1) = CI'_R(V,0) \supsetneq CI_R(V,2) = CI'_R(V,1) \supsetneq CI_R(V,3) = CI'_R(V,2) \supsetneq \dots$$

and by Theorem 4.2.1, none of them admits the structure of a left nearring and a right nearring.

4.3 The Semigroups $I_R(V,k)$ and $I'_R(V,k)$

From Proposition 2.12 (ii) if $\dim_R V$ is infinite, then for a nonzero cardinal number k with $k < \dim_R V, I_R(V,k)$ is not equal to $K_R(V,l)$ and $CI_R(V,l)$ for any cardinal number $l \leq \dim_R V$. This is also true for $I'_R(V,k), K'_R(V,l)$ and $CI'_R(V,l)$ where $0 < k \leq \dim_R V$ and $0 \leq l < \dim_R V$. Then characterizing when $I_R(V,k)$ admits the structure of a left nearring or a right nearring should be also considered.

Theorem 4.3.1. *Let k be a cardinal number with $k \leq dim_R V$.*

(1) $I_R(V,k)$ admits the structure of a left nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $k = dim_R V$.

(iii) k is an infinite cardinal number.

(2) $I_R(V,k)$ admits the structure of a right nearring if and only if one of the following statements holds.

(i) $k = 0$.

(ii) $k = dim_R V$.

(iii) k is an infinite cardinal number.

Proof. To prove sufficiency of (1) and (2), assume that (i), (ii) or (iii) holds.

(i) If $k = 0$, then $I_R(V,k) = I_R(V,0) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq 0\} = \{0\}$, so $I_R(V,k) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

(ii) If $k = \dim_R V$, then

$$\begin{aligned} I_R(V,k) &= I_R(V, \dim_R V) \\ &= \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq \dim_R V\} \\ &= L_R(V), \end{aligned}$$

so $I_R(V,k) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

(iii) Assume that k is an infinite cardinal number. Then $k + k = k$. We know that for $\alpha, \beta \in L_R(V)$, Im $(\alpha + \beta) \subseteq$ Im $\alpha +$ Im β and Im $(-\alpha) =$ Im α where $+$ is the usual addition on $L_R(V)$. Thus for $\alpha, \beta \in I_R(V, k)$,

$$\begin{aligned} \dim_R \operatorname{Im}\,(\alpha - \beta) &\leq \dim_R \operatorname{Im}\,\alpha + \dim_R \operatorname{Im}\,\beta \\ &\leq k + k = k. \end{aligned}$$

So $I_R(V, k)$ is a subring of $(L_R(V), +, \circ)$. Hence $I_R(V, k) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $I_R(V, k) \in \mathcal{LNR} \cup \mathcal{RNR}$. To show that one of (i), (ii) and (iii) must hold, suppose on the contrary that (i), (ii) and (iii) are all false. Then $0 < k < \dim_R V$ and k is finite. Let B be a basis of V, $B_1 \subseteq B$ such that $|B_1| = k$ and $I = B \smallsetminus B_1$. Since $k < \dim_R V$, $I \neq \varnothing$. Let $u \in I$ be fixed, define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B \smallsetminus B_1 \\ v & 0 \end{pmatrix}_{v \in B_1} \quad \text{and} \quad \beta = \begin{pmatrix} u & B \smallsetminus \{u\} \\ u & 0 \end{pmatrix}. \tag{1}$$

Then Im $\alpha = \langle B_1 \rangle$ and Im $\beta = \langle u \rangle$. So $\dim_R$ Im $\alpha = |B_1| = k$ and $\dim_R$ Im $\beta = |\{u\}| = 1 \leq k$. These imply that $\alpha, \beta \in I_R(V, k)$. Obviously,

$$\alpha^2 = \alpha, \;\; \beta^2 = \beta, \;\; \text{and} \;\; \alpha\beta = \beta\alpha = 0.$$

Suppose that $(I_R(V, k), \oplus, \circ)$ is a left nearring for some operation $\oplus$ on $I_R(V, k)$, then

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha^2 \oplus \alpha\beta = \alpha \oplus 0 = \alpha, \\ \beta(\alpha \oplus \beta) &= \beta\alpha \oplus \beta^2 = 0 \oplus \beta = \beta. \end{aligned} \tag{2}$$

Let $\lambda = \alpha \oplus \beta \in I_R(V, k)$. Then from (2), $\alpha\lambda = \alpha$ and $\beta\lambda = \beta$. We therefore from these equalities and (1) that

$$\begin{aligned} \text{for every } v \in B_1, v\lambda = v\alpha\lambda = v\alpha = v \;\; \text{and} \\ u\lambda = u\beta\lambda = u\beta = u. \end{aligned}$$

So Im $\lambda \supseteq \langle B_1 \cup \{u\} \rangle$. Then $\dim_R$ Im $\lambda \geq |B_1 \cup \{u\}| = k + 1 > k$ since k is finite. Hence $\lambda \notin I_R(V, k)$, a contradiction.

