BLOW-UP PHENOMENA OF A NONLINEAR PSEUDO-PARABOLIC EQUATION

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ปรากฏการณ์ระเบิดของสมการเชิงพาราโบลาเทียมไม่เชิงเส้น

นายพุทธา สักกะพลางกูร

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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ในวิทยานิพนธ์ฉบับนี้เราศึกษาสมการเชิงพาราโบลาเทียมไม่เชิงเส้น

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \Delta u}{\partial t} = \Delta u + a(x)u^{p}, \\ u(x,0) = u_0(x) \end{cases}$$

โดยที่ $1 \le p < \infty, k > 0$ เป็นค่าคงที่ u_0 เป็นฟังก์ชันที่ไม่เป็นลบ ไม่ใช่ฟังก์ชันศูนย์ และ มี compact support และ $a(x) > 0$ เป็นฟังก์ชันต่อเนื่องซึ่งสอดคล้องกับ

$$c\,|x|^{\sigma} \le a(x) \le c^{*}\,|x|^{\sigma} \qquad (x \in \mathbb{R}^{n})$$

สำหรับบางค่าคงที่ $c, c^{*} > 0$ และ $\sigma > -2$ ซึ่ง จากการศึกษาพบว่ามีค่าของ p ที่น่าสนใจ 2 จุดคือ

$$p_1 = 1 + \left(\frac{\sigma}{n}\right)_{+} \qquad p_2 = 1 + \frac{\sigma+2}{n}$$

โดยที่ $m_{+} = m$ เมื่อ $m > 0$ และ $m_{+} = 0$ เมื่อ $m < 0$ โดยลักษณะของผลเฉลยขึ้นอยู่กับค่าของ p กล่าวคือ ผลเฉลยของปัญหาที่เป็นฟังก์ชันไม่เป็นลบเมื่อ $p_1 < p < p_2$ ผลเฉลยจะระเบิดในเวลาจำกัด โดยที่ไม่ขึ้นกับค่าของเงื่อนไขเริ่มต้น

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In this thesis, we study the pseudo-parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \triangle u}{\partial t} = \triangle u + a(x)u^p, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $1 \leq p < \infty, k > 0$, $a(x) > 0$ is a continuous function satisfying

$$c|x|^\sigma \leq a(x) \leq c^*|x|^\sigma, \qquad (x \in \mathbb{R}^n)$$

for some constants $c, c^* > 0$, $\sigma > -2$ are constants and u_0 is nontrivial, nonnegative as well as u_0 has a compact support.

The characteristic of solutions depend on the value of p and the initial condition u_0. The nonnegative classical solution blows up in a finite time when

$$1 + \left(\frac{\sigma}{n}\right)_+ < p < 1 + \left(\frac{\sigma+2}{n}\right)$$

regardless of the initial condition u_0.

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CONTENTS

CHAPTER I

INTRODUCTION

The study of nonlinear heat equation has attracted attention for many decades. In general, people consider the following initial boundary value problem of the nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u + f(Du, u, x, t), & x \in \Omega, t > 0 \\ u(x,t) = g(x,t), & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

where $u(x,t)$ is an unknown function defined on a domain $\Omega \subset \mathbb{R}^n$, $t \in [0,T]$, f and g are given functions, and u_0 is the initial state. This problem is used as a mathematical model for many physical systems involving reaction diffusion type phenomena. In 1960's, H. Fujita, in his original paper [2], considered the model case $f = u^p$ on $\Omega = \mathbb{R}^n$, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u + u^p & x \in \mathbb{R}^n, t \in (0,T] \\ u(x,0) = u_0(x) \end{cases} \tag{1.2}$$

where $p > 1$ is a constant and u_0 is a nonnegative, nontrivial, bounded smooth function in $\mathbb{R}^n$. He proposed the concept of critical exponents during the discussion of the heat conduction equation with a nonlinear source. In his study, he showed that there is a critical value p_c of the exponent p such that the solution u of (1.2) behaves differently as p increases:

1. if $1 < p < 1+2/n$ the solution for any nontrivial initial data u_0 blows up in a finite time, i.e. the supremum norm over $x \in \mathbb{R}^n$ of the solution is infinite in a finite time T.

2. if $p > 1 + 2/n$ the equation admits both a global (in time) solution if the initial value u_0 is sufficiently small and a blowing-up solution if u_0 is sufficiently large.

We remark that if $0 \leq p \leq 1$, then there exist global (in time) solutions for each initial datum.

The behavior of solutions to the problem (1.2) when $p = 1+2/n$ was shown to belong to the blow-up case (i.e. the first case above) by Hayakawa for $n = 1, 2$, and by Kobayashi et.al. for general $n \geq 1$. Same result as Kobayashi et.al. was also derived by Weissler using different approach. Nowadays, the value $p = 1 + 2/n$ is called the *Fujita critical exponent* for the nonlinear heat equation (1.2).

Nowadays, this classical work by H. Fujita has initiated several other works on blow-up phenomenon in many directions such as

1. the equation related to wave equations ($u_{tt} = c\triangle u + u^p$), Quantum mechanics ($iu_t = \triangle u + |u|^{p-1}u$), or higher order equations

2. the heat equation on manifold by changing the Laplace operator to Laplace-Beltrami operator

3. the reaction and/or convection terms to u^p into $a(x)u^p$ (autonomous equation) or $a(x,t)u^p$ (non-autonomous equation)

4. the geometry of domains such as bounded domain Ω or exterior of a bounded domain

5. the boundary condition

6. the condition of solutions not necessary positive.

See [3] and [6] for more examples and some intuitive discussion of this result based upon the observations of Fujita and others. Moreover, the study of many important questions such as existence, uniqueness, continuous dependence of solution and asymptotic properties has been carried out by many mathematicians.

A related result to that of Fujita was the work by C.P. Wang and S.N. Zheng [7] in 2006 which also served as a motivation for our work. The authors studied the critical Fujita exponent for the initial-value problem of the degenerate and a parabolic equation of the form

$$\begin{cases} |x|^{\lambda_1}\dfrac{\partial u}{\partial t} = \triangle u^m + |x|^{\lambda_2}u^p, \qquad x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), \end{cases}$$

where $m \geq 1$, $p > m$ and $0 \leq \lambda_1 \leq \lambda_2 < p(\lambda_1 + 1) - 1$. Their work was motivated by a flow model in a channel of a fluid whose viscosity is temperature dependent. For this equation, they have found the critical exponent to be

$$p_c = m + \frac{2 + \lambda_2}{n + \lambda_1},$$

and the solution always blows up if $p \leq p_c$ regardless of nontrivial u_0, while if $p > p_c$, the solution can be either global (with sufficiently small initial data) or blow-up (with sufficiently large initial data).

