เรขาคณิตของเซตจูเลียของพหุนามเชิงซ้อน z^n+c

นางสาวแคทลียา ดาวสุด

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2545

ISBN 974-17-1577-3

GEOMETRY OF JULIA SETS OF COMPLEX POLYNOMAIL z^n+c

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2002

ISBN 974-17-1577-3.

Thesis Title	Geometry of Julia sets of Complex polynomails z^n+c
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Field of study	Mathematics
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แคทลียา ดาวสุด : เรขาคณิตของเซตจูเลียของพหุนามเชิงซ้อน z^n+c. (GEOMETRY OF JULIA SETS OF COMPLEX POLYNOMIALS z^n+c) อ. ที่ปรึกษา : อ.ณัฐพันธ์ กิติสิน, 24 หน้า ISBN 974-17-1577-3

วิทยานิพนธ์เล่มนี้มีจุดมุ่งหมายเพื่อที่จะประมาณค่าขอบเขตบนของ $|c|$ ที่ทำให้เซตจูเลียของพหุนามเชิงซ้อนที่อยู่ในรูป z^n+c เป็นโค้งปิดเชิงเดียว เมื่อ n = 2, 3, 4, ... นอกจากนี้เราจะศึกษาสมบัติทางเรขาคณิตของเซตจูเลีย เราทราบแล้วว่า เซตจูเลียของพหุนามเชิงซ้อนที่อยู่ในรูป z^2+c เป็นโค้งปิดเชิงเดียว เมื่อ $|c| < \frac{1}{4}$ เราคาดว่า ผลลัพธ์ที่ได้จะคล้ายเดิม นั่นคือเซตจูเลียของพหุนามเชิงซ้อนที่อยู่ในรูป z^n+c เป็นโค้งปิดเชิงเดียว ถ้า $|c|$ มีค่าเล็กพอ อย่างไรก็ตามทุกจุดบนโค้งปิดเชิงเดียวดังกล่าวเราไม่สามารถหาอนุพันธ์ได้

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

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ปีการศึกษา 2545

4372225123 : MAJOR MATHEMATICS

KEY WORDS : JULIA SETS

KATTHALEEYA DAOWSUD : GEOMETRY OF JULIA SETS OF COMPLEX POLYNOMALS z^n+c. THESIS ADVISOR : NATAPHAN KITISIN, Ph.D., 24 pp. ISBN 974-17-1577-3.

This thesis is intended to estimate the upper bounded of $|c|$ such that Julia sets of complex polynomials of the form z^n+c are simple closed curves when n = 2, 3, 4, ... Moreover, we study the geometric properties of these Julia sets. We know that Julia sets of complex polynomials of the form z^2+c are simple closed curves provided $|c| < \frac{1}{4}$. We expect the same phenomenon, i.e. Julia sets of complex polynomials of the form z^n+c are simple closed curves if $|c|$ is small enough. However, they are far from being smooth; indeed, they contain no smooth arcs at all.

Department **Mathematics** Student's signature.......................................

Field of study **Mathematics** Advisor's signature.......................................

Academic year **2002**

ACKNOWLEDGMENTS

I never complete this thesis without the assistance of Dr.Nataphan Kitisin, my thesis advisor. I gratefully acknowledge his invaluable advice and inspiration. I must not forget to thank his patience in reading and revising the manuscript. I feel that it is not possible to adequately express my gratitude to all of his teachings throughout my studies at Chulalongkorn University. Thanks are also due to Assoc. Prof. Dr.Kritsana Neammanee and Dr.Phichet Chaoha for serving in committee and making useful comments. Besides, I feel thankful to all of my teachers who have taught me for my knowledge and skills. I would particularly like to thank my friends and my family for their sincere encouragement.

CONTENTS

CHAPTER I

Introduction

Let f be a function from $\mathbb{C}$ to $\mathbb{C}$ and $w \in \mathbb{C}$. We denote the iterations of a function f by $f^1 = f$ and $f^k = f^{k-1} \circ f$. We call w a *fixed point* of f provided $f(w) = w$. If $f^p(w) = w$ for some integer $p \geqslant 1$, then w is a *periodic point* of f. The least p such that $f^p(w) = w$ is called the *period* of w. Suppose f is holomorphic in a neighborhood of w and w is a periodic point of period p, with $(f^p)'(w) = \lambda$, where the prime denotes complex differentiation. The point w is called *attractive* if $|\lambda| < 1$, *repelling* if $|\lambda| > 1$, and *indifferent* if $|\lambda| = 1$. If w is an attractive fixed point of f, we write $A(w) = \{z \in \mathbb{C} : f^k(z) \to w \text{ as } k \to \infty\}$ for the *basin of attraction* of w. We define the basin of attraction of infinity, $A(\infty)$, in the same way. The *Julia set* $J(f)$ *of a complex polynomial* f is the closure of the set of repelling periodic points of f. The complement of the Julia set of a complex polynomial is called the *Fatou set* or *stable set* $F(f)$.

