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D'ALEMBERT'S FUCNTIONAL EQUATIONS ON COMPACT HOMOGENEOUS SPACES

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In this work, we introduce and solve a new functional equation on compact homogeneous spaces based on the d'Alembert's functional equation. Our work extends the result of D.Yang, in which the d'Alembert functional equation on compact groups was solved with tools from harmonic analysis.

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CHAPTER I

INTRODUCTION

Let G be a Hausdoff topological group. A continuous function $f: G \to \mathbb{C}$ is said to satisfy the *d'Alembert's functional equation* if

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \ \forall x, y \in G. \qquad (1.1)$$

On $G = (\mathbb{R}, +)$, this functional equation is obviously satisfied by the cosine since we have the following identity:

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y .$$

Hence, (1.1) has an alternative name *cosine equation*. In 1968, Kannapan proved that the nonzero solutions of this equation on Abelian group G are of the form

$$f(x) = \frac{g(x) + g(x)^{-1}}{2}, \ \forall x \in G, \qquad (1.2)$$

where $g: G \to \mathbb{C}^*$ is a continuous homomorphism. Consequently, the current interest is in determining the solutions on non-abelian groups. In 2006, Yang used the structure theory of semisimple Lie groups to find nonzero solutions on compact connected groups, which is either in form of (1.2) or is factored through the map $\frac{1}{2}\mathrm{tr}: SU_2(\mathbb{C}) \to \mathbb{C}$. Then, Davison in his paper [1] simplified the result by using elementary group theory and topology. Most recently in [4], Yang provided a new insight in solving this equation as she employed the techniques from harmonic analysis on compact groups.

In this work, we introduce and solve a new functional equation on compact homogeneous spaces based on the d'Alembert's functional equation. Because any compact homogeneous space derives its harmonic analysis from the underlying compact group, we can turn this problem into that of Yang in [4], which allows us to reapply her arguments. Accordingly, our result is made as a simple generalization of Yang's to compact homogeneous spaces.

CHAPTER II

PRELIMINARIES

In this chapter, we give basic definitions for compact groups, Haar measure and homogeneous spaces. In addition, we give references to important fundamental results such as the existence of Haar measure and its uniqueness (up to scalar multiplication) on locally compact groups, and an integration formula for G-invariant radon measure on homogeneous spaces.

2.1 Haar Measures on Compact Groups

Definition 2.1 A *topological group* is a group G equipped with a topology with respect to which the group operations are continuous; that is the maps

$$(x, y) \mapsto xy, \quad G \times G \to G,$$
$$x \mapsto x^{-1}, \quad G \to G,$$

are continuous. This amount to saying that

$$(x, y) \mapsto xy^{-1}, \quad G \times G \to G,$$

is continuous.

Definition 2.2 A *(locally) compact group* is a group G equipped with a (locally) compact Hausdorff topology.

Definition 2.3 Suppose f is a function on the topological group G and $y \in G$. We define the *left* and *right translates* of f through y by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy). \tag{2.1}$$

The reason for using y^{-1} in L_y and y in R_y is to make the maps $y \mapsto L_y$ and $y \mapsto R_y$ group homomorphism:

$$L_{yz} = L_y L_z, \qquad R_{yz} = R_y R_z. \tag{2.2}$$

Definition 2.4 Let G be a locally compact group. A *left* (resp. *right*) *Haar measure* on G is a nonzero Radon measure μ on G that satisfies $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for every Borel set $E \subset G$ and every $x \in G$.

Proposition 2.5 Let μ be a Radon measure on the locally compact group G, then

1. μ is left Haar measure if and only if $\int_G L_y f d\mu = \int_G f d\mu$ for all $f \in C_c(G)$.
2. μ is right Haar measure if and only if $\int_G R_y f d\mu = \int_G f d\mu$ for all $f \in C_c(G)$

Proof. See [3, Proposition 2.9] for the left Haar measure case. The proof for the right Haar measure case is similar. ❑

Theorem 2.6 Every locally compact group G possesses a left Haar measure μ. This measure is unique up to a positive factor.

Proof. See [3, Theorem 2.10, 2.20]. ❑

When μ is a left Haar measure and let $\tilde{\mu}(E) = \mu(E^{-1})$, it is not hard to see that $\tilde{\mu}$ is a right Haar measure. So the existence of left Haar measure on locally compact group G implies that of right Haar measure.

On the other hand, if right Haar measures λ, μ are given then $\tilde{\mu}(E) = \mu(E^{-1})$ and $\tilde{\lambda}(E) = \lambda(E^{-1})$ define left Haar measure $\tilde{\lambda}, \tilde{\mu}$. Uniqueness in the above theorem means there exists a constant $c \in \mathbb{R}^+$ such that $\tilde{\lambda} = c\tilde{\mu}$. Hence, $\lambda(E) = \tilde{\lambda}(E^{-1}) = c\tilde{\mu}(E^{-1}) = c\mu(E)$. So the same statement as in Theorem 2.6 also applies to the right Haar measure.

Let G be a locally compact group and μ a left Haar measure. For each fixed $x \in G$, let $\mu_x(Ex) = \mu(E)$ for every Borel set $E \subset G$; then μ_x is also left Haar measure. Indeed, $\mu_x(yE) = \mu((yE)x) = \mu(y(Ex)) = \mu(Ex) = \mu_x(E)$. By the uniqueness theorem (Theorem 2.6), there is a number $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$. To verify that Δ is independent of the original choice of μ, let μ' be any left Haar measure. Theorem 2.6 say that there exists $c \in \mathbb{R}^+$ for which $\mu' = c\mu$. But then $\mu'_x(E) = \mu'(Ex) = c\mu(Ex) = c(\Delta(x)\mu(E)) = \Delta(x)\mu'(E)$.

Definition 2.7 The function $\Delta : G \to \mathbb{R}^{+}$ is called the *modular function* of *G*. If $\Delta \equiv 1$ then G is called *unimodular*.

