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POLYNOMIALS WITH NONNEGATIVE REAL COEFFICIENTS

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งานวิจัยในวิทยานิพนธ์ คือ การพิสูจน์สมบัติบางประการของพหุนามที่มีสัมประสิทธิ์เป็นจำนวนจริงไม่เป็นลบ ซึ่งเป็นการขยายผลงานของ รอยมาน และ รูบินสไตย์ ที่ตีพิมพ์ในปี ค.ศ. 1992 อาทิ การหาจำนวนจริงบวก r ที่น้อยที่สุดที่ทำให้ พหุนาม (x-r)f(x) มีสัมประสิทธิ์นำเป็นจำนวนจริงบวก และสัมประสิทธิ์อื่นๆ เป็นจำนวนจริงไม่เป็นบวก เมื่อกำหนดพหุนามที่มีสัมประสิทธิ์เป็นจำนวนจริง f(x) มาให้

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Certain properties of polynomials with nonnegative real coefficients are proved generalizing those of Roitman and Rubinstein in 1992. The results include finding the smallest positive real number r such that the polynomial $(x-r)f(x)$ has a positive leading coefficient and other coefficients being nonnegative when $f(x)$ is a given real polynomial.

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CONTENTS

CHAPTER I

INTRODUCTION

1.1 Polynomials with nonnegative real coefficients

Polynomials with nonnegative real coefficients have been subject to a good deal of investigations for a long time, see e.g., [2], [3], [4], [5]. In this chapter, we collect propositions and theorems about polynomials with nonnegative real coefficients. In order to state relevant known results, we need some notation and terminology. Denote by

$$\Pi := \left\{ f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}[x] : \ n \in \mathbb{N} \cup \{0\}, \ c_n > 0, \ c_i \geq 0 \ \ (0 \leq i \leq n-1) \right\},$$

$$Q := \left\{ f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}[x] : \ n \in \mathbb{N} \cup \{0\}, \ c_n > 0, \ c_i \leq 0 \ \ (0 \leq i \leq n-1) \right\},$$

$$\mathbb{R}^+[x] := \left\{ f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}[x] : \ n \in \mathbb{N} \cup \{0\}, \ c_i > 0 \ \ (0 \leq i \leq n) \right\}.$$

For $r > 0$, let

$$Q(r) := \left\{ f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}[x] : \ n \in \mathbb{N} \cup \{0\}, \ (x-r)f(x) \in Q \right\},$$

$$Q^+(r) := \left\{ f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}^+[x] : \ n \in \mathbb{N} \cup \{0\}, \ (x-r)f(x) \in Q \right\}.$$

We say that $f(x) \in (x-r)Q(r)$ if r is a zero of $f(x)$ and $f(x)/(x-r) \in Q(r)$.

In 1992, Roitman and Rubinstein [7] gave the following properties of polynomials in Q, Π and $Q(r)$.

Proposition 1.1. *([7, p. 151]) Let* $r > 0$ *and* $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}[x] \smallsetminus \{0\}$. *Then*

1) $f(x) \in Q(r) \Longleftrightarrow 0 \le c_0,\ c_i \le rc_{i+1}$ *for all* $i \in \{0, 1, 2, \ldots, n-1\}$,

   i.e., $f(x) \in Q(r) \Longleftrightarrow 0 \le c_0 \le rc_1 \le r^2 c_2 \le \ldots \le r^n c_n$*;*

2) $f(x) \in Q(1) \Longleftrightarrow$ *the sequence* (c_i) *is nonnegative and nondecreasing;*

3) *if* $0 < a < b$, *then* $Q(a) \subsetneq Q(b)$*;*

4) *we have* $Q(r) \subsetneq \Pi$*;*

5) *if* $f(x) \in Q$ *and* $c_i \ne 0$ *for some* $i \in \{0, 1, 2, \ldots, n-1\}$, *then* $f(x)$ *has a unique positive zero;*

6) *we have*

$$\bigcup_{r>0} (x-r)Q(r) = \{f(x) \in Q :\ c_i \ne 0 \ \text{for some}\ i \in \{0, 1, 2, \ldots, n-1\}\};$$

7) *if* $f(x) \in \Pi$, *then* $f(x^k) \in \Pi$ *for all* $k \in \mathbb{N}$*; and*

   if $f(x) \in Q$, *then* $f(x^k) \in Q$ *for all* $k \in \mathbb{N}$.

Proof. 1) We have

$$(x-r)f(x) = c_n x^{n+1} + (c_{n-1} - rc_n)x^n + \cdots + (c_0 - rc_1)x - rc_0. \tag{1.1}$$

Assume that $f(x) \in Q(r)$. Then $(x-r)f(x) \in Q$, and from (1.1), we have $0 \le c_0$ and $c_i \le rc_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$.

Conversely, assume that $0 \le c_0$ and $c_i \le rc_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. Then $-rc_0 \le 0$ and $c_i - rc_{i+1} \le 0$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. It remains to show that

c_n is positive. We observe that

$$0 \le c_0 \le rc_1 \le r^2 c_2 \le \ldots \le r^{n-1} c_{n-1} \le r^n c_n.$$

Since $f \not\equiv 0$, we get $0 < c_n$. Hence $f(x) \in Q(r)$.

2) We have immediately from part 1) that

$$f(x) \in Q(1) \Longleftrightarrow 0 \le c_0 \le c_1 \le \ldots \le c_n.$$

Thus, $f(x) \in Q(1)$ if and only if the sequence (c_i) is nonnegative and nondecreasing.

3) Let $0 < a < b$ and $f(x) = \sum_{i=0}^{n} c_i x^i \in Q(a)$. By part 1),

$$0 \le c_0 \le ac_1 \le a^2 c_2 \le \ldots \le a^{n-1} c_{n-1} \le a^n c_n.$$

Since $0 < a < b$, it follows that

$$0 \le c_0 \le bc_1 \le b^2 c_2 \le \ldots \le b^{n-1} c_{n-1} \le b^n c_n.$$

That is $Q(a) \subseteq Q(b)$.

To verify that $Q(a) \ne Q(b)$, consider $g(x) = x^2 + (a+b)x/2 + ba$. Observe that

$$(x-b)g(x) = x^3 - \frac{(b-a)}{2}x^2 - \frac{b(b-a)}{2}x - b^2 a.$$

Since $0 < a < b$,

$$-\frac{(b-a)}{2} < 0, \; -\frac{b(b-a)}{2} < 0 \text{ and } -b^2 a < 0,$$

i.e., $(x-b)g(x) \in Q$ and so $g(x) \in Q(b)$. But $g(x) \notin Q(a)$, because $0 < a < b$ and

$$(x-a)g(x) = x^3 + \frac{(b-a)}{2}x^2 + \frac{a(b-a)}{2}x - ba^2 \notin Q.$$

4) If $f(x) = \sum_{i=0}^{n} c_i x^i \in Q(r)$, then $(x-r)f(x) \in Q$. From (1.1), we have $c_n > 0$ and

$$0 \leq c_0 \leq rc_1 \leq r^2c_2 \leq \ldots \leq r^{n-1}c_{n-1} \leq r^n c_n.$$

Since $r > 0$, we have $c_i \geq 0$ for all $i \in \{0, 1, 2, \ldots, n\}$. Then $f(x) \in \Pi$.

Next, we consider $h(x) = x^2 + 1 \in \Pi$. Since $r > 0$, we have

$$(x-r)h(x) = (x-r)(x^2+1) = x^3 - rx^2 + x - r \notin Q,$$

i.e., $Q(r) \neq \Pi$.

5) Suppose that $f(x) = \sum_{i=0}^{n} c_i x^i \in Q$ and $c_i \neq 0$ for some $i \in \{0, 1, 2, \ldots, n-1\}$. Let $m = \min\{i \in \{0, 1, 2, \ldots, n-1\} : \; c_i \neq 0\}$. From

$$\begin{aligned} f(x) &= c_n x^n + c_{n-1}x^{n-1} + \cdots + c_{m+1}x^{m+1} + c_m x^m \\ &= x^m \left(c_n x^{n-m} + c_{n-1}x^{n-m-1} + \cdots + c_{m+1}x + c_m\right), \end{aligned}$$

let $h(x) = c_n x^{n-m} + c_{n-1}x^{n-m-1} + \cdots + c_{m+1}x + c_m \in \mathbb{R}[x]$. Then $f(x) = x^m h(x)$ and all zeros of $h(x)$ are zeros of $f(x)$. Since $f(x) \in Q$, we have

$$c_n > 0 \;\text{ and }\; c_i \leq 0 \;\text{ for all } i \in \{0, 1, 2, \ldots, n-1\} \tag{1.2}$$

and so $h(0) = c_m < 0$. Since the polynomial $h(x)$ is a continuous function and $c_n > 0$, we have

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} x^{n-m}\left[c_n + \frac{c_{n-1}}{x} + \frac{c_{n-2}}{x^2} + \cdots + \frac{c_{m+1}}{x^{n-m-1}} + \frac{c_m}{x^{n-m}}\right] > 0.$$

Then $h(x)$ has a positive zero and so $f(x)$. By Descartes' rule of sign, any real polynomial cannot have more positive zeros (counting multiplicity) than there are sign changes in its coefficients, and by (1.2), we get $f(x)$ has a unique positive zero.

6) Let

$$A := \left\{\sum_{i=0}^{n} c_i x^i \in Q :\ c_i \neq 0 \text{ for some } i \in \{0,1,2,\ldots,n-1\}\right\}.$$

Suppose that $f(x) = \sum_{i=0}^{n} c_i x^i \in A$. Then $f(x) \in Q$ and $c_i \neq 0$ for some $i \in \{0,1,2,\ldots,n-1\}$. From part 5), there exists a positive number t such that $f(t) = 0$.

