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เทียบกับฟังก์ชันความหนาแน่นเอกซ์โพเนนเชียล

นายกำธร ไชยลึก

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A POINTWISE BOUND FOR A HOLOMORPHIC FUNCTION WHICH IS SQUARE-INTEGRABLE WITH RESPECT TO AN EXPONENTIAL DENSITY FUNCTION

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กำธร ไชยลึก : ขอบเขตรายจุดของฟังก์ชันโฮโลมอร์ฟิกซึ่งกำลังสองอินทิเกรตได้เทียบกับฟังก์ชันความหนาแน่นเอกซ์โพเนนเชียล.(A POINTWISE BOUND FOR A HOLOMORPHIC FUNCTION WHICH IS SQUARE-INTEGRABLE WITH RESPECT TO AN EXPONENTIAL DENSITY FUNCTION) อ. ที่ปรึกษา : อ.ดร.วิชาญ ลิ่วกีรติยุตกุล, 24 หน้า. ISBN 974-03-1711-1.

ให้ φ เป็นฟังก์ชันค่าจริงบน C ซึ่งสอดคล้องกับเงื่อนไข $0 \le \Delta\varphi \le M$ สำหรับบาง $M > 0$ และให้ $HL^2(C, e^{-\varphi})$ แทนปริภูมิของฟังก์ชันโฮโลมอร์ฟิกบน C ซึ่งกำลังสองอินทิเกรตได้เทียบกับเมเซอร์ $e^{-\varphi(z)}dz$ ในวิทยานิพนธ์นี้ เราหาขอบเขตรายจุดของฟังก์ชันในปริภูมินี้ ซึ่งได้ว่าจะมีค่าคงตัวK ซึ่งขึ้นกับ φ เท่านั้นที่ทำให้

$$|f(z)|^2 \le Ke^{\varphi(z)}\|f\|^2_{L^2(C,e^{-\varphi})}$$

สำหรับ f ใดๆ ใน $HL^2(C, e^{-\varphi})$ และ z ใดๆ ใน C

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต……………………………………………

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา…………………………..

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Let φ be a real-valued function on C satisfying $0 \le \Delta\varphi \le M$ for some $M > 0$. Denote by $HL^2(C, e^{-\varphi})$ the space of all holomorphic functions on C which are square-integrable with respect to the measure $e^{-\varphi(z)}dz$. In this work, we obtain a pointwise bound of any function in this space. We can show that there exists a constant K depending only on φ such that

$$|f(z)|^2 \le K e^{\varphi(z)} \|f\|^2_{L^2(C, e^{-\varphi})}$$

for any f in $HL^2(C, e^{-\varphi})$ and any $z \in C$.

Department **Mathematics** Student's signature……………………………

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Chapter 1

Introduction

Let U be a non-empty open subset of $\mathbb{C}$. Denote by $\mathcal{H}L^2(U, \alpha)$ the space of all holomorphic functions on U which are square-integrable with respect to the measure $\alpha(\omega)\, d\omega$.

For any $t > 0$, consider the Gaussian measure

$$d\mu_t(z) \;=\; \frac{1}{\pi t} e^{-|z|^2/t}\, dz.$$

Then the space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$ is called the Segal-Bargmann space. In this space, it is well-known that a pointwise bound for any function $f \in \mathcal{H}L^2(\mathbb{C}, \mu_t)$ is given by

$$|f(z)|^2 \;\leq\; e^{|z|^2/t} \|f\|^2_{L^2(\mathbb{C},\mu_t)}. \tag{1.1}$$

Indeed, for any space $\mathcal{H}L^2(U, \alpha)$, there exists a function $K(z, \omega)$ on $U \times U$, called the *reproducing kernel*, such that

$$|f(z)|^2 \;\leq\; K(z, z) \|f\|^2_{L^2(U,\alpha)} \tag{1.2}$$

for any $f \in \mathcal{H}L^2(U, \alpha)$ and any $z \in U$. The pointwise bound (1.1) for $\mathcal{H}L^2(\mathbb{C}, \mu_t)$ follows from the following formula of the reproducing kernel for the Segal-Bargmann space:

$$K(z, \omega) \;=\; e^{z\overline{\omega}/t}. \tag{1.3}$$

In this work, we study a pointwise bound for a function in a more general holomorphic function space. Note that $\Delta\left(|z|^2/t\right)$ is a positive constant, so we first replace the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C},\mu_t)$ by $\mathcal{H}L^2(\mathbb{C},e^{-\varphi})$, where $\Delta\varphi$ is a positive constant. The technique used will be that of holomorphic equivalence. Two holomorphic function spaces $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are holomorphically equivalent if there exists a nowhere zero holomorphic function ϕ on U such that

$$\beta(z) = \frac{\alpha(z)}{|\phi(z)|^2} \qquad \text{for all } z \in U.$$

We will show that if $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are holomorphically equivalent spaces, then

$$\alpha(z)K_\alpha(z,z) \;=\; \beta(z)K_\beta(z,z) \tag{1.4}$$

where K_α and K_β are their respective reproducing kernels. If $\Delta\varphi = c > 0$, then $\mathcal{H}L^2(\mathbb{C},e^{-\varphi})$ is holomorphically equivalent to the Segal-Barmann space $\mathcal{H}L^2(\mathbb{C},\mu_t)$ where $t = 4/c$. It follows from (1.2), (1.3) and (1.4) that

$$|f(z)|^2 \;\le\; \frac{c}{4\pi}e^{\varphi(z)}\|f\|^2_{L^2(\mathbb{C},e^{-\varphi})},$$

for any $f \in \mathcal{H}L^2(\mathbb{C},e^{-\varphi})$ and any $z \in \mathbb{C}$.