In the next proposition, $\mathbb{R}_{\times}$ denote the multiplication group of positive real numbers.

Proposition 2.8 Δ is a continuous homomorphism from G to $\mathbb{R}_{\times}$. Moreover, for any $f \in L^{1}(\mu)$,

$$\int_{G} R_{y} f d\mu = \Delta(y^{-1}) \int_{G} f d\mu .$$

Proof. See [3, Proposition 2.24]. ❑

Proposition 2.9 If K is any compact subgroup of G then $\Delta \mid K \equiv 1$.

Proof. $\Delta(K)$ is a compact subgroup of $\mathbb{R}_{\times}$, hence equal to $\{1\}$. ❑

Corollary 2.10 If G is compact, then G is unimodular.

If the locally compact group G is unimodular, then it is clear from Proposition 2.5 and 2.8 that any left Haar measure μ is also a right Haar measure. On the other hand, if μ is a right Haar measure then $\tilde{\mu} = \mu((\cdot)^{-1})$ is a left Haar measure which, in turn, is a right Haar measure by the same argument. This helps to show that $\mu = \tilde{\mu}((\cdot)^{-1})$ is a left Haar measure. To conclude, any Haar measures on unimodular groups are both left and right.

By Corollary 2.10, any Haar measure on a compact group is both left and right.

2.2 Case of Group which is Open in $\mathbb{R}^{m}$

Assume that *G* is realized as an open set in $\mathbb{R}^{m}$. Recall *the change of variable formula* for a differentiable map $\varphi : G \to \mathbb{R}^{m}$:

$$\int_{\varphi(U)} f(y) dy = \int_{U} f(\varphi(x)) \,|\det(D\varphi)(x)|\, dx \tag{2.3}$$

for every continuous function *f* on *G* with support contained in $\varphi(U)$.

Assume further that the left and right translations:

$$T^{l}(g) : x \mapsto gx \qquad\qquad T^{r}(g) : x \mapsto xg \tag{2.4}$$

be restrictions of linear or affine linear transformations. We will show how to construct Haar measures, if exists, of the form:

$$\mu(dx) = h(x)dx, \tag{2.5}$$

where dx is the Lesbegue measure on $\mathbb{R}^m$.

Let's start by assuming that μ is a left Haar measure. For each fixed g define $\varphi := T^l$ (g), then (2.3) becomes

$$\int_{gE} h(y)dy = \int_E h(gx) \,|\det(DT^l(g))|\, dx, \tag{2.6}$$

for every measurable set $E \subseteq G$. In this case, *the Jacobian determinant* is independent of the point x because $T^l(g)$ is either affine or linear.

By definition of μ in (2.5) the result of L.H.S. of (2.6) is $\mu(gE)$. Since μ is a left Haar measure, $\mu(gE) = \mu(E) = \int_E h(x)dx$. Equating this to the R.H.S., we get

$$\int_E h(x)dx = \int_E h(gx)\,|\det(DT^l(g))|\,dx.$$

A sufficient condition for the last equality to hold is the equality (2.7)

$$h(x) = h(gx)\,|\det(DT^l(g))| \tag{2.7}$$

By plugging $x = e$ into both sides, we get $h(g) = h(e)(|\det(DT^l(g))(e)|)^{-1}$, $\forall g \in G$. Conversely, if we simply define

$$h(x) = |\det(DT^l(x))|^{-1} \ \forall x \in G. \tag{2.8}$$

Then,

$$\begin{aligned} h(gx)\,|\det(DT^l(g))| \ &= \ (|\det(DT^l(g))(x)|)(|\det(DT^l(gx))|)^{-1}, \\ &= \ (|\det(DT^l(g))|)(|\det(D(T^l(g)\circ T^l(x)))|)^{-1}, \\ &= \ (|\det(DT^l(g))|)(|\det(D(T^l(g)))|\cdot|\det(D(T^l(x)))|)^{-1}, \\ &= \ (|\det(D(T^l(x)))|)^{-1} = h(x). \end{aligned}$$

Since this definition of h satisfies (2.7), μ defined as in (2.5) is a left Haar measure. Similarly, we can construct a right Haar measure μ by using

$$h(x) = |\det(DT^r(x))|^{-1} \ \forall x \in G. \tag{2.9}$$

Example 2.11 Let G be the group of affine linear transformations of the real line. It can be identified with the subgroup $GL(2,\mathbb{R})$ consisting of the matrices

$$g=\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},\ a,b\in\mathbb{R},$$

and is homeomorphic to the open set in $\mathbb{R}^2$:

$$\{(a,b)\in\mathbb{R}^2 \mid a\neq 0\}=\mathbb{R}^*\times\mathbb{R}\,.$$

The operation of G is given by $(a_1,b_1)\cdot(a_2,b_2)=(a_1a_2,a_1b_2+b_1)$ because

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a_1a_2 & a_1b_2+b_1 \\ 0 & 1 \end{pmatrix}.$$

By restricting transformations in (2.4) to G, if $g=(a,b)$ and $x=(u,v)$,

$$T^l(g)x=(au,av+b) \quad\text{, and}\quad T^r(g)x=(au,bu+v)\,.$$

So,

$$DT^l(g)=\begin{bmatrix} \dfrac{\partial(au)}{\partial u} & \dfrac{\partial(au)}{\partial v} \\ \dfrac{\partial(av+b)}{\partial u} & \dfrac{\partial(av+b)}{\partial v} \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \text{ and } |\det(DT^l(g))(x)|=a^2\text{; while}$$

$$DT^r(g)=\begin{bmatrix} \dfrac{\partial(au)}{\partial u} & \dfrac{\partial(au)}{\partial v} \\ \dfrac{\partial(bu+v)}{\partial u} & \dfrac{\partial(bu+v)}{\partial v} \end{bmatrix}=\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}, \text{ and } |\det(DT^l(g))(x)|=|a|\,.$$

Using (2.8) and (2.9), the measure μ_l,μ_r given by

$$\int_G f(x)\mu_l(dx)=\int_{\mathbb{R}^*+\mathbb{R}} f(u,v)\frac{dudv}{u^2} \text{ and} \int_G f(x)\mu_l(dx)=\int_{\mathbb{R}^*+\mathbb{R}} f(u,v)\frac{dudv}{|u|}$$

are a left Haar measure and a right Haar measure, respectively.