Next, assume that $(I_R(V, k), \oplus, \circ)$ is a right nearring for some operation $\oplus$ on $I_R(V, k)$, then

$$\begin{aligned} (\alpha \oplus \beta)\alpha &= \alpha^2 \oplus \beta\alpha = \alpha \oplus 0 = \alpha, \\ (\alpha \oplus \beta)\beta &= \alpha\beta \oplus \beta^2 = 0 \oplus \beta = \beta. \end{aligned} \tag{3}$$

Let $\lambda = \alpha \oplus \beta \in I_R(V,k)$. Then by (3), $\lambda\alpha = \alpha$ and $\lambda\beta = \beta$. We therefore from these equalities and (1) that

$$\begin{aligned} \text{for every } v \in B_1,\ v\lambda\alpha = v\alpha = v \ \text{ and} \\ u\lambda\beta = u\beta = u, \end{aligned} \tag{4}$$

$$\begin{aligned} \text{for every } v \in B_1,\ v\lambda\beta = v\beta = 0 \ \text{ and} \\ u\lambda\alpha = u\alpha = 0. \end{aligned} \tag{5}$$

Claim that for every $v \in B_1$, $v\lambda \in \langle B_1 \rangle$ and $u\lambda \in \langle u \rangle$. If $I = \{u\}$, then by (1) and (5) we have that

$$\begin{aligned} \text{for every } v \in B_1, v\lambda \in \text{Ker } \beta = \langle B \smallsetminus \{u\} \rangle = \langle B_1 \rangle \quad \text{and} \\ u\lambda \in \text{Ker } \alpha = \langle B \smallsetminus B_1 \rangle = \langle I \rangle = \langle u \rangle . \end{aligned}$$

If $I \smallsetminus \{u\} \neq \varnothing$, then for each $w \in I \smallsetminus \{u\}$, define $\beta_w \in L_R(V)$ by

$$\beta_w = \begin{pmatrix} w & B \smallsetminus \{w\} \\ w & 0 \end{pmatrix}. \tag{6}$$

Then if $w \in I \smallsetminus \{u\}$, we have Im $\beta_w = \langle w \rangle$, so $\dim_R \text{Im } \beta_w = |\{w\}| = 1 \leq k$. Thus $\beta_w \in I_R(V,k)$ for every $w \in I \smallsetminus \{u\}$. Obviously,

$$\text{for every } \ w \in I \smallsetminus \{u\},\ \alpha\beta_w = \beta\beta_w = 0.$$

Since $(I_R(V,k), \oplus, \circ)$ is a right nearring,

$$\text{for every } w \in I \smallsetminus \{u\},\ \ \lambda\beta_w = (\alpha \oplus \beta)\beta_w = \alpha\beta_w \oplus \beta\beta_w = 0 \oplus 0 = 0.$$

Thus if $w \in I \smallsetminus \{u\}$, then for every $v \in B$, $v\lambda\beta_w = 0$. From (1), (5), (6) and this fact, we have that for every $v \in B_1$,

$$\begin{aligned} v\lambda &\in \text{Ker } \beta \bigcap \left(\bigcap_{w \in I \smallsetminus \{u\}} \text{Ker } \beta_w \right) \\ &= \langle B \smallsetminus \{u\} \rangle \bigcap \left(\bigcap_{w \in I \smallsetminus \{u\}} \langle B \smallsetminus \{w\} \rangle \right) \\ &= \langle B \smallsetminus \{u\} \rangle \cap \langle B \smallsetminus (I \smallsetminus \{u\}) \rangle \qquad \text{by Proposition 2.9} \end{aligned}$$

$$= \langle B \setminus \{u\} \rangle \cap \langle B \setminus ((B \setminus B_1) \setminus \{u\}) \rangle$$
$$= \langle B \setminus \{u\} \rangle \cap \langle B_1 \cup \{u\} \rangle$$
$$= \langle B_1 \rangle \qquad \text{by Proposition 2.8}$$

and $u\lambda \in \text{Ker } \alpha \bigcap \left(\bigcap_{w \in I \setminus \{u\}} \text{Ker } \beta_w \right)$

$$= \langle B \setminus B_1 \rangle \bigcap \left(\bigcap_{w \in I \setminus \{u\}} \langle B \setminus \{w\} \rangle \right)$$
$$= \langle B \setminus B_1 \rangle \cap \langle B \setminus (I \setminus \{u\}) \rangle \qquad \text{by Proposition 2.9}$$
$$= \langle B \setminus B_1 \rangle \cap \langle B \setminus ((B \setminus B_1) \setminus \{u\}) \rangle$$
$$= \langle B \setminus B_1 \rangle \cap \langle B_1 \cup \{u\} \rangle$$
$$= \langle u \rangle . \qquad \text{by Proposition 2.8}$$