Next, we turn to the space $\mathcal{H}L^2(\mathbb{C},e^{-\varphi})$, where $\Delta\varphi$ is positive and bounded, i.e. $0 \le \Delta\varphi \le M$ for some $M > 0$. This space is not holomorphically equivalent to a Segal-Bargmann space, so we cannot apply the same technique here. Our proof can be divided into the following steps:

First, at $z = 0$, we show that for any $f \in \mathcal{H}L^2(\mathbb{C},e^{-\varphi})$,

$$|f(0)|^2 \;\le\; Ce^{\varphi(0)}\int_{D(0,1)} |f(\omega)|^2 e^{-\varphi(\omega)}\,d\omega$$

for some C depending only on M. Next, by translation to any point $z \in \mathbb{C}$, we have

$$|f(z)|^2 \;\le\; Ce^{\varphi(z)}\int_{D(z,1)} |f(\omega)|^2 e^{-\varphi(\omega)}\,d\omega.$$

Finally, a pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $0 \le \Delta\varphi \le M$ is given by

$$|f(z)|^2 \;\le\; Ce^{\varphi(z)}\|f\|^2_{L^2(\mathbb{C},e^{-\varphi})}.$$

Here is a brief summary of this work. In Chapter 2, we study basic properties of a holomorphic function space. After that, we introduce the concept of holomorphic equivalence and establish a necessary and sufficient condition for two spaces to be holomorphically equivalent in Chapter 3. In the remaining two chapters, we estimate a pointwise bound for functions in some holomorphic function spaces. In Chapter 4, we use some properties in Chapter 3 to estimate a pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $\Delta\varphi$ is a positive constant. Finally, in Chapter 5, we use the technique outlined above to estimate a pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $0 \le \Delta\varphi \le M$ for some $M > 0$.

Chapter 2

Holomorphic function spaces

Let U be a non-empty open subset of $\mathbb{C}$. Denote by $\mathcal{H}(U)$ the space of all holomorphic functions on U. If α is a strictly positive function on U, let $L^2(U,\alpha)$ be the space of all functions on U which are square-integrable with respect to the measure $\alpha(\omega)\,d\omega$. That is,

$$L^2(U,\alpha) = \left\{ f\colon U \to \mathbb{C} \,\middle|\, \int_{\mathbb{C}} |f(\omega)|^2 \alpha(\omega)\, d\omega < \infty \right\}.$$

Then $L^2(U,\alpha)$ is a Hilbert space. Let $\mathcal{H}L^2(U,\alpha) = \mathcal{H}(U) \cap L^2(U,\alpha)$. Then $\mathcal{H}L^2(U,\alpha)$ is a closed subspace of $L^2(U,\alpha)$ and hence a Hilbert space. Moreover, it is well-known that $\mathcal{H}L^2(U,\alpha)$ is separable.

Definition 2.1. A Segal-Bargmann space is the space $\mathcal{H}L^2(\mathbb{C},\mu_t)$, where

$$\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t}$$

for some $t > 0$.

Theorem 2.2. *Let $z \in U$ and $s > 0$ be such that $\overline{D(z,s)} \subset U$. Then*

$$|f(z)|^2 \leq \frac{1}{(\pi s^2)^2} \left\| \chi_{D(z,s)} \frac{1}{\alpha} \right\|^2_{L^2(U,\alpha)} \|f\|^2_{L^2(U,\alpha)},$$

for all $f \in \mathcal{H}L^2(U,\alpha)$.

Proof. Let $z \in U$ and $s > 0$ be such that $\overline{D(z,s)} \subset U$. We claim that

$$f(z) = \frac{1}{\pi s^2} \int_{D(z,s)} f(\omega)\, d\omega.$$

Since f is holomorphic on U, we can expand f in a Taylor series at $\omega = z$, that is,

$$f(\omega) = f(z) + \sum_{n=1}^{\infty} a_n (\omega - z)^n,$$

for all $\omega \in U$. This series converges uniformly to f on the compact set $\overline{D(z,s)}$. Thus

$$\begin{aligned} \int_{D(z,s)} f(\omega)\, d\omega &= \int_{D(z,s)} f(z)\, d\omega + \int_{D(z,s)} \sum_{n=1}^{\infty} a_n (\omega - z)^n\, d\omega \\ &= \pi s^2 f(z) + \sum_{n=1}^{\infty} a_n \int_{D(z,s)} (\omega - z)^n\, d\omega. \end{aligned}$$

If we use polar coordinates with the origin at z, then $(\omega - z)^n = r^n e^{in\theta}$. Hence, for $n \geq 1$,

$$\begin{aligned} \int_{D(z,s)} f(\omega)\, d\omega &= \pi s^2 f(z) + \sum_{n=1}^{\infty} a_n \int_0^s \int_0^{2\pi} r^n e^{in\theta} r\, d\theta\, dr \\ &= \pi s^2 f(z) + \sum_{n=1}^{\infty} a_n \int_0^s r^{n+1} \int_0^{2\pi} e^{in\theta}\, d\theta\, dr \\ &= \pi s^2 f(z). \end{aligned}$$