2.3 Homogeneous Spaces

Suppose H is a subgroup of the topological group G. Let G/H be the space of left cosets of H, and let $q: G \to G/H$ be the canonical quotient map. By imposing the quotient topology on G/H, we see that q is a surjective continuous function.

Let G be a locally compact group and H its subgroup. Then, the quotient topology for G/H is also locally compact. Moreover, if H is closed, then this topology is Hausdorff. See [3, Proposition 2.2] for the proofs.

Especially, when G is a compact group, then the quotient topology for G/H is also compact Hausdorff. Indeed, G/H is an image of compact group G under the map q and hence compact.

Definition 2.12 Let G be a locally compact group and S a locally compact Hausdorff space. A left action of G on S is a continuous map $(x,s) \mapsto xs$ from $G \times S$ to S such that

1. $s \mapsto xs$ is a homeomorphism of S for each $x \in G$.
2. $x(ys) = (xy)s$ for all $x, y \in G$ and $s \in S$.

A space S equipped with an action of G is called a *G-space*. A G-space is called *transitive* if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.

Example 2.13 (Transitive G-spaces) The quotient space G/H where H is a closed subgroup, and G is a locally compact group acting on G/H by left multiplication, i.e., $(x, yH) \mapsto (xy)H$. In fact, quotient spaces are almost the only examples of homogeneous spaces.

Suppose S is a transitive G-space. Pick $s_0 \in S$ and define $\varphi: G \to S$ by $\varphi(x) = xs_0$. Then, $H = \{x \in G : xs_0 = s_0\}$ is called the *isotropic group* of s_0. Note that $H = \varphi^{-1}(\{s_0\})$ is a closed subgroup of G, and by transitivity φ is a continuous surjection of G onto S that is constant on the left cosets of H. Recall that $q: G \to G/H$ is the canonical quotient map and observe that φ is constant on $q^{-1}(\{u\})$ for all $u \in G/H$. Hence, φ induces a continuous function $\Phi: G/H \to S$ such that $\Phi \circ q = \varphi$. This map is injective since $\Phi(xH) = \Phi(yH)$ implies $xs_0 = ys_0$, but then $(x^{-1}y)s_0 = s_0$ and $xH = yH$. It is also surjective since φ is surjective.

Now we have a bijection Φ between S and G/H. If we can always show that Φ^{-1} is continuous then Φ is a homeomorphism, and then every transitive *G*-space would be topologically equivalent to some quotient space. Unfortunately, this cannot be true, as illustrated in example 2.14. By the way, Proposition 2.15 give a sufficient condition for determining whether Φ is a homeomorphism.

Example 2.14 Let $G=(\mathbb{R},+)$ be equipped with the discrete topology and $S=\mathbb{R}$ with the usual topology. Let *G* act on *S* by translations, $(x,s)\mapsto x+s$. Pick any fixed $s_0\in S$. It's clear that $H=\{x\in G: x+s_0=s_0\}=\{0\}$. G/H has discrete topology, so there is no homeomorphism Φ between G/H and S.

Proposition 2.15 With notation as above, If *G* is σ-compact then Φ is a homoemorphism.

Proof. See [3, Proposition 2.44]. ❑

Definition 2.16 With notation as above, a transitive *G*-space is called a *homogeneous space* if it is isomorphic to a quotient space G/H - that is, the map Φ is a homeomorphism. Generally, we shall identify a homogeneous S with G/H as they are isomorphic via Φ.

This identification depends on the choice of a base point $s_0\in S$. But if we choose a different base point $s'_0=x_0s_0$, we only need to replace H with $x_0Hx_0^{-1}$; and the map $x\mapsto x_0xx_0^{-1}$ induces a homeomorphism between G/H and G/H'.

When *G* is compact group, it follows from Proposition 2.15 that every transitive *G*-space is isomorphic to some a quotient space G/H, thereby being a homogeneous space.

2.4 *G*-invariant Radon Measures

Definition 2.17 Let *G* be a locally compact group and *H* its closed subgroup. A *G-invariant Radon measure* on homogeneous space G/H is a nonzero Radon measure μ that satisfies $\mu(xE)=\mu(E)$ for any Borel set $E\subset G/H$ and every $x\in G$. Note that $xE=\{xs: s\in E\}$.

In what follows, *G* is a locally compact group with a left Haar measure d_lx,; and *H* is a closed subgroup of *G* with left Haar measure d_lh. Both Δ_G and Δ_H

are the modular functions of *G* and *H*, respectively. Recall that $q: G \to G/H$ denotes the canonical quotient map $q(x) = xH$. We can define a map $P: C_c(G) \to C_c(G/H)$ by

$$Pf(xH) = \int_H f(xh)d_l h \,. \tag{2.10}$$

If $x' = xh'$ with $h' \in H$, then, $\int_H f(xh'h)d_l h = \int_H f(xh)d_l h$ by left-invariant of $d_l h$. Hence, *Pf* is well-defined and it's not hard to see that $\text{supp}(Pf) \subset q(\text{supp} f)$.

The following theorem give a necessary and sufficient condition for the existence of *G*-invariant Radon measure on homogeneous space G/H.

Theorem 2.18 The *G*-invariant Radon measure μ on G/H exists if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor, and if this factor is suitably chosen we have

$$\int_G f(x)d_l x = \int_{G/H} Pf d\mu = \int_{G/H}\int_H f(xh)d_l h\, d\mu(xH)\,, \tag{2.11}$$

for every $f \in C_c(G)$.

