CHAPTER III

STAR-CONGRUENCES ON STAR-SEMIGROUPS

The purpose of this chapter is to introduce well-known congruences on *-semigroups which are *-congruences.

If ρ is a congruence on a *-semigroup S such that for all a, b $\in$ S, $a\rho b$ implies $a^*\rho b^*$, then ρ is said to be a *-congruence on S or preserve * on S.

If ρ is a *-congruence on a *-semigroup, then S/ρ is a *-semigroup with an involution * on S/ρ defined by $(a\rho)^* = a^*\rho$. Moreover, if ρ is a *-congruence on a regular-* semigroup, then the quotient semigroup S/ρ is also regular-* because for all a $\in$ S, $(a\rho)(a\rho)^*(a\rho) = (a\rho)(a^*\rho)(a\rho) = (aa^*a)\rho = a\rho$. But for a *-congruence ρ on a *-semigroup S which is *-regular, the semigroup S/ρ is not necessarily *-regular under the involution defined by $(a\rho)^* = a^*\rho$.

Example. Let I be a set such that $|I| = 2$, $\mathbb{Z}$ the set of all integers and $S = I\times\mathbb{Z}\times I$. Let $P : I\times I \to \mathbb{Z}$ be the map such that

$$(a, b)P = p_{ab} \quad \text{where} \quad p_{ab} = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{if } a \neq b. \end{cases}$$

Define a multiplication on S by

$$(a, n, b)(c, m, d) = (a, n+p_{bc}+m, d).$$

Then S is a semigroup. Define the map * on S by $(a, n, b)^* = (b, n, a)$. We have shown in Chapter I that the map * is an involution on S and under this involution, S is a *-regular semigroup. Define a relation ρ on S by

$$(a, n, b)\rho(c, m, d) \iff a = c \text{ and } b = d.$$

Obviously, ρ is a *-congruence on S and hence S/ρ is a *-semigroup under the map * defined by $(a\rho)^* = a^*\rho$. Next, we show that under the involution defined by $(a\rho)^* = a^*\rho$, S/ρ is not a *-regular semigroup. Let x, y be two distinct elements in I. From $((x, n, y)\rho)^*(x, n, y)\rho = ((x, n, y)\rho)^*(y, m, y)\rho = ((y, m, y)\rho)^*(x, n, y)\rho = ((y, m, y)\rho)^*(y, m, y)\rho$ but $(x, n, y)\rho \neq (y, m, y)\rho$ for $n, m \in \mathbb{Z}$, so S/ρ is not a *-regular semigroup with $((a, n, b)\rho)^* = (a, n, b)^*\rho$. #

The first theorem gives necessary and sufficient conditions for a Rees congruence on a *-semigroup S to be a *-congruence on S.

3.1 <u>Theorem</u>. Let A be an ideal of a *-semigroup S. Then the Rees congruence ρ_A is a *-congruence on S if and only if $A^* \subseteq A$.

<u>Proof</u> : Assume $A^* \subseteq A$. If $a, b \in S$ such that $a\rho_A b$, then $a, b \in A$ or $a = b$, so $a^*, b^* \in A^* \subseteq A$ or $a^* = b^*$, hence $a^*\rho_A b^*$.

Conversely, assume ρ_A is a *-congruence on S. Suppose $A^* \nsubseteq A$. Then there exists $x \in A^*$ but $x \notin A$. Thus $x^* \in A$. Let a be an element of A. Then $a^* \in A^*$. Since A is an ideal of S, $aa^* \in A$. Thus $x^*\rho_A aa^*$. Since ρ_A is a *-congruence on S, $x\rho_A aa^*$ which implies $x, aa^* \in A$ or $x = aa^*$. Hence $x \in A$, a contradiction. #

A congruence ρ on a semigroup S is a semilattice congruence on S (that is, S/ρ is a semilattice) if and only if $a\rho a^2$ and $ab\rho ba$ for all a, b $\in$ S. Then, for any congruence ρ on a semilattice S, the semigroup S/ρ is a semilattice.

