Chapter 2

Gaussian Perturbations

In this chapter, we describe briefly the theory of cosmological perturbations in order to illustrate that the CMB anisotropies imprint the cosmological perturbations generated during inflation, an epoch in the early universe. The main points of this chapter are that the standard inflation generates Gaussian perturbations, then properties of Gaussian statistics can be preserved through the linear evolutions until the last-scattering time, i.e. the CMB anisotropy keep the primordial Gaussian statistical properties.

2.1 Inflation

According to the inflationary model, the universe has an epoch in the early time, before the nucleosynthesis epoch, when its expansion was accelerated and very fast. During inflation, the expansion was exponential with nearly constant Hubble expansion rate, $H \equiv \mathrm{d}\ln a/\mathrm{d}t$, with

$$a(t) = a(t_0)\exp\left(\int_{t_0}^{t} H(t')\mathrm{d}t'\right) \approx a(t_0)\exp\left(H_I(t-t_0)\right), \tag{2.1}$$

where t_0 is a given beginning time and H_I is the approximately constant Hubble expansion rate during inflation. The exponential expansion in such epoch has solved the flatness, the horizon and the monopole problems of the standard Big Bang cosmology. In other words, the inflation is the process for creation of the initial conditions that the Big Bang cosmology has had to assume to obtain the present universe which is consistent with the observations. Note that the inflation has not disturbed the successes of the Big Bang model that have the observational supports such as the discovery of the expansion of the universe and the cosmic microwave background radiation and light element abundances [9]. It is only an add-on.

A period of inflation can occur only when the pressure P is negative such that

$$P < -\frac{\rho}{3}, \tag{2.2}$$

ρ is the energy density. Neither matter with $P \approx 0$ nor radiation with $P = \rho/3$ can create it (see appendix A). One reasonable possibility is that such a period happened when the universe was dominated by the vacuum energy density of the inflaton field, the scalar field associated with driving inflation. The energy density and the pressure of a homogeneous scalar field ϕ are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.3}$$

and

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.4}$$

respectively, where $V(\phi)$ is the potential of ϕ and dots denote differentiation with respect to cosmic time t. For the inflaton field, we require that $\dot{\phi}^2 < V(\phi)$ sufficiently to drive inflation that gives the perturbations in the large scale structure and CMB anisotropies and the observable universe consistent with the observations. In an extreme case called a *de-Sitter stage*, $P_\phi \simeq -\rho_\phi$ resulted from that the kinetic term is negligible to the potential one, $\dot{\phi}^2 \ll V(\phi)$, we can see this by using Eq. (2.3) and Eq. (2.4). Consequently, during this stage both the energy density ρ and the Hubble expansion rate H are nearly constant in time, according to the continuity equation and the Friedmann equation. The results from recent observations of CMB anisotropy tell us that the working inflationary model seems close to this extreme case.

The inflation results from the kinematics of inflaton on its potential, namely slowly rolling down its potential that is nearly flat. Consider the equation of motion of a scalar field in the Friedmann-Robertson-Walker universe (see appendix A):

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{\partial V}{\partial \phi} = 0, \tag{2.5}$$

where ∇^2 denotes the Laplacian with respect to the comoving coordinate $\mathbf{x}$. Since any pre-inflation inhomogeneity, if it exists, moved away quite immediately from being observable not long after the inflation start, the inflaton field is really homogeneous at least in the region corresponding to the present observable universe. We can therefore neglect the Laplacian term in the above equation. In fact, there are the inhomogeneities due to quantum fluctuations but they are very very small relative to the homogeneous part. Thus, the equation of motion of the field becomes

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{2.6}$$

To achieve a de-Sitter stage, when $\dot{\phi}^2 \ll V(\phi)$, Eq. (2.6) implies that the potential $V(\phi)$ have to be sufficiently flat so that the friction term $3H\dot{\phi}$ dominates over the driving term for the motion of ϕ, $\partial V(\phi)/\partial\phi$. As a result, the inflaton field slowly rolls down the potential with a nearly constant velocity, i.e. $\ddot{\phi} \simeq 0$ and then $\ddot{\phi} \ll 3H\dot{\phi}$. These conditions for achieving the de-Sitter stage, $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$, are called the slow-roll approximation. Indeed, the inflation can occur not only when the slow-roll approximation holds; these conditions is sufficient but not necessary for inflation. We demonstrate this extreme case to convince how the slowly rolling down the nearly flat potential of the inflaton cause the inflation.