It follows that

$$\begin{aligned} f(z) &= \frac{1}{\pi s^2} \int_{D(z,s)} f(\omega)\, d\omega \\ &= \frac{1}{\pi s^2} \int_U \chi_{D(z,s)}(\omega) \frac{1}{\alpha(\omega)} f(\omega) \alpha(\omega)\, d\omega \\ &= \frac{1}{\pi s^2} \left\langle \chi_{D(z,s)} \frac{1}{\alpha}, f \right\rangle. \end{aligned}$$

By the Schwarz inequality, we have

$$|f(z)|^2 \leq \frac{1}{(\pi s^2)^2} \left\| \chi_{D(z,s)} \frac{1}{\alpha} \right\|_{L^2(U,\alpha)}^2 \|f\|_{L^2(U,\alpha)}^2.$$

□

By Theorem 2.2, we have that the pointwise evaluation is continuous. That is, for each $z \in U$, the map that takes a function $f \in \mathcal{H}L^2(U, \alpha)$ to the number $f(z)$ is a continuous linear functional on $\mathcal{H}L^2(U, \alpha)$. Then, by the Riesz representation theorem, this linear functional can be represented uniquely as an inner product with some $\phi_z \in \mathcal{H}L^2(U, \alpha)$. That is,

$$f(z) \ = \ \langle \phi_z, f \rangle = \int_U \overline{\phi_z(\omega)} f(\omega)\alpha(\omega)\, d(\omega),$$

Define $K(z, \omega) = \overline{\phi_z(\omega)}$ for any $z, \omega \in U$. We call K the *reproducing kernel* for the space $\mathcal{H}L^2(U, \alpha)$.

We summarize important properties of the reproducing kernel in the next theorem. The proof can be found in [H].

Theorem 2.3. *Let $\mathcal{H}L^2(U, \alpha)$ be defined as above. Then there exists a function $K(z, \omega)$, where $z, \omega \in U$, with satisfies the following properties :*

(i) *$K(z, \omega)$ is holomorphic in the first variable and anti-holomorphic in the second variable, and*

$$K(z, \omega) = \overline{K(\omega, z)}.$$

(ii) *For each $f \in \mathcal{H}L^2(U, \alpha)$,*

$$f(z) = \int_U K(z, \omega) f(\omega)\alpha(\omega)\, d\omega.$$

(iii) *For each $f \in L^2(U, \alpha)$, the orthogonal projection of f onto $\mathcal{H}L^2(U, \alpha)$, denoted by Pf, is*

$$Pf(z) = \int_U K(z, \omega) f(\omega)\alpha(\omega)\, d\omega.$$

(iv) *For each $z, u \in U$,*

$$K(z, u) = \int_U K(z, \omega) K(\omega, u)\alpha(\omega)\, d\omega.$$

(v) *For each* $z \in U$,

$$|f(z)|^2 \leq K(z,z)\|f\|^2_{L^2(U,\alpha)}, \tag{2.1}$$

and the constant $K(z,z)$ *is optimal in the sense that for each* $z \in U$ *there exists a nonzero function* $f_z \in \mathcal{H}L^2(U,\alpha)$ *for which equality holds.*

(vi) *For each* $z \in U$, *if* $\phi_z \in \mathcal{H}L^2(U,\alpha)$ *satisfies*

$$f(z) = \int_U \overline{\phi_z(\omega)} f(\omega)\alpha(\omega)\, d\omega$$

for all $f \in \mathcal{H}L^2(U,\alpha)$, *then* $\overline{\phi_z(\omega)} = K(z,\omega)$.

Corollary 2.4. *Let* $K(z,\omega)$ *be the reproducing kernel for* $\mathcal{H}L^2(U,\alpha)$. *Then for each* $z \in U$,

$$K(z,z) = \sup_{\|f\|_{L^2(U,\alpha)}=1} |f(z)|^2.$$

Proof. It follows from inequality (2.1) that

$$\sup_{\|f\|_{L^2(U,\alpha)}=1} |f(z)|^2 \leq K(z,z).$$

Since for each $z \in U$ there exists a nonzero function $f_z \in \mathcal{H}L^2(U,\alpha)$ such that

$$|f_z(z)|^2 = K(z,z)\|f_z\|^2_{L^2(U,\alpha)},$$

we see that $g_z = \frac{f_z}{\|f_z\|} \in \mathcal{H}L^2(U,\alpha)$ satisfies

$$\|g\|_{L^2(U,\alpha)} = 1 \quad \text{and} \quad |g_z(z)|^2 = K(z,z).$$

Hence,

$$K(z,z) = \sup_{\|f\|_{L^2(U,\alpha)}=1} |f(z)|^2.$$

□

By inequality (2.1), we obtain a pointwise bound for a holomorphic function $f \in \mathcal{H}L^2(U,\alpha)$ from the reproducing kernel. Next, we express the reproducing kernel K in terms of an orthonormal basis for the Hilbert space $\mathcal{H}L^2(U,\alpha)$.