Let

$$g(x) = \frac{f(x)}{x-t} = \sum_{i=0}^{n-1} a_i x^i \in \mathbb{R}[x].$$

By direct calculation, we have

$$a_{n-1} = tc_n,\ a_{n-2} = c_{n-1} + tc_n,\ a_{n-3} = c_{n-2} + t(c_{n-1} + tc_n), \ldots,$$

$$a_i = c_{i+1} + t(c_{i+2} + \cdots + t(c_{n-1} + tc_n)\ldots) \text{ for all } i \in \{0,1,2,\ldots,n-2\}.$$

Since $f(x) \in Q$, we have $c_n > 0$ and $c_i \leq 0$ for all $i \in \{0,1,2,\ldots,n-1\}$. It follows that $a_0 = g(0) = c_0/(-t) \geq 0$. Thus, for all $i \in \{0,1,2,\ldots,n-2\}$

$$a_i = c_{i+1} + t(c_{i+2} + \cdots + t(c_{n-1} + tc_n)\ldots) \leq t(c_{i+2} + \cdots + t(c_{n-1} + tc_n)\ldots) = ta_{i+1}.$$

From part 1), we have $g(x) \in Q(t)$, i.e., $f(x) \in (x-t)Q(t)$.

Conversely, let

$$p(x) = \sum_{j=0}^{m} b_j x^j \in \bigcup_{r>0} (x-r)Q(r).$$

There exists a positive number $k > 0$ such that $p(x) \in (x-k)Q(k)$. Thus $p(k) = 0$ and $p(x)/(x-k) \in Q(k)$, and so

$$p(x) = (x-k)\frac{p(x)}{x-k} \in Q.$$

That is, $b_m > 0$ and $b_j \leq 0$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. Assume that $b_j = 0$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. Then $p(x) = b_m x^m$, which is a contradiction, because $p(k) = 0$ and $k \neq 0$. Then there exists some $j \in \{0, 1, 2, \ldots, m-1\}$ such that $b_j \neq 0$, i.e., $p(x) \in A$.

7) Let $f(x) = \sum_{i=0}^{n} c_i x^i \in \Pi$. Then $c_n > 0$ and $c_i \geq 0$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. It is easy to see that for any $k \in \mathbb{N}$,

$$f(x^k) = c_n x^{nk} + c_{n-1} x^{(n-1)k} + c_{n-2} x^{(n-2)k} + \cdots + c_1 x^k + c_0 \in \Pi.$$

Similarly, if $f(x) \in Q$, then $f(x^k) \in Q$ for any $k \in \mathbb{N}$. □

Theorem 1.2. *([7, Lemma 1]) Let $f(x)$ be a polynomial in Q such that $f(0) \neq 0$ and $f(x) = \widetilde{f}(x^k)$ with $k \geq 1$ maximal. Assume that s is a positive zero of $f(x)$.*

1) *For any zero ω of $f(x)$, we have $s \geq |\omega|$.*

2) *If ω is a zero of $f(x)$ with $|\omega| = s$, then ω is a simple zero and ω/s is a k^{th} root of unity (that is, $\omega^k = s^k$).*

3) *If $f(s) = f(s\epsilon) = 0$, where $\epsilon^d = 1$ with $d \geq 1$ minimal, then $f(x)$ has no zeros of the form $t\gamma$ where $0 < t < s$ and $\gamma^d = 1$.*

Theorem 1.3. *([7, Lemma 2]) Let $r_1, r_2, \ldots, r_s$ be positive real numbers and $f_k(x) \in Q(r_k)$ for all $k \in \{1, 2, 3, \ldots, s\}$. Then*

$$\prod_{k=1}^{s} f_k(x) \in Q\left(\sum_{k=1}^{s} r_k\right).$$

Theorem 1.4. *([7, Lemma 3]) If z is a complex number which is not real positive, then z is a zero of a polynomial in $Q(r)$ for any $r > |z|$.*

Theorem 1.5. *([7, Lemma 4]) Any polynomial $f(x)$ of positive degree with no positive zeros divides a polynomial in $Q(r)$ for any positive number $r > \max\{|\omega| \ : \ f(\omega) = 0\}$.*

Part 1) of Proposition 1.1 is closely related to the classical Enström-Kakeya theorem ([8, Theorem 1.1]). It is an effective criterion to test whether a real polynomial has all its zeros in the unit disk.

Theorem 1.6. *(Eneström-Kakeya Theorem) If $f(x) \in Q^+(1)$, then $f(x)$ has all its zeros in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.*

Example 1.7. Let $f(x) = 4x^3 + 3x^2 + 2x$. From Proposition 1.1 part 1), $f(x) \in Q^+(1)$. All zeros of $f(x)$ are 0, $\left(-3 - i\sqrt{23}\right)/8$ and $\left(-3 + i\sqrt{23}\right)/8$ belonging to the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Anderson, Saff and Varga [1] extended Eneström-Kakeya Theorem to a polynomial in $Q(r)$ by proving the following result.

Theorem 1.8. *([1, Theorem 1]) Let $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}^+[x] \setminus \mathbb{R}$. If*

$$\alpha[f] := \min_{0 \leq i \leq n-1} \left(\frac{c_i}{c_{i+1}}\right), \quad \beta[f] := \max_{0 \leq i \leq n-1} \left(\frac{c_i}{c_{i+1}}\right),$$

then $f(x) \in Q^+(\beta[f])$ and all the zeros of $f(x)$ are contained in the annulus

$$\{z \in \mathbb{C} : \ \alpha[f] \leq |z| \leq \beta[f]\}.$$

Example 1.9. Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. We have $\alpha[f] = 1 = \beta[f]$ and all zeros of $f(x)$ are

$$-1,\ \frac{1+\sqrt{3}i}{2},\ \frac{-1+\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}.$$

1.2 Objectives

In the next chapter, Chapter II, we answer the following questions:

1. For $f(x) \in \mathbb{R}[x]$, find the smallest positive real number r such that $f(x) \in Q(r)$;

2. For positive real numbers $r_1, r_2, \ldots, r_s$ and $f_k(x) \in Q(r_k)$ for all $k \in \{1, 2, 3, \ldots, s\}$, find conditions such that

$$\prod_{k=1}^{s} f_k(x) \in Q\left(r\right)$$

for some positive real number $r < r_1 + r_2 + \cdots + r_s$.

3. For positive real numbers r_1, r_2, $f_1(x) \in Q(r_1)$ and $f_2(x) \in Q(r_2)$, find a positive real number r such that $f_1(x) + f_2(x) \in Q(r)$.

4. For $z_1, z_2, \ldots, z_k \in \mathbb{C} \setminus \mathbb{R}^+$, find $r > 0$ and a polynomial $f(x) \in Q(r)$ such that $z_1, z_2, \ldots, z_k$ are zeros of $f(x)$.

In Chapter III, we give conditions such that product of two polynomials is not in $Q(r)$ for any positive real number r. We investigate the lower and upper Eneström-Kakeya quotients and their connection with reciprocal polynomials in Chapter IV. Finally, we give a connection between linear recursions and polynomials in $Q(r)$ in Chapter V.

CHAPTER II

BASIC PROPERTIES

2.1 The smallest positive real number r

Given a polynomial in some $Q(r)$, by Proposition 1.1 part 3), this polynomial also belongs to $Q(s)$ for all $s \geq r$, a natural question is to find the least possible value of r. We answer it in the next proposition.

Proposition 2.1. *Let $r > 0$ and assume $f(x) = \sum_{i=0}^{n} c_i x^i \in Q(r)$.*

1) If $c_i = 0$ for some $i \in \{0, 1, 2, \ldots, n-1\}$, then $c_j = 0$ for all $j \in \{0, 1, 2, \ldots, i\}$.

2) If

$$M[f] := \max_{\substack{0 \leq i \leq n-1 \\ c_{i+1} \neq 0}} \left(\frac{c_i}{c_{i+1}} \right), \tag{2.1}$$

then $M[f] \geq 0$.

3) If $M[f] > 0$, then the smallest $r > 0$ such that $f(x) \in Q(r)$ is $M[f]$.

4) If $M[f] = 0$, then $f(x) \in Q(u)$ for all $u > 0$.

5) We have $M[af] = M[f]$ for all $a > 0$.

Proof. By Proposition 1.1 part 4), we have $f(x) \in \Pi$. Then

$$c_n > 0 \ \text{ and } \ c_i \geq 0 \ \text{ for all } \ i \in \{0, 1, 2, \ldots, n-1\}. \tag{2.2}$$

1) From $f(x) \in Q(r)$ and Proposition 1.1 part 1), we get

$$0 \leq c_0 \leq rc_1 \leq r^2c_2 \leq \ldots \leq r^{n-1}c_{n-1} \leq r^n c_n.$$

It is easy to see that if $c_i = 0$ for some $i \in \{0, 1, 2, \ldots, n-1\}$, then $c_j = 0$ for all $j \in \{0, 1, 2, \ldots, i\}$.

2) Let $M[f] = \max\limits_{\substack{0 \leq i \leq n-1 \\ c_{i+1} \neq 0}} \left(\dfrac{c_i}{c_{i+1}} \right)$. From (2.2), we have $M[f] \geq 0$.

3) Suppose that $M[f] > 0$. By (2.1) and (2.2), we get $c_0 \geq 0$ and $c_i \leq M[f]c_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. From Proposition 1.1 part 1), $f(x) \in Q(M[f])$. Since $f(x) \in Q(r)$ and Proposition 1.1 part 1),

$$r \geq \max_{\substack{0 \leq i \leq n-1 \\ c_{i+1} \neq 0}} \left(\frac{c_i}{c_{i+1}} \right) = M[f].$$

Then the smallest $r > 0$ such that $f(x) \in Q(r)$ is $M[f]$.

4) Suppose that $M[f] = 0$. By (2.1) and part 1), we get $0 = c_0 = c_1 = \ldots = c_{n-1}$. Then $f(x) = c_n x^n$. From (2.2), we have $c_n > 0$ and

$$(x-u)f(x) = (x-u)c_n x^n = c_n x^{n+1} - uc_n x^n \in Q \text{ for all } u > 0.$$

Then $f(x) \in Q(u)$ for all $u > 0$.