A semilattice congruence on a *-semigroup need not be a *-congruence. A counter example is given as follows :

Example. Let S = {0, a, b}, define the operation on S by

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Then S is a semilattice. Define the map * : S $\to$ S by 0* = 0, a* = b, b* = a. Then S is a *-semigroup. Let A = {0, a}. Then A is an ideal of S, and so the Rees congruence ρ_A induced by A is a semilattice congruence on S. Because $A^* \not\subseteq A$, it follows by Theorem 3.1 that ρ_A is not a *-congruence on S. #

Every semigroup S has a minimum semilattice congruence which is the intersection of all semilattice congruences on S.

Let S be a semigroup. A subsemigroup F of S is a filter of S if for all a, b $\in$ S, ab $\in$ F implies a, b $\in$ F. For x $\in$ S, let N(x) be the smallest filter of S containing x, that is, N(x) is the intersection of all filters of S containing x. For x $\in$ S, let

$$N_x = \{y \in S \mid N(x) = N(y)\}.$$

It is proved in [11, Proposition II.2.9] that if η is the minimum semilattice congruence on S, then

$$\eta = \{(x, y) \in S \times S \mid N_x = N_y\}.$$

In the next theorem, we show that the minimum semilattice congruence, η, on a *-semigroup S is always a *-congruence. The following lemma is required first :

3.2 Lemma. Let S be a *-semigroup. Then the following hold :

(i) If F is a filter of S, then so is F*.

(ii) $(N(x))^* = N(x^*)$ for all $x \in S$.

(iii) $(N_x)^* = N_{x^*}$ for all $x \in S$.

Proof : (i) Let F be a filter of S. Since F is a subsemigroup of S, F* is a subsemigroup of S. Suppose $a, b \in S$ such that $ab \in F^*$. Then $b^*a^* = (ab)^* \in F$, so $b^* \in F$ and $a^* \in F$ because F is a filter of S. Hence $a \in F^*$ and $b \in F^*$. This proves that F* is a filter of S.

(ii) Let $x \in S$. Because N(x) is the smallest filter of S containing x, it follows by (i) that $(N(x))^*$ is a filter of S containing x^*. To show that $(N(x))^*$ is the smallest filter of S which contains x^*, let F be a filter of S containing x^* such that $F \subseteq (N(x))^*$. Then $F^* \subseteq N(x)$. By (i), F* is a filter of S containing x, so $F^* = N(x)$. Hence $F = (N(x))^*$. This proves that $(N(x))^*$ is the smallest filter of S containing x^*. Hence $(N(x))^* = N(x^*)$.

(iii) Let $x \in S$ and $y \in (N_x)^*$. Then $y^* \in N_x$, so $N(y^*) = N(x)$. By (ii), $N(y^*) = (N(y))^*$ and $N(x^*) = (N(x))^*$, so we have that $N(x^*) = (N(x))^* = (N(y^*))^* = ((N(y))^*)^* = N(y)$. Thus $y \in N_{x^*}$. Hence $(N_x)^* \subseteq N_{x^*}$. This proves that $(N_a)^* \subseteq N_{a^*}$ for all $a \in S$. Therefore $(N_{x^*})^* \subseteq N_{(x^*)^*} = N_x$. But $N_x = ((N_x)^*)^* \subseteq (N_{x^*})^*$, it follows that $(N_{x*})^* = N_x$, so $N_{x*} = (N_x)^*$. #

3.3 Theorem. The minimum semilattice congruence η, on any *-semigroup is a *-congruence.

Proof : By [11, Proposition II.2.9], $\eta = \{(x, y) \in S\times S \mid N_x = N_y\}$. By Lemma 3.2(iii), $(N_x)^* = N_{x*}$ for all $x \in S$. It then follows that η is a *-congruence on S. #

A congruence ρ on a semigroup S is an idempotent-separating congruence on S if every ρ-class contains at most one idempotent. Howie has shown in [6] that the maximum idempotent-separating congruence μ on an inverse semigroup exists and

$$\mu = \{(a, b) \in S\times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\};$$

or equivalently,

$$\mu = \{(a, b) \in S\times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\}.$$

Let S be an inverse semigroup which is a *-semigroup. From $(E(S))^* = E(S)$ and $(a^{-1})^* = (a^*)^{-1}$ for all $a \in S$, it follows that if $a\mu b$ in S, then $a^*\mu b^*$. Hence μ is a *-congruence.