Let U be an open set in $\mathbb{C}$, and let $\{g_k : U \to \mathbb{C}\}$ be a family of complex holomorphic functions. The family $\{g_k\}$ is said to be *normal* on U if every sequence of functions selected from $\{g_k\}$ has a subsequence which converges uniformly on every compact subset of U, either to a bounded holomorphic function or to ∞. The family $\{g_k\}$ is *normal at the point* w of U if there is some open subset V of U containing w such that $\{g_k\}$ is a normal family on V. Define

$$(1.1) \qquad J_0(f) = \{z \in \mathbb{C} : \text{the family } \{f^k\}_{k \geqslant 1} \text{ is not normal at } z\}$$

and

$$F_0(f) \equiv \mathbb{C} \setminus J_0(f)$$
$$= \{z \in \mathbb{C} \text{ such that there is an open set } V \text{ with}$$
$$z \in V \text{ and } \{f^k\} \text{ normal on } V\}.$$

In this work, we estimate the upper bounded of $|c|$ such that Julia sets of complex polynomials of the form $z^n + c$, when $n = 2, 3, 4, \ldots$ are simple nowhere differentiable closed curves. We know that for $n = 2$, this upper bounded is $\frac{1}{4}$ (See [2]).

For $n = 3$, we use the cubic formula in the estimate step. Note that $\partial A(w) = J(f)$ (See Ch. II, Lem. 2.6). The cubic formula is used to find fixed points of complex polynomials of the form $z^3 + c$. Furthermore, we obtain that one fixed point is attractive and others are repelling. If the polynomials of the form $z^3 + c$ have exactly one attractive fixed point, we will show that their Julia sets are simple closed curves. Moreover, if c is a complex number which is not real, then their Julia sets contain no smooth arcs.

In general cases, we can not use the same method because there is no general formula for solving an algebraic equation of degree $n \geqslant 5$. Note that for $c = 0$, the complex polynomials of the form z^n have one attractive fixed at $z = 0$ and $n - 1$ repelling fixed points on the unit circle. If $|c|$ is small enough, we expect the result would resemble the case $c = 0$, namely, these polynomials also have one attractive fixed point near the point $z = 0$ and $n - 1$ repelling fixed points. To prove our main theorem, we will apply Rouché's theorem (See Appendix). We use it to compare the zeros of complex polynomials of the form $z^n - z + c$ with the zeros of complex polynomials of the form $z^n - z$. Consequently, we can estimate the upper bound of $|c|$ such that the complex polynomials of the form $z^n + c$ have exactly one attractive fixed points. Finally, we will show that if the complex polynomials of the form $z^n + c$ have only one attractive fixed point, then their Julia sets are simple closed curves. Moreover, if c is a complex number which is not real, then their Julia sets are nowhere differentiable.

In Chapter II we present some basic properties of the Julia sets. Although our definition of $J(f)$ is intuitively more appealing, $J_0(f)$ is rather easier to work with, since complex variable techniques are more readily applicable. We derive some basic properties of $J_0(f)$ and prove that $J(f) = J_0(f)$.

In Chapter III we study the particular case when $n = 3$. We show that the upper bounded of $|c|$ such that Julia sets of complex polynomials of the form $z^3 + c$ are simple closed curves. Moreover, if c is a complex number which is not real, then their Julia sets are nowhere differentiable. First, we introduce the cubic formula. We then use it to estimate the upper bounded of $|c|$ so that complex polynomials of the form $z^3 + c$ have exactly one fixed point. Finally, we show that their Julia sets are simple closed curves.

In Chapter IV we prove our main theorem by showing that if $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$, then the Julia sets of complex polynomials of the form $z^n + c$ are simple closed curves when $n = 2, 3, 4, \ldots$ Moreover, if c is a complex number which is not real, then their Julia sets are nowhere differentiable.

CHAPTER II

Properties of Julia set.

In this chapter we shall discuss the essential properties of Julia set. In fact (1.1) is often taken as the definition of the repelling periodic points, $J(f)$. Although our definition of $J(f)$ is intuitively more appealing, $J_0(f)$ is rather easier to work with, since complex variable techniques are more readily applicable. We derive some basic properties of $J_0(f)$, with the eventual aim of showing that $J(f) = J_0(f)$. We note that f is a complex polynomial of degree $n \geqslant 2$. For further reference, see [3].