5) Let $a > 0$. Since $f(x) \in Q(r)$ and $af(x) = \sum_{i=0}^{n} ac_i x^i$, we get $af(x) \in Q(r)$. From (2.1), we have

$$M[af] = \max_{\substack{0 \leq i \leq n-1 \\ ac_{i+1} \neq 0}} \left(\frac{ac_i}{ac_{i+1}} \right) = \max_{\substack{0 \leq i \leq n-1 \\ c_{i+1} \neq 0}} \left(\frac{c_i}{c_{i+1}} \right) = M[f].$$

$\square$

From Theorem 1.8 and Proposition 2.1, we have the following corollary:

Corollary 2.2. *Let $r > 0$ and $f(x) \in \mathbb{R}[x]$.*

1) If $f(x) \in \mathbb{R}^+[x] \setminus \mathbb{R}$, then $\beta[f] = M[f]$.

2) If $f(x) \in Q(r)$, then all zeros of $f(x)$ lie in the close circle with radius $M[f]$.

Proposition 2.3. *Let $a_1,\ a_2,\ a_3,\ \ldots\ , a_m$ be positive real numbers. Then*

$$M\left[\prod_{i=1}^{m}(x+a_i)\right] = \sum_{i=1}^{m} a_i.$$

Proof. Since $(x - a_i)(x + a_i) = x^2 - a_i^2 \in Q$, we have $x + a_i \in Q(a_i)$ for all $i \in \{1, 2, 3, \ldots, m\}$. By Theorem 1.3, we have

$$\prod_{i=1}^{m}(x+a_i) \in Q\left(\sum_{i=1}^{m} a_i\right).$$

By Proposition 2.1 part 3), we get

$$M\left[\prod_{i=1}^{m}(x+a_i)\right] \le \sum_{i=1}^{m} a_i.$$

Since

$$\prod_{i=1}^{m}(x+a_i) = x^m + \left(\sum_{i=1}^{m} a_i\right) x^{m-1} + \cdots + \prod_{i=1}^{m} a_i$$

and (2.1), we have

$$M\left[\prod_{i=1}^{m}(x+a_i)\right] \ge \sum_{i=1}^{m} a_i.$$

Then

$$M\left[\prod_{i=1}^{m}(x+a_i)\right] = \sum_{i=1}^{m} a_i.$$

□

Example 2.4. Let $f(x) = (2x+4)(4x+1)(4x+3)$. Then

$$f(x) = 32(x+2)(x+1/4)(x+3/4).$$

By Proposition 2.3, we get

$$M\left[(x+2)(x+\frac{1}{4})(x+\frac{3}{4})\right] = 2 + \frac{1}{4} + \frac{3}{4} = 3.$$

By Proposition 2.1 part 5), we have $M[f] = 3$.

2.2 Product of polynomials in $Q(r)$

The result in Theorem 1.3 tells us that multiplying a polynomial in $Q(r_1)$ by a polynomial in $Q(r_2)$ generally resulting in a polynomial in $Q(r_1 + r_2)$. Another natural question is to ask in which situation the resulting polynomial remains in the old class. This question is treated in the next proposition.

Proposition 2.5. *Let $r > 0$ and assume $f(x) = \sum_{i=0}^{n} c_i x^i \in Q(r)$.*

1) For $w > 0$, we have $x + w \in Q(w)$ and

$$(x+w)f(x) \in Q(r) \Longleftrightarrow c_{n-1} \le (r-w)c_n.$$

2) For $w < 0$, we have $(x+w)f(x) \notin Q(r)$.

Proof. 1) Let $w > 0$. Then $x + w \in Q(w)$. Since $f(x) \in Q(r)$, by Proposition 1.1 part 1), we have

$$0 \le c_0,\ c_i \le rc_{i+1} \text{ for all } i \in \{0, 1, 2, \ldots, n-1\} \tag{2.3}$$

Consider

$$(x-r)(x+w)f(x) = (x-r)(x+w)\sum_{i=0}^{n} c_i x^i$$
$$= c_n x^{n+2} + (c_{n-1} - rc_n + wc_n)x^{n+1} + (c_{n-2} - rc_{n-1} + w(c_{n-1} - rc_n))x^n$$
$$+ \cdots + (c_0 - rc_1 + w(c_1 - rc_2))x^2 + (-rc_0 + w(c_0 - rc_1))x - wrc_0. \qquad (2.4)$$

Since $r > 0$, $w > 0$ and (2.3), we have $-wrc_0 \leq 0$, $-rc_0 + w(c_0 - rc_1) \leq 0$ and $c_i - rc_{i+1} + w(c_{i+1} - rc_{i+2}) \leq 0$ for all $i \in \{0, 1, 2, \ldots, n-2\}$. Then (2.4) and Proposition 1.1 part 1) show that

$$(x-w)f(x) \in Q(r) \Longleftrightarrow c_{n-1} - rc_n + wc_n \leq 0 \Longleftrightarrow c_{n-1} \leq (r-w)c_n.$$

2) Suppose that $w < 0$. Let $(x-r)f(x) = \sum_{i=0}^{n+1} a_i x^i$. Since $f(x) \in Q(r)$, we get $a_{n+1} > 0$ and $a_i \leq 0$ for all $i \in \{0, 1, 2, \ldots, n\}$. Assume that $(x+w)f(x) \in Q(r)$. Then

$$(x-r)(x+w)f(x) = (x+w)\sum_{i=0}^{n+1} a_i x^i = a_{n+1}x^{n+2} + (a_n + wa_{n+1})x^{n+1} +$$
$$+ (a_{n-1} + wa_n)x^n + \cdots + (a_0 + wa_1)x + wa_0 \in Q.$$

We have $wa_0 \leq 0$ and $a_{i-1} + wa_i \leq 0$ for all $i \in \{1, 2, 3, \ldots, n+1\}$. From $w < 0$, we get $a_0 \geq 0$. Since $0 \leq a_0 \leq -wa_1$, we get $a_1 \geq 0$. Since $0 \leq a_1 \leq -wa_2$, we get $a_2 \geq 0$. Similarly, we have $a_i \geq 0$ for all $i \in \{0, 1, 2, \ldots, n\}$. Since $0 \leq a_i \leq 0$ for all $i \in \{0, 1, 2, \ldots, n\}$, we have $a_i = 0$ for all $i \in \{0, 1, 2, \ldots, n\}$ and $(x-r)f(x) = a_{n+1}x^{n+1}$. Hence $r > 0$ is a zero of $a_{n+1}x^{n+1}$, which is a contradiction. □

Example 2.6. Let $r = 4$, $w = 2$ and $f(x) = x^2 + 2x + 8$. By Proposition 2.1 part 2) and part 3), we get $M[f] = 4$ and $f(x) \in Q(4)$. Since $c_1 = 2 \leq (4-2)(1) = (r-w)c_2$, by Proposition 2.5 part 1), we have

$$(x+2)f(x) = x^3 + 4x^2 + 12x + 16 \in Q(4).$$

Theorem 2.7. *Let* $r > 0$*,* $w > 0$*,* $m \in \mathbb{N}$ *and* $f(x) = \sum_{i=0}^{n} c_i x^i \in Q(r)$*. Then* $(x+w)^m \in Q(mw)$ *and*

$$(x+w)^m f(x) \in Q(r) \Leftrightarrow c_{n-1} \leq (r - mw)c_n. \tag{2.5}$$

Proof. From $w > 0$, we have $x + w \in Q(w)$. By Theorem 1.3, we have $(x+w)^m \in Q(mw)$. Next, we prove (2.5) by induction on m. The case $m = 1$ is done by using Proposition 2.5 part 1). Assume that (2.5) is true for m. We claim that

$$(x+w)^{m+1} f(x) \in Q(r) \Leftrightarrow c_{n-1} \leq (r - (m+1)w)c_n.$$

Let $(x+w)^{m+1} f(x) \in Q(r)$. Since

$$(x+w)^{m+1} f(x) = c_n x^{m+n+1} + (c_{n-1} + (m+1)wc_n)x^{m+n} + \cdots + c_0 w^{m+1} \in Q(r),$$

by Proposition 1.1 part 1), we get $c_{n-1} + (m+1)wc_n \leq rc_n$. Then

$$c_{n-1} \leq (r - (m+1)w)c_n.$$

Conversely, suppose that $c_{n-1} \leq (r - (m+1)w)c_n$. Then $c_{n-1} \leq (r - mw)c_n$. Since (2.5) is true for m, we have $(x+w)^m f(x) \in Q(r)$. Let $F(x) = (x+w)^m f(x)$. Then

$$F(x) = c_n x^{m+n} + (c_{n-1} + mwc_n)x^{m+n-1} + \cdots + c_0 w^m \in Q(r).$$

Since $c_{n-1} \le (r-(m+1)w)c_n$, we have

$$c_{n-1} + mwc_n \le (r-w)c_n.$$

From Proposition 2.5 part 1), we have $(x+w)F(x) \in Q(r)$, i.e.,

$$(x+w)^{m+1}f(x) \in Q(r).$$

□

Example 2.8. Let $w=7$, $m=5$ and $f(x) = 2x^4 + 10x^3 + 400x^2 + 37x + 15$. From Proposition 2.1 part 2) and part 3), we have $M[f] = 40$ and $f(x) \in Q(40)$. Since

$$c_3 = 10 \le (40-(5)(7))(2) = (M[f]-mw)c_4,$$

by Proposition 2.7, we have

$$(x+7)^5 f(x) = 2x^9 + 80x^8 + 1730x^7 + 25797x^6 + 255620x^5 + 1544319x^4 \\ + 5104330x^3 + 7218435x^2 + 801934x + 252105 \in Q(40).$$

Theorem 2.9. *Let r_1, r_2 be positive real numbers and let $f_1(x) = \sum_{j=0}^{J} c_j x^j \in Q(r_1)$ and $f_2(x) = \sum_{k=0}^{K} d_k x^k \in Q(r_2)$. If $c_{J-1} \le (r_1 - r_2)c_J$, then $f_1(x)f_2(x) \in Q(r_1)$.*

Proof. Since $f_1(x) \in Q(r_1) \subseteq \Pi$ and $f_2(x) \in Q(r_2) \subseteq \Pi$, by Proposition 1.1 part 1), we have

$$0 \le c_j,\ 0 < c_J,\ c_j \le r_1 c_{j+1} \text{ for all } j \in \{0,1,2,\ldots,J-1\}, \tag{2.6}$$

$$0 \le d_k,\ 0 < d_K,\ d_k \le r_2 d_{k+1} \text{ for all } k \in \{0,1,2,\ldots,K-1\}. \tag{2.7}$$