In 2009, C.P. Wang, Y. Cao and J. Yin studied in [2] the following third order equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \triangle u}{\partial t} = \triangle u + u^p, \qquad x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $p > 0$, $k > 0$, and $u_0(x)$ is nonnegative and smooth. In [2], the authors have been able to find the critical exponent of the above nonlinear equation to be

$$p_0 = 1 \qquad p_c = 1 + \frac{2}{n}.$$

The results of this paper are

1. there exists a global solution for any initial data u_0 where $0 < p \le 1$;

2. the solution blows up in a finite time if $1 < p \le p_c$;

3. in case $p > p_c$, the solution can be either global (with sufficiently small initial data) or blow-up (with sufficiently large initial data).

Moreover, some asymptotic properties of global solutions and blow-up time estimations are also derived.

The equation (1.3) can also be regarded as a *pseudo-parabolic equation*, *a Sobolev type equation* or *a Sobolev-Galpern type equation* (for more details see the next chapter). Equations of the form (1.4) are usually used to explain mathematical and physical processes. For example, we can use them as mathematical models to explain behaviors of liquids through a layer of rocks, to interpret the propagation of population or to account the nonstationary process in semiconductors etc.

In my thesis, motivated by [2] and [7], we consider the Cauchy problem of the following semilinear pseudo-parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \triangle u}{\partial t} = \triangle u + a(x)u^p, \qquad x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.4}$$

where $1 \le p < \infty, k > 0$ and $a(x) > 0$ is a continuous function satisfying

$$c|x|^\sigma \le a(x) \le c^*|x|^\sigma, \qquad (x \in \mathbb{R}^n)$$

for some constants $c, c^* > 0$, u_0 is nonnegative and appropriately smooth as well as $\sigma > -2$ are constants. In the case $a(x) = 1$ and $k = 0$, the equation (1.4) is the well-known Cauchy problem studied by H. Fujita in [5] and is also the problem studied by Wang et. al. if $a(x) = 1$.

In this thesis, we extend the main result of Wang et. al. [2] in to the more general form (1.4). The appearance of the term $a(x)$ for the Cauchy problem (1.4) makes our proof more complicated than the proof of Wang et. al. for the case $a(x) = 1$. We have to create inequalities in order to bound the terms having the appearance of $a(x)$. More importantly, we find the two interesting points of the above equation, namely

$$p_{c_1} = 1 + \left(\frac{\sigma}{n}\right)_+ \qquad p_{c_2} = 1 + \left(\frac{\sigma + 2}{n}\right),$$

where $m_+ = m$ if $m > 0$ and $m_+ = 0$ if $m < 0$. When $\sigma = 0$, p_{c_1} is the same point as Wang et. al., so does the point p_{c_2}. Furthermore, we study the behavior of solutions for the Cauchy problem (1.4) in the cases:

$$p_{c_1} < p < p_{c_2}.$$

In this case, any nontrivial classical solution (1.4) blows up in a finite time regardless of the initial condition. The main result is as follows:

Theorem 1. *Let* $1+\left(\frac{\sigma}{n}\right)_+ < p < 1+\frac{\sigma+2}{n}$ *and* $\sigma > -2$. *Then for any nontrivial* $0 \leq u_0 \in C^{2+\alpha}(\mathbb{R}^n)$ *with a compact support, the nonnegative classical solution of the Cauchy problem blows up in a finite time.*

We conjecture that p_{c_1}, p_{c_2} are critical exponents and the behavior of solutions for (1.4) are different from Wang et. al. However, to answer these questions requires more techniques and time. We hope that this thesis will be the beginning of the study to extend the work of Wang et. al. We also expect this thesis will give ideas to other mathematicians who are curious about extending the work of Wang et. al. or studying blow-up problems to different types of PDEs.

This paper is arranged into four chapters as follows.

In chapter 2, we introduce some fundamental facts, definitions and theories in Partial Differential Equations as well as solutions of linear pseudo-parabolic equations. We begin constructing the important inequalities essential to prove the integral estimate in chapter 3. Finally, chapter 4 contains our main results.

CHAPTER II

PRELIMINARIES

In this chapter, we give some basic concepts in PDEs which are omitted the details of proofs. The proof can be found in common PDEs textbooks. They will be used in this work and be presented as the following:

2.1 Basic Knowledge

Pseudoparabolic Equation

A pseudoparabolic equation is an arbitrary higher-order partial differential equation with the first-order derivative with respect to time:

$$\frac{\partial}{\partial t}(A(u)) + B(u) = 0$$

where $A(u)$ and $B(u)$ are elliptic operators, the differential operators that generalize the Laplace operator

Example 1. *The Barenblatt-Zheltov-Kochina equation*

$$\frac{\partial}{\partial t}(\triangle u + cu) + \triangle u = 0, \qquad c \in \mathbb{R} - \{0\}.$$

This equation is considered a linear equation and it describes nonstationary filtering process in fissured-porous media.

Example 2. *The Showalter equation*

$$\frac{\partial}{\partial t}(\triangle u + \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u) + \alpha\triangle u + \alpha \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

In some works, pseudoparabolic equations are equations in which the operator $A(u)$ has a continuous inverse operator in appropriate Banach spaces; in the opposite case, the equation is said to be a Sobolev-type equation.

The terms for a solution of pseudoparabolic equations (1.4)

$$\frac{\partial u}{\partial t} - k\frac{\partial \triangle u}{\partial t} = \triangle u + a(x)u^p,\ \ x \in \mathbb{R}^n,\ t \geq 0$$

are usually found as follow:

Definition 1. *Classical Solution*

A classical solution is the solution of which all derivatives appearing in the equation (1.4) exist and are continuous.