Proposition 2.1. *If f is a polynomial, then $J_0(f)$ is compact.*

Proposition 2.2. *$J_0(f)$ is non-empty.*

Proposition 2.3. *$J_0(f)$ is forward and backward invariant, i.e. $J_0 = f(J_0) = f^{-1}(J_0)$.*

Proposition 2.4. *$J_0(f^p) = J_0(f)$ for every positive integer p.*

Lemma 2.5. *Let f be a polynomial, let $w \in J_0(f)$ and let U be any neighborhood of w. Then $W \equiv \bigcup_{k=1}^{\infty} f^k(U)$ is the whole of $\mathbb{C}$, except possibly for a single point. Any such exceptional point is not in $J_0(f)$, and is independent of w and U.*

Corollary 2.6.

(a) The following holds for all $z \in \mathbb{C}$ with, at most, one exception: if U is an open set intersecting $J_0(f)$ then $f^{-k}(z)$ intersects U for infinitely many values of k.

(b) If $z \in J_0(f)$ *then* $J_0(f)$ *is the closure of* $\bigcup_{k=1}^{\infty} f^{-k}(z)$.

Corollary 2.7. *If* f *is a polynomial,* $J_0(f)$ *has empty interior.*

Proposition 2.8. $J_0(f)$ *is a perfect set (i.e. closed and with no isolated points) and is therefore uncountable.*

Theorem 2.9. *If* f *is a polynomial,* $J(f) = J_0(f)$.

Lemma 2.10. *Let* w *be an attractive fixed point of* f. *Then* $\partial A(w) = J(f)$. *The same is true if* $w = \infty$.

CHAPTER III

Geometry of Julia sets of complex polynomials $z^3 + c$

In this chapter we study the particular case when $n = 3$. Before that, we recall the cubic formula, which can be funded in [4]. We will use it to estimate the upper bounded of $|c|$ such that Julia sets of complex polynomials of the form $z^3 + c$ are simple closed curves. Moreover, If c is a complex number which is not real, their Julia sets contain no smooth arcs.

Let $f(z) = z^3 + qz + r$ and let u be a root of $f(z)$ and choose numbers y and x with $u = y + x$.

Then

$$u^3 = (x + y)^3 = x^3 + y^3 + 3(x^2y + xy^2) = x^3 + y^3 + 3uxy.$$

Therefore,

$$x^3 + y^3 + (3xy + q)u + r = 0. \tag{3.1}$$

So far we have imposed only one constraint on x and y, namely, $u = x + y$. We note that if u, v are numbers, then there exist (possibly complex) numbers x and y such that $x + y = u$ and $xy = v$. Thus, we may impose a second constraint:

$$xy = -\frac{q}{3} \tag{3.2}$$

so that, in the equation (3.1), the linear term in u vanishes. We now have

$$x^3 + y^3 = -r$$

and

$$x^3y^3 = -\frac{q^3}{27}.$$

These two equations can be solved for x^3 and y^3. In detail,

$$x^3 - \frac{q^3}{27x^3} = -r,$$

and hence

$$x^6 + rx^3 - \frac{q^3}{27} = 0.$$

The quadratic formula gives

$$x^3 = \frac{1}{2}\left(-r + \sqrt{r^2 + \frac{4q^3}{27}}\right), \tag{3.3}$$

and the equation (3.2) give $y = -\frac{q}{3x}$. Having found one root $u = x + y$ of $f(x)$, one can find the other two as the roots of the quadratic $f(z)/(z - u)$.

Here is an explicit formula for the other two roots, in contrast to the method just described for finding them. If $\omega = e^{2\pi/3}$ is a cube root of unity, then there are three values for x; one is given by the equation (3.3); the other two are ωx and $\omega^2 x$. The corresponding mates are

$$-\frac{q}{3\omega x} = \left(\frac{1}{\omega}\right) y = \omega^2 y$$

and

$$-\frac{q}{3\omega^2 x} = \left(\frac{1}{\omega^2}\right) y = \omega y.$$

We conclude that the roots of the cubic polynomial are given by the *cubic formula*:

$$z_1 = x + y;\ z_2 = \omega x + \omega^2 y;\ z_3 = \omega^2 x + \omega y; \tag{3.4}$$

here $x^3 = \frac{1}{2}(-r + \sqrt{R})$ and $R = r^2 + \frac{4q^3}{27}$.