Theorem 2.5. *Let* $\{e_i\}_{i=0}^{\infty}$ *be an orthonormal basis for* $\mathcal{H}L^2(U,\alpha)$*. Then for all* $z,\omega \in U$*,*

$$\sum_{i=0}^{\infty} \left| e_i(z)\overline{e_i(\omega)} \right| < \infty$$

and the reproducing kernel for this space is given by

$$K(z,\omega) = \sum_{i=0}^{\infty} e_i(z)\overline{e_i(\omega)}. \tag{2.2}$$

Chapter 3

Holomorphic equivalence

Definition 3.1. Holomorphic function spaces $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are said to be *holomorphically equivalent* spaces if there exists a nowhere zero holomorphic function ϕ on U such that

$$\beta(z) = \frac{\alpha(z)}{|\phi(z)|^2} \qquad \text{for all } z \in U.$$

Proposition 3.2. *Let $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ be holomorphically equivalent spaces and ϕ defined as above. Let $\Lambda\colon \mathcal{H}L^2(U,\alpha) \to \mathcal{H}L^2(U,\beta)$ be defined by $\Lambda f = \phi f$. Then Λ is unitary.*

Proof. Let $g \in \mathcal{H}L^2(U,\beta)$. Then g/ϕ is holomorphic. Since

$$\int_U \frac{|g(\omega)|^2}{|\phi(\omega)|^2}\alpha(\omega)\,d\omega \;=\; \int_U |g(\omega)|^2\beta(\omega)\,d\omega \;<\; \infty,$$

$g/\phi \in \mathcal{H}L^2(U,\alpha)$. Thus Λ is onto. Then for any $f \in \mathcal{H}L^2(U,\alpha)$,

$$\begin{aligned}\int_U |f(\omega)|^2\alpha(\omega)\,d\omega \;&=\; \int_U |\phi(\omega)|^2|f(\omega)|^2\frac{\alpha(\omega)}{|\phi(\omega)|^2}\,d\omega \\ &=\; \int_U |\Lambda f(\omega)|^2\beta(\omega)\,d\omega.\end{aligned}$$

Hence, Λ is unitary. $\square$

Theorem 3.3. *Let $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ be holomorphically equivalent spaces. Let K_α and K_β be their respective reproducing kernels. Then for each $z \in U$,*

$$\alpha(z)K_\alpha(z,z) = \beta(z)K_\beta(z,z).$$

Proof. Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}L^2(U,\alpha)$. Since any unitary map preserves an orthonormal basis, $\{\phi e_i\}_{i=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}L^2(U,\beta)$. Then, by Theorem 2.5,

$$\begin{aligned} K_\beta(z,\omega) &= \sum_{i=0}^{\infty} \phi(z)e_i(z)\overline{\phi(\omega)e_i(\omega)} \\ &= \phi(z)\overline{\phi(\omega)}\sum_{i=0}^{\infty} e_i(z)\overline{e_i(\omega)} \\ &= \phi(z)\overline{\phi(\omega)}K_\alpha(z,\omega). \end{aligned}$$

Hence,

$$\begin{aligned} K_\beta(z,z) &= \phi(z)\overline{\phi(z)}K_\alpha(z,z) \\ &= |\phi(z)|^2 K_\alpha(z,z) \\ &= \frac{\alpha(z)}{\beta(z)}K_\alpha(z,z). \end{aligned}$$

Therefore, $\alpha(z)K_\alpha(z,z) = \beta(z)K_\beta(z,z)$. □

The next goal in this chapter is to establish a necessary and sufficient condition for two spaces to be holomophically equivalent. This is given in Theorem 3.8. Before that, let us recall some facts from complex analysis.

Definition 3.4. Let $z = x + iy \in \mathbb{C}$ and $f(z)$ be a complex-valued function in an open set U such that f_{xx} and f_{yy} exist at every point of U. Then the *Laplacian* of f is defined by

$$\Delta f = f_{xx} + f_{yy}.$$

In the $(z, \overline{z})$-coordinate, the Laplacian is given by the formula

$$\Delta f \;=\; \frac{4\partial^2}{\partial z \partial \overline{z}}\, f.$$

If f is continuous and $\Delta f = 0$ at every point of an open set U, then f is said to be *harmonic* on U.

Proposition 3.5. *If a function $f(z) = u(x, y) + iv(x, y)$ is holomorphic on an open set U, then Ref and Imf are harmonic on U. Conversely, if $u \colon U \to \mathbb{R}$ is harmonic on a simply connected domain U, then there is a holomorphic function f on U such that $u = Ref$.*

Proposition 3.6. *The following assertions are equivalent :*

(1) U is an open simply connected set in $\mathbb{C}$;

(2) If $h \in \mathcal{H}(U)$ and $\frac{1}{h} \in \mathcal{H}(U)$, then there exists $g \in \mathcal{H}(U)$ such that $h = e^g$;

(3) If $h \in \mathcal{H}(U)$ and $\frac{1}{h} \in \mathcal{H}(U)$, then there exists $f \in \mathcal{H}(U)$ such that $h = f^2$.

Proof. See [R], page 274. □

Lemma 3.7. *Let U be an open simply connected set in $\mathbb{C}$ and α a strictly positive smooth function on U. Then there exists a holomorphic function ϕ such that $|\phi|^2 = \alpha$ if and only if $\log \alpha$ is harmonic.*

Proof. ($\Rightarrow$) Since $\phi \in \mathcal{H}(U)$, by Proposition 3.6, there exists a function $\theta \in \mathcal{H}(U)$ such that $\phi = e^\theta$. Let $u = \operatorname{Re}\theta$. Thus, $|\phi| = e^u$ and hence $\alpha = e^{2u}$. Then $\log \alpha = 2u$, which implies that $\Delta \log \alpha = \Delta 2u = 0$.
($\Leftarrow$) Assume that $u = \log \alpha$ is harmonic. Then, by Proposition 3.5, there exists a holomorphic function f such that $u = \operatorname{Re} f$. Then e^f is also holomorphic. Thus, by Proposition 3.6, there exists $\phi \in \mathcal{H}(U)$ such that $e^f = \phi^2$. Hence, $\alpha = e^u = |e^f| = |\phi^2| = |\phi|^2$. □

