CHAPTER III

A FINITE INVERSE BIPREFIX CODE WHOSE SYNTACTIC MONOID HAS TWO $\mathcal{D}$ - CLASSES

M.P. Schützenberger (see [4]) has studied a prefix code C whose syntactic monoid $M(C^*)$ is a group. It is obvious that $M(C^*)$ has only one $\mathcal{D}$ - class.

In this chapter, the existence of a finite inverse biprefix code whose syntactic moniod has exactly two nonzero $\mathcal{D}$ - classes is given.

Now, we refer to the inverse biperfix code constructed by K. Jantarakhajorn (see [2]) :

For a given $n \geq 3$, let $A = \{ a_1, a_2, \ldots, a_n \}$ be an alphabet and let

$$C_1 = \{ a_i a_{i+1} \ldots a_{n-1} a_1 a_2 \ldots a_i \mid i = 1, 2, \ldots, n-1 \},$$

$$C_2 = \{ a_i a_{i+1} \ldots a_n a_2 a_3 \ldots a_{i-1} \mid i = 3, 4, \ldots, n-1 \},$$

and $C_3 = \{ a_1 a_2 \ldots a_n a_2 a_3 \ldots a_{n-1}, a_n a_2 a_3 \ldots a_n a_2 a_3 \ldots a_{n-1}, a_n a_2 a_3 \ldots a_{n-1} a_1, a_2 a_3 \ldots a_n \}$.

Theorem 3.1. *The code $C = C_1 \cup C_2 \cup C_3$ is a finite inverse biprefix code.*

Proof. See [2]. □

In [2], K. Jantarakhajorn labelled the tree representation of C^* as follows:

The top and the end points of the tree are labelled 1.

For each $i \in \{1, 2, \ldots, n-1\}$, $j \in \{1, 2, \ldots, n-1\}$ with $i \leq j$,

the node associated with $a_i a_{i+1} \ldots a_j$ is labelled

$$(n-2)i + j - (n-3).$$

For each $i \in \{ 2, 3, \ldots, n-1 \}$, $j \in \{ 1, 2, \ldots, i-1 \}$,

the node associated with $a_i a_{i+1} \ldots a_{n-1} a_1 a_2 \ldots a_j$ is labelled

$$(n-2)i + j + 2.$$

Since each of the remaining unlabelled nodes has the same subtree as one of the above labelled nodes, they have the same name. Hence $P_{C^*}^{(r)}$ has been constructed.

The corresponding syntactic monoid $M(C^*)$ is generated by

$$\{ \tau(a_i) \mid i = 1, 2, \ldots, n \}.$$

Notations:

(i) For each i, let $\triangle_i$ denote the domain of $\tau(a_i)$ and

∇_i denote the image of $\tau(a_i)$.

(ii) For each $i \in \{ 2, 3, \ldots, n \}$, let

$$A(i) = \{ (n-2)(k-1) + i \mid k = 1, 2, \ldots, i-1 \},$$

$$B(i) = \{ (n-2)k + i + 1 \mid k = i, i+1, \ldots, n-1 \},$$

and $A(1) = \{ (n-2)k + 2 \mid k = 1, 2, \ldots, n-1 \}$.

We have a few remarks on the code defined above.

Remark 3.2. (i) $\triangle_1 = \triangle_n = \{ 1 \} \cup A(1)$.

$\triangle_i = \{ 1 \} \cup A(i) \cup B(i)$ *for all* $i \geq 2$.

(ii) $\nabla_1 = \nabla_n = \triangle_2$.

$\nabla_i = \triangle_{i+1}$ *for all* $i \in \{ 2, 3, \ldots, n-1 \}$.

It follows that

$$Dom\ \tau(a_i a_{i+1} \ldots a_j) = \triangle_i \quad and \quad Im\ \tau(a_i a_{i+1} \ldots a_j) = \nabla_j$$

for all $i, j \in \{ 1, 2 \ldots, n-1 \}$ *and* $i \leq j$.

(iii) $\tau(a_i)$' *s are defined as follows :*

$$s\tau(a_1) = \begin{cases} 2 & \textit{if } s = 1 \\ 1 & \textit{if } s = n \\ s+1 & \textit{otherwise,} \end{cases}$$

$$s\tau(a_n) = \begin{cases} 2 & \textit{if } s = 1 \\ 1 & \textit{if } s = 2n-2 \\ (n-2)(n-1)+3 & \textit{if } s = n \\ s-(n-3) & \textit{otherwise,} \end{cases}$$

$$s\tau(a_i) = \begin{cases} (n-1)i-(n-3) & \textit{if } s = 1 \\ 1 & \textit{if } s = (n-1)i+1 \\ s+1 & \textit{otherwise.} \end{cases}$$

Before proving the next proposition, we need the following lemmas.