Specially, if $q = -1$ and $r = c$, applying the equation (3.4) we have

$$z_1 = x + \frac{1}{3x},\ z_2 = \omega x + \frac{\omega^2}{3x} \text{ and } z_3 = \omega^2 x + \frac{\omega}{3x} \text{ where } x = \left(\frac{1}{2}\left(-c + \sqrt{c^2 - \frac{4}{27}}\right)\right)^{\frac{1}{3}}$$

are roots of a complex polynomial of the form $z^3 - z + c$. It follows that z_1, z_2 and z_3 are fixed points of a complex polynomial of the form $z^3 + c$.

Lemma 3.1. *Suppose c is a complex number such that $|c| < \dfrac{1}{100}$ and let $f_c(z) = z^3 + c$. Then z_1, z_2 are repelling and z_3 is attractive.*

Proof. Assume that $|c| < \dfrac{1}{100}$. From the above results, we get z_1, z_2 and z_3 are fixed points of f_c. Let $x = re^{i\theta}$. By interchanging the parameter of x, we have

$$|z_1|^2 = r^2 + \frac{2}{3}\cos(2\theta) + \frac{1}{9r^2} \tag{3.4}$$

$$|z_2|^2 = r^2 + \frac{2}{3}\cos\left(2\theta - \frac{2\pi}{3}\right) + \frac{1}{9r^2} \tag{3.5}$$

$$|z_3|^2 = r^2 + \frac{2}{3}\cos\left(2\theta + \frac{2\pi}{3}\right) + \frac{1}{9r^2}. \tag{3.6}$$

Consider $x = \left(\dfrac{1}{2}\left(-c + \sqrt{c^2 - \dfrac{4}{27}}\right)\right)^{\frac{1}{3}}$.

Then

$$\left|c^2 - \frac{4}{27}\right| = \left|\frac{4}{27} - c^2\right| \geqslant \left|\left|\frac{4}{27}\right| - |c|^2\right| > \left|\frac{4}{27} - \frac{1}{10000}\right| = 0.1480$$

and

$$\left|c^2 - \frac{4}{27}\right| \leqslant |c|^2 + \left|\frac{4}{27}\right| < \frac{1}{10000} + \frac{4}{27} = 0.1482.$$

Thus,

$$0.3847 < \left|c^2 - \frac{4}{27}\right|^{\frac{1}{2}} < 0.3850 \text{ and } 0 < \arg\left(c^2 - \frac{4}{27}\right) < \pi.$$

Compute

$$\begin{aligned} r^3 &= \left|\frac{1}{2}(-c + \sqrt{c^2 - \frac{4}{27}}\right| \\ &\geqslant \frac{1}{2}\left|\left|c^2 - \frac{4}{27}\right|^{\frac{1}{2}} - |c|\right| \\ &> 0.1874, \end{aligned}$$

and

$$\begin{aligned} r^3 &= \left|\frac{1}{2}(-c + \sqrt{c^2 + \frac{4}{27}}\right| \\ &\leqslant \frac{1}{2}\left(\left|c^2 - \frac{4}{27}\right|^{\frac{1}{2}} + |c|\right) \\ &< 0.1975. \end{aligned}$$

Hence, $0.5723 < r < 0.5824$ and $0 < \arg(x) < \frac{\pi}{3}$. By using equation (3.4), (3.5) and (3.6) we get

$$|z_1|^2 > (0.5723)^2 + \frac{2}{3}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{9(0.5824)^2} > 0.3,$$

$$|z_2|^2 > (0.5723)^2 + \frac{2}{3}\cos\left(-\frac{2\pi}{3}\right) + \frac{1}{9(0.5824)^2} > 0.3,$$

and
$$|z_3|^2 < (0.5824)^2 + \frac{2}{3}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{9(0.5723)^2} < 0.3.$$

Hence, $|f_c'(z_1)| = |3z_1^2| > 1$, $|f_c'(z_2)| = |3z_2^2| > 1$ and $|f_c'(z_3)| = |3z_3^2| < 1$. This implies that z_1, z_2 are repelling and z_3 is attractive. □

Let $[\alpha, \beta]$ be a compact interval in $\mathbb{R}$. A *curve* γ with parameter interval $[\alpha, \beta]$ is a continuous function $\gamma : [\alpha, \beta] \to \mathbb{C}$. It has *initial point* $\gamma(\alpha)$ and *final point* $\gamma(\beta)$, and is *closed* if $\gamma(\alpha) = \gamma(\beta)$. It is *simple* if $\alpha \leqslant s < t \leqslant \beta$ implies $\gamma(s) \neq \gamma(t)$ for $t - s < \beta - \alpha$. A curve γ is said to be *smooth* if the function γ has a continuous derivative on its parameter interval $[\alpha, \beta]$. For brevity, we term a smooth, closed, simple curve in the complex plane a *loop.* We refer to the parts of $\mathbb{C}$ inside and outside such a curve as the *interior* and *exterior* of the loop.