Definition 2. *Weak Solution*

A function $u(x,t) \in L^2(\mathbb{R}^n \times [0,T))$ is called a weak solution of the initial value problem (1.4) if and only if the equation

$$\begin{aligned} -\int_{\mathbb{R}^n} u_0\psi(x,0)\,dx - \int_0^T\int_{\mathbb{R}^n} u\partial_t\psi\,dxds + k\int_{\mathbb{R}^n} u_0\triangle\psi(x,0)\,dx \\ +k\int_0^T\int_{\mathbb{R}^n} u\partial_t\triangle\psi\,dxds = \int_0^T\int_{\mathbb{R}^n} u\triangle\psi\,dxds + \int_0^T\int_{\mathbb{R}^n} a(x)u^p\psi\,dxds \end{aligned} \tag{2.1}$$

is valid for all smooth functions $\psi \in C_c^\infty(\mathbb{R}^n \times [0,T))$ where $0 < T < \infty$.

We observe that the integrands in (2.1) do not involve any derivatives of u. That is the equation (2.1) remains well defined even if u or its derivatives have discontinuities.

Definition 3. *(Cao. Y et. al., 2009) Mild Solution*

A solution $u \in C([0,T]; C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of the integral equation

$$\begin{aligned} u(x,t) &= G(t)u_0(x) + \int_0^t G(t-\tau)Bu^p(x,\tau)\ d\tau \\ &= \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy + \int_0^t d\tau\int_{\mathbb{R}^n} H(x-y,t-\tau)u^p(y,\tau)\,dy \end{aligned}$$

where

$$G(x,t) = (2\pi)^{\frac{n}{2}} e^{\frac{-t}{k}} \sum_{m=0}^{\infty} \frac{k^{-m}t^m}{m!} B_m(x) \geq 0, \qquad x \in \mathbb{R}, t \geq 0$$

$$H(x,t) = (2\pi)^{\frac{n}{2}} e^{\frac{-t}{k}} \sum_{m=0}^{\infty} \frac{k^{-m}t^m}{m!} B_{m+1}(x) \geq 0, \qquad x \in \mathbb{R}, t \geq 0,$$

with B_m being the Bessel-Macdonald kernel is called a mild solution of the Cauchy problem (1.4) in $[0,T]$.

Definition 4.

1. *A mild solution u of the Cauchy problem (1.4) is said to be a local solution if the domain is $\mathbb{R}^n \times [0,T)$ where $T < \infty$*
2. *A mild solution u of the Cauchy problem (1.4) is said to be a global solution if the domain is $\mathbb{R}^n \times [0,\infty)$*
3. *A mild solution u of the Cauchy problem (1.4) is said to be a blow-up solution in a finite time if there exist $0 < T < \infty$ such that*

$$\lim_{t\to T^-} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = \infty.$$

Theorem 2. *Divergence Theorem*

If $u, \psi : U \to \mathbb{R}$ and $\psi \in C^1$, $u \in C^2$ and $U \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary, then

$$\int_U \psi \triangle u \, dx = -\int_U \nabla\psi \cdot \nabla u \, dx + \int_{\partial U} \psi \partial_\nu u \, d\sigma$$

Here ∂_ν denotes the outward directional derivative of the functions on the boundary, and $d\sigma$ is the surface measure of ∂U.

Theorem 3. *Gauss-Green Theorem*

Let U be a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $u \in C^1(\overline{U})$. Then

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu^i\,d\sigma \quad (i = 1, 2, 3, ..., n)$$

Theorem 4. *Integration-by-parts formula*

Let U be a bounded, open subset of $\mathbb{R}^n$ and ∂U is C^1. Let $u, v \in C^1(\overline{U})$. Then

$$\int_U u_{x_i} v\,dx = -\int_U u v_{x_i}\,dx + \int_{\partial U} uv\nu^i\,d\sigma \quad (i = 1, 2, 3, ..., n)$$

An **eigenfunction** of an operator T is a function f such that the application of T on f gives f again, times a constant, k,

$$Tf = kf.$$

f is said to be an eigenfunction of T with the eigenvalue k.

A **radial function** on $\mathbb{R}^n$ is a function whose value at each point depends only on the distance between that point and the origin.

2.2 Solutions of Linear Pseudo-Parabolic Equations

In this section, we consider the solution of a linear pseudo-parabolic equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \triangle u}{\partial t} = \triangle u, & x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = u_0(x), \end{cases} \tag{2.2}$$

where $k > 0$ and u_0 is nonnegative and appropriately smooth. We begin considering this linear equation before studying the nonlinear equation. At the beginning, we will introduce some basic knowledge about the existence of weak solutions for linear elliptic equations.

Theorem 5. *[4] Third Existence Theorem for weak solution*

There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary-value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0, & \text{on } \partial U \end{cases}$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

Theorem 6. *[4] Boundedness of the inverse*

If $\lambda \notin \Sigma$, there exists a constant C such that $\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)}$, where $f \in L^2(U)$ and $u \in H_0^1(U)$ is the unique weak solution of

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0, & \text{on } \partial U \end{cases}$$

The constant C depends only on λ, U and the coefficients of L.

The equation $Lu = \lambda u + f$ can be seen in view of the operator. That is $(L - \lambda)u = f$ where $L - \lambda : H_0^1 \to L^2$ is the operator. From this theorem, we get that $(L - \lambda)^{-1}$ exist because f has the unique weak solution u. The operator $(L - \lambda)^{-1}$ is also bounded because u is bounded by f.

Moreover, we can prove that the operator

$$e^{t\mathcal{A}} = \sum_{n=0}^{\infty} \frac{t^n \mathcal{A}^n}{n!} = 1 + t\mathcal{A} + \frac{t^2 \mathcal{A}^2}{2!} + \frac{t^3 \mathcal{A}^3}{3!} + \cdots + \frac{t^n \mathcal{A}^n}{n!} + \ldots$$

is bounded at each t if $\mathcal{A} : X \to Y$ is a bounded linear operator in Banach spaces X and Y as the following proposition

Proposition 1. *Let $\mathcal{A}$ be a bounded linear operator. The operator $e^{t\mathcal{A}}$ is bounded at each t.*

Proof. Consider

$$\left\|e^{t\mathcal{A}}\right\| = \left\|\sum_{n=0}^{\infty} \frac{t^n \mathcal{A}^n}{n!}\right\| \leq \sum_{n=0}^{\infty} \frac{|t|^n \, \|\mathcal{A}\|^n}{n!}.$$

Since the operator $\mathcal{A}$ is bounded and t is fixed, we get that the series $\displaystyle\sum_{n=0}^{\infty} \frac{|t|^n \; \|\mathcal{A}\|^n}{n!}$ converges for each time t. Consequently, the operator $e^{t\mathcal{A}}$ is bounded. □

The boundedness in Proposition (1) implies that the operator $e^{t\mathcal{A}}$ is continuous. In fact, the operator $e^{t\mathcal{A}}$ is called a uniformly continuous semigroup, the map $t \to T(t)$ is continuous from $[0, \infty)$ to the Banach space $L(X, Y)$.