Suppose that $c_{J-1} \le (r_1 - r_2)c_J$. From (2.7), we have

$$r_2 \ge \max_{\substack{0 \le k \le K-1 \\ d_{k+1} \ne 0}} \left(\frac{d_k}{d_{k+1}} \right).$$

Then for all $k \in \{0, 1, \ldots, K-1\}$ with $d_{k+1} \ne 0$, we have

$$c_{J-1} \le (r_1 - r_2)c_J \le (r_1 - \frac{d_k}{d_{k+1}})c_J.$$

By Proposition 2.1 part 1) and (2.7), we have

$$c_J d_k + c_{J-1} d_{k+1} \le r_1 c_J d_{k+1} \text{ for all } k \in \{0, 1, 2, \ldots, K-1\}. \tag{2.8}$$

If $J \ge K$, then $J = K + T$ for some $T \in \mathbb{N} \cup \{0\}$. We have

$$f_1(x) f_2(x) = \sum_{i=0}^{J+K} a_i x^i$$

where

$$\begin{aligned} a_k &= c_k d_0 + c_{k-1} d_1 + \cdots + c_1 d_{k-1} + c_0 d_k, \quad (0 \le k \le K), \\ a_{K+t} &= c_{K+t} d_0 + c_{K+t-1} d_1 + \cdots + c_{t+1} d_{K-1} + c_t d_K, \quad (0 \le t \le T), \\ a_{K+T+k} &= c_{K+T} d_k + c_{K+T-1} d_{k+1} + \cdots + c_{T+k+1} d_{K-1} + c_{T+k} d_K, \quad (0 \le k \le K). \end{aligned}$$

Since $c_0 \ge 0$ and $d_0 \ge 0$, we have $a_0 = c_0 d_0 \ge 0$. From (2.6), we have for all $k \in \{0, 1, 2, \ldots, K-1\}$

$$a_k = c_k d_0 + c_{k-1} d_1 + \cdots + c_1 d_{k-1} + c_0 d_k \le r_1 (c_{k+1} d_0 + c_k d_1 + \cdots + c_1 d_k + c_0 d_{k+1}) = r_1 a_{k+1}$$

and for all $t \in \{0, 1, 2, \ldots, T-1\}$

$$a_{K+t} = c_{K+t}d_0 + c_{K+t-1}d_1 + \cdots + c_t d_K \leq r_1(c_{K+t+1}d_0 + c_{K+t}d_1 + \cdots + c_{t+1}d_K) = r_1 a_{K+t+1}.$$

From (2.6) and (2.8), we have for all $k \in \{0, 1, 2, \ldots, K-1\}$

$$\begin{aligned} a_{K+T+k} &= [c_{K+T}d_k + c_{K+T-1}d_{k+1}] + [c_{K+T-2}d_{k+2} + \cdots + c_{T+k+1}d_{K-1} + c_{T+k}d_K] \\ &\leq r_1 [c_{K+T}d_{k+1}] + r_1 [c_{K+T-1}d_{k+2} + \cdots + c_{T+k+2}d_{K-1} + c_{T+k+1}d_K] \\ &= r_1 a_{K+T+k+1}. \end{aligned}$$

Then $0 \leq a_0$ and $a_i \leq r_1 a_{i+1}$ for all $i \in \{0, 1, 2, \ldots, J+K-1\}$. By Proposition 1.1 part 1), we have $f_1(x)f_2(x) \in Q(r_1)$.

If $J < K$, then $K = J + S$ for some $S \in \mathbb{N}$. We have

$$f_1(x)f_2(x) = \sum_{i=0}^{J+K} b_i x^i$$

where

$$\begin{aligned} b_j &= c_0 d_j + c_1 d_{j-1} + \cdots + c_{j-1}d_1 + c_j d_0, \quad (0 \leq j \leq J), \\ b_{J+s} &= c_0 d_{J+s} + c_1 d_{J+s-1} + \cdots + c_{J-1}d_{s+1} + c_J d_s, \quad (0 \leq s \leq S), \\ b_{J+S+j} &= c_j d_{J+S} + c_{j+1} d_{J+S-1} + \cdots + c_{J-1} d_{S+j+1} + c_J d_{S+j}, \quad (0 \leq j \leq J). \end{aligned}$$

Since $c_0 \geq 0$ and $d_0 \geq 0$, we have $b_0 = c_0 d_0 \geq 0$. From (2.6), we have for all $j \in \{0, 1, 2, \ldots, J-1\}$

$$b_j = c_0 d_j + c_1 d_{j-1} + \cdots + c_{j-1} d_1 + c_j d_0 \leq r_1(c_0 d_{j+1} + c_1 d_j + \cdots + c_j d_1 + c_{j+1} d_0) = r_1 b_{j+1}$$

and from (2.7) and $0 < r_2 \leq r_1$, we have for all $s \in \{0, 1, 2, \ldots, S-1\}$

$$b_{J+s} = c_0 d_{J+s} + c_1 d_{J+s-1} + \cdots + c_J d_s \leq r_2(c_0 d_{J+s+1} + c_1 d_{J+s} + \cdots + c_J d_{s+1})$$
$$\leq r_1(c_0 d_{J+s+1} + c_1 d_{J+s} + \cdots + c_J d_{s+1}) = r_1 b_{J+s+1}.$$

From (2.6) and (2.8), we have for all $k \in \{0, 1, 2, \ldots, K-1\}$

$$b_{J+S+j} = [c_j d_{J+S} + c_{j+1} d_{J+S-1} + \cdots + c_{J-2} d_{S+j+2}] + [c_{J-1} d_{S+j+1} + c_J d_{S+j}]$$
$$\leq r_1 \left[c_{j+1} d_{J+S} + c_{j+2} d_{J+S-1} + \cdots + c_{J-1} d_{S+j+2}\right] + r_1 \left[c_J d_{S+j+1}\right]$$
$$= r_1 b_{J+S+j+1}.$$

Then $0 \leq b_0$ and $b_i \leq r_1 b_{i+1}$ for all $i \in \{0, 1, 2, \ldots, J+K-1\}$. By Proposition 1.1 part 1), we have $f_1(x)f_2(x) \in Q(r_1)$. □

Example 2.10. Let $f_1(x) = 4x^3 + x^2 + 8x$ and $f_2(x) = 6x^4 + 3x^3 + 3x^2 + 2x + 1$. By Proposition 2.1 part 3), we have $M[f_1] = 8$ and $M[f_2] = 1$. Since

$$c_2 = 1 < (8-1)(4) = (M[f_1] - M[f_2])c_3,$$

by Theorem 2.9, we get

$$f_1(x)f_2(x) = 24x^7 + 18x^6 + 63x^5 + 35x^4 + 30x^3 + 17x^2 + 8x \in Q(M[f_1]).$$

Remark 2.11. The converse of Theorem 2.9 is not true. For example, let $f_1(x) = 2x^3 + 4x^2 + x + 5$ and $f_2(x) = x^2 + x + 4$. Then $M[f_1] = 5$ and $M[f_2] = 4$. Since

$$f_1(x)f_2(x) = 2x^5 + 6x^4 + 13x^3 + 22x^2 + 9x + 20,$$

we have $M[f_1f_2]=3$. By Proposition 1.1 part 3), we get $f_1(x)f_2(x)\in Q(M[f_1])$. But

$$c_2=4>(5-4)2=(M[f_1]-M[f_2])\,c_3.$$

Corollary 2.12. *Keeping the notation of Theorem 2.9, if*

$$M[f_1]=\frac{c_{J-1}}{c_J}+M[f_2]\ \text{ and }\ M[f_2]=\frac{d_{K-1}}{d_K},$$

then

$$M[f_1f_2]=M[f_1]=\frac{c_{J-1}}{c_J}+\frac{d_{K-1}}{d_K}.$$

Proof. Since

$$M[f_1]=\frac{c_{J-1}}{c_J}+M[f_2],$$

we have $c_{J-1}=(M[f_1]-M[f_2])\,c_J$. By Theorem 2.9, we get $f_1(x)f_2(x)\in Q\,(M[f_1])$. Then $M[f_1f_2]\leq M[f_1]$. Since

$$f_1(x)f_2(x)=c_Jd_Kx^{J+K}+(c_{J-1}d_K+c_Jd_{K-1})\,x^{J+K-1}+\cdots+c_0d_0\ \text{ and }\ M[f_2]=\frac{d_{K-1}}{d_K},$$

we have

$$M[f_1f_2]\geq\frac{c_{J-1}d_K+c_Jd_{K-1}}{c_Jd_K}=\frac{c_{J-1}}{c_J}+\frac{d_{K-1}}{d_K}=\frac{c_{J-1}}{c_J}+M[f_2]=M[f_1].$$

Then

$$M[f_1f_2]=M[f_1]=\frac{c_{J-1}}{c_J}+\frac{d_{K-1}}{d_K}.$$

□

Example 2.13. Let $f_1(x) = 3x^3 + 3x^2 + 9x$ and $f_2(x) = 2x^4 + 4x^3 + 6x^2 + 7x + 2$. By Proposition 2.1, we have $M[f_1] = 3$ and $M[f_2] = 2$. Since

$$M[f_1] = 3 = \frac{3}{3} + 2 = \frac{c_2}{c_3} + M[f_2] \ \ and \ \ M[f_2] = 2 = \frac{4}{2} = \frac{d_3}{d_4},$$

by Corollary 2.12, we get

$$M[f_1 f_2] = M[6x^7 + 18x^6 + 48x^5 + 75x^4 + 81x^3 + 69x^2 + 18x] = M[f_1] = 3.$$