Lemma 3.2. *Let C be a loop in the complex plane. Suppose c is a complex number and let $f_c = z^3 + c$. If c is inside C, then $f_c^{-1}(C)$ is a loop, with the inverse image of interior of C as the interior of $f_c^{-1}(C)$.*

Proof. Suppose that c is inside C. Note that $f_c^{-1}(C) = (z-c)^{1/3}$ and $(f_c^{-1})'(z) = \frac{1}{3}(z-c)^{-2/3}$, which is finite and non-zero if $z \neq c$. Hence, if we select one of the three branch of f_c^{-1}, the set $f_c^{-1}(C)$ is locally a smooth curve, provided $c \notin C$.

Take an initial point w on C and choose one of the three values for $f_c^{-1}(w)$. Allowing $f_c^{-1}(z)$ to vary continuously as z moves around C, the point $f_c^{-1}(z)$ traces out a smooth curve. When z returns to w, however, $f_c^{-1}(w)$ takes its second value. As z traverses C again, $f_c^{-1}(z)$ continues on its smooth path, which closes as z returns to w the second time. Next, when z returns to w the third time, $f_c^{-1}(w)$

takes its third value. As z traverses C again, $f_c^{-1}(z)$ continues on its smooth path, which closes as z returns to w the third time. Since $c \notin C$, we have $0 \notin f_c^{-1}(C)$, so $f_c{}'(z) \neq 0$ on $f_c^{-1}(C)$. This f_c is locally a smooth bijective transformation near points on $f_c^{-1}(C)$. In particular $z \in f_c^{-1}(C)$ cannot be a point of self-intersection of $f_c^{-1}(C)$, otherwise $f_c(z)$ would be at a self-intersection of C.

Since f_c is a continuous function that maps the loop $f_c^{-1}(C)$ and no other points onto the loop C, the polynomial f_c must map the interior and exterior of $f_c^{-1}(C)$ into the interior and exterior of C, respectively. Hence f_c^{-1} maps the interior of C to the interior of $f_c^{-1}(C)$. □

Lemma 3.3. *Suppose c is a complex number such that $|c| < \dfrac{1}{100}$ and let $f_c = z^3 + c$. If C_0 is the circle $|z| = \dfrac{1}{\sqrt{3}}$, then $C_k = f_c^{-k}(C_0)$ is a loop surrounding (= enclosing, possibly, touching) $C_{k-1} = f_c^{-k+1}(C_0)$ where $k = 1, 2, 3, \ldots$*

Proof. Write $C_k = f_c^{-k}(C_0)$ where $k = 1, 2, 3, \ldots$ Let C_0 be the circle $|z| = \dfrac{1}{\sqrt{3}}$.

Since
$$
\begin{aligned}
|f_c^{-1}(z)| &\geqslant ||z| - |c||^{\frac{1}{3}} \\
&> \left| \frac{1}{\sqrt{3}} - \frac{1}{100} \right|^{\frac{1}{3}} \\
&= 0.8278 \\
&> \frac{1}{\sqrt{3}} \\
&= |z| \qquad \text{, for all } z \in C_0,
\end{aligned}
$$

and by Lemma 3.2., C_1 is a loop surrounding C_0.

Next, we will show that C_2 is a loop surrounding C_1. Assume that for each $z, w \in C_0$ such that $\arg(f_c^{-2}(z)) = \theta = \arg(f_c^{-1}(w))$ and $|f_c^{-2}(z)| < |f_c^{-1}(w)|$. Hence, $|(f_c^{-2}(z))^3| < |(f_c^{-1}(w))^3|$ and $\arg((f_c^{-2}(z))^3) = 3\theta = \arg((f_c^{-1}(w))^3)$. Therefore, for a fixed c, we have $(f_c^{-2}(z))^3 + c \in C_1$, $(f_c^{-1}(w))^3 + c \in C_0$, and $|(f_c^{-2}(z))^3 + c| < |(f_c^{-1}(w))^3 + c|$, which contradicts the above result. Thus

$|f_c^{-2}(z)| \geqslant |f_c^{-1}(w)|$ for all $z, w \in C_0$. This implies C_2 is surrounding C_1. By Lemma 3.2., C_2 is a loop surrounding C_1. By the same argument, we have that C_k is a loop surrounding C_{k-1} where k = 1,2,3,... □

Lemma 3.4. *Let c be a complex number such that* $|c| < \dfrac{1}{100}$, $f_c = z^3 + c$, *and* C_0 *be the circle* $|z| = \dfrac{1}{\sqrt{3}}$. *Then, for each* $k > 1$, $|f_c^{-k}(z) - f_c^{-k+1}(z)| < \alpha\gamma^k$ *for some constants* α *and* $\gamma > 1$.