Proposition 2. *Let $\mathcal{A}$ be a bounded linear operator. The Cauchy problem*

$$\begin{cases} v_t = \mathcal{A}v \\ v(0) = v_0. \end{cases} \tag{2.3}$$

has a unique solution

$$v = e^{t\mathcal{A}} v_0.$$

Proof. It can be shown that $e^{t\mathcal{A}}v_0$ is a solution of the problem (2.3). For the uniqueness, assume $g, f \in X$ are the initial conditions of the following equations

$$\begin{cases} v_t = \mathcal{A}v \\ v(0) = f \end{cases} \quad \begin{cases} v_t = \mathcal{A}v \\ v(0) = g. \end{cases}$$

The solutions of two problems are $v_1(t) = e^{t\mathcal{A}}f$ and $v_2(t) = e^{t\mathcal{A}}g$ respectively. Consider,

$$\|v_1 - v_2\| = \left\|e^{t\mathcal{A}}(f - g)\right\| \leq \left\|e^{t\mathcal{A}}\right\| \cdot \|f - g\| \, .$$

Since the operator $\mathcal{A}$ is bounded, if $f = g$, we see that $v_1(t) = v_2(t)$. Therefore the solution of the problem (2.3) is unique. □

Theorem 7. *For each $u_0 \in L^2(\mathbb{R}^n) \cap H^2$, the Cauchy problem (2.2) has a unique solution.*

Proof. From the equation (2.2), we define $v = u - k\triangle u = (I - k\triangle)u$. By theorem (6), letting $L = -k\triangle$ and $\lambda = -1$, we get that $(I - k\triangle)^{-1}$ exist and is bounded. Then,

$$\begin{aligned} \frac{dv}{dt} = \frac{d}{dt}[u - k\triangle u] &= \triangle(I - k\triangle)^{-1} v \\ &= \frac{1}{k}[I - I(I - k\triangle)](I - k\triangle)^{-1} v \\ &= \left(\frac{1}{k}(I - k\triangle)^{-1} - \frac{1}{k} I\right) v. \end{aligned} \tag{2.4}$$

Let $\mathcal{A} = \frac{1}{k}(I - k\triangle)^{-1} - \frac{1}{k} I$. The equation (2.4) can be written as

$$\begin{cases} v_t = \mathcal{A} v \\ v(x, 0) = (I - k\triangle) u_0(x) \end{cases} \tag{2.5}$$

The operator $\mathcal{A}$ is bounded and linear, hence we get that

$$v(x, t) = e^{t\mathcal{A}}(I - k\triangle) u_0(x)$$

by the proposition (2). Since $v = u - k\triangle u = (I - k\triangle)u$, we get

$$u(x, t) = (I - k\triangle)^{-1} e^{t\mathcal{A}}(I - k\triangle) u_0(x)$$

is a solution of the Cauchy problem (2.2). □

CHAPTER III

INTEGRAL ESTIMATES

In this chapter, we derive a crucial integral estimate for solutions of the following initial-valued problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial \triangle u}{\partial t} = \triangle u + a(x)u^p, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.4}$$

where $p > 1, k > 0$ are constants, $a(x) > 0$ is a continuous function satisfying

$$c|x|^\sigma \leq a(x) \leq c^*|x|^\sigma, \qquad (x \in \mathbb{R}^n)$$

for some constants $c, c^* > 0$, and u_0 is an initial condition.

We assume u is a nonnegative classical solution, i.e. it is C^2 in x and is C^1 in t. u_0 has a compact support. Throughout this chapter, let $B_1 := B_1(0)$ denote the unit disk in $\mathbb{R}^n$.

Let $\varphi : B_1 \to \mathbb{R}$ be the principal eigenfunction of the Laplacian $-\triangle$ in B_1 with the homogeneous Dirichlet boundary condition. In other words, φ is a nontrivial C^2 function on B_1 satisfying

$$\begin{cases} \triangle\varphi + \lambda\varphi = 0 & \text{in } B_1, \\ \varphi|_{\partial B_1} = 0 \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant which is smallest so that such a function exists.

It is a basic fact that $\varphi = \varphi(|x|)$, i.e. φ is a radial function, and φ can be normalized so that

$$\varphi(0) = 1, \qquad \varphi'(0) = 0, \qquad \varphi'(t) < 0 \quad \text{for all } 0 < t \leq 1,$$

where we have regarded φ as a function of one variable. In addition, we set

$$\varphi(t) = 0 \quad \text{if } t > 1, \qquad \varphi(t) = 1 \quad \text{if } t \le 0.$$

Test Functions. For $\ell > 0$, let $\psi_\ell : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\psi_\ell(x) = \varphi\Big(\frac{|x|}{\ell} - 1\Big) \tag{3.1}$$

It is important to note that $\psi_\ell \in C^1$ on $B_{2\ell}(0)$, but it is not C^1 on $\mathbb{R}^n$; in fact, $\nabla\psi_\ell$ does not exist on

$$\partial B_{2\ell}(0).$$

The fact that ψ_ℓ is not C^1 on the whole $\mathbb{R}^n$ will be reflected in the next chapter.

This proof is arranged into four steps as follows:

3.1 Employing the Divergence Theorem

Now let $u = u(x,t)$ be a classical solution of (1.4). For $\ell > 0$, we multiply the equation (1.4) with the test function (3.1), $\psi_\ell(x)$, and then integrate over $B_R(0)$ for $R > 2\ell$ and $\ell > 1$ to get

$$\frac{d}{dt}\int_{B_R(0)} (u - k\triangle u)\psi_\ell \, dx = \int_{B_R(0)} (\triangle u + a(x)u^p)\psi_\ell \, dx. \tag{3.2}$$