Every inverse semigroup is an orthodox semigroup. Orthodox semigroups need not be inverse. Meakin has proved in [8] that the maximum idempotent-separating congruence μ on an orthodox semigroup S exists and

$$\mu = \{(a, b) \in S\times S \mid \text{there are } a' \in V(a),\ b' \in V(b) \text{ such that } a'ea = b'eb \text{ and } aea' = beb' \text{ for all } e \in E(S)\}.$$

For a *-semigroup S, $(E(S))^* = E(S)$ and $(V(a))^* = V(a^*)$ for all $a \in S$, it follows that the maximum idempotent-separating congruence μ on an orthodox semigroup S which is a *-semigroup preserves * on S.

3.4 Theorem. The maximum idempotent-separating congruence on an orthodox semigroup which is a *-semigroup is a *-congruence. In particular, the maximum idempotent-separating congruence on an inverse semigroup which is a *-semigroup is a *-congruence.

Is it true that an idempotent-separating congruence on a *-semigroup is *-congruence? To answer this question, the following example is given.

Example. Let S = {0, a, b} be a zero semigroup with zero 0. Define the map * on S by $0^* = 0$, $a^* = b$, $b^* = a$. The map * is an involution on S, so S is a *-semigroup. Let A = {0, a}. Then A is an ideal of S and the Rees congruence ρ_A is an idempotent-separating congruence on S. Because $A^* \nsubseteq A$, by Theorem 3.1, ρ_A is not a *-congruence.

This proves that an idempotent-separating congruence on a *-semigroup is not necessary to preserve *. #

Next, we study an inverse congruence on a *-semigroup. Every semilattice is an inverse semigroup. Then every semilattice congruence on a semigroup S is an inverse congruence on S. The first example of this chapter shows that a semilattice congruence on a *-semigroup need not be a *-congruence. Thus that example also shows that an inverse congruence on a *-semigroup need not be a *-congruence.

It has been shown by Hall in [4, Theorem 3] that the maximum inverse congruence $\mathcal{Y}$ on an orthodox semigroup S exists and

$$\mathcal{Y} = \{(a, b) \in S\times S \mid V(a) = V(b)\}.$$

In any semigroup S, $(V(a))^* = V(a^*)$ for all $a \in S$. Hence the minimum inverse congruence on an inverse semigroup which is a *-semigroup preserves *.

3.5 Theorem. The minimum inverse congruence on an orthodox semigroup which is a *-semigroup is a *-congruence.

It has been proved by Munn in [9] that the minimum group congruence σ on an inverse semigroup S always exists and

$$\sigma = \{(a, b) \in S\times S \mid ea = eb \text{ for some } e \in E(S)\};$$

or equivalently,

$$\sigma = \{(a, b) \in S\times S \mid ae = be \text{ for some } e \in E(S)\}.$$

Let S be an inverse semigroup which is a *-semigroup. If $a, b \in S$ such that $a\sigma b$, then $ae = be$ for some $e \in E(S)$, so $e^*a^* = e^*b^*$ and $e^* \in E(S)$ which implies $a^*\sigma b^*$. Hence we have the following theorem :

3.6 Theorem. The minimum group congruence on an inverse semigroup which is a *-semigroup is a *-congruence.

The Green's relation $\mathcal{H}$ on a semigroup S need not be a congruence on S.

Let S be a regular semigroup, $a, b \in S$. Suppose that there are $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$ and $a'a = b'b$. Since $a\mathcal{R}aa'$, $b\mathcal{R}bb'$, $a'a\mathcal{L}a$ and $b'b\mathcal{L}b$, it follows that $a\mathcal{R}b$ and $a\mathcal{L}b$, and hence $a\mathcal{H}b$.

Assume $a\mathcal{H}b$ in a regular semigroup S. Then by [2, Section 2.3] there exist $a' \in V(a)$, $b' \in V(b)$ such that $a'\mathcal{H}b'$. From $a\mathcal{H}b$, we have that $aa'\mathcal{R}bb'$ and $a'a\mathcal{L}b'b$. From $a'\mathcal{H}b'$, we have that $aa'\mathcal{L}bb'$ and $a'a\mathcal{R}b'b$. Hence $aa'\mathcal{H}bb'$ and $a'a\mathcal{H}b'b$ which implies $aa' = bb'$ and $a'a = b'b$ since $aa', bb', a'a, b'b \in E(S)$.