Proof. Since $(f_c)'(z) = 3z^2$ and C_0 is the circle $|z| = \dfrac{1}{\sqrt{3}}$, there is a positive number $r > 1$ such that $|(f_c)'(z)| > r$ for all z outside C_0. Thus, $|(f_c^{-1}(z))'| \leqslant \dfrac{1}{|(f_c)'(z)|} < \dfrac{1}{r}$ for all z outside C_0. For each two points z_1, z_2 outside C, let $\beta : [0,1] \to \mathbb{C} \setminus C$ be the straight line joining z_1 to z_2.

$$\text{Then} \quad \begin{aligned} |f_c^{-1}(z_2) - f_c^{-1}(z_1)| &= \left| \int_\beta (f_c^{-1}(z))' dz \right| \\ &\leqslant \int_{z_1}^{z_2} |(f_c^{-1}(\beta(t)))'||d\beta(t)| \\ &< \frac{1}{r} \int_{z_1}^{z_2} |d\beta(t)| \\ &= \frac{1}{r}|z_1 - z_2|. \end{aligned}$$

By direct calculation, for each $z \in C_0$ and $k \in \mathbb{N}$, $f_c^{-k}(z)$ is outside C_0. Applying the above inequality, we get that

$$\begin{aligned} |f_c^{-k}(z) - f_c^{-k+1}(z)| &< \left(\frac{1}{r}\right) |f_c^{-k+1}(z) - f_c^{-k+2}(z)| \\ &< \left(\frac{1}{r}\right)^2 |f_c^{-k+2}(z) - f_c^{-k+3}(z)| \\ &\vdots \\ &< \left(\frac{1}{r}\right)^{k-2} |f_c^{-2}(z) - f_c^{-1}(z)|. \end{aligned}$$

Hence, for each $k > 1$ and $z \in C_0$, $|f_c^{-k}(z) - f_c^{-k+1}(z)| < \alpha\gamma^k$ where $\gamma = \left(\dfrac{1}{r}\right)$ and $\alpha = r^2|f_c^{-2}(z) - f_c^{-1}(z)|$, as required. □

Lemma 3.5. *Let* $\{\psi_k(\theta)\}_{k=0}^{\infty}$ *be a sequence of continuous functions on an open domain* U *such that there is a positive number* $\gamma < 1$ *such that for each* $n \in \mathbb{N}$,

$$|\psi_k(\theta) - \psi_{k-1}(\theta)| < (\gamma)^k.$$

Then $\psi_k(\theta)$ *converges uniformly to a continuous function* $\psi(\theta)$ *as* $k \to \infty$.

Proof. Let $f_k(\theta) = \psi_k(\theta) - \psi_{k-1}(\theta)$. Then for each $k \in \mathbb{N}$, $|f_k(\theta)| < (\gamma)^k$. By Weierstrass M-test, $\psi_k(\theta) - \psi_0(\theta)$ converges uniformly to a continuous function $\phi(\theta)$ as k$\to \infty$. Let $\psi(\theta) = \phi(\theta) + \psi_0(\theta)$. Then $\psi(\theta)$ is also a continuous function. Hence $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as $k \to \infty$. □

Theorem 3.6. *Let* c *be a complex number such that* $|c| < \dfrac{1}{100}$ *and* $f_c(z) = z^3 + c$. *Then* $J(f_c)$ *is a simple closed curve.*

Proof. Let C_0 be the circle $|z| = \dfrac{1}{\sqrt{3}}$. By Lemma 3.1., c and the attractive fixed point of f_c are inside C_0. By Lemma 3.2. and 3.3., the inverse image $C_1 = f_c^{-1}(C_0)$ is a loop surrounding C_0. Let A_1 be the annular region between C_0 and C_1. We fill A_1 by a continuum of curves, called trajectories, which leave C_0 and reach C_1 perpendicularly. For each θ, let $\psi_1(\theta)$ be the point on C_1 at the end of the trajectory leaving C_0 at $\psi_0(\theta) = \dfrac{1}{\sqrt{3}}\, e^{i\theta}$. Let A_2 be the inverse image $f_c^{-1}(A_1)$ which is the annular region with outer boundary the loop $C_2 = f_c^{-1}(C_1)$ and inner boundary C_1, with f_c mapping A_2 to A_1 in a three-to-1 manner. Then the inverse image of the trajectories joining C_0 to C_1 provides a family of trajectories joining C_1 to C_2. Let $\psi_2(\theta)$ be the point on C_2 at the end of the trajectory leaving C_1 at $\psi_1(\theta)$. Continuing this process, we get a sequence of loops C_k, each surrounding its predecessor, and families of trajectories joining the points $\psi_k(\theta)$ on C_k to $\psi_{k+1}(\theta)$ on C_{k+1} for each k.