Consider the Laplacian terms, using the divergence theorem we have

$$\int_{B_R(0)} \psi_\ell \triangle u \, dx = \int_{B_{2\ell}(0)} \psi_\ell \triangle u \, dx = -\int_{B_{2\ell}(0)} \nabla\psi_\ell \cdot \nabla u \, dx + \int_{\partial B_{2\ell}(0)} \psi_\ell \partial_\nu u \, d\sigma$$

where we have used the fact that ψ_ℓ is the test function, i.e., $\psi_\ell \in C^1$ on $B_{2\ell}(0)$ and $\psi_\ell(x) = 0$ if $|x| > 2\ell$. Since $\psi_\ell = 0$ on $\partial B_{2\ell}(0)$, we find that

$$\int_{B_R(0)} \psi_\ell \triangle u \, dx = -\int_{B_{2\ell}(0)} \nabla\psi_\ell \cdot \nabla u \, dx. \tag{3.3}$$

To proceed further, we must be careful because $\nabla\psi_\ell$ is not C^1 on $B_{2\ell}(0)$. We split $B_{2\ell}(0)$ into two sets

$$B_{2\ell}(0) = U_1 \cup U_2$$

where

$$U_1 := \Big\{|x| \le \ell\Big\}, \quad U_2 := \Big\{\ell \le |x| \le 2\ell\Big\}.$$

It is immediate that $\nabla\psi_\ell = 0$ on U_1 since ψ_ℓ is constant there. Thus

$$-\int_{U_1} \nabla\psi_\ell \cdot \nabla u \, dx = 0.$$

On U_2, $\psi_\ell(x) = \varphi(|x|/\ell - 1)$ and φ is C^∞ there, hence applying the divergence theorem to get

$$\begin{aligned} -\int_{U_2} \nabla\psi_\ell \cdot \nabla u \, dx &= \int_{U_2} u\triangle\varphi(|x|/\ell - 1)\, dx - \int_{\partial U_2} u\partial_\nu\varphi(|x|/\ell - 1)\, d\sigma \\ &= -\frac{\lambda}{\ell^2}\int_{U_2} u\varphi(|x|/\ell - 1)\, dx - \frac{1}{\ell}\int_{|x|=2\ell} u\varphi'(|x|/\ell - 1)\, d\sigma. \end{aligned}$$

We have used the fact that $\triangle\varphi(x) = -\lambda\varphi(x)$.

Therefore, we arrive at the following conclusion.

Lemma 1. *For all $t \ge 0$ and $R > 2\ell$, we have*

$$\int_{B_R(0)} \psi_\ell \triangle u \, dx = -\frac{\lambda}{\ell^2}\int_{\ell\le|x|\le 2\ell} u\varphi\left(\frac{|x|}{\ell} - 1\right) dx - \frac{1}{\ell}\int_{|x|=2\ell} u\varphi'\left(\frac{|x|}{\ell} - 1\right) dx.$$

3.2 Integrating w.r.t. Time

Next we integrate the equation (3.2) with respect to time over $[0, t]$ to get

$$\int_{B_R(0)} (u - k\triangle u)\psi_\ell \, dx - \int_{B_R(0)} (u_0 - k\triangle u_0)\psi_\ell \, dx = \int_0^t \int_{B_R(0)} (\triangle u + a(x)u^p)\psi_\ell \, dxds$$

Rewrite it to find that

$$\begin{aligned} \int_{B_R(0)} u\psi_\ell \, dx = k\int_{B_R(0)} \psi_\ell \triangle u \, dx + \int_{B_R(0)} (u_0 - k\triangle u_0)\psi_\ell \, dx \\ + \int_0^t \int_{B_R(0)} \psi_\ell \triangle u \, dxds + \int_0^t \int_{B_R(0)} a(x)u^p\psi_\ell \, dxds. \end{aligned}$$

We estimate the terms on the right hand side as follows.

The term of u_0: Let ℓ be sufficiently large so that $\operatorname{supp} u_0 \subset B_{2\ell}(0)$. Then by Lemma 1 above, we get

$$\begin{aligned}\int_{B_R(0)} (u_0 - k\triangle u_0)\psi_\ell \, dx &= \int_{B_R(0)} u_0\psi_\ell \, dx + \frac{k\lambda}{\ell^2}\int_{\ell\le|x|\le 2\ell} u_0\varphi \, dx + \frac{1}{\ell}\int_{|x|=2\ell} u_0\varphi' \, dx \\ &\ge \int_{B_R(0)} u_0\psi_\ell \, dx.\end{aligned}$$

The forth term is zero because u_0 has a compact support.

The first Laplacian of u: By lemma 1, then

$$\begin{aligned}\int_{B_R(0)} \psi_\ell \triangle u \, dx &= -\frac{\lambda}{\ell^2}\int_{\ell\le|x|\le 2\ell} u\varphi \, dx - \frac{1}{\ell}\int_{|x|=2\ell} u\varphi' \, dx \\ &\ge -\frac{\lambda}{\ell^2}\int_{\ell\le|x|\le 2\ell} u\varphi \, dx \\ &\ge -\frac{\lambda}{\ell^2}\int_{B_R(0)} u\psi_\ell \, dx,\end{aligned}$$

since $\varphi' \le 0$ at $|x| = 2\ell$.

The second Laplacian of u: As the equation (3.3), we use Divergence theorem again as

$$\int_{B_R(0)} \psi_\ell \triangle u \, dx = \int_{B_{2\ell}(0)} u\triangle\psi_\ell \, dx - \frac{1}{\ell}\int_{|x|=2\ell} u\varphi'(|x|/\ell - 1) \, d\sigma$$

Using the fact that $\varphi' \le 0$, we get

$$\int_{B_R(0)} \psi_\ell \triangle u \, dx \ge \int_{B_{2\ell}(0)} u\triangle\psi_\ell \, dx.$$

So

$$\int_0^t \int_{B_R(0)} \psi_\ell \triangle u \, dx \ge -\int_0^t \int_{B_R(0)} u|\triangle\psi_\ell| \, dx.$$

Combining the above three estimates, we see that

$$\int_{B_R} u\psi_\ell \, dx \ge \frac{1}{1+kC_0\ell^{-2}}\bigg(\int_{B_R} u_0\psi_\ell \, dx - \int_0^t \int_{B_R} u|\triangle\psi_\ell| \, dxds + \int_0^t \int_{B_R} a(x)u^p\psi_\ell \, dxds\bigg).$$

Taking $R \to \infty$ and using that $a(x) \ge c|x|^\sigma$, it follows that

Lemma 2. *Suppose $\ell > 0$ is sufficiently large so that* $\operatorname{supp} u_0 \subset B_{2\ell}(0)$*. Then*

$$\int_{\mathbb{R}^n} u\psi_\ell \, dx \geq \frac{1}{1+kC_0\ell^{-2}}\Big(\int_{\mathbb{R}^n} u_0\psi_\ell \, dx - \int_0^t \int_{\mathbb{R}^n} u|\triangle\psi_\ell| \, dxds + c\int_0^t \int_{\mathbb{R}^n} |x|^\sigma u^p \psi_\ell \, dxds\Big)$$

where $c, C_0 > 0$ are constants.