Theorem 2.14. *Let $r_1, r_2, r_3, \ldots, r_s$ be positive real numbers and assume*

$$f_k(x) = \sum_{j=0}^{J_k} c_{(k,j)} x^{(k,j)} \in Q(r_k)$$

for all $k \in \{1, 2, 3, \ldots, s\}$. If

$$c_{(1,J_1-1)} \leq (r_1 - \sum_{k=2}^{s} r_k) c_{(1,J_1)},$$

then

$$\prod_{k=1}^{s} f_k(x) \in Q(r_1).$$

Proof. We prove by induction on s. The case $s = 1$ is obvious. The case $s = 2$ is done by Theorem 2.9. Assume the assertion is true for $k \in \{1, 2, 3, \ldots, s\}$. Let

$$c_{(1,J_1-1)} \leq (r_1 - \sum_{k=2}^{s+1} r_k) c_{(1,J_1)}.$$

From $0 < r_{s+1}$ and $0 < c_{(1,J_1)}$, we have

$$c_{(1,J_1-1)} \leq (r_1 - \sum_{k=2}^{s} r_k) c_{(1,J_1)}.$$

Thus,

$$F(x) := \prod_{k=1}^{s} f_k(x) = C_J x^J + C_{J-1}x^{J-1} + \cdots + C_0 \in Q(r_1)$$

where $J = J_1 + J_2 + \cdots + J_s$, $C_J = c_{(1,J_1)}c_{(2,J_2)} \cdots c_{(s,J_s)}$, $C_0 = c_{(1,J_0)}c_{(2,J_0)} \cdots c_{(s,J_0)}$ and

$$C_{J-1} = c_{(1,J_1-1)}c_{(2,J_2)} \cdots c_{(s,J_s)} + c_{(1,J_1)}c_{(2,J_2-1)} \cdots c_{(s,J_s)} + \cdots + c_{(1,J_1)}c_{(2,J_2)} \cdots c_{(s,J_s-1)}.$$

We claim that $C_{J-1} \leq (r_1 - r_{s+1})C_J$. Since

$$f_k(x) = \sum_{j=0}^{J_k} c_{(k,j)}x^{(k,j)} \in Q(r_k)$$

for all $k \in \{2, 3, 4, \ldots, s\}$ and

$$c_{(1,J_1-1)} \leq (r_1 - \sum_{k=2}^{s+1} r_k)c_{(1,J_1)},$$

we have

$$\begin{aligned} c_{(1,J_1-1)} &\leq (r_1 - r_2 - r_3 - \cdots - r_s - r_{s+1})c_{(1,J_1)} \\ c_{(1,J_1-1)} &\leq (r_1 - \frac{c_{(2,J_2-1)}}{c_{(2,J_2)}} - \frac{c_{(3,J_3-1)}}{c_{(3,J_3)}} - \cdots - \frac{c_{(s,J_s-1)}}{c_{(s,J_s)}} - r_{s+1})c_{(1,J_1)} \\ C_{J-1} &\leq (r_1 - r_{s+1})C_J. \end{aligned}$$

By Theorem 2.9, we get $F(x)f_{s+1}(x) \in Q(r_1)$, i.e, $\prod_{k=1}^{s+1} f_k(x) \in Q(r_1)$. □

2.3 Sum of polynomials in $Q(r)$

Theorem 1.3 treats the product of polynomials in $Q(r)$. We next consider the sum of polynomials in $Q(r)$.

Proposition 2.15. *Let* $0 < r_2 \le r_1$, $f(x) = \sum_{i=0}^{n} a_i x^i \in Q(r_1)$ *and* $g(x) = \sum_{j=0}^{m} b_j x^j \in Q(r_2)$.

1) If $n = m$, *then* $f(x) + g(x) \in Q(r_1)$.

2) If $n > m$, *then* $f(x) + g(x) \in Q(r_1)$ *if and only if* $a_m + b_m \le r_1 a_{m+1}$.

3) If $n < m$, *then* $f(x) + g(x) \in Q(r_1)$ *if and only if* $a_n + b_n \le r_1 b_{n+1}$.

Proof. Since $f(x) \in Q(r_1)$ and $g(x) \in Q(r_2)$ and Proposition 1.1 part 1), we have

$$0 \le a_0,\ a_i \le r_1 a_{i+1} \text{ for all } i \in \{0, 1, 2, \ldots, n-1\}, \tag{2.9}$$

$$0 \le b_0,\ b_j \le r_2 b_{j+1} \text{ for all } j \in \{0, 1, 2, \ldots, m-1\}. \tag{2.10}$$

1) Suppose that $n = m$. Then

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Since $0 < r_2 \le r_1$ and (2.10), we have $b_i \le r_1 b_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. From (2.9), we get $0 \le a_0 + b_0$ and $a_i + b_i \le r_1(a_{i+1} + b_{i+1})$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. From Proposition 1.1 part 1), we have $f(x) + g(x) \in Q(r_1)$.

2) Suppose that $n > m$. Then

$$f(x) + g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_{m+1}x^{m+1} + (a_m + b_m)x^m$$
$$+ (a_{m-1} + b_{m-1})x^{m-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Since $0 < r_2 \leq r_1$ and (2.10), we have $b_j \leq r_1 b_{j+1}$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. From (2.9), we have $0 \leq a_0 + b_0$ and $a_j + b_j \leq r_1(a_{j+1} + b_{j+1})$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. By Proposition 1.1 part 1) and (2.9), we have $f(x) + g(x) \in Q(r_1)$ if and only if $a_m + b_m \leq r_1 a_{m+1}$.

3) Suppose that $n < m$. Then

$$f(x) + g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_{n+1}x^{n+1} + (a_n + b_n)x^n$$
$$+ (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Since $0 < r_2 \leq r_1$ and (2.10), we have $b_i \leq r_1 b_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. From (2.9), we have $0 \leq a_0 + b_0$ and $a_i + b_i \leq r_1(a_{i+1} + b_{i+1})$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. By Proposition 1.1 part 1) and (2.10), we have $f(x) + g(x) \in Q(r_1)$ if and only if $a_n + b_n \leq r_1 b_{n+1}$. □

Example 2.16. Let $f(x) = x^2 + 2x + 6$ and $g(x) = 2x^2 + 4x + 3$. From Proposition 2.1 part 3), we have $f(x) \in Q(3)$ and $g(x) \in Q(2)$. By Proposition 2.15 part 1),

$$f(x) + g(x) = 3x^2 + 6x + 9 \in Q(3).$$

Example 2.17. Let $f(x) = 4x^3 + 2x^2 + 8x + 12$ and $g(x) = 5x^2 + 8x + 24$. From Proposition 2.1 part 3), we have $f(x) \in Q(4)$ and $g(x) \in Q(3)$. Since

$$a_2 + b_2 = 2 + 5 \leq (4)(4) = (r_1)(a_3),$$

by Proposition 2.15 part 2), we have

$$f(x) + g(x) = 4x^3 + 7x^2 + 16x + 36 \in Q(4).$$

Example 2.18. Let $f(x) = 2x^2 + 8x + 7$ and $g(x) = 3x^3 + 4x^2 + 8x + 10$. From Proposition 2.1 part 3), we have $f(x) \in Q(4)$ and $g(x) \in Q(2)$. Since

$$a_2 + b_2 = 2 + 4 \leq (4)(3) = (r_1)(b_3),$$

by Proposition 2.15 part 3), we have

$$f(x) + g(x) = 3x^3 + 6x^2 + 16x + 17 \in Q(4).$$

Corollary 2.19. *Keeping the notation of Proposition 2.15. Let*

$$m[f] := \min_{\substack{0 \leq i \leq n-1 \\ a_{i+1} \neq 0}} \left(\frac{a_i}{a_{i+1}} \right) \text{ and } m[g] := \min_{\substack{0 \leq j \leq m-1 \\ b_{j+1} \neq 0}} \left(\frac{b_j}{b_{j+1}} \right).$$

1) If $n = m$ and $M[f] = M[g] = m[g]$, then $M[f+g] = M[f]$.

2) If $n > m$, $M[f] = M[g] = m[g]$ and $a_m + b_m \leq a_{m+1} M[f]$, then $M[f+g] = M[f]$.

3) If $n < m$, $M[f] = M[g] = m[f]$ and $a_n + b_n \leq b_{n+1} M[f]$, then $M[f+g] = M[f]$.

Proof. 1) By Proposition 2.15 part 1) and Proposition 2.1 part 3), we have $M[f+g] \leq M[f] = m[g]$. Then

$$b_{i+1} M[f+g] \leq b_i \text{ for all } i \in \{0, 1, 2, \ldots, n-1\}.$$

Since

$$(a_{i+1} + b_{i+1}) M[f+g] \geq (a_i + b_i) \text{ for all } i \in \{0, 1, 2, \ldots, n-1\},$$

we have

$$a_{i+1} M[f+g] \geq a_i \text{ for all } i \in \{0, 1, 2, \ldots, n-1\}.$$

By Proposition 2.1 part 3), we get $M[f+g] \geq M[f]$. Hence $M[f+g] = M[f]$.