Proof. See [3, Theorem 2.49]. ❑

When the group *G* is compact, the Haar measure is said to be *normalized* if

$$\int_G d\mu = 1\,.$$

In the above discussion, suppose *G* is a compact group with closed (thus compact) subgroup *H*, and both $d_l x$, $d_l h$ are normalized. Let $x \in G$ and $F \in C_c(G/H)$. Observe that, $(F \circ q)(xh) = F(xH)$ for all $h \in H$. Hence,

$$P(F \circ q)(xH) = \int_H (F \circ q)(xh)d_l h = \int_H F(xH)d_l h = F(xH)\int_H d_l h = F(xH)\,.$$

By (2.11),

$$\int_G F \circ q(x)d_l x = \int_{G/H} P(F \circ q)d\mu = \int_{G/H} F(xH)\, d\mu(xH)\,. \tag{2.12}$$

CHAPTER III

HARMONIC ANALYSIS ON COMPACT GROUPS

As a tool for solving d'Alembert functional equation on compact groups in the next chapter, we introduces in this chapter some basic concepts in harmonic analysis on compact groups. The main ingredient is the Fourier expansion of integrable functions on compact groups.

The notion of unitary representation is introduced in section 3.1. In section 3.2, the Hilbert space direct sum decomposition of the L^2-function space of a compact group is given. An explicit formula of the Fourier series of L^2-functions is then derived as a consequence of the decomposition. Section 3.3 presents a sufficient condition under which the Fourier series converges uniformly and absolutely to a continuous function. The materials in this chapter can be found in [2].

3.1 Unitary Representations

Definition 3.1 Let G be a topological group and $\mathcal{V}$ a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ ($\mathcal{V} \neq \{0\}$). Let $\mathcal{L}(\mathcal{V})$ denote the algebra of bounded operators on $\mathcal{V}$. A *representation* of G on $\mathcal{V}$ is a map

$$\begin{aligned} \pi : G &\to \mathcal{L}(\mathcal{V}), \\ g &\mapsto \pi(g), \end{aligned}$$

such that

1. $\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \pi(e) = I$,
2. for every $v \in \mathcal{V}$, the map

$$\begin{aligned} G &\to \mathcal{V} \\ g &\mapsto \pi(g)v, \end{aligned}$$

is continuous.

Definition 3.2 Let $(\pi, \mathcal{V})$ be a representation of G. A subspace $\mathcal{W} \subseteq \mathcal{V}$ is said to be *invariant* if, for every $g \in G$, $\pi(g)\mathcal{W} = \mathcal{W}$. Putting $\pi_0(g) = \pi(g)|_{\mathcal{W}}$, the restriction of $\pi(g)$ to $\mathcal{W}$, we get a representation of G on $\mathcal{W}$. One says that π_0 is a *subrepresentation* of π.

The representation π is said to be *irreducible* if the only invariant closed subspace are $\{0\}$ and $\mathcal{V}$. Observe that one dimensional representation $\mathcal{V}$ $(\dim \mathcal{V} = 1)$ is always irreducible.

Definition 3.3 Let $(\pi, \mathcal{V})$ and $(\pi', \mathcal{V}')$ be two representations of G. If a continuous linear map $A : \mathcal{V} \to \mathcal{V}'$ satisfies the relation

$$A\pi(g) = \pi'(g)A,$$

for every $g \in G$, one says that A is an *intertwining operator* or that A *intertwines* the representations π and π'.

Definition 3.4 The representations $(\pi, \mathcal{V})$ and $(\pi', \mathcal{V}')$ of G are said to be *equivalent* if there exists an isomorphism $A : \mathcal{V} \to \mathcal{V}'$ which intertwines the representations π and π'. We write $(\pi, \mathcal{V}) \sim (\pi', \mathcal{V}')$ if $(\pi, \mathcal{V})$ and $(\pi', \mathcal{V}')$ are equivalent. The following theorem verifies that $\sim$ is an equivalence relation.

Theorem 3.5 $\sim$ in Definition 3.4 defines an equivalence relation.

Proof. The verifications for reflexivity and symmetry are trivial. Now suppose we have an isomorphism $A : \mathcal{V} \to \mathcal{V}'$ which intertwines π and π', as well as an isomorphism $B : \mathcal{V}' \to \mathcal{V}''$ which intertwines π' and π''. Then for each $g \in G$

$$BA\pi(g) = B\pi'(g)A = \pi''(g)BA.$$

Accordingly, an isomorphism BA intertwines π and π''. It follows that the two representations are equivalent. This asserts transitivity of the relation. ❑

Definition 3.6 Let $\mathcal{H}$ be a Hilbert space. A representation π of G on $\mathcal{H}$ is said to be *unitary* if, for every $g \in G$, $\pi(g)$ is a unitary operator; this can be written

$$\forall g \in G, \forall v \in \mathcal{H}, \|\pi(g)v\| = \|v\|.$$

Definition 3.7 The unitary representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of G are said to be *unitary equivalent* if there exists a unitary map $U : \mathcal{H} \to \mathcal{H}'$ which intertwines the representations π and π'.

To show that unitary equivalence is an equivalence relation on the set of unitary representations of G, one can follow the steps appeared in theorem 3.5 with a few modifications.

Theorem 3.8 Every irreducible unitary representation $(\pi, \mathcal{H})$ of a compact group is finite dimensional ($\dim \mathcal{H} < \infty$).

Proof See [2, theorem 6.3.2]. ❑

Theorem 3.8 says that any irreducible unitary representation on a compact group G, is a finite dimensional representation. Usually, we will denote $d_\pi =$ $\dim \mathcal{H}$.

3.2 Peter-Weyl Theorem

In this secton, a vector space $\mathcal{V}$ always represent a Hilbert space equipped with norm induced by an inner product. Moreover, the integrable function spaces $L^p(G), p \geq 1,$ in this section refer to $L^p(G, \mu)$, where μ is a Haar measure (both left and right).

Definition 3.9 Let $(\pi, \mathcal{V})$ be a representation of G. The function on G defined by

$$\pi_{u,v}(g) = \langle \pi(g)u, v \rangle \quad (u, v \in \mathcal{V})$$

are called *matrix coefficients* of π. Observe that $\pi_{u,v}$ is continuous, by continuity of the representation π.