3.3 Using Hölder's Inequality

Now we estimate

$$\int_{\mathbb{R}^n} u|\triangle\psi_\ell| \, dx$$

by using the Hölder's inequality. Recall the constant p in the nonlinear term of our equation satisfies $p > 1$. To justify the following calculation, let

$$\Omega = \{x : \ell < |x| < 2\ell\}.$$

Then $\psi_\ell > 0$ in Ω. By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} u|\triangle\psi_\ell| \, dx &= \int_\Omega u|\triangle\psi_\ell| \, dx \\ &\leq \Big(\int_\Omega |x|^\sigma u^p \psi_\ell \, dx\Big)^{1/p} \Big(\int_\Omega |x|^{-\sigma/(p-1)} |\triangle\psi_\ell|^{p/(p-1)} \psi_\ell^{-1/(p-1)} \, dx\Big)^{(p-1)/p}\end{aligned}$$

We make the following claim.

Claim. There is a constant $C > 0$ such that

$$|\triangle\psi_\ell|^{p/(p-1)} \psi_\ell^{-1/(p-1)} \leq C/\ell^{2p/(p-1)} \quad \text{if } x \in \Omega,$$

Proof of Claim. In Ω we have $\psi_\ell(x) = \varphi\Big(\frac{|x|}{\ell} - 1\Big)$. So the required inequality is true on $\ell < |x| < 2\ell$ since

$$\triangle\varphi(|x|/\ell - 1)| = \frac{1}{\ell^2}\lambda\varphi(x) \leq \frac{\lambda}{\ell^2}\varphi^{1/p}.$$

the claim then follows. □

Using the claim, then

$$\begin{aligned}\int_\Omega |x|^{-\sigma/(p-1)}|\triangle\psi_\ell|^{p/(p-1)}\psi_\ell^{-1/(p-1)}\,dx&\\ &\le C\ell^{-2p/(p-1)}\int_{\ell<|x|<2\ell}|x|^{-\sigma/(p-1)}\,dx\\ &\le C\ell^{-2p/(p-1)}\int_\ell^{2\ell} r^{n-1-\sigma/(p-1)}dr.\end{aligned}$$

From that

$$\int_\ell^{2\ell} r^{n-1-\sigma/(p-1)}dr \le \begin{cases} C\ell^{(n-\frac{\sigma}{p-1})}, & n\neq \frac{\sigma}{p-1},\\ C, & n=\frac{\sigma}{p-1}\end{cases}$$

hence,

$$\int_\Omega |x|^{-\sigma/(p-1)}|\triangle\psi_\ell|^{p/(p-1)}\psi_\ell^{-1/(p-1)}\,dx \le C\ell^{(n-\frac{\sigma+2p}{p-1})}.$$

Therefore, we have

$$\int_{\mathbb{R}^n} u|\triangle\psi_\ell|\,dx \le C\Big(\ell^{(n-\frac{\sigma+2p}{p-1})}\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |x|^\sigma u^p\psi_\ell\,dx\Big)^{1/p}$$

From lemma(2), we get

$$\int_{\mathbb{R}^n} u\psi_\ell\,dx \ge \frac{1}{1+kC_0\ell^{-2}}\Big\{\int_{\mathbb{R}^n} u_0\psi_\ell\,dx + \int_0^t\Big[c\int_{\mathbb{R}^n}|x|^\sigma u^p\psi_\ell\,dx - C\Big(\ell^{n-\frac{\sigma+2p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |x|^\sigma u^p\psi_\ell\,dx\Big)^{1/p}\Big]ds\Big\}$$

We summarize our calculation above.

Lemma 3. *If $\ell>0$ is sufficiently large so that $\operatorname{supp} u_0\subset B_{2\ell}(0)$, then*

$$\int_{\mathbb{R}^n} u\psi_\ell\,dx \ge \frac{1}{1+kC_0\ell^{-2}}\Big\{\int_{\mathbb{R}^n} u_0\psi_\ell\,dx + \int_0^t\Big(\int_{\mathbb{R}^n}|x|^\sigma u^p\psi_\ell\,dx\Big)^{1/p}\Big[c\Big(\int_{\mathbb{R}^n}|x|^\sigma u^p\psi_\ell\,dx\Big)^{(p-1)/p} - C\Big(\ell^{n-\frac{\sigma+2p}{p-1}}\Big)^{\frac{p-1}{p}}\Big]ds\Big\}$$

3.4 Last Estimate

We further estimate the term

$$\int_{\mathbb{R}^n} |x|^\sigma u^p \psi_\ell \, dx$$

as follows. Using Hölder's inequality then

$$\int_{\mathbb{R}^n} |x|^\sigma u^p \psi_\ell \, dx \geq \Big(\int_{\mathbb{R}^n} u \psi_\ell \, dx \Big)^p \Big(\int_{\mathbb{R}^n} |x|^{-\sigma/(p-1)} \psi_\ell \, dx \Big)^{-(p-1)} \tag{3.4}$$

and if $p > 1 + \dfrac{\sigma}{n}$, then

$$\int_{\mathbb{R}^n} |x|^{-\sigma/(p-1)} \psi_\ell \, dx = \int_0^{2\ell} r^{n-1-\sigma/(p-1)} \, dr = C \ell^{(n - \frac{\sigma}{p-1})}$$

Thus

$$\int_{\mathbb{R}^n} |x|^\sigma u^p \psi_\ell \, dx \geq C_1 \Big(\int_{\mathbb{R}^n} u \psi_\ell \, dx \Big)^p \ell^{\sigma - n(p-1)}. \tag{3.5}$$

Now we define

$$w_\ell(t) = \int_{\mathbb{R}^n} u(x,t) \psi_\ell(x) \, dx, \quad t \geq 0.$$

Using this estimate and the previous lemma, we get

$$w_\ell(t) \geq \frac{1}{1 + k C_0 \ell^{-2}} \Big\{ w_\ell(0) + \Big(C_1 \ell^{\sigma - n(p-1)} \Big)^{\frac{1}{p}} \int_0^t w_\ell(s) \Big[c C_1^{\frac{p-1}{p}} w_\ell^{p-1}(s) \Big(\ell^{\sigma - n(p-1)} \Big)^{\frac{p-1}{p}} - C \Big(\ell^{n - \frac{\sigma + 2p}{p-1}} \Big)^{\frac{p-1}{p}} \Big] ds \Big\} \tag{3.6}$$