2) Since $n > m$, we have

$$f(x) + g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_{m+1}x^{m+1} + (a_m + b_m)x^m$$
$$+ (a_{m-1} + b_{m-1})x^{m-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0). \tag{2.11}$$

By Proposition 2.15 part 2) and Proposition 2.1 part 3), we have $M[f+g] \leq M[f] = m[g]$. Then

$$b_{i+1}M[f+g] \leq b_i \text{ for all } i \in \{0, 1, 2, \ldots, m-1\}.$$

Since

$$(a_{i+1} + b_{i+1})M[f+g] \geq (a_i + b_i) \text{ for all } i \in \{0, 1, 2, \ldots, m-1\},$$

we have

$$a_{i+1}M[f+g] \geq a_i \text{ for all } i \in \{0, 1, 2, \ldots, m-1\}. \tag{2.12}$$

Since $a_{m+1}M[f+g] \geq a_m + b_m$ and $b_m > 0$, we have

$$a_{m+1}M[f+g] \geq a_m. \tag{2.13}$$

By Proposition 2.1 part 3) and (2.11), we get

$$a_{i+1}M[f+g] \geq a_i \text{ for all } i \in \{m+1, m+2, m+3, \ldots, n-1\}. \tag{2.14}$$

By Proposition 2.1 part 3) and (2.12) – (2.14), we have $M[f+g] \geq M[f]$. Hence $M[f+g] = M[f]$. The proof of part 3) is similar to part 2). □

Example 2.20. Let $f(x) = 7x^3 + 4x^2 + 6x + 12$ and $g(x) = x^2 + 2x + 4$. Then $M[f] = M[g] = m[g] = 2$ and $a_2 + b_2 = 5 \leq 14 = a_3M[f]$. By Corollary 2.19, we get

$$M[f+g] = M[7x^3 + 5x^2 + 8x + 16] = 2 = M[f].$$

Corollary 2.21. *Let $0 < r_s \leq r_{s-1} \leq \ldots \leq r_1$. If $f_k(x) \in Q(r_k)$ have the same degree for all $k \in \{1, 2, 3, \ldots, s\}$, then*

$$\sum_{k=1}^{s} f_k(x) \in Q(r_1).$$

Example 2.22. Let $f_1(x) = x^3 + 4x^2 + 11x + 7$, $f_2(x) = x^3 + 3x^2 + 2x + 1$ and $f_3(x) = 2x^3 + 3x^2 + 6x + 2$. By Proposition 2.1 part 3), we have $f_1(x) \in Q(4)$, $f_2(x) \in Q(3)$ and $f_3(x) \in Q(2)$. By Corollary 2.21, we have

$$f_1(x) + f_2(x) + f_3(x) = 4x^3 + 10x^2 + 19x + 10 \in Q(4).$$

2.4 Zeros of a polynomial in $Q(r)$

Our next result provides equivalent conditions for polynomials with real coefficients to have zeros which are not real and positive.

Theorem 2.23. *Let $z_1, z_2, \ldots, z_k$ be complex numbers. Then the following statements are equivalent:*

1) all elements $z_1, z_2, \ldots, z_k$ are in $\mathbb{C} \setminus \mathbb{R}^+$;

2) there exists $p(x) \in Q(r)$ for any $r > |z_1| + \cdots + |z_k|$, such that

$$p(z_1) = p(z_2) = \cdots = p(z_k) = 0;$$

3) there exists $g(x) \in \Pi$ such that $g(z_1) = g(z_2) = \cdots = g(z_k) = 0$;

4) there exists $q(x) \in Q$ and $w \in \mathbb{R}^+ \setminus \{z_1, \ldots, z_k\}$ such that

$$q(w) = q(z_1) = q(z_2) = \cdots = q(z_k) = 0.$$

Proof. (1) $\Rightarrow$ (2). If $r = |z_1| + |z_2| + \cdots + |z_k| + \epsilon$, $\epsilon > 0$, then by Theorem 1.4, there exists $p_j(x) \in Q(|z_j| + \epsilon/k)$ such that $p_j(z_j) = 0$ for all $j \in \{1, 2, 3, \ldots, k\}$. Let $p(x) = \prod_{j=1}^{k} p_j(x)$. Clearly, $p(z_1) = p(z_2) = \cdots = p(z_k) = 0$. From Theorem 1.3, we know that

$$p(x) \in Q\left(|z_1| + |z_2| + \cdots + |z_k| + \epsilon\right) = Q(r).$$

(2) $\Rightarrow$ (3) Suppose that $p(x) \in Q(r)$ for any $r > |z_1| + \cdots + |z_k|$, such that

$$p(z_1) = p(z_2) = \cdots = p(z_k) = 0\ .$$

From Proposition 1.1 part 4), we have $Q(r) \subseteq \Pi$. Then $p(x) \in \Pi$.

(3) $\Rightarrow$ (4). Assume that there exists

$$g(x) = g_m x^m + g_{m-1} x^{m-1} + \cdots + g_1 x + g_0 \in \Pi$$

such that $g(z_1) = g(z_2) = \cdots = g(z_k) = 0$. Since $g(x) \in \Pi$, we have $g_m > 0$ and $g_t \geq 0$ for all $t \in \{0, 1, 2, \ldots, m-1\}$ and so none of $z_1, z_2, \ldots, z_k$ can be real and positive. If all the elements $z_1, z_2, \ldots, z_k$ are equal to 0, choosing $q(x) = x^2 - x \in Q$, we see that (iv) holds with $w = 1$. If $z_j \neq 0$ for some $j \in \{1, 2, 3, \ldots, k\}$, choose

$$q(x) = (g_m x^m)^2 - (g_{m-1} x^{m-1} + g_{m-2} x^{m-2} + \cdots + g_1 x + g_0)^2 \in Q.$$

Since $g(x)$ is a factor of $q(x)$, we have $q(z_1) = q(z_2) = \cdots = q(z_k) = 0$. From $z_j \neq 0$ and Proposition 1.1 part 5), we know that $q(x)$ has a unique positive zero, say w, which must then be distinct from all $z_1, z_2, \ldots, z_k$, as desired.

(4) $\Rightarrow$ (1) follows directly from Proposition 1.1 part 5). $\square$

Example 2.24. Let $z_1 = -2,\ z_2 = -i,\ z_3 = (-1+i\sqrt{3})/2$. Then $|z_1|+|z_2|+|z_3| = 4$. Taking

$$p(x) = (x+2)(x^2+x+1)(x^3+x^2+x+1) = x^6+4x^5+7x^4+9x^3+8x^2+5x+2 \in \Pi,$$

we see that $p(z_1) = p(z_2) = p(z_3) = 0$. Since

$$\begin{aligned}(x-4)p(x) &= (x-4)(x^6+4x^5+7x^4+9x^3+8x^2+5x+2)\\ &= x^7 - 9x^5 - 19x^4 - 28x^3 - 27x^2 - 18x - 8 \in Q,\end{aligned}$$

we have $p(x) \in Q(4)$. From Proposition 1.1 part 3), $p(x) \in Q(r)$ for all $r > 4$. This agrees with Theorem 2.23 parts ii) and iii). To verify Theorem 2.23 part iv), take

$$\begin{aligned}q(x) = (x^6)^2 - (4x^5+7x^4+9x^3+8x^2+5x+2)^2 = x^{12} - 16x^{10} - 56x^9\\ - 121x^8 - 190x^7 - 233x^6 - 230x^5 - 182x^4 - 116x^3 - 57x^2 - 20x - 4.\end{aligned}$$

Clearly, $q(x) \in Q$, and by direct computation we find $q(w) = q(z_1) = q(z_2) = q(z_3) = 0$, where $w \approx 5.59114$.

A simple necessary condition for a real polynomial to belong to $Q(r)$ is given in the next lemma, which will be used in the next chapter.

Lemma 2.25. *Let* $r > 0$. *If*

$$\mathcal{F}(x) = F_m x^m + F_{m-1} x^{m-1} + \cdots + F_2 x^2 + F_1 x + F_0 \in Q(r),$$

then $\mathcal{F}(1) - F_m \leq r(\mathcal{F}(1) - F_0)$.

Proof. Since $\mathcal{F}(x) \in Q(r)$ and Proposition 1.1 part 1), we have $0 \leq F_0$ and $F_i \leq rF_{i+1}$ for all $i \in \{0, 1, 2, \ldots, m-1\}$. Thus,

$$\mathcal{F}(1) - F_m = F_0 + F_1 + \cdots + F_{m-1} \leq r(F_1 + \cdots + F_m) = r(\mathcal{F}(1) - F_0).$$

□

Example 2.26. Let $\mathcal{F}(x) = 3x^4 + 2x^3 + 4x^2 + 5x + 1$. From Proposition 2.1 part 3), we have $\mathcal{F}(x) \in Q(2)$. By Lemma 2.25, we get

$$\mathcal{F}(1) - F_m = 12 \leq (2)(14) = r(\mathcal{F}(1) - F_0).$$

CHAPTER III

PRODUCT OF TWO POLYNOMIALS

Given $f(x), d(x) \in \mathbb{R}[x] \smallsetminus \mathbb{R}$. This chapter is devoted to the problem of finding conditions ensuring that $f(x)d(x) \notin Q(r)$. The next theorem is a simple application of Lemma 2.25.

Theorem 3.1. *Let $r > 0$ and let*

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{R}[x] \smallsetminus \mathbb{R},$$
$$d(x) = d_m x^m + d_{m-1}x^{m-1} + \cdots + d_1 x + d_0 \in \mathbb{R}[x] \smallsetminus \mathbb{R}.$$

If $f(1)d(1)(1-r) > c_n d_m - rc_0 d_0$, then $f(x)d(x) \notin Q(r)$.

Proof. Assume that $f(x)d(x) \in Q(r)$. From Lemma 2.25, we know that

$$f(1)d(1) - c_n d_m \leq r(f(1)d(1) - c_0 d_0),$$

i.e., $f(1)d(1)(1-r) \leq c_n d_m - rc_0 d_0$, contradicting the hypothesis $f(1)d(1)(1-r) > c_n d_m - rc_0 d_0$. $\square$

Example 3.2. Let $r = 1/2$, $f(x) = 2x^3 + 4x^2 + x + 1$ and $d(x) = x^2 + 2x + 1$. Then

$$f(1)d(1)(1-r) = (8)(4)\left(1 - \frac{1}{2}\right) > (2)(1) - \left(\frac{1}{2}\right)(1)(1) = c_3 d_2 - rc_0 d_0.$$

By Theorem 3.1, we have $f(x)d(x) = 2x^5 + 8x^4 + 11x^3 + 7x^2 + 3x + 1 \notin Q(1/2)$.

Example 3.3. Let $r = 1$, $f(x) = x^2 + x + 1$ and $d(x) = x^3 + 1$. Then

$$f(1)d(1)(1 - r) = (3)(1)\,(1 - 1) = 0 = (1)(1) - (1)(1)(1) = c_2d_3 - rc_0d_0.$$

By Proposition 2.1 part 3), we have $f(x)d(x) = x^5 + x^4 + x^3 + x^2 + x + 1 \in Q(1)$.

Remark 3.4. The converse of Theorem 3.1 is not true. For example, let $r = 3$, $f(x) = x^2 + 2x$ and $d(x) = x + 3$. By Proposition 2.1 part 3), we have

$$f(x)d(x) = x^3 + 5x^2 + 6x \notin Q(3).$$

But $f(1)d(1)(1 - r) = (3)(4)(1 - 3) < (1)(1) - (3)(0)(3) = c_2d_1 - rc_0d_0$.