Let G be a compact group. Fix $(\pi, \mathcal{V})$, a unitary irreducible representation of G. We denote the linear span of matrix elements of π by $\mathcal{M}_\pi$. $\mathcal{M}_\pi$ is a subspace of $C(G)$; hence also of $L^p(G)$ for $p \geq 1$.

Theorem 3.10 $\mathcal{M}_\pi = \mathcal{M}_{\pi'}$ if $(\pi, \mathcal{V}) \sim (\pi', \mathcal{V}')$.

Proof. Suppose that $(\pi, \mathcal{V})$ and $(\pi', \mathcal{V}')$ are unitary representations of the same class. Then there must be a unitary $T : \mathcal{V} \to \mathcal{V}'$ which intertwines π and π'.

Let $u, v \in \mathcal{V}$, then $\pi_{u,v}(g) = \langle \pi(g)u, v \rangle = \langle T\pi(g)u, Tv \rangle' = \langle \pi'(g)Tu, Tv \rangle' = \pi'_{Uu,Uv}(g)$, for all $g \in G$. Hence, $\mathcal{M}_\pi \subseteq \mathcal{M}_{\pi'}$.

It's not hard to see that $T^{-1} : \mathcal{V}' \to \mathcal{V}$ is an intertwining operator for π' and π. Similarly, $\mathcal{M}_{\pi'} \subseteq \mathcal{M}_\pi$. ❑

Let $\{e_1,\ldots,e_{d_\pi}\}$ be an orthonomal basis of $\mathcal{V}$ $(d_\pi = \dim \mathcal{V})$. For $i,j \in \{1,2,\ldots,d_\pi\}$, we define the matrix coefficient π_{ij} by

$$\pi_{ij}(g) = \langle \pi(g)e_j, e_i \rangle \quad \forall g \in G.$$

Then, the matrix coefficients $\{\pi_{ij}\}_{i,j\in\{1,2,\ldots,d_\pi\}}$ span $\mathcal{M}_\pi$. For each $j \in \{1,2,\ldots d_\pi\}$, $\mathcal{M}_\pi^{(j)}$ denotes the subspace of $\mathcal{M}_\pi$ spanned by the entries of the j^{th} row, that is the functions π_{jk}, for $k = 1,\ldots,d_\pi$. We then have the following theorem.

Theorem 3.11 (Peter-Weyl Theorem) Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of the compact group G. Here, we use the notations in the above discussion. Let R denote the right regular representation of G on $L^2(G)$:

$$(R(g)f)(x) = f(xg), \quad f \in L^2(G), \; g,x \in G.$$

Then for each $\pi \in \widehat{G}$ and each $j \in \{1,2,\ldots,d_\pi\}$, $\mathcal{M}_\pi^{(j)}$ is an invariant subspace of $L^2(G)$ under the representation R of G and the restriction of R to $\mathcal{M}_\pi^{(j)}$ is equivalent to π. Moreover, we have the *Hilbert space direct sum decomposition*:

$$\begin{aligned} L^2(G) &= \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \mathcal{M}_\pi \\ &= \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \left(\bigoplus_{[\pi]\in\widehat{G}}^{d_\pi} \mathcal{M}_\pi^{(j)} \right). \\ &\cong \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \left(\bigoplus_{[\pi]\in\widehat{G}}^{d_\pi} \pi \right). \end{aligned}$$

Recall that

$$\widehat{\bigoplus_{[\pi]\in\widehat{G}}} \mathcal{M}_\pi$$

denotes the closure in $L^2(G)$ of

$$\mathcal{M} = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{M}_\pi,$$

which is the space of finite linear combinations of matrix coefficients of finite dimensional representations of G.

Proof. See [2, theorem 6.4.1]. ❑

Definition 3.12 Let G be a compact group. For each $[\pi] \in \widehat{G}$, we choose a representative $(\pi, \mathcal{V}_\pi)$. When $f \in L^1(G)$, we define its *Fourier coefficient* $\pi(f)$ to be the operator on $\mathcal{V}_\pi$ such that

$$\langle \pi(f)v, w \rangle = \int_G f(g) \langle \pi(g)v, w \rangle \, dg \qquad \text{for all } v, w \in \mathcal{V}_\pi .$$

The map $\pi(f)$ is well-defined for every $f \in L^1(G)$ because for $f \in L^1(G)$ if we set $B_f(v,w) = \int_G f(g) \langle \pi(g)v, w \rangle \, dg$, then $B_f(v,w)$ is linear in v and conjugate linear in w. Moreover,

$$\begin{aligned} \left| B_f(v,w) \right| &\le \int_G |f(g)| \|\pi(g)v\| \|w\| \, dg, \\ &\le \|f\|_1 \|v\| \|w\|, \end{aligned}$$

which implies that B_f is a bounded sesquilinear functional and hence the consequence of the Riesz representation theorem implies that there is a unique bounded linear operator $\pi(f)$ on $\mathcal{V}_\pi$ such that $B_f(v,w) = \langle \pi(f)v, w \rangle$ for all $v, w \in \mathcal{V}_\pi$. Recall that Haar measure of G is finite. By Hölder's inequality, $L^2(G) \subseteq L^1(G)$, and thus $\pi(f)$ is well-defined when $f \in L^2(G)$

Formally, the Fourier coefficient $\pi(f)$ can be expressed in the term of the operator-valued integration

$$\pi(f) = \int_G f(g) \pi(g) dg \tag{3.1}$$

Next, theorem 3.13 asserts that the direct sum of $L^2(G)$ in Peter-Weyl theorem is actually orthogonal. Then, therorem 3.15, which is also known as the *Plancherel's thorem*, can be obtained as a consequence.

Theorem 3.13 (Shur Orthogonality Relations) Let π and π' be irreducible unitary representations of G, and consider $\mathcal{M}_\pi$ and $\mathcal{M}_{\pi'}$ as subspace of $L^2(G)$.