As k $\to \infty$, the curve C_k approach the boundary of the basin of attraction fixed point of f_c, that is, they approach the Julia set $J(f_c)$. By Lemma 3.4., we get the length of the trajectory joining $\psi_k(\theta)$ to $\psi_{k+1}(\theta)$ converges to 0 at a geometric rate as k $\to \infty$. By Lemma 3.5., $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as k $\to \infty$. Hence $J(f_c)$ is the closed curve given by $\psi(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$.

Now, it remains to show that $\psi(\theta)$ represents a simple curve. Assume that $\psi(\theta_1) = \psi(\theta_2)$. Let D be the region bounded by C_0 and the two trajectories joining $\psi_0(\theta_1)$ and $\psi_0(\theta_2)$ to this common point. The boundary of D remains bounded under iterates of f_c , so by the maximum modulus Theorem (See Appendix), D remains bounded under iterates of f_c. By Lemma 2.5., we have the interior of D cannot contain any points of $J(f_c)$. Hence $\psi(\theta_1) = \psi(\theta) = \psi(\theta_2)$ for all θ between θ_1 and θ_2. It follows that $\psi(\theta)$ has no point self-intersection. Therefore $J(f_c)$ is a simple closed curve. □

Proposition 3.7. *Let $f_c(z) = z^3 + c$. Suppose c is a complex number which is not real and $|c| < \dfrac{1}{100}$. Then $J(f_c)$ is a simple nowhere differentiable closed curve.*

Proof. By Theorem 3.6., $J(f_c)$ is a simple closed curve. Let z_1 be a repelling fixed point of f_c. It is easy to check that $f_c'(z_1)$ is a complex number which is not real. We will show that z_1 does not lie in a smooth arc in $\psi(\theta)$. Suppose not. Since $J(f_c)$ is invariant under f_c, the image of $\psi(\theta)$ would also be a smooth arc in $J(f_c)$ passing through z_1. Since $arg(f_c'(z_1)) \neq 0$ and $\neq \pi$, the tangents to these two curves would not be parallel. Hence, $\psi(\theta)$ would not be simple at z_1, which ia a contradiction. This implies z_1 does not lie in a smooth arc in $\psi(\theta)$. By Proposition 2.6.(b), the preimages of z_1 are dense in $J(f_c)$. It follows that $J(f_c)$ contains no smooth arcs. □

CHAPTER IV

Geometry of Julia sets of complex polynomials $z^n + c$

The inspiration behind this chapter is the desire for an answer to the following question: How can we estimate the upper bounded of $|c|$ so that Julia sets of complex polynomials of the form $z^n + c$ when $n = 2, 3, 4, \ldots$ are simple closed curves? From the previous chapter, using the cubic formula, we found that this upper bounded of $|c|$ is $\frac{1}{100}$, when $n = 3$. In this chapter we cannot use the same method because there is no general formula for solving an algebraic equation of degree $n \geqslant 5$. Consequently, we have to use the theorem from Complex Analysis, namely The Rouché's Theorem (See appendix), to estimate the upper bounded of $|c|$ for arbitrary $n \in \mathbb{N}, n \geqslant 2$.

Let $f_{n,c}(z) = z^n + c$ and $\widetilde{f}_{n,c}(z) = z^n - z + c$ when $n = 2, 3, 4, \ldots$

Lemma 4.1. *If* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$, *then* $f_{n,c}(z)$ *has exactly one attractive fixed point in* $D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$.

Proof. Let $g(z) = z^n - z$. Consider $\xi \in \partial D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$.

Then
$$|\widetilde{f}_{n,c}(\xi) - g(\xi)| = |(\xi^n - \xi - c) - (\xi^n - \xi)| = |c| < \frac{n-1}{n\sqrt[n-1]{n}},$$

and
$$|g(\xi)| = |\xi^n - \xi| \geqslant ||\xi|^n - |\xi|| = \left|\left(\frac{1}{\sqrt[n-1]{n}}\right)^n - \left(\frac{1}{\sqrt[n-1]{n}}\right)\right|$$

$$= \left| \frac{1}{n \sqrt[n-1]{n}} - \frac{1}{\sqrt[n-1]{n}} \right|$$
$$= \frac{n-1}{n \sqrt[n-1]{n}} \ .$$