Next we compare the two important terms

$$\ell^{(\sigma - n(p-1))\frac{p-1}{p}} := \ell^A \quad \Rightarrow \quad A = (\sigma - n(p-1)) \frac{p-1}{p}$$

and the other

$$\ell^{(n - \frac{\sigma+2p}{p-1})\frac{p-1}{p}} = \ell^{\frac{n(p-1) - \sigma - 2p}{p}} := \ell^B \quad \Rightarrow \quad B = \frac{n(p-1) - \sigma - 2p}{p}.$$

From the equation (3.6), we consider p in the cases $1 + \dfrac{\sigma}{n} < p < 1 + \dfrac{\sigma + 2}{n}$.

Proposition 3.

If $1+\frac{\sigma}{n}<p<1+\frac{\sigma+2}{n}$ *then* $B<A$.

Proof. To prove the proposition, we first show that the assumption $p<1+\frac{\sigma+2}{n}$ implies $B<A$. Consider

$$\begin{aligned} A-B &= \frac{p-1}{p}(\sigma-n(p-1))-\frac{n(p-1)-\sigma-2p}{p} \\ &= n\Big(1+\frac{\sigma+2}{n}-p\Big)>0. \end{aligned}$$

This proves the proposition. □

CHAPTER IV

THEOREM ON CRITICALS EXPONENTS

In this chapter, we prove the main result on the critical exponents for classical solutions of the Cauchy problem. According to the Introduction in chapter 1, the main theorem is stated as follows .

Theorem 1 *Let* $1+\left(\frac{\sigma}{n}\right)_+ < p < 1+\frac{\sigma+2}{n}$ *and* $\sigma > -2$. *Then for any nontrivial* $0 \le u_0 \in C^{2+\alpha}(\mathbb{R}^n)$ *with a compact support, the nonnegative classical solution of the Cauchy problem blows up in a finite time.*

Proof. We will split this theorem into several parts in order to make it easy to understand and follow.

From the inquality (3.6),

$$w_\ell(t) \ge \frac{1}{1+kC_0\ell^{-2}}\left\{w_\ell(0) + \left(C_1\ell^{\sigma-n(p-1)}\right)^{\frac{1}{p}} \int_0^t w_\ell(s)\left[cC_1^{\frac{p-1}{p}} w_\ell^{p-1}(s)\left(\ell^{\sigma-n(p-1)}\right)^{\frac{p-1}{p}} - C\left(\ell^{n-\frac{\sigma+2p}{p-1}}\right)^{\frac{p-1}{p}}\right]ds\right\}$$

For convenience, we let

$$R = \frac{1}{1+kC_0} \qquad R_2 = cC_1^{\frac{p-1}{p}}\left(\ell^{\sigma-n(p-1)}\right)^{\frac{p-1}{p}}$$
$$R_1 = \left(C_1\ell^{\sigma-n(p-1)}\right)^{\frac{1}{p}} \qquad R_3 = C\left(\ell^{n-\frac{\sigma+2p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Thus the inequality (3.6) can be rewritten

$$w_\ell(t) \ge R\left(w_\ell(0) + R_1\int_0^t w_\ell(s)\left[R_2 w_\ell^{p-1}(s) - R_3\right]ds\right). \tag{4.1}$$

Step 1

In this step, we give a condition to show that the classical solution blows up in a finite time. In the proposition (3), we get

$$\frac{n(p-1)-\sigma-2p}{p} < \frac{p-1}{p}(\sigma - n(p-1))$$

Since $w_\ell(t)$ is continuous, there is $t_0 > 0$ such that

$$w_\ell(t_0) > Rw_\ell(0) \quad \text{and} \quad w_\ell(t) \geq Rw_\ell(0) \ \forall t \in [0, t_0].$$

Let $t \in [0, t_0]$. From (4.1),

$$\begin{aligned} w_\ell(t) &\geq R\Bigg(w_\ell(0) + R_1 \int_0^t w_\ell(s) \Big[R_2 w_\ell^{p-1}(s) - R_3 \Big] ds \Bigg) \\ &\geq R\Bigg(w_\ell(0) + R_1 \int_0^t w_\ell(s) \Big[R^{p-1} R_2 w_\ell^{p-1}(s) - R_3 \Big] ds \Bigg). \end{aligned} \tag{4.2}$$

As the classical solution u is nonnegative, we also get that $w_\ell(t) \geq 0$. From the proposition (3), we fix $\ell > 0$ sufficiently large such that

$$\ell \geq 1 + \left(\frac{2C_2(1+kC_0)^{p-1}}{cC_1^{\frac{p-1}{p}} w_\ell^{p-1}(0)} \right)^{\frac{1}{\sigma+2+n-np}}. \tag{4.3}$$

Then, from (4.2),

$$\begin{aligned} w_\ell(t) &\geq R\Bigg(w_\ell(0) + R_1 \int_0^t w_\ell(s) \Big[-R_3 + \frac{1}{2} R_2 w_\ell^{p-1}(s) + \frac{1}{2} R_2 w_\ell^{p-1}(s) \Big] ds \Bigg) \\ &> R\left(w_\ell(0) + \frac{1}{2} R_1 R_2 \int_0^t w_\ell^p(s) ds \right) \qquad \forall t \in [0, t_0]. \end{aligned} \tag{4.4}$$

Step 2 Construct the Lemma

This lemma is essential to construct the inequality related to prove that the solution is blow up at a finite time.