Corollary 3.5. *Keeping the notation of Theorem 3.1, if $c_n > 0$, $c_0 > 0$ and*

$$f(1) - c_n \geq rf(1),$$

then $f(x)d(x) \notin Q(r)$ for all $d(x) \in \mathbb{R}^+[x] \setminus \mathbb{R}$.

Proof. Assume that there exists a polynomial $d(x) \in \mathbb{R}^+[x] \setminus \mathbb{R}$ such that $f(x)d(x) \in Q(r)$. Then we have $d(1) > d_m > 0$ and $d_0 > 0$. Since $f(1) - c_n \geq rf(1)$, we get $f(1)(1 - r) \geq c_n$. From $d(1) > 0$, we have

$$f(1)d(1)(1 - r) \geq c_nd(1).$$

From $d(1) > d_m$ and $c_n > 0$, we get $c_nd(1) > c_nd_m$. Then

$$f(1)d(1)(1 - r) > c_nd_m,$$

Since $rc_0d_0 > 0$, we have

$$f(1)d(1)(1-r) > c_nd_m - rc_0d_0,$$

contradicting Theorem 3.1. □

Example 3.6. Let $r = 12$, $f(x) = 2x^2 - 3x + 27$ and $d(x) = x + 5$. Then

$$f(1) - c_2 = 24 < (12)(26) = rf(1).$$

By Proposition 2.1 part 3), we have $f(x)d(x) = 2x^3 + 7x^2 + 12x + 135 \in Q(12)$.

3.1 Brunotte exponents

Conditions ensuring that a product of two real polynomials does not belong to $Q(r)$ can also be derived using the following lemma of Brunotte, [3, Lemma 2].

Lemma 3.7. *([3, Lemma 2]) Let $s > 0$. If $d(x) \in \mathbb{R}[x]$ is a monic polynomial having no nonnegative roots, then there exists an $h \in \mathbb{N}$ bounded by an effectively computable constant such that $(x+s)^h d(x)$ has only positive coefficients. (We call the parameter $h = h_{s,d}$ the Brunotte exponent of $d(x)$ with respect to s.)*

Proposition 3.8. *Let $r > 0$ and $s > 0$,*

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \in \mathbb{R}[x] \setminus \mathbb{R},$$
$$d(x) = x^m + d_{m-1}x^{m-1} + \cdots + d_1x + d_0 \in \mathbb{R}[x] \setminus \mathbb{R}.$$

Assume that $d(x)$ has no nonnegative real roots with $h = h_{s,d}$ being its Brunotte exponent of $d(x)$ with respect to s. If $c_0 > 0$, $f(1) - 1 \geq rf(1)$ and $c_{n-1} + d_{m-1} - r + hs < 0$, then $f(x)d(x) \notin Q(r)$.

Proof. Since $(x-r)f(x)d(x) = x^{n+m+1} + (c_{n-1} + d_{m-1} - r)x^{n+m} + \cdots + (-rc_0d_0)$, the hypothesis $c_{n-1} + d_{m-1} - r + hs < 0$ and Theorem 2.7 show that $(x+s)^h f(x)d(x) \in Q(r)$. Since $(x+s)^h d(x) \in \mathbb{R}^+[x]$, the hypothesis $f(1) - 1 \geq rf(1)$ and Corollary 3.5 imply that $f(x)d(x) \notin Q(r)$. □

3.2 Eneström-Kakeya like conditions

In this section, we derive some Eneström-Kakeya like conditions which are necessarily for a product of two polynomials not to be in $Q(r)$.

Theorem 3.9. *Let $r > 0$ and $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}^+[x]$. If*

$$c_i > rc_{i+1} \quad for\ all \quad i \in \{0, 1, 2, \ldots, n-1\}, \tag{3.1}$$

then $f(x)d(x) \notin Q(r)$ for all $d(x) \in \mathbb{R}[x] \setminus \{0\}$.

Proof. Let

$$d(x) = \sum_{j=0}^{m} d_j x^j \in \mathbb{R}\,[x] \setminus \{0\}.$$

If $f(x)d(x) \in Q(r)$, then Proposition 1.1 part 1) gives

$$0 \leq c_0d_0 \tag{3.2}$$

$$c_0d_0 \leq r(c_1d_0 + c_0d_1) \tag{3.3}$$

$$c_1d_0 + c_0d_1 \leq r(c_2d_0 + c_1d_1 + c_0d_2) \tag{3.4}$$

$$\vdots$$

$$c_{m-1}d_0 + c_{m-2}d_1 + \cdots + c_0d_{m-1} \leq r(c_md_0 + c_{m-1}d_1 + \cdots + c_0d_m) \tag{3.5}$$

$$c_md_0 + c_{m-1}d_1 + \cdots + c_0d_m \leq r(c_{m+1}d_0 + c_md_1 + \cdots + c_1d_m), \tag{3.6}$$

where we adopt the convention that $c_i = 0$ for all $i > n$, and $d_j = 0$ for all $j > m$.

From $c_0 > 0$ and (3.2), we get $d_0 \geq 0$. From (3.3) and (3.1), we have

$$(c_0 - rc_1)d_0 \leq rc_0 d_1 \quad \text{and} \quad d_1 \geq 0.$$

From (3.4) and (3.1), we get

$$(c_1 - rc_2)d_0 + (c_0 - rc_1)d_1 \leq rc_0 d_2,$$

which together with previous results yield $d_2 \geq 0$. Continuing in the same manner up to (3.5), we get $d_3 \geq 0, d_4 \geq 0, \ldots, d_m \geq 0$. Thus,

$$(c_m - rc_{m+1})d_0 + (c_{m-1} - rc_m)d_1 + \cdots + (c_1 - rc_2)d_{m-1} + (c_0 - rc_1)d_m \geq 0. \quad (3.7)$$

Since $c_i > rc_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$, the left hand expression in (3.7) can be 0 only when $d_0 = d_1 = \cdots = d_m = 0$, i.e., $d(x) \equiv 0$, which is not possible. Thus, the strict inequality holds in (3.7), which contradicts (3.6). $\square$

Example 3.10. Let $r = 3$ and $f(x) = x^3 + 6x^2 + 19x + 60$. Then the condition (3.1) is true. Choose $d(x) = x^2 + 2x + 3$. By Theorem 3.9, we have

$$f(x)d(x) = x^5 + 8x^4 + 38x^3 + 116x^2 + 177x + 180 \notin Q(3).$$

CHAPTER IV

ENESTRÖM-KAKEYA QUOTIENTS

In this chapter, we investigate the lower and upper Eneström-Kakeya quotients and their connection with the reciprocal polynomials.

Definition 4.1. Let $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}^+[x]$ be a non-constant polynomial. We define its **lower and upper Eneström-Kakeya quotients**, respectively, by

$$\alpha[f] := \min\left\{\frac{c_0}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_{n-2}}{c_{n-1}}, \frac{c_{n-1}}{c_n}\right\}$$

and

$$\beta[f] := \max\left\{\frac{c_0}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_{n-2}}{c_{n-1}}, \frac{c_{n-1}}{c_n}\right\}.$$

Proposition 4.2. *Let $f(x) \in \mathbb{R}^+[x]$ be a non-constant polynomial.*

1) *The upper Eneström-Kakeya quotient $\beta[f]$ is the smallest $r > 0$ such that $f(x) \in Q^+(r)$.*

2) *The lower Eneström-Kakeya quotient has the property that if $p(x) \in Q^+(r)$ with $0 < r < \alpha[f]$, then $f(x) \nmid p(x)$ over $\mathbb{R}[x]$.*

Proof. 1) The first part is obtained directly by Proposition 2.1 part 3) and Corollary 2.2 part 1). Next, we prove part 2). Let $p(x) \in Q^+(r)$ with $0 < r < \alpha[f]$. By Definition 4.1, we have

$$r < \frac{c_i}{c_{i+1}} \quad \text{for all} \ \ i \in \{0, 1, 2, \ldots, n-1\}.$$

Since $f(x) \in \mathbb{R}^+[x]$, we have $c_i > rc_{i+1}$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. Assume that $f(x) \mid p(x)$ over $\mathbb{R}[x]$. There exists some polynomial $d(x) \in \mathbb{R}[x]$ such that $p(x) = f(x)d(x) \in Q^+(r)$. Then $d(x) \neq 0$, contradicting with Theorem 3.9. □

Polynomials having positive real coefficients with equal upper and lower Eneström-Kakeya quotients are of very special form which are intimately connected to Eneström-Kakeya theorem as analyzed by Hurwitz, [1].

Proposition 4.3. *Let* $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{R}^+[x]$ *be a non-constant polynomial. Then the lower and upper Eneström-Kakeya quotients of* $f(x)$ *are equal, i.e.,* $\alpha[f] = \beta[f]$ *if and only if* $f(x)$ *is of the form* $f(x) = c_n\left(x^n + tx^{n-1} + t^2x^{n-2} + \cdots + t^{n-1}x + t^n\right)$ *for some positive real number* t.

Proof. Suppose that $\alpha[f] = \beta[f]$. Then

$$\frac{c_0}{c_1} = \frac{c_1}{c_2} = \cdots = \frac{c_{n-3}}{c_{n-2}} = \frac{c_{n-2}}{c_{n-1}} = \frac{c_{n-1}}{c_n},$$

and so

$$c_{n-2} = \frac{c_{n-1}^2}{c_n},\ c_{n-3} = \frac{c_{n-2}^2}{c_{n-1}} = \frac{c_{n-1}^3}{c_n^2},\ \ \ldots\ ,\ c_1 = \frac{c_{n-1}^{n-1}}{c_n^{n-2}},\ c_0 = \frac{c_{n-1}^n}{c_n^{n-1}},$$

and so

$$\begin{aligned} f(x) &= c_nx^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + \cdots + c_1x + c_0 \in \mathbb{R}^+[x] \setminus \mathbb{R} \\ &= c_n\left(x^n + tx^{n-1} + t^2x^{n-2} + \cdots + t^{n-1}x + t^n\right) \end{aligned}$$

where $t = c_{n-1}/c_n > 0$. The converse is trivial. □

The upper and lower Eneström-Kakeya quotients are inverse of each other for a special class of polynomials known as self-reciprocal polynomials.