Lemma 3.3. *For each* $i \in \{ 2, 3, \ldots, n-1 \}$, $A(1) \cap [A(i) \cup B(i)] = \varnothing$.

Proof. Let $i \in \{ 2, 3, \ldots, n-1 \}$.

Suppose $A(1) \cap [A(i) \cup B(i)] \neq \varnothing$.

Let $s \in A(1) \cap [A(i) \cup B(i)]$. There are two cases to be considered :

Case 1 : $s \in A(1) \cap A(i)$. Then

$$s = (n-2)k_1 + 2 \quad \text{and} \quad s = (n-2)(k_2 - 1) + i$$

for some $k_1 \in \{ 1, 2, \dots, n-1 \}$, $k_2 \in \{ 1, 2, \dots, i-1 \}$.

This yields

$$i = (n-2)(k_1 - k_2 + 1) + 2.$$

By considering all possibilities of k_1 and k_2, we get a contradiction.

Case 2 : $s \in A(1) \cap B(i)$.

If $i = n-1$, then $B(i) = \{ (n-1)(n-2)+n \}$. Since $s \in A(1)$, $s = (n-2)k_1+2$ for some $k_1 \in \{ 1, 2, \dots, n-1 \}$. Then

$$(n-2)k_1 + 2 = (n-1)(n-2) + n,$$

and so $k_1 = n$, which is a contradiction.

Assume that $i \neq n-1$. Since $s \in B(i)$,

$$s = (n-2)k_2 + i + 1$$

for some $k_2 \in \{ i, i+1, \dots, n-1 \}$. Thus

$$i = (n-2)(k_1 - k_2) + 1,$$

which is a contradiction. □

Lemma 3.4. *Let $i, j \in \{ 2, 3, \dots, n-1 \}$ be such that $i \neq j$. Then the following statements hold:*

(i) $A(i) \cap A(j) = \varnothing$.

(ii) $B(i) \cap B(j) = \varnothing$.

(iii) $A(i) \cap B(j) = \varnothing$.

Moreover, $[A(i) \cup B(i)] \cap [A(j) \cup B(j)] = \varnothing$.

Proof. Let $i, j \in \{ 2, 3, \ldots, n-1 \}$ be such that $i \neq j$. Without loss of generality, we may assume that $i < j$ and let $j = i + l$ for some $l \in \mathbb{N}$.

To prove (i), assume on the contrary that $s \in A(i) \cap A(j)$.

Then

$$s = (n-2)k_1 + (i-1) - (n-3) \quad \text{and} \quad s = (n-2)k_2 + (j-1) - (n-3)$$

for some $k_1 \in \{ 1, 2, \ldots, i-1 \}$, $k_2 \in \{ 1, 2, \ldots, j-1 \}$.

This yields

$$l = j - i = (k_1 - k_2)(n-2).$$

This contradicts to the condition on l that $0 < l < n-3$.

Hence $A(i) \cap A(j) = \varnothing$.

(ii) Similar to the statement (i).

To prove (iii), assume on the contrary that $s \in A(i) \cap B(j) = A(i) \cap B(i+l)$.

Then

$$s = (n-2)k_1 + (i-1) - (n-3) \quad \text{and} \quad s = (n-2)k_2 + i + l + 1$$

for some $k_1 \in \{ 1, 2, \ldots, i-1 \}$, $k_2 \in \{ i+l, i+l+1, \ldots, n-1 \}$.

Consider

$$\begin{aligned} s &= (n-2)k_2 + i + l + 1 \\ &> (n-2)k_2 + i \\ &> (n-2)k_1 + i \\ &> (n-2)k_1 + i - (n-2) \\ &= s. \end{aligned}$$

This is a contradiction.