Theorem 3.8. *Let U be an open simply connected set in $\mathbb{C}$ and α, β strictly positive smooth functions on U. Then $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are holomorphically equivalent spaces if and only if $\Delta\log\alpha(z) = \Delta\log\beta(z)$.*

Proof. The following statements are equivalent :

$\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are holomorphically equivalent spaces

$$\begin{aligned}
&\iff \exists\phi \in \mathcal{H}(U) \text{ such that } \phi \neq 0 \text{ and } |\phi(z)|^2 = \frac{\alpha(z)}{\beta(z)} \\
&\iff \log\frac{\alpha(z)}{\beta(z)} \text{ is harmonic} \\
&\iff \Delta(\log\alpha(z) - \log\beta(z)) = 0 \\
&\iff \Delta\log\alpha(z) = \Delta\log\beta(z).
\end{aligned}$$

□

This immediately implies the following corollary:

Corollary 3.9. *A holomorphic function space $\mathcal{H}L^2(\mathbb{C},\alpha)$, where α is a strictly positive smooth function on $\mathbb{C}$, is holomorphically equivalent to one of the Segal-Bargmann spaces if and only if $\Delta\log\alpha = c < 0$. In particular, if φ is a smooth function and $\Delta\varphi$ is a positive constant, then the space $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ is holomophically equivalent to a Segal-Bargmann space.*

Proof. Note that if

$$\mu_t(z) \;=\; \frac{1}{\pi t}e^{-|z|^2/t},$$

then

$$\Delta\log\mu_t(z) \;=\; -\Delta\frac{|z|^2}{t} \;=\; -\frac{4}{t}\frac{\partial^2}{\partial z\overline{z}}(z\overline{z}) \;=\; -\frac{4}{t} \;<\; 0.$$

Thus if $\mathcal{H}L^2(\mathbb{C},\alpha)$ is holomorphically equivalent to the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C},\mu_t)$, then $\Delta\log\alpha = \Delta\log\mu_t < 0$.

Conversely, if $\Delta \log \alpha = c < 0$, then $\Delta \log \alpha = \Delta \log \mu_t$ where $t = -4/c$. Therefore, $\mathcal{H}L^2(\mathbb{C}, \alpha)$ is holomorphically equivalent to the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$, where $t = -4/c$. $\square$

Chapter 4

Pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $\Delta\varphi$ is constant

In this chapter, we obtain a pointwise bound for any function in the holomorphic function space $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$, where $\Delta\varphi$ is constant. First, we recall the pointwise bound for a Segal-Bargmann space. In [H], we have

$$\left\{ \frac{z^n}{\sqrt{n!t^n}} \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\}$$

is an orthonormal basis for the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$. Then, by Theorem 2.5,

$$\begin{aligned} K(z, w) &= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!t^n}} \overline{\frac{\omega^n}{\sqrt{n!t^n}}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z\overline{\omega}}{t} \right)^n \\ &= e^{z\overline{\omega}/t}. \end{aligned}$$

Thus,

$$K(z, z) = e^{|z|^2/t}.$$

By Theorem 2.3, we have a pointwise bound for functions in $\mathcal{H}L^2(\mathbb{C}, \mu_t)$.

Theorem 4.1. *For any $f \in \mathcal{H}L^2(\mathbb{C}, \mu_t)$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \ \le \ e^{|z|^2/t} \|f\|^2_{L^2(\mathbb{C},\mu_t)}. \tag{4.1}$$

Next, we will obtain a pointwise bound for a function in a holomorphic function space $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $\Delta\varphi$ is a positive constant. This is a generalization of Theorem 4.1 since $\Delta|z|^2/t = 4/t > 0$. Note also that, by Corollary 3.9, $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ is holomorphically equivalent to $\mathcal{H}L^2(\mathbb{C}, \mu_t)$. Hence, we can obtain a pointwise bound estimation for functions in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ from the pointwise bound estimation for functions in $\mathcal{H}L^2(\mathbb{C}, \mu_t)$.

Theorem 4.2. *Let φ be a smooth function such that $\Delta\varphi = c$ where c is a positive constant. Then, for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \ \le \ \frac{c}{4\pi} e^{\varphi(z)} \|f\|^2_{L^2(\mathbb{C},e^{-\varphi})}. \tag{4.2}$$

Proof. By Corollary 3.9, $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ is holomorphically equivalent to $\mathcal{H}L^2(\mathbb{C}, \mu_t)$, where $t = 4/c$.

Then, by Proposition 3.3,

$$\begin{aligned} K_{e^{-\varphi}}(z,z) \ &= \ \frac{\frac{1}{\pi t} e^{-|z|^2/t}}{e^{-\varphi(z)}} e^{|z|^2/t} \\ &= \ \frac{1}{\pi t} e^{\varphi(z)} \\ &= \ \frac{c}{4\pi} e^{\varphi(z)}. \end{aligned}$$

By Theorem 2.3, we have

$$|f(z)|^2 \ \le \ \frac{c}{4\pi} e^{\varphi(z)} \|f\|^2_{L^2(\mathbb{C},e^{-\varphi})},$$

for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$. □

Corollary 4.3. *Let φ be a smooth function such that $\Delta\varphi = c$ where c is a positive constant. Then, for any $f \in \mathcal{H}L^2(\mathbb{C}, \frac{c}{4\pi} e^{-\varphi})$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \ \le \ e^{\varphi(z)} \|f\|^2_{L^2(\mathbb{C},\frac{c}{4\pi}e^{-\varphi})}.$$

In the remaining of this chapter, we give an alternative proof of a pointwise bound for a Segal-Bargmann space. This method uses the estimate in Theorem 2.2 and avoid an explicit formula for an orthonormal basis for $\mathcal{H}L^2(\mathbb{C}, \mu_t)$. Although this method usually gives a less sharp estimate, it is more applicable because in the next chapter we will establish a pointwise bound for a wider class of holomorphic function spaces and we generally do not have explicit formulas for orthonormal bases of these spaces.