1. If $[\pi] \neq [\pi']$ then, $\mathcal{M}_\pi \perp \mathcal{M}_{\pi'}$.
2. If $\{e_j\}$ is any orthonormal basis of $\mathcal{H}_\pi$, then $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \ldots, d_\pi\}$ is an orthonormal basis for $\mathcal{M}_\pi$.

Proof. See [3, theorem 5.8]. ❑

Theorem 3.14 (Plancherel's Theorem) Let $f \in L^2(G)$. Then f is equal in $L^2(G)$ to the sum of its *Fourier series*:

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right). \tag{3.2}$$

Proof. Denote by $\langle\cdot,\cdot\rangle_2$ the standard inner product in $L^2(G)$. It follows from theorem 3.11, 3.13 that for any $f \in L^2(G)$,

$$\begin{aligned} f(x) &= \sum_{[\pi]\in\widehat{G}} \sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left\langle f, \sqrt{d_\pi}\overline{\pi}_{ij}\right\rangle_2 \sqrt{d_\pi}\overline{\pi}_{ij}(x) \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left\langle f, \overline{\pi}_{ij}\right\rangle_2 \overline{\pi_{ij}(x)} \end{aligned} \tag{3.3}$$

, as $\{\sqrt{d_\pi}\,\overline{\pi_{ij}} : i, j = 1, \ldots, d_\pi\}_{[\pi]\in\widehat{G}}$ forms an orthonomal basis of $L^2(G)$. Then,

$$\begin{aligned} \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left\langle f, \overline{\pi_{ij}}\right\rangle_2 \overline{\pi_{ij}(x)} &= \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left(\int_G f(g)\pi_{ij}(g)dg \right) \overline{\pi_{ij}(x)} \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \int_G f(g) \left(\sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \pi_{ij}(g)\overline{\pi_{ij}(x)} \right) dg \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \int_G f(g) \left(\sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left\langle \pi(g)e_j, e_i\right\rangle \overline{\left\langle \pi(x)e_j, e_i\right\rangle} \right) dg \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \int_G f(g) \left(\sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \left\langle \pi(g)e_j, e_i\right\rangle \left\langle e_i, \pi(x)e_j\right\rangle \right) dg \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \int_G f(g) \left(\sum_{j=1}^{d_\pi} \left\langle \pi(g)e_j, \pi(x)e_j\right\rangle \right) dg \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi} \left(\int_G f(g) \left\langle \pi(g)e_j, \pi(x)e_j\right\rangle dg \right). \end{aligned}$$

Moreover,

$$\sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\left(\int_G f(g)\left\langle \pi(g)e_j, \pi(x)e_j\right\rangle dg\right) = \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\left\langle \pi(f)e_j, \pi(x)e_j\right\rangle$$
$$= \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{j=1}^{d_\pi}\left\langle \pi(x^{-1})\pi(f)e_j, e_j\right\rangle$$
$$= \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right).$$

The first equality comes from Definition 3.12. ❑

3.3 Absolute Convergence of Fourier series

Definition 3.15 Let $\mathcal{V}$ be a finite dimensional Hilbert space and $A \in \mathcal{L}(\mathcal{V})$. The *Hilbert-Schmidt* of A is defined by

$$|||A|||^2 = \operatorname{tr}\left(AA^*\right).$$

Theorem 3.16 Let f be a continuous function on a compact group G such that

$$\sum_{[\pi]\in\widehat{G}} d_\pi^{3/2}\, |||\pi(f)||| < \infty,$$

then

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right),$$

the convergence is absolute and uniform on G.

Proof. See [2, Proposition 6.6.1]. ❑

CHAPTER IV

HARMONIC ANALYSIS ON COMPACT HOMOGENEOUS SPACES

Let U be a compact group and H its closed subgroup. In this chapter we are interested in a compact homogeneous space of the form U/H, by which we denote the space of left cosets of U modulo H.

Materials in this chapter will serve as a tool for solving our d'Alembert functional equation on the compact homogeneous space U/H in the next chapter.

4.1 $L^2(U/H)$ and $L^2(U)^H$

Let $d_l x$ be a normalized Haar measure on U. Denote $d\mu(xH)$ a U-invariant Radon measure normalized so that (2.10) in theorem 2.18 holds. In this specific case, we prefer to normalize $d_l h$ that appear in the theorem as well. Then, we arrive at (2.11) which states an integration formula

$$\int_U F \circ q(x) d_l x = \int_{U/H} F(xH)\, d\mu(xH),$$

for every $F \in C_c(U/H)$. Note that $C_c(U/H)$ dense in $C(U/H)$, which, in turn, dense in $L^2(U/H)$. So, this integration formula extends to every $F \in L^2(U/H)$ as well, in which case, $q^* : F \mapsto F \circ q$ maps $L^2(U/H)$ injectively into $L^2(U)$. The image of this map equals the space $L^2(U)^H$ of right H-invariant square-integrable functions on U. Via the topological linear isomorphism q^* we shall identify $L^2(U/H)$ with $L^2(U)^H$.

4.2 H-Spherical Representations

As in Chapter 3, we denote by $\widehat{U}$ the set of equivalence classes of irreducible unitary representations of U. For each $[\pi] \in \widehat{U}$, we can always choose a fixed representative $(\pi, \mathcal{V}_\pi)$.

Definition 4.1 For each irreducible unitary representation $(\pi, \mathcal{V}_\pi)$. Define $\mathcal{V}_\pi^H$ the space of H-fixed vectors in $\mathcal{V}_\pi$ by

$$\mathcal{V}_\pi^H = \{ v \in \mathcal{V}_\pi \mid \forall h \in H, \pi(h)v = v \}.$$

Denote c_π for the dimension of $\mathcal{V}_\pi^H$. Note that π is said to be a H-spherical representation if $\mathcal{V}_\pi^H \neq \{0\}$.
Moreover,

$$\widehat{(U/H)} = \{[\pi] \in \widehat{U} \mid \mathcal{V}_\pi^H \neq \{0\}\}$$

denotes the set of equivalence classes of H-spherical representation of U/H.