In fact, the inflationary stage arises when the rolling down the potential is slow enough. It is important to point out that the slowness of the rolling down is related to the flatness of the potential. Cosmologists have defined the slow-roll parameters for specifying how slow the inflaton rolls down its potential, or how flat the potential is. For instance,

$$\epsilon(\phi) \equiv \frac{M_{Pl}^2}{2}\left[\frac{1}{V}\left(\frac{\partial V}{\partial \phi}\right)\right]^2, \tag{2.7}$$

$$\eta(\phi) \equiv M_{Pl}^2\left[\frac{1}{V}\frac{\partial^2 V}{\partial \phi^2}\right], \tag{2.8}$$

where $M_{Pl} \equiv (8\pi G)^{-1/2}$ is the *reduced Planck mass* [9]; G is the well-known Newtonian's Gravitational constant, are the slow-roll parameters where $\epsilon \ll 1$ and $|\eta| \ll 1$ is equivalent to the slow-roll approximation. Notice that ϵ is positive by definition. Since the acceleration

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2, \tag{2.9}$$

the parameter ϵ can be written as $\epsilon = -\dot{H}/H^2$. From this equation, we can see that inflation can be attained only if $\epsilon < 1$.

Hence, inflation ends when $\epsilon < 1$ is violated. After the end of inflation, the scalar field rolls fast toward the minimum of its potential and then oscillates around it. Then the energy density of the universe being dominated in the form of coherent oscillations of inflaton decays into particles and radiations due to the interactions of inflaton field with other lighter particles; the universe is eventually thermalized. This stage is called the *reheating*. Before the reheating occur, any other contribution to the energy density and entropy of the universe has been redshifted away by inflationary expansion. Hence, we require to have the reheating stage after inflation in order to achieve the universe's history well explained by the standard Big Bang scenario.

2.2 The Horizon

In the theory of cosmological perturbation, a discussion about the horizon is necessary. Any perturbation can grow to form structure via gravitational instability if its wavelength is well smaller than the horizon; on the contrary, it cannot if its wavelength is larger than the horizon. Because the universe has the expanding background and the receding velocity, observed by an observer, of a distant region is larger than that of a less distant region following the Hubble's law, $v = Hr$, where v is the receding velocity of the region away from the observer with the proper distance r, the region at $r > H^{-1}$ (we have used the convention: $c = 1$) recedes with the velocity more than that of light so that it is not able to be observed; this can also be thought of as the consequence of infinite redshift. Hence, H^{-1} can be used to be the horizon, so-called the Hubble horizon, that is the characteristic scale in which two points can have a causal communication. In fact, there are several definitions of the horizon, but it is most convenient, in the consideration about cosmological perturbations, to use the Hubble horizon since it has already been in the equations of the dynamics of the universe.

From the Friedmann equation neglecting the curvature term[1],

$$H^2 = \frac{8\pi G}{3}\rho, \qquad (2.10)$$

where G is the Newtonian gravitational constant and ρ is the energy density, we can see that $H^{-1} \propto \rho^{-1/2}$. In order of time, the laws of evolution of H^{-1} are different in different epochs:

- **Inflationary epoch:** Because ρ is nearly constant in time, H^{-1} is approximately constant with the very small value of order 10^{-30} m, which is the scale that the quantum effect dominates. Note that the WMAP data provides that 5.8×10^{-31} m $< H^{-1} < 6.6 \times 10^{-29}$ m during inflation [15].
- **Radiation-dominated epoch:** H^{-1} expands with the rate $H^{-1} \propto a^2$ because $\rho \propto a^{-4}$ in this epoch.
- **Matter-dominated epoch:** $H^{-1} \propto a^{3/2}$ when $\rho \propto a^{-3}$ in this epoch.

The proper wavelength λ of the perturbations expands with $\lambda \propto a$ for all epochs. We can identify λ with the comoving wavenumber k, with $\lambda = 2\pi a/k$. We can say that a mode of perturbations is within the horizon, called subhorizon, if $k/aH > 1$ while the mode is outside the horizon, called superhorizon, if $k/aH < 1$. During inflation, the modes of vacuum fluctuations of the scalar field with subhorizon scale are stretched rapidly, with the exponential expansion, becoming superhorizon perturbations. This process, called the first horizon-crossing, turns the quantum fluctuations into the frozen classical perturbations. After the inflationary epoch, during the radiation-dominated and the matter-dominated epochs, H^{-1} expands faster than the stretch of the perturbations' wavelength. Consequently, the modes which have the first horizon-crossing later, with less stretching of their wavelengths, would have the second horizon-crossing before the others which have the first horizon-crossing prior, with more stretching of their wavelengths. Note that some modes have not yet reentered the horizon until now; they remain nearly frozen, in amplitude not in wavelength, since the first horizon-crossing.