Lemma 4.4. *Let* $\mathcal{U} = D(0,1)$*. For any space* $\mathcal{H}L^2(\mathcal{U}, \mu_t)$*, there exists a constant* C *depending only on* t *such that for any* $f \in \mathcal{H}L^2(\mathcal{U}, \mu_t)$*,*

$$|f(0)|^2 \;\leq\; C \int_{D(0,1)} |f(\omega)|^2 \mu_t(\omega)\, d\omega.$$

Proof. By Theorem 2.2, for any $f \in \mathcal{H}L^2(\mathcal{U}, \mu_t)$ and s such that $0 < s < 1$,

$$|f(0)|^2 \leq (\pi s^2)^{-2} \left\| \chi_{D(0,s)} \frac{1}{\mu_t} \right\|^2_{L^2(\mathcal{U},\mu_t)} \|f\|^2_{L^2(\mathcal{U},\mu_t)}.$$

Consider

$$\begin{aligned} \left\| \chi_{D(0,s)} \frac{1}{\mu_t} \right\|^2 &= \int_{\mathcal{U}} \left| \chi_{D(0,s)} \frac{1}{\mu_t(\omega)} \right|^2 \mu_t(\omega)\, d(\omega) \\ &= \int_{D(0,s)} \frac{1}{\mu_t(\omega)}\, d\omega \\ &= \int_{D(0,s)} \pi t e^{|\omega|^2/t}\, d\omega \\ &= \pi t \int_0^{2\pi} \int_0^s e^{r^2/t} r\, dr\, d\theta \\ &= 2\pi^2 t \int_0^s \frac{t}{2}\, de^{r^2/t} \\ &= \pi^2 t^2 (e^{s^2/t} - 1). \end{aligned}$$

Let $C = \frac{t^2}{s^4}(e^{s^2/t} - 1)$. If $0 < s < 1$ is fixed, then

$$|f(0)|^2 \;\leq\; C\|f\|^2_{L^2(\mathcal{U},\mu_t)} \;=\; C \int_{D(0,1)} |f(\omega)|^2 \mu_t(\omega)\, d\omega,$$

for any $f \in \mathcal{H}L^2(\mathcal{U}, \mu_t)$. □

Theorem 4.5. *For any Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$. There exists a constant C depending only on t such that, for any $f \in \mathcal{H}L^2(\mathbb{C}, \mu_t)$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \ \leq \ Ce^{|z|^2/t}\|f\|^2_{L^2(\mathbb{C},\mu_t)}. \tag{4.3}$$

Proof. Let K_{μ_t} be the reproducing kernel for $\mathcal{H}L^2(\mathcal{U}, \mu_t)$. By the previous lemma and Theorem 2.4,

$$K_{\mu_t}(0,0) \leq C.$$

Let $z \in \mathbb{C}$ and $\beta_z(\omega) = \frac{1}{\pi t}e^{-|z+\omega|^2/t}$. Then $\Delta\frac{|z+\omega|^2}{t} = \Delta\frac{|\omega|^2}{t}$. Hence $\mathcal{H}L^2(\mathcal{U}, \mu_t)$ and $\mathcal{H}L^2(\mathcal{U}, \beta_z)$ are holomorphically equivalent spaces. Let K_{β_z} be the reproducing kernel for $\mathcal{H}L^2(\mathcal{U}, \beta_z)$. Then

$$\begin{aligned} K_{\beta_z}(0,0) \ &= \ \frac{\mu_t(0)}{\beta_z(0)}K_{\mu_t}(0,0) \\ &= \ e^{|z|^2/t}K_{\mu_t}(0,0) \\ &\leq Ce^{|z|^2/t}. \end{aligned}$$

Let $f \in \mathcal{H}L^2(\mathbb{C}, \mu_t)$ and $g_z(\omega) = z + \omega$. Then $g_z \in \mathcal{H}(\mathbb{C})$ and $f \circ g_z \in \mathcal{H}(\mathbb{C})$. Hence, $h = f \circ g_z\big|_{\mathcal{U}} \in \mathcal{H}L^2(\mathcal{U}, \beta_z)$. Then

$$\begin{aligned} |f(z)|^2 \ = \ |f \circ g_z(0)|^2 \ = \ |h(0)|^2 \ &\leq \ Ce^{|z|^2/t}\|h\|^2_{L^2(\mathcal{U},\beta_z)} \\ &= \ Ce^{|z|^2/t}\int_{D(0,1)} |h(\omega)|^2\frac{1}{\pi t}e^{-|z+\omega|^2/t}\,d\omega \\ &= \ Ce^{|z|^2/t}\int_{D(0,1)} |f \circ g_z(\omega)|^2\frac{1}{\pi t}e^{-|z+\omega|^2/t}\,d\omega \\ &= \ Ce^{|z|^2/t}\int_{D(0,1)} |f(z+\omega)|^2\frac{1}{\pi t}e^{-|z+\omega|^2/t}\,d\omega \\ &= \ Ce^{|z|^2/t}\int_{D(z,1)} |f(\omega)|^2\frac{1}{\pi t}e^{-|\omega|^2/t}\,d\omega \\ &\leq \ Ce^{|z|^2/t}\int_{\mathbb{C}} |f(\omega)|^2\frac{1}{\pi t}e^{-|\omega|^2/t}\,d\omega \\ &= \ Ce^{|z|^2/t}\|f\|^2_{L^2(\mathbb{C},\mu_t)}. \end{aligned}$$