From (i)-(iii), we get $[A(i) \cup B(i)] \cap [A(j) \cup B(j)] = \varnothing$. $\square$

Proposition 3.5. (i) *For each* $j \in \{ 1, 2, \ldots, n \}$,

$$\nabla_{n-1} \cap \triangle_j = \begin{cases} \triangle_j & \text{if } j = n \text{ or } j = 1, \\ \{ 1 \} & \text{otherwise.} \end{cases}$$

(ii) *For each* $i, j \in \{ 1, 2, \ldots, n \}$ *and* $i \neq n-1$,

$$\nabla_i \cap \triangle_j = \begin{cases} \triangle_j & \text{if } j = i+1 \text{ *}, \\ \{ 1 \} & \text{otherwise.} \end{cases}$$

Proof. (i) It follows from Remark 3.2 and Lemma 3.3.

(ii) Let $i, j \in \{ 1, 2, \ldots, n \}$ and $i \neq n-1$.

Since $\nabla_n = \nabla_1$, $\nabla_n \cap \triangle_j = \nabla_1 \cap \triangle_j$. It suffices to prove only the case $i \neq n$.

If $j = i+1$, then by Remark 3.2(ii), $\nabla_i = \triangle_j$ and so $\nabla_i \cap \triangle_j = \triangle_j$.

Assume that $j \neq i+1$. It is clear that $1 \in \nabla_i \cap \triangle_j$.

But by Lemma 3.3 and Lemma 3.4, $(\nabla_i \cap \triangle_j) - \{ 1 \} = \varnothing$, thus $\nabla_i \cap \triangle_j = \{ 1 \}$. □

Proposition 3.6. *For each* $\alpha \in M(C^*)$, *rank* $\alpha = n$ *or rank* $\alpha = 1$. *Moreover, if rank* $\alpha = n$, *then Dom* $\alpha = \triangle_i$ *and Im* $\alpha = \nabla_j$ *for some* $i, j \in \{ 1, 2, \ldots, n \}$.

Proof. Let $\alpha = \tau(a_i a_{i_1} a_{i_2} \ldots a_{i_k})$ for some $i \in \{ 1, 2, \ldots, n \}$, and $k \in \mathbb{N}$.

Assume that rank $\alpha = r \neq 1$. Then

$$\text{rank } \tau(a_i a_{i_1}) \neq 1 \quad \text{and} \quad \text{rank } \tau(a_{i_l} a_{i_{l+1}}) \neq 1$$

for all $l \in \{ 1, 2, \ldots, k-1 \}$.

Thus

$$|\ \nabla_i \cap \triangle_{i_1}\ | > 1 \quad \text{and} \quad |\ \nabla_{i_l} \cap \triangle_{i_{l+1}}\ | > 1$$

*If $i = n$, then $i + 1 = 2$.

for all $l \in \{ 1, 2, \ldots, k-1 \}$ (since rank $\alpha\beta = (\text{Im } \alpha \cap \text{Dom } \beta)\alpha^{-1}$).

By Proposition 3.5,

$$\nabla_i \cap \triangle_{i_1} = \triangle_{i_1}, \quad \nabla_{i_l} \cap \triangle_{i_{l+1}} = \triangle_{i_{l+1}}$$

for all $l \in \{ 1, 2, \ldots, k-1 \}$, and $a_i a_{i_1} a_{i_2} \ldots a_{i_k}$ is one of these forms :

(i) $a_i a_{i+1} \ldots a_j$ for some j, $1 \leq i \leq j \leq n-1$.

(ii) $a_i a_{i+1} \ldots a_{n-1} u_1 a_2 a_3 \ldots a_{n-1} u_2 a_2 \ldots u_s a_2 a_3 \ldots a_j$ for some $s \in \mathbb{N}$, for some $j \in \{ 2, 3, \ldots, n-1 \}$, and $u_t \in \{ a_1, a_n \}$.

In any cases, we have

$$\text{Dom } \alpha = \triangle_i \quad \text{and} \quad \text{Im } \alpha = \nabla_j$$

for some $i, j \in \{ 1, 2, \ldots, n \}$.

Therefore, rank $\alpha = n$. □

Before proving the next proposition, we need the following lemma.

Lemma 3.7. (i) *For each $i \in \{ 1, 2, \ldots, n \}$, if $s \in \triangle_i$, then there exists $\eta \in M(C^*)$ such that $1\eta = s$.*

(ii) *There exists $\iota \in M(C^*)$,*

$$\iota = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Proof. (i) Let $i \in \{ 1, 2, \ldots, n \}$ and assume that $s \in \triangle_i$. There are two cases to be considered :

Case 1 : $i = 1$.

Set $\delta = \tau(a_1 a_2 \ldots a_{n-1})$.

By Remark 3.2, Dom $\delta = \triangle_1$ and Im $\delta = \nabla_{n-1} = \triangle_1$.