Lemma 4.2 Let $f \in L^2(U)^H$ and $(\pi, \mathcal{V}_\pi)$ be an irreducible unitary representation of U. Suppose that $\pi(f) \in \mathcal{L}(\mathcal{V}_\pi)$ is a nonzero linear transformation. Then, $[\pi] \in \widehat{(U/H)}$.

Proof. We start with a prelude. Assume that $f \in L^2(U)$ and f is right H-invariant. Fix an irreducible unitary representation $(\pi, \mathcal{V}_\pi)$. For each $v \in \mathcal{V}_\pi$, we formally get by (3.1)

$$\begin{aligned}\pi(f)v &= \int_U f(x)\pi(x)v \, dx \\ &= \int_U f(xh^{-1})\pi(x)v \, dx \\ &= \int_U f(x)\pi(xh)v \, dx \\ &= \int_U f(x)\pi(x)\pi(h)v \, dx,\end{aligned}$$

for all $h \in H$. Integrating over H, we formally have

$$\begin{aligned}\pi(f)v &= \int_H \int_U f(x)\pi(x)\pi(h)v \, dxdh \\ &= \int_U f(x)\pi(x)\left(\int_H \pi(h)v \, dh \right) dx.\end{aligned} \tag{4.1}$$

We can verify the validity of the above equations by taking the inner product with $w \in \mathcal{V}_\pi$ as in Definition 3.12, i.e., the above calculations are justified by the computations in the weak sense.

By setting for each $v \in \mathcal{V}_\pi$,

$$P_{\mathcal{V}_\pi^H}(v) = \int_H \pi(h)v\, dh .,$$

(4.1) implies $\pi(f)v = \pi(f)P_{\mathcal{V}_\pi^H}(v)$. Moreover, for every $k \in H$, we have

$$\pi(k)\left(\int_H \pi(h)v\, dh\right) = \int_H \pi(k)\pi(h)v\, dh = \int_H \pi(kh)v\, dh = \int_H \pi(h)v\, dh,$$

which implies $P_{\mathcal{V}_\pi^H}(v) \in \mathcal{V}_\pi^H$ for all $v \in \mathcal{V}_\pi$.

Next, we prove the lemma by contrapositive. Let $(\pi, \mathcal{V}_\pi)$ be an irreducible unitary representation of U. Suppose that $[\pi] \notin (\widehat{U/H})$. Then, $\mathcal{V}_\pi^H = \{0\}$ by the definition, and $\pi(f)v = \pi(f)P_{\mathcal{V}_\pi^H}(v) = 0$ for all $v \in \mathcal{V}_\pi$.

Theorem 4.4 Let $f \in L^2(U)^H$. Then f is equal in $L^2(U)$ to the sum

$$f(x) = \sum_{[\pi]\in\widehat{U/H}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right) \tag{4.2}$$

Proof. Since for each class $[\pi] \notin (\widehat{U/H})$ we have $\pi(f) = 0$, the expansion (3.2) of $f \in L^2(U)$ becomes

$$f(x) = \sum_{[\pi]\in\widehat{U/H}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right),$$

Recall that $(\widehat{U/H}) = \{[\pi] \in \widehat{U} \mid \mathcal{V}_\pi^H \neq \{0\}\}$, by Definition 4.1. ❑

CHAPTER V

D'ALEMBERT'S FUNCTIONAL EQUATIONS ON COMPACT HOMOGENEOUS SPACES

In this chapter, we define the d'Alembert's functional equation on compact homogeneous spaces in Definition 5.1. Our main result is theorem 5.4, where we present a method for solving the aforementioned functional equation.

5.1 D'Alembert's Functional Equation on U / H

Definition 5.1 Let U / H be a compact homogenous space. A continuous function F with domain U / H and codomain the field $\mathbb{C}$ of complex numbers is said to satisfy the *d'Alembert's functional equation on* U / H if, for all $x, y \in U$,

$$F(xyH)+F(xy^{-1}H)=2F(xH)F(yH), \tag{5.1}$$

and $F(H)=1$.

5.2 Solving the Functional Equation

Lemma 5.2 Let $A \in SL(2)$. Then $A+A^{-1}=\mathrm{tr}(A)I_2$. Suppose, in addition, that A is unitary. Then, $\mathrm{tr}(A)=2$ if and only if $A=I_2$.

Proof. Let $A \in SL(2)$. Since $\det(A)=1$, the characteristic polynomial of A is $p_A(x)=\det(xI-A)=x^2-\mathrm{tr}(A)x+1$. By Cayley-Hamilton theorem, we have $0=p_A(A)=A^2-tr(A)A+I_2$. So, $0=A^{-1}(p_A(A))=A+tr(A)I_2+A^{-1}$, and $A+A^{-1}=\mathrm{tr}(A)I_2$.

Next, assume that A is unitary. Suppose $\mathrm{tr}(A)=2$. Then, $p_A(x)=x^2-2x+1$ $=(x-1)^2$. So $\lambda=1$ is the only eigenvalue of A. Note that A is diagonalizable by Spectral theorem. Hence, there exists a unitary matrix T such that $A=T\,\mathrm{diag}(1,1)T^*=TT^*=I_2$. The converse is obvious. ❑

This lemma is the main step in the work of Yang ([4]).

Lemma 5.3 Let G be a compact group, and $\pi : U \to \mathcal{V}$ be a unitary irreducible representation of dimension n. Suppose that there exists a nonzero vector $v \in \mathcal{V}$ such that

$$(\pi(x) + \pi(x)^{-1})v \in \mathbb{C}v, \forall x \in U. \tag{5.2}$$

Then either $n = 1$ and π is a unitary group character, or $n = 2$ and $\pi(U) \subseteq SU(2)$.

Proof. See [4]. ❑

Finally, we give our main result in the following theorem.