□

Chapter 5

Pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ where $0 \le \Delta\varphi \le M$

We recall that the function Γ defined by

$$\Gamma(z) = \frac{1}{2\pi} \log |z|$$

is the *fundamental solution* for the Laplace's equation on $\mathbb{R}^2$. Thus if $\psi \in C_c^\infty(\mathbb{C})$, then

$$\Phi(z) \;=\; \Gamma * \psi(z) \;=\; \int_{\mathbb{C}} \Gamma(\zeta)\psi(z-\zeta)\, d\zeta$$

satisfies $\Delta\Phi = \psi$.

Proposition 5.1. *Let K be a compact subset of $\mathbb{C}$ and O an open set containing K. Then there exists a function $g \in C_c^\infty(\mathbb{C})$ such that $0 \le g \le 1$, $g = 1$ on K and $g = 0$ outside O.*

Proof. See [F], page 245. $\square$

Lemma 5.2. *Let $\varphi \in C^\infty(\mathbb{C})$ satisfying $0 \le \Delta\varphi \le M$. Then there exists a constant C depending only on M such that for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$,*

$$|f(0)|^2 \le Ce^{\varphi(0)} \int_{D(0,1)} |f(\omega)|^2 e^{-\varphi(\omega)}\, d\omega.$$

Proof. By Proposition 5.1, there exists a function $g \in C_c^\infty(\mathbb{C})$ such that $g = 1$ on $\overline{D(0,1)}$ and $g = 0$ outside $D(0,2)$. Let $\psi = g \cdot \Delta\varphi$. Then $\psi \in C_c^\infty(\mathbb{C})$, $\psi = \Delta\varphi$ on $\overline{D(0,1)}$ and $\psi = 0$ outside $D(0,2)$. Thus $\Phi = \Gamma * \psi$ satisfies

$$\Delta\Phi(z) = \psi(z) = \Delta\varphi(z) \tag{5.1}$$

for all $z \in D(0,1)$. Let $h \in \mathcal{H}L^2(\mathcal{U}, e^{-\Phi})$. It follows from Theorem 2.2 that for all $0 < s < 1$,

$$|h(0)|^2 \;\leq\; (\pi s^2)^{-2}\big\|\chi_{D(0,s)}e^{\Phi}\big\|^2_{L^2(\mathcal{U},e^{-\Phi})}\|h\|^2_{L^2(\mathcal{U},e^{-\Phi})}.$$

Let $0 < s < 1$ and $\omega \in D(0,s)$. Then

$$\begin{aligned}
\Phi(\omega) &= \int_{\mathbb{C}} \Gamma(\zeta)\psi(\omega-\zeta)\,d\zeta \\
&= \int_{D(\omega,2)} \Gamma(\zeta)\psi(\omega-\zeta)\,d\zeta \\
&= \int_{D(\omega,2)\setminus D(0,1)} \Gamma(\zeta)\psi(\omega-\zeta)\,d\zeta + \int_{D(0,1)} \Gamma(\zeta)\psi(\omega-\zeta)\,d\zeta \\
&\leq \int_{D(\omega,2)\setminus D(0,1)} \Gamma(\zeta)\psi(\omega-\zeta)\,d\zeta \\
&\leq \int_{D(\omega,2)\setminus D(0,1)} M\Gamma(\zeta)\,d\zeta \\
&= M\int_{D(\omega,2)\setminus D(0,1)} \Gamma(\zeta)\,d\zeta \\
&= \frac{M}{2\pi}\int_{D(\omega,2)\setminus D(0,1)} \log|\zeta| d\zeta.
\end{aligned}$$

Because $\int_{D(\omega,2)\setminus D(0,1)} \log|\zeta|\,d\zeta$ is a function which is bounded above on $D(0,1)$ and so is Φ. Let $C_1 = \sup_{\omega\in D(0,1)} \Phi(\omega)$. We note that C_1 depends only on M. It follows that

$$\begin{aligned}
\big\|\chi_{D(0,s)}e^{\Phi}\big\|^2_{L^2(\mathcal{U},e^{-\Phi})} &= \int_{D(0,s)} e^{\Phi(\omega)}\,d\omega \\
&\leq \int_{D(0,s)} e^{C_1}\,d\omega \\
&= e^{C_1}\pi s^2.
\end{aligned}$$

Thus

$$|h(0)|^2 \ \leq \ \frac{e^{C_1}}{\pi s^2}\|h\|^2_{L^2(\mathcal{U},e^{-\Phi})}$$

for all $h \in \mathcal{H}L^2(\mathcal{U}, e^{-\Phi})$. Therefore, by Theorem 2.4,

$$K_{e^{-\Phi}}(0,0) \leq \frac{e^{C_1}}{\pi s^2}$$

where $K_{e^{-\Phi}}$ is the reproducing kernel for $\mathcal{H}L^2(\mathcal{U}, e^{-\Phi})$.