Then for every $v \in B_1$, $v\lambda \in \langle B_1 \rangle$ and $u\lambda \in \langle u \rangle$. Hence the claim is proved.

By the claim, (1) and (4), we have that

$$\text{for every } v \in B_1, v\lambda = v\lambda\alpha = v \text{ and}$$
$$u\lambda = u\lambda\beta = u.$$

Then $\text{Im } \lambda \supseteq \langle B_1 \cup \{u\} \rangle$. Since k is finite, $\dim_R \text{Im } \lambda \geq |B_1 \cup \{u\}| = k + 1 > k$. Hence $\lambda \notin I_R(V, k)$, a contradiction.

Therefore the theorem is proved. □

Corollary 4.3.2. *Let k be a cardinal number with $0 < k \leq dim_R V$. Then the following statements hold.*

(i) $I'_R(V, k)$ admits the structure of left nearring if and only if either $k = 1$ or k is an infinite cardinal number.

(ii) $I'_R(V, k)$ admits the structure of right nearring if and only if either $k = 1$ or k is an infinite cardinal number.

Proof. Assume that $k = 1$ or k is an infinite cardinal number. If $k = 1$, then $I'_R(V, k) = I'_R(V, 1) = I_R(V, 0) = \{0\}$. So $I'_R(V, k) \in \mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$. If k is an infinite cardinal number, then $k + k = k$. For $\alpha, \beta \in I'_R(V, k), \dim_R \text{Im } \alpha < k$

and $\dim_R \mathrm{Im}\ \beta < k$. So

$$\begin{aligned}\dim_R \mathrm{Im}\ (\alpha - \beta) &\leq \dim_R \mathrm{Im}\ \alpha + \dim_R \mathrm{Im}\ \beta \\ &< k + k = k.\end{aligned}$$

Thus $(I'_R(V,k), +, \circ)$ is a ring where $+$ is the usual addition of linear transformations. Hence $I'_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $1 < k$ and k is finite. Then $I'_R(V,k) = I_R(V,k-1)$, $0 < k-1 < \dim_R V$ and $k-1$ is finite. By Theorem 4.3.1, $I_R(V,k-1) \notin \mathcal{LNR} \cup \mathcal{RNR}$. Hence $I'_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$. □

Theorem 4.3.1 and Corollary 4.3.2 and their proofs yield the following results.

Corollary 4.3.3. *For a cardinal number k with $k \leq \dim_R V$, the semigroup $I_R(V,k)$ admits a ring structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = \dim_R V$.

(iii) *k is an infinite cardinal number.*

Corollary 4.3.4. *For a cardinal number k with $0 < k \leq \dim_R V$, the semigroup $I'_R(V,k)$ admits a ring structure if and only if either $k = 1$ or k is an infinite cardinal number.*

Remark 4.3.5. Assume that $\dim_R V$ is infinite and let B a basis of V. Then B contains a subset $\{u_n \mid n \in \mathbb{N}\}$ where $u_n \neq u_m$ if $n \neq m$. For each positive integer n, let $\alpha_n \in L_R(V)$ be define by

$$\alpha_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n & B \smallsetminus \{u_1, u_2, \dots, u_n\} \\ u_1 & u_2 & \dots & u_n & 0 \end{pmatrix}.$$

Then $\dim_R \mathrm{Im}\ \alpha_n = \dim_R \langle u_1, \dots, u_n \rangle = n$ for every $n \in \mathbb{N}$, so $\alpha_n \in I_R(V,n) \smallsetminus I_R(V,n-1)$ for every $n \geq 1$. Consequently,

$$I_R(V,1) = I'_R(V,2) \supsetneq I_R(V,2) = I'_R(V,3) \supsetneq I_R(V,3) = I'_R(V,4) \supsetneq \dots$$

and Theorem 4.3.1 shows that none of these semigroups admits the structure of a left nearring and a right nearring.

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