Definition 4.4. Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with $\deg(f) = n$. The **reciprocal polynomial** of $f(x)$ is defined as

$$f^*(x) = x^n f(1/x) = c_n + c_{n-1}x + \cdots + c_1 x^{n-1} + c_0 x^n,$$

and we say that $f(x)$ is **self-reciprocal** if $f(x) = f^*(x)$.

Proposition 4.5. *If $f(x) \in \mathbb{R}^+[x]$ is a non-constant polynomial, then $\beta[f^*] = 1/\alpha[f]$, and $f^*(x) \in Q^+(1/\alpha[f])$. Moreover, if $f(x)$ is self-reciprocal, then $\beta[f] = 1/\alpha[f]$ and $f(x) \in Q^+(1/\alpha[f])$.*

Proof. Writing $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{R}^+[x] \smallsetminus \mathbb{R}$, we have

$$f^*(x) = x^n f(1/x) = c_n + c_{n-1}x + \cdots + c_1 x^{n-1} + c_0 x^n$$

and so

$$\beta[f^*] = \max\left\{\frac{c_1}{c_0}, \frac{c_2}{c_1}, \ldots, \frac{c_{n-1}}{c_{n-2}}, \frac{c_n}{c_{n-1}}\right\} = 1/\alpha[f].$$

From Proposition 4.2 part 1), we have $f^*(x) \in Q^+(\beta[f^*]) = Q^+(1/\alpha[f])$. If $f(x)$ is self-reciprocal, then clearly $\beta[f] = 1/\alpha[f]$ and $f(x) \in Q^+(1/\alpha[f])$. □

Proposition 4.6. *If $f(x) \in \mathbb{R}^+[x]$ is a non-constant self-reciprocal polynomial, then*

1) $\alpha[f] \le 1$, *and*

2) $\alpha[f] = 1$ *if and only if* $\beta[f] = 1$.

Proof. 1) If $\alpha[f] > 1$, then $1/\alpha[f] < 1 < \alpha[f]$. Since f is self-reciprocal, by Proposition 4.5, we get $f(x) \in Q^+(1/\alpha[f])$, which contradicts Proposition 4.2 part 2).

2) follows readily from Proposition 4.5. □

Remark 4.7. From Proposition 4.3 and Proposition 4.6 part 2), a nonconstant self-reciprocal polynomial $f(x)$ with equal lower and upper Eneström-Kakeya quotients is of the from

$$f(x) = c_n \left(x^n + x^{n-1} + x^{n-2} + \cdots + x + 1 \right).$$

CHAPTER V

CONNECTION WITH LINEAR RECURSIONS

In this chapter, we give a connection between linear recursions and polynomials in $Q(r)$.

Definition 5.1. Let k be a positive integer. We say that the sequence $(u_n)_{n\geq 0} \subseteq \mathbb{C}$ satisfies a **linear recursion of order** k if there exist complex numbers $b_1, \ldots, b_k \neq 0$ such that

$$u_{n+k} = b_1 u_{n+k-1} + b_2 u_{n+k-2} + \cdots + b_k u_n \quad (n \geq 0).$$

By Theorem A.7 [6], we have the following theorem:

Theorem 5.2. *Let $(u_n)_{n\geq 0}$ be a sequence of complex numbers. The following two assertions are equivalent:*

1) $(u_n)_{n\geq 0}$ satisfies a linear recursion of order k, i.e., there exist complex numbers $b_1, b_2, \ldots, b_k \neq 0$ such that

$$u_{n+k} = b_1 u_{n+k-1} + b_2 u_{n+k-2} + \cdots + b_k u_n \quad (n \geq 0).$$

2) the power series $\sum_{n=0}^{\infty} u_n x^n$ has positive radius of convergence and there exists a polynomial $f(x)$ of degree k such that the product $f(x)\sum_{n=0}^{\infty} u_n x^n$ is a polynomial of degree $< k$ with complex coefficients.

In 1992, Roitman and Rubinstein characterized linear recursions which imply a linear recursion with nonnegative coefficients.

Theorem 5.3. *[7, Theorem 5]) Let $a_1, a_2, \ldots, a_k$ be given complex numbers, and let $P(x) = x^k - a_1x^{k-1} - \cdots - a_k$. Then the conditions (A), (B) and (C) below are equivalent:*

(A) Any infinite sequence $(u_n)_{n\geq 0}$ of complex numbers which satisfies the recursion

$$u_{n+k} = a_1u_{n+k-1} + a_2u_{n+k-2} + \cdots + a_ku_n \quad (n \geq 0),$$

for $n \geq 0$ satisfies also a linear recursion with nonnegative coefficients.

(B) The polynomial $P(x)$ divides a polynomial in Q.

(C) In Case the polynomial $P(x)$ has a positive zero s, then all conditions 1)–4) below are satisfied:

1) $s \geq |\omega|$ for any zero ω of $P(x)$;

2) if $|\omega| = s$ for some zero ω of $P(x)$, then ω/s is a zero of unity;

3) all zero of $P(x)$ with absolute value s are simple;

4) if $P(s) = P(s\epsilon) = 0$, where $\epsilon^k = 1$ with $k \geq 1$ minimal, then $P(x)$ has no zeros of the form $t\gamma$ where $0 < t < s$ and $\gamma^k = 1$.

Next, we show that polynomials in $Q(r)$ are related to sequences satisfying certain linear recursions.

Definition 5.4. Let $r > 0$ and $(u_n)_{n\geq 0}$ be a sequence in $\mathbb{C}$. We say that a sequence $(u_n)_{n\geq 0}$ belongs to the set $CQ^+(r)$ if it satisfies a linear recursion of the form

$$u_{n+t} = -q_{t-1}u_{n+t-1} - q_{t-2}u_{n+t-2} - \cdots - q_0u_n \quad (n \geq 0), \tag{5.1}$$

for some fixed $t \in \mathbb{N}$, where its characteristic polynomial

$$q(x) = x^t + q_{t-1}x^{t-1} + \cdots + q_1 x + q_0$$

is in $Q^+(r)$.

Theorem 5.5. *Let $k \in \mathbb{N}$, $r > 0$, and $p(x) = x^k - p_1 x^{k-1} - \cdots - p_k \in \mathbb{C}[x]$. Then the following assertions (A) and (B) are equivalent.*

(A) Any infinite sequence $(u_n)_{n\geq 0}$ of complex numbers which satisfies the recursion

$$u_{n+k} = p_1 u_{n+k-1} + p_2 u_{n+k-2} + \cdots + p_k u_n \quad (n \geq 0), \tag{5.2}$$

belongs to the set $CQ^+(r)$.

(B) The polynomial $p(x)$ divides a polynomial in $Q^+(r)$.

Proof. We have

$$p^*(x) = x^k p\left(\frac{1}{x}\right) = 1 - p_1 x - \cdots - p_{k-1}x^{k-1} - p_k x^k \in \mathbb{C}[x].$$

(A) $\Rightarrow$ (B). Consider a power series $1/p^*(x)$. Write $1/p^*(x) = \sum_{n=0}^{\infty} u_n x^n$. We have that $p^*(x) \sum_{n=0}^{\infty} u_n x^n = 1$ which is a constant polynomial. By Theorem 5.2, we have the power series $1/p^*(x)$ satisfies a linear recursion (5.2). From (A), the power series $1/p^*(x)$ satisfies also a linear recursion with coefficients in $CQ^+(r)$. There exists some $t \in \mathbb{N}$ and positive real number $q_1, q_2, \ldots, q_t$ such that the power series $1/p^*(x)$ satisfies also a linear recursion

$$u_{n+t} = -q_{t-1}u_{n+t-1} - q_{t-2}u_{n+t-2} - \cdots - q_0 u_n \quad (n \geq 0),$$

and its characteristic polynomial is

$$q(x) = x^t + q_{t-1}x^{t-1} + \cdots + q_1 x + q_0 \in Q^+(r).$$

We have

$$q^*(x) = x^t q\left(\frac{1}{x}\right) = 1 + q_{t-1}x + \cdots + q_1 x^{t-1} + q_0 x^t.$$

Thus,

$$\frac{q^*(x)}{p^*(x)} = q^*(x) \sum_{n=0}^{\infty} u_n x^n$$

is a polynomial of degree $< t$, i.e., $p^*(x)|q^*(x)$. Then

$$p(x) = x^k p^*\left(\frac{1}{x}\right) | x^t q^*\left(\frac{1}{x}\right) = q(x)$$

so $p(x)|q(x)$.

(B) $\Rightarrow$ (A). Let $q(x) = q_t x^t + q_{t-1}x^{t-1} + \cdots + q_1 x + q_0 \in Q^+(r)$ be divisible by $p(x)$. We can assume that $q_t = 1$. Then there is a polynomial $g(x)$ of degree s such that $p(x)g(x) = q(x)$. Thus,

$$p^*(x)g^*(x) = x^k p\left(\frac{1}{x}\right) \cdot x^s g\left(\frac{1}{x}\right) = x^{k+s} q\left(\frac{1}{x}\right) = q^*(x),$$

where $k + s = t$. Assume that an infinite sequence (u_n) of complex numbers satisfies the recursion (5.2). Thus, $p^*(x)\sum_{n\geq 0} u_n x^n$ is a polynomial of degree $< k$. Consequently,

$$q^*(x) \sum_{n\geq 0} u_n x^n = g^*(x) p^*(x) \sum_{n\geq 0} u_n x^n$$

is a polynomial of degree $< k + s$. By Theorem 5.2, we have

$$u_{n+t} = -q_{t-1}u_{n+t-1} - q_{t-2}u_{n+t-2} - \cdots - q_0 u_n$$

for all $n \geq 0$, showing that the sequence $(u_n)_{\geq 0}$ satisfies a recursion of the form (5.1). Since $q(x) \in Q^+(r)$ and Definition 5.4, we have $(u_n)_{n\geq 0} \in CQ^+(r)$. $\square$

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