Let $K_{e^{-\varphi}}$ be the reproducing kernel for $\mathcal{H}L^2(\mathcal{U}, e^{-\varphi})$. Then, by equation (5.1) and Theorem 3.8, $\mathcal{H}L^2(\mathcal{U}, e^{-\varphi})$ and $\mathcal{H}L^2(\mathcal{U}, e^{-\Phi})$ are holomorphically equivalent and hence, by Theorem 3.3,

$$\begin{aligned} K_{e^{-\varphi}}(0,0) \ &= \ \frac{e^{-\Phi(0)}}{e^{-\varphi(0)}} K_{e^{-\Phi}}(0,0) \\ &\leq \ \frac{e^{C_1}}{\pi s^2} e^{-\Phi(0)} e^{\varphi(0)}. \end{aligned}$$

Let $C = \frac{e^{C_1}}{\pi s^2} e^{-\Phi(0)}$. Thus

$$|h(0)|^2 \leq C e^{\varphi(0)} \|h\|^2_{L^2(\mathcal{U},e^{-\varphi})},$$

for any $h \in \mathcal{H}L^2(\mathcal{U}, e^{-\varphi})$.

It remains to show that C depends only on M. Now, consider

$$\begin{aligned} \Phi(0) \ &= \ \int_{\mathbb{C}} \Gamma(\zeta)\psi(-\zeta)\, d\zeta \\ &= \ \int_{D(0,2)} \Gamma(\zeta)\psi(-\zeta)\, d\zeta \\ &= \ \int_{D(0,1)} \Gamma(\zeta)\psi(-\zeta)\, d\zeta + \int_{D(0,2)\setminus D(0,1)} \Gamma(\zeta)\psi(-\zeta)\, d\zeta \\ &\geq \ \int_{D(0,1)} \Gamma(\zeta)\psi(-\zeta)\, d\zeta \qquad (5.2) \\ &\geq \ M \int_{D(0,1)} \Gamma(\zeta)\, d\zeta \qquad (5.3) \\ &= \ -\frac{M}{4}. \qquad (5.4) \end{aligned}$$

For inequality (5.2), we use the fact that $\int_{D(0,2)\setminus D(0,1)} \Gamma(\zeta)\psi(-\zeta)\,d\zeta \geq 0$. For inequality (5.3), we use the fact that Γ is negative on $D(0,1)$ and $0 \leq \psi \leq M$ on $D(0,1)$. Equation (5.4) follows from the computation below:

$$\begin{aligned}
\int_{D(0,1)} \Gamma(\zeta)\,d\zeta &= \frac{1}{2\pi}\int_{D(0,1)} \log|\zeta|\,d\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 r\log r\,dr\,d\theta \\
&= \int_0^1 r\log r\,dr \\
&= \int_0^1 d\left(\frac{r^2}{2}(\log r - \frac{1}{2})\right) \\
&= \left(\frac{1}{2}(\log 1 - \frac{1}{2}) - \lim_{t\to 0}\frac{t^2}{2}(\log t - \frac{1}{2})\right) \\
&= -\frac{1}{4}.
\end{aligned}$$

Thus $e^{-\Phi(0)}$ depends only on M and so does C.

Let $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ and $h = f\big|_{\mathcal{U}}$. Then $h \in \mathcal{H}L^2(\mathcal{U}, e^{-\varphi})$ and

$$\begin{aligned}
|f(0)|^2 &= |h(0)|^2 \\
&\leq Ce^{\varphi(0)}\int_{D(0,1)} |h(\omega)|^2 e^{\varphi(\omega)}\,d\omega \\
&= Ce^{\varphi(0)}\int_{D(0,1)} |f(\omega)|^2 e^{\varphi(\omega)}\,d\omega.
\end{aligned}$$

□

Theorem 5.3. *Let $\varphi \in C^\infty(\mathbb{C})$ satisfying $0 \leq \Delta\varphi \leq M$. Then there exists a constant C depending only on M such that for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \leq Ce^{\varphi(z)}\|f\|^2_{L^2(\mathbb{C},e^{-\varphi})}.$$

Proof. Let $z \in \mathbb{C}$ and $g_z(\omega) = z + \omega$. Then $0 \le \Delta(\varphi \circ g_z) \le M$. Let $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi})$ and $h = f \circ g_z$. Then $h \in \mathcal{H}L^2(\mathbb{C}, e^{-\varphi \circ g_z})$ and by Lemma 5.2,

$$\begin{aligned}
|f(z)|^2 &= |f \circ g_z(0)|^2 \\
&= |h(0)|^2 \\
&\le Ce^{\varphi \circ g_z(0)} \int_{D(0,1)} |h(\omega)|^2 e^{-\varphi \circ g_z(\omega)}\, d\omega \\
&= Ce^{\varphi(z)} \int_{D(0,1)} |f \circ g_z(\omega)|^2 e^{-\varphi \circ g_z(\omega)}\, d\omega \\
&= Ce^{\varphi(z)} \int_{D(0,1)} |f(z+\omega)|^2 e^{-\varphi(z+\omega)}\, d\omega \\
&= Ce^{\varphi(z)} \int_{D(z,1)} |f(\omega)|^2 e^{-\varphi(\omega)}\, d\omega \\
&\le Ce^{\varphi(z)} \int_{\mathbb{C}} |f(\omega)|^2 e^{-\varphi(\omega)}\, d\omega \\
&= Ce^{\varphi(z)} \|f\|^2_{L^2(\mathbb{C}, e^{-\varphi})}.
\end{aligned}$$

□

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