[1]The inflation makes the universe extremely flat.

This issue is important in the study of CMB anisotropy. Eq. 2.1 shows the first and second horizon crossing of the perturbations A, B and C where the co-moving wavenumber $k_{\mathrm{A}} < k_{\mathrm{B}} < k_{\mathrm{C}}$; the scale of A, B and C correspond to the angular scale more than about 2^o, about 2^o and less than about 2^o in the CMB sky, respectively. We can see from the figure that during inflation A left the Hubble horizon before B and C, respectively. As mentioned in chapter 1, the angular scale corresponding to the horizon size at decoupling is about 2^o. The perturbation B thus crossed the Hubble horizon at decoupling, while A had not reentered the horizon yet and C had already reentered the horizon at that time. Consequently, this means that the perturbation C had obtained the evolution resulting from the dynamics in the photon-baryon fluid but A had not yet. As a result, the CMB anisotropies corresponding to the perturbation A directly imprints the perturbation in the early universe. This discussion shows the relation between the horizon and the evolution of the perturbations that appears as the CMB anisotropies.

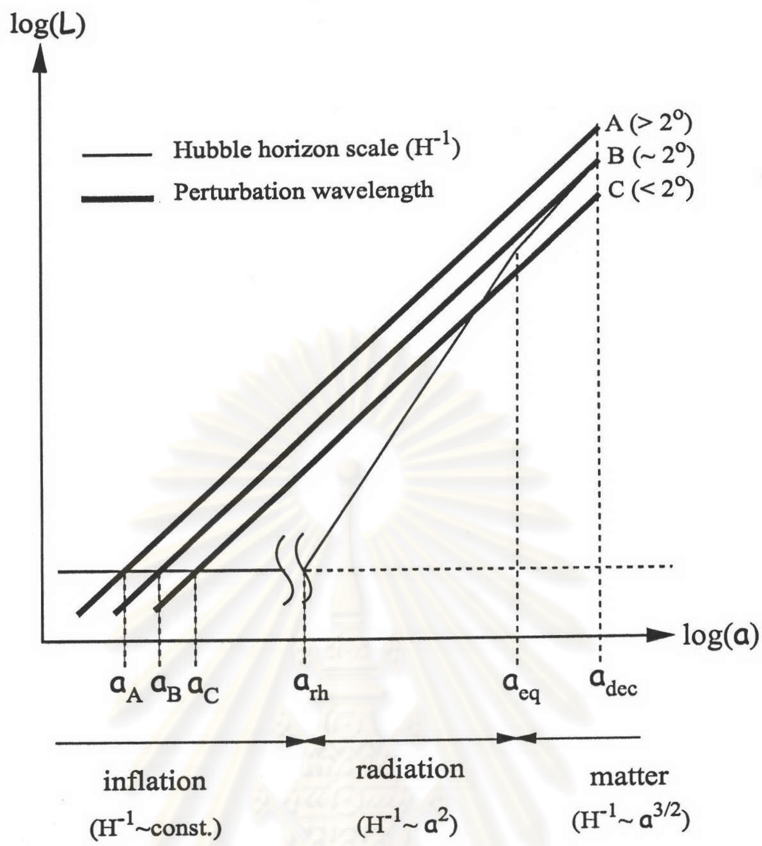

Figure 2.1: Horizon crossing of the perturbations A, B and C with different wavelengths corresponding to the angular scale on the CMB sky $> 2^o$, $\sim 2^o$ and $< 2^o$, respectively. The thin line represents the evolution of the Hubble horizon $H^{-1}(a)$, the thick lines draw the evolution of the perturbation wavelengths, $2\pi a/k$, where a is the scale factor. H^{-1} is nearly constant during inflation, grows as a^2 in the radiation era, and grows as $a^{3/2}$ in the matter era. We use the scale factor as a time scale: $a_{\rm A}$, $a_{\rm B}$ and $a_{\rm C}$ for the first horizon-crossing time of A, B and C, respectively, $a_{\rm rh}$ for the reheating epoch, $a_{\rm eq}$ for the matter-radiation equality, and $a_{\rm dec}$ for the decoupling time.