In [2], δ is a cycle of rank n. Since $1 \in \triangle_1 =$ Dom δ and $s \in \triangle_1 =$ Im δ, there exists $l \in \mathbb{N}$ such that $1\delta^l = s$.

Case 2 : $i \neq 1$.

Set $\theta_i = \tau(a_1a_2 \ldots a_{i-1})$.

By Remark 3.2, Dom $\theta_i = \triangle_1$ and Im $\theta_i = \nabla_{i-1} = \triangle_i$.

Since $s \in \triangle_i =$Im θ_i, there exists $s_1 \in$ Dom $\theta_i = \triangle_1$ such that $s_1\theta_i = s$.

By Case 1, there exists $\eta_1 \in M(C^*)$ such that $1\eta_1 = s_1$.

Let $\eta = \eta_1\theta_i$. Then $\eta \in M(C^*)$.

Hence

$$1\eta = (1\eta_1)\theta_i = s_1\theta_i = s.$$

(ii) Let $\iota = \tau(a_1a_1)\tau(a_1)^{-1}$. Then

$$\iota = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

□

It is known that, in general, if α, β are in a proper subsemigroup of $\mathcal{I}(X)$ with rank $\alpha =$ rank β, then α and β may not be $\mathcal{D}-$ related. However, the next proposition shows that this is true in the proper subsemigroup $M(C^*)$ of $\mathcal{I}(X)$.

Proposition 3.8. *Let $\alpha, \beta \in M(C^*)$.*

$$\alpha\mathcal{D}\beta \quad \textit{if and only if} \quad \textit{rank } \alpha = \textit{rank } \beta.$$

Proof. The sufficient part follows from Corollary 2.2.

For the necessary part, assume that rank $\alpha =$ rank $\beta = r$. Then, by Proposition 3.5, there are two cases to be considered:

Case 1 : $r = n$.

Let Im $\alpha = \triangle_i$ and Dom $\beta = \triangle_j$ for some $i, j \in \{ 1, 2, \ldots, n \}$.

By Remark 3.2, Dom $\tau(a_i a_{i+1} \ldots a_{n-1}) = \triangle_i$, Im$\tau(a_i a_{i+1} \ldots a_{n-1}) = \nabla_{n-1} = \triangle_1$, Dom $\tau(a_1 a_2 \ldots a_{j-1}) = \triangle_1$ and Im $\tau(a_1 a_2 \ldots a_{j-1}) = \nabla_{j-1} = \triangle_j$.

Hence

$$\text{Dom } \tau(a_i a_{i+1} \ldots a_{n-1} a_1 a_2 \ldots a_{j-1}) = \triangle_i$$

and

$$\text{Im } \tau(a_i a_{i+1} \ldots a_{n-1} a_1 a_2 \ldots a_{j-1}) = \nabla_{j-1}.$$

Set $\gamma = \alpha\tau(a_i a_{i+1} \ldots a_{n-1} a_1 a_2 \ldots a_{j-1})\beta$. Then Dom γ = Dom α and Im γ =Im β.

By Theorem 2.1, $\alpha\mathcal{R}\gamma$ and $\gamma\mathcal{L}\beta$. Therefore, $\alpha\mathcal{D}\beta$.

Case 2 : $r = 1$.

Let $\alpha = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ and $\beta = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ for some $s_1 \in \triangle_i, s_2 \in \triangle_j, t_1 \in \triangle_k, t_2 \in \triangle_l$.

By Lemma 3.7, there exist $\eta_{s_1}, \eta_{t_2} \in M(C^*)$ such that $1\eta_{s_1} = s_1$ and $1\eta_{t_2} = t_2$.

Again ,by Lemma 3.7 , let $\iota = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Set $\gamma = \eta_{s_1}^{-1}\iota\eta_{t_2}$. Then $\gamma \in M(C^*)$. Since rank $\iota = 1$, so is γ.

It is clear that $\gamma = \begin{pmatrix} s_1 \\ t_2 \end{pmatrix}$.

By Theorem 2.1, $\alpha\mathcal{R}\gamma$ and $\gamma\mathcal{L}\beta$. Therefore, $\alpha\mathcal{D}\beta$. □

Theorem 3.9. *The syntactic monoid of C^*, $M(C^*)$, has exactly two nonzero $\mathcal{D}$ - classes.*

Proof. The theorem is obtained directly from Proposition 3.6 and Proposition 3.8. $\square$