Therefore, in a regular semigroup S,

$$\mathcal{H} = \{(a, b) \in S\times S \mid aa' = bb' \text{ and } a'a = b'b \text{ for some } a' \in V(a), b' \in V(b)\}.$$

For any *-semigroup S, $(v(a))^* = V(a^*)$ for all $a \in S$. Then if S is a regular semigroup which is a *-semigroup, then $a\mathcal{H}b$ in S implies $a^*\mathcal{H}b^*$.

For idempotents e, f in a semigroup S, we define $e \leq f$ if $e = ef = fe$. If S is a *-semigroup, then for $e, f \in E(S)$, $e \leq f$ implies $e^* \leq f^*$.

It has been proved by Hall in [5] that the maximum congruence contained in $\mathcal{H}$ on any regular semigroup S, δ say, is given by

$$\delta = \{(a, b) \in S\times S \mid \text{for some } a' \in V(a), b' \in V(b), aa' = bb', a'a = b'b \text{ and } a'ea = b'eb \text{ for each idempotent } e \leq aa'\}.$$

3.7 <u>Lemma</u>. Let S be a regular semigroup, $a, b \in S$. Assume that $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$ and $a'a = b'b$. Then $a'ea = b'eb$ for each idempotent $e \leq aa'$ if and only if $afa' = bfb'$ for each idempotent $f \leq a'a$.

Proof : From Result 3 of [5], we have that

$$aa'xaa' = x \qquad (*)$$

for all $x \in aa'Saa'$, and

$$a'aya'a = y \qquad (**)$$

for all $y \in a'aSa'a$. Assume that $a'ea = b'eb$ for all $e \in E(S)$ such that $e \leq aa'$. Let $f \in E(S)$ such that $f \leq a'a = b'b$. Then $f = fa'a = a'af = fb'b = b'bf$, so $b'bfb'b = f$. Since $bfb' = bb'bfb'bb' \in bb'Sbb' = aa'Saa'$, from (*), we obtain

$$aa'bfb'aa' = bfb' \qquad (I)$$

Because $(bfb')(bfb') = bf(b'bf)b' = bffb' = bfb'$, $bfb' \in E(S)$. Also, $(bfb')(bb') = bfb' = (bb')(bfb')$, hence $bfb' \leq bb' = aa'$. By assumption, $a'bfb'a = b'bfb'b$ and thus $a'bfb'a = f$. From (I), we get $afa' = bfb'$.

A proof of the converse is given similarly by using (**). #

By Hall in [5] and Lemma 3.7, we have that for a regular semigroup S, the maximum congruence δ of S contained in $\mathcal{H}$ is given by

$$\begin{aligned}\delta &= \{(a, b) \in S\times S \mid \text{for some } a' \in V(a),\ b' \in V(b),\\ &\qquad aa' = bb',\ a'a = b'b \text{ and } a'ea = b'eb \text{ for each}\\ &\qquad \text{idempotent } e \leq aa'\},\\ &= \{(a, b) \in S\times S \mid \text{for some } a' \in V(a),\ b' \in V(b),\\ &\qquad aa' = bb',\ a'a = b'b \text{ and } aea' = beb' \text{ for each}\\ &\qquad \text{idempotent } e \leq a'a\}.\end{aligned}$$

3.8 <u>Theorem</u>. If S is a regular semigroup which is a *-semigroup, then the maximum congruence δ of S contained in $\mathscr{H}$ is a *-congruence on S.

<u>Proof</u> : Let $(a, b) \in \delta$. Then there are $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$ and $a'ea = b'eb$ for each idempotent $e \leq aa'$. Thus $(a')^* \in V(a^*)$, $(b')^* \in V(b^*)$, $a^*(a')^* = b^*(b')^*$, $(a')^*a^* = (b')^*b^*$ and $a^*e^*(a')^* = b^*e^*(b')^*$ for each idempotent $e \leq aa'$. If $e \in E(S)$ such that $e \leq (a')^*a^*$, then $e^* \leq aa'$, so $a^*(e^*)^*(a')^* = b^*(e^*)^*(b')^*$ which implies $a^*e(a')^* = b^*e(b')^*$. This proves that $a^*e(a')^* = b^*e(b')^*$ for all $e \in E(S)$ such that $e \leq (a')^*a^*$. Hence $a^* \delta b^*$. #