2.3 Gaussian Perturbations

In the present, the inflation has become the dominant paradigm to understand the creation of the perturbations in the early universe which are the initial conditions for structure formation and CMB anisotropy, even though the first model by Guth in 1981 [16] was proposed in order to solve such problems as mentioned above. Since the inflationary expansion had redshifted any pre-inflation inhomogeneities away completely, they were not the initial conditions for CMB anisotropies. We can say that the primeval perturbations were created from nothing, namely the vacuum, during inflation. This is a wonderful result from inflation. The inflation turns the vacuum fluctuations of any scalar field[2], including inflaton, into the frozen fluctuations, which appears like a classical field. This is the generation of the perturbations, which is the starting point in the description of cosmological perturbation.

Taking into account the vacuum fluctuations, we can split the inflaton field $\phi(t, \mathbf{x})$ into two parts as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \varphi(t, \mathbf{x}), \tag{2.11}$$

where $\phi_0(t)$ is the homogeneous part, with infinite wavelength, whose potential energy drive inflation as mentioned in last section, $\varphi(t, \mathbf{x})$ represents the quantum fluctuation part of the field, and $\mathbf{x}$ is the comoving coordinate. By perturbing Eq. (2.5) in first order, we obtain the equation describing the dynamics of fluctuations written as

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2\varphi}{a^2} + m_\phi^2\varphi = 0, \tag{2.12}$$

where ∇^2 denotes the Laplacian with respect to the comoving coordinate $\mathbf{x}$, and $m_\phi^2 \equiv \partial^2 V/\partial\phi^2$ where m_ϕ can be thought of as the inflaton's mass. Since this equation is linear, we can describe the dynamics of the fluctuations in each Fourier mode φ_k, with

$$\varphi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\, \varphi_\mathbf{k}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.13}$$

separately using the equation

$$\ddot{\varphi}_k + 3H\dot{\varphi}_k + \left[\frac{k^2}{a^2} + m_\phi^2\right]\varphi_k = 0, \tag{2.14}$$

[2]The vacuum fluctuation is the quantum fluctuation around its vacuum state. Its existence is predicted by the quantum field theory.

where we assume isotropy so that we can express the wave number $\mathbf{k}$ just as a scalar k.

Since the inflaton is a quantum field, we have to quantize the field, which is the so-called second quantization, in order to describe the behavior of the inflaton field. The quantization is described in appendix C.

The standard inflation predicts that the CMB anisotropy map is Gaussian. This statement is the consequence of three causes:

1. The vacuum fluctuations $\varphi(t, \mathbf{x})$ have random values following Gaussian distribution law.

   In quantum field theory, the theory in which quantum mechanics and special relativity are successfully reconciled, the uncertainty principle provides the commutation relation between scalar field $\varphi(\mathbf{x}, t)$ and the momentum density $\pi(\mathbf{x}, t)$ as

   $$[\varphi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\,\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad . \tag{2.15}$$

   This implies that at a particular $\mathbf{x}$ the scalar field $\varphi(\mathbf{x})$ does not have a well-defined value, that is its value will be random following Gaussian probability. This probabilistic nature is the consequence of quantum mechanics. Indeed, the probability of getting value of a free massive scalar field to be Gaussian can be manifested in the path-integral approach of quantum field theory. We can, however, explain in another way that it is the result from that every Fourier mode $\mathbf{k}$ of the field have random phase, hence they are independent of each other. From the central limit theorem, we obtain that $\varphi(\mathbf{x})$ has Gaussian statistical properties [9].

2. The process during inflation that turns the vacuum fluctuations into the frozen fluctuations on superhorizon scale is linear.

   Since the vacuum fluctuations are very small, the linear equation (2.12) well describe the dynamics of the vacuum fluctuations of inflaton field. Hence, the independence between modes of fluctuations is preserved in the process during inflation. The phase of different modes of frozen fluctuations have no correlation. From the central limit theorem, the frozen fluctuation field remains Gaussian.

3. The evolution of the perturbations during the time between when the fluctuations are frozen and the last-scattering time can be well described by the first order general-relativistic cosmological perturbation theory.

   The perturbations are of order 10^{-5} which is small. Consequently, we can use the first order equation, obtaining from the theory of cosmological perturbation, in order to describe the evolution of perturbations [15]. The linear evolution implies that each mode of perturbations evolves independently. This keep the characteristic of mode independence. With the same reason as above, the CMB anisotropy map is a Gaussian random field.