Theorem 5.4 Suppose F is a nonzero solution of the d'Alembert's functional equation on a compact homogeneous space U / K. Then, there is an *H*-spherical representation $\varphi : U \to SU(2)$ such that

$$F(xH) = \frac{\chi_\varphi(x)}{2}, \forall x \in U, \tag{5.3}$$

and $H \leq \ker \varphi$.

Proof. First, let $q : U \mapsto U / H$ be the canonical projection $q(u) = uH, \forall u \in U$, and let $F \in C(U / K)$ be a nonzero solution to (5.1). Obviously, $f = F \circ q \in C(U)$ is a right *H*-invariant L^2-function. Indeed, let $x \in U$ then $f(xh) = F \circ q(xh)$ $= F((xh)H) = F(xH) = f(x), \forall h \in H$. Since f is continuous and U is compact, f is bounded. It follows that $f \in L^2(U)^H$. We obtain

$$f(x) = \sum_{[\pi] \in \widehat{U/H}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\pi(f)\right),$$

where the equality holds in L^2 sense. Suppose $\pi(f) = 0$ for every class $[\pi] \in \widehat{U / H}$. By theorem 3.16, the series on the right converges uniformly to both 0 and f. This contradicts the assumption that f is nonzero. Thus, we can pick a *H*-spherical representation $(\pi, \mathcal{V}_\pi)$ with $\pi(f) \neq 0$.

To show that f satisfies d'Alembert's functional equation on U, let $x, y \in U$. Then,

$$\begin{aligned} 2f(x)f(y) &= 2F(xH)F(yH) \\ &= F(xyH) + F(xy^{-1}H) \\ &= f(xy) + f(xy^{-1}). \end{aligned}$$

Note that *f* satisfies the d'Alembert's functional equation on a compact group. To obtain solutions for *f*, we will reapply the idea of Yang in [4].

Let $y \in U$ be arbitrarily fixed. We can rewrite the last equation as

$$R_y f + R_{y^{-1}} f = 2f(y)f\ .$$

Remember that $\pi(y)$ is a unitary operator, and recall the formula (3.1) for calculating the Fourier coefficient. Then, by calculating the Fourier coefficient at π of L.H.S., we obtain,

$$\begin{aligned} \pi(R_y f + R_{y^{-1}} f) &= \int_U f(gy)\pi(g)dg + \int_U f(gy^{-1})\pi(g)dg \\ &= \int_U f(g)\pi(gy^{-1})dg + \int_U f(g)\pi(gy)dg \\ &= \int_U f(g)\pi(g)\pi(y^{-1})dg + \int_U f(g)\pi(g)\pi(y)dg \\ &= \left(\int_U f(g)\pi(g)dg\right)\pi(y^{-1}) + \left(\int_U f(g)\pi(g)dg\right)\pi(y) \\ &= \pi(f)\left(\pi(y)^{-1} + \pi(y)\right). \end{aligned}$$

Comparing to RHS.'s, we get $\pi(f)\left(\pi(y^{-1}) + \pi(y)\right) = \pi(2f(y)f) = 2f(y)\pi(f)$.
By applying an adjoint operator to both sides, we get

$$\begin{aligned} 2f(y)\pi(f)^* &= \left(\pi(f)\left(\pi(y)^{-1} + \pi(y)\right)\right)^* \\ &= \left(\pi(y)^{-1} + \pi(y)\right)^* \pi(f)^* \\ &= \left(\pi(y)^{-1} + \pi(y)\right)^* \pi(f)^* \\ &= \left(\pi(y) + \pi(y)^{-1}\right)\pi(f)^*. \end{aligned}$$

Next, since $\pi(f)^* \neq 0$, there exists a vector $v \in \mathcal{V}_\pi$ such that $\pi(f)^* v \neq 0$.

By applying both sides of the above equation to v, we get, for all $y \in U$,

$$2f(y)\left(\pi(f)^* v\right) = \left(\pi(y) + \pi(y)^{-1}\right)\left(\pi(f)^* v\right). \tag{5.5}$$

We conclude from Lemma 5.3 that either $d_\pi = 1$, or $d_\pi = 2$ and $\pi(U) \subseteq SU(2)$.

Case $d_\pi = 1$. Then (5.5) becomes

$$2f(y)\left(\pi(f)^* v\right) = \left(\pi(y) + \overline{\pi(y)}\right)\left(\pi(f)^* v\right), \forall y \in U.$$

Hence, $f = \frac{1}{2}(\pi + \overline{\pi})$. Let φ be the direct sum of π and $\overline{\pi}$: $\varphi = \pi \oplus \overline{\pi}$ and $f = \frac{\chi_\varphi}{2}$. It's not hard to verify that $\varphi(U) \subseteq SU(2)$.

Case $d_\pi = 2$ and $\pi(U) \subseteq SU(2)$, then (5.5) becomes

$$2f(y)\left(\pi(f)^* v\right) = \chi_\pi(y)\left(\pi(f)^* v\right), \forall y \in U,$$

since $\pi(x) + \pi(x)^{-1} = \mathrm{tr}(\pi(x))$ by Lemma 5.2. Hence, $f = \frac{\chi_\pi}{2}$. Letting $\varphi := \pi$.

Either case, φ is a *H*-spherical representation $\varphi : U \to SU(2)$ such that for all $x \in U$

$$F(xH) = f(x) = \frac{\chi_\varphi(x)}{2}.$$

Let $h \in H$. Then $1 = F(H) = f(h) = \frac{\chi_\varphi(h)}{2}$. So, $h = I_{2\times 2}$ by Lemma 5.2. This shows $H \le \ker \varphi$. ❑

By taking $H = \{e\}$, our result extends one given below:

Corollary 5.5 (D.Yang, [4]) Suppose f is a nonzero solution of the d'Alembert's functional equation on a compact group *U*. Then, there is a representation $\varphi : U \to SU(2)$ such that

$$f(x) = \frac{\chi_\varphi(x)}{2}, \forall x \in U. \tag{5.6}$$

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