Lemma 4. *$w_\ell(t) \geq Rw_\ell(0)$ $\forall t \geq 0$ where R is a constant independent of ℓ.*

Proof. We have already known that $w_\ell(t_0) > Rw_\ell(0)$ and $w_\ell(t) \geq Rw_\ell(0)$ $\forall t \in [0, t_0]$ for some $t_0 > 0$. Define

$$A = \{t \geq 0 | \, w_\ell(t) \geq Rw_\ell(0)\} \text{ and } T_M = \sup\{t \geq 0 | \, [0,t] \subseteq A\}.$$

We see that $T_M \geq 0$. As $T_M - \frac{1}{n} < T_M$, we get there exist $t_n \in \{t \geq 0 | \, [0,t] \subseteq A\}$ such that $T_M - \frac{1}{n} < t_n < T_M$. We also see that $t_n \nearrow T_M$ and $w_\ell(t_n) \geq Rw_\ell(0)$, $\forall n$. From (4.4),

$$\begin{aligned} w_\ell(t_n) &\geq R\left\{w_\ell(0) + \frac{1}{2}R_1R_2\int_0^{t_n} w_\ell^p(s)\,ds\right\} \\ &\geq R\left\{w_\ell(0) + \frac{1}{2}R_1R_2\int_0^{t_n} (Rw_\ell(0))^p\,ds\right\} \\ &\geq R\left\{w_\ell(0) + \frac{1}{2}R_1R_2(Rw_\ell(0))^p t_n\right\} \end{aligned}$$

Since w_ℓ is continuous and $t_n \nearrow T_M$, we get that

$$w_\ell(T_M) \geq R\left\{w_\ell(0) + \frac{1}{2}R_1R_2(Rw_\ell(0))^p T_M\right\} > Rw_\ell(0).$$

If T_M were finite, then $[0, T_M + \frac{1}{n}] \nsubseteq A$. We get there exist $x_n \in (T_M, T_M + \frac{1}{n}]$ such that $w_\ell(x_n) < Rw_\ell(0)$. So,

$$w_\ell(x_n) < Rw_\ell(0) < w_\ell(T_M).$$

Since $x_n \in (T_M, T_M + \frac{1}{n}]$, $x_n \to T_M$. We get $w_\ell(T_M) \leq Rw_\ell(0) < Rw_\ell(0)$, a contradiction. This means that T_M is infinite, i.e., $w_\ell(t) \geq Rw_\ell(0)$ $\forall t \geq 0$. $\square$

Step 3 ODE inequality

We will set up the inequality which results in blowing up of the solution. By the above lemma and the selected ℓ in (4.3), we get

$$\frac{1}{2}R_2 w_\ell^{p-1}(s) - R_3 \geq 0 \qquad \forall s \in [0,t], \ \forall t \geq 0.$$

Then

$$\begin{aligned} w_\ell(t) &\geq R\bigg(w_\ell(0) + R_1 \int_0^t w_\ell(s)\Big[R_2 w_\ell^{p-1}(s) - R_3\Big]ds\bigg) \\ &= R\bigg(w_\ell(0) + R_1 \int_0^t w_\ell(s)\Big[-R_3 + \frac{1}{2}R_2 w_\ell^{p-1}(s) + \frac{1}{2}R_2 w_\ell^{p-1}(s)\Big]ds\bigg) \\ &\geq R\left(w_\ell(0) + \frac{1}{2}R_1R_2 \int_0^t w_\ell^p(s)ds\right) \qquad\qquad \forall t \geq 0. \end{aligned} \tag{4.5}$$

Let $\delta = \frac{1}{2}R_1R_2 = \frac{1}{2}cC_1\ell^{\sigma-n(p-1)}$ and

$$v(m) = w_\ell(0) + \delta \int_0^m w_\ell^p(s)\,ds,\ \forall m \geq 0.$$

By Fundamental Theorem of Calculus, we get

$$\begin{aligned} \frac{d}{dm}v(m) = \delta w_\ell^p(m) &\geq \delta\{Rv(m)\}^p \\ \int_0^t \frac{1}{v^p}\,dv &\geq \int_0^t \delta R^p\,dm \\ \frac{v^{1-p}}{1-p}\bigg|_{m=0}^{m=t} &\geq \delta R^p m\bigg|_{m=0}^{m=t} \\ \frac{v^{1-p}(t)}{1-p} &\geq \delta t R^p + \frac{v^{1-p}(0)}{1-p} = \delta t R^p + \frac{w_\ell^{1-p}(0)}{1-p} \\ v^{1-p}(t) &\leq (1-p)\delta t R^p + w_\ell^{1-p}(0) \qquad ; 1 < p \end{aligned}$$

Thus,

$$v(t) \geq \left(\frac{1}{(1-p)\delta t R^p + w_\ell^{1-p}(0)}\right)^{\frac{1}{p-1}}. \tag{4.6}$$

If $(1-p)\delta T^* R^p + w_\ell^{1-p}(0) = 0$, we get

$$\begin{aligned} T^* &= \frac{w_\ell^{1-p}(0)}{(p-1)\delta R^p} = \frac{w_\ell^{1-p}(0)}{(p-1)(1/2)R_1R_2R^p} \\ &= \frac{2}{(p-1)R^pR_1R_2w_\ell^{p-1}(0)} \\ &= \frac{2(1+kC_0)^p}{(p-1)cC_1w_\ell^{p-1}(0)} \cdot \ell^{np-n-\sigma} \end{aligned}$$

From (4.6), if $t \to T^*$, we get the right hand side of the inequality is infinite. This implies that $\lim_{t\to T^*} v(t) = \infty$. It leads to

$$w_\ell(t) = \int_{\mathbb{R}^n} u(x,t)\psi_\ell(x)\ dx \to \infty \quad \text{as} \quad t \to T^*,$$

which implies $\|u(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to T^*$ because $\displaystyle\int_{\mathbb{R}^n} u(x,t)\psi_\ell(x)\ dx \to \infty$ as $t \to T^*$ and $\|\psi_\ell(x)\|_{L^\infty(\mathbb{R}^n)} \leq 1$, if we assume that $u(x,t)$ were not blow up as $t \to T^*$, then $|u(x,t)| \leq M$ for some $M > 0$ and for all $x \in \mathbb{R}^n$ and $t \leq T^*$. Consider

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} u(x,t)\psi_\ell(x)\,dx\right| &= \left|\int_{B_{2\ell}} u(x,t)\psi_\ell(x)\,dx\right| \\
&\leq \int_{B_{2\ell}} |u(x,t)\psi_\ell(x)|\ dx \\
&\leq \int_{B_{2\ell}} M \cdot 1\,dx \quad \leq M \cdot Vol(B_{2\ell})\ < \infty,
\end{aligned}$$

a contradiction the fact that $w_\ell(t) = \displaystyle\int_{\mathbb{R}^n} u(x,t)\psi_\ell(x)\ dx \to \infty$ as $t \to T^*$. So $u(x,t)$ blows up in a finite time. $\square$

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