Thus $|\widetilde{f}_{n,c}(\xi) - g(\xi)| < |g(\xi)| \quad \forall \xi \in \partial D\left(0; \frac{1}{\sqrt[n-1]{n}}\right)$. By Rouché's Theorem, $\widetilde{f}_{n,c}$ and g have the same number of zeros in $D\left(0; \frac{1}{\sqrt[n-1]{n}}\right)$. Since g has only one zero in $D\left(0; \frac{1}{\sqrt[n-1]{n}}\right)$, so does $\widetilde{f}_{n,c}$. Let z_0 be a zero of $\widetilde{f}_{n,c}$ in $D\left(0; \frac{1}{\sqrt[n-1]{n}}\right)$. Since $|f'_{n,c}(z_0)| = |n(z_0)^{n-1}| < n\left(\frac{1}{\sqrt[n-1]{n}}\right)^{n-1} = 1$, z_0 is attractive. This implies $f_{n,c}$ has exactly one attractive fixed point in $D\left(0; \frac{1}{\sqrt[n-1]{n}}\right)$. □

Lemma 4.2. *Let C be a loop in the complex plane. If c is inside C, then $f_{n,c}^{-1}(C)$ is a loop, with the inverse image of interior of C as the interior of $f_{n,c}^{-1}(C)$.*

Proof. Suppose that c is inside C. Note that $f_{n,c}^{-1}(C) = (z-c)^{1/n}$ and $(f_{n,c}^{-1})'(z) = \frac{1}{n}(z-c)^{-(n-1)/n}$, which is finite and non-zero if $z \neq c$. Hence, if we select one of the n branches of $f_{n,c}^{-1}$, the set $f_{n,c}^{-1}(C)$ is locally a smooth curve, provided $c \notin C$.

Take an initial point w on C and choose one of the n values for $f_{n,c}^{-1}(w)$. Allowing $f_{n,c}^{-1}(z)$ to vary continuously as z moves around C, the point $f_{n,c}^{-1}(z)$ traces out a smooth curve. When z returns to w, however, $f_{n,c}^{-1}(w)$ takes its second value. As z traverses C again, $f_{n,c}^{-1}(z)$ continues on its smooth path, which closes as z returns to w the second time. We continue in this way until z returns to w the n^{th} time. Since $c \notin C$, we have $0 \notin f_{n,c}^{-1}(C)$, so $f_{n,c}{}'(z) \neq 0$ on $f_{n,c}^{-1}(C)$. This $f_{n,c}$ is locally a smooth bijective transformation near points on $f_{n,c}^{-1}(C)$. In particular $z \in f_{n,c}^{-1}(C)$ cannot be a point of self-intersection of $f_{n,c}^{-1}(C)$, otherwise $f_{n,c}(z)$ would be at a self-intersection of C.

Since $f_{n,c}$ is a continuous function that maps the loop $f_{n,c}^{-1}(C)$ and no other points onto the loop C, the polynomial $f_{n,c}$ must map the interior and exterior of $f_{n,c}^{-1}(C)$ into the interior and exterior of C, respectively. Hence $f_{n,c}^{-1}$ maps the interior of C to the interior of $f_{n,c}^{-1}(C)$. $\square$

Lemma 4.3. *Assume that* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$ *when* $n = 2, 3, 4, \ldots$ *If* C_0 *is the circle* $|z| = \dfrac{1}{\sqrt[n-1]{n}}$, *then* $C_k = f_{n,c}^{-k}(C_0)$ *is a loop surrounding (=enclosing, possibly, touching)* $C_{k-1} = f_{n,c}^{-k+1}(C_0)$ *where* $k = 1, 2, 3, \ldots$

Proof. Write $C_k = f_{n,c}^{-k}(C_0)$ where $k = 1, 2, 3, \ldots$ Let C_0 be the curve $|z| = \dfrac{1}{\sqrt[n-1]{n}}$.

Since
$$\begin{aligned} |f_{n,c}^{-1}(z)| &\geqslant ||z| - |c||^{\frac{1}{n}} \\ &> \left| \frac{1}{\sqrt[n-1]{n}} - \frac{n-1}{n\sqrt[n-1]{n}} \right|^{\frac{1}{n}} \\ &= \frac{1}{\sqrt[n-1]{n}} \\ &= |z| \qquad , \text{ for all } z \in C_0, \end{aligned}$$

and by Lemma 4.2., C_1 is a loop surrounding C_0.

Next, we will show that C_2 is a loop surrounding C_1. Assume that for each $z, w \in C_0$ such that $\arg(f_{n,c}^{-2}(z)) = \theta = \arg(f_{n,c}^{-1}(w))$, $|f_{n,c}^{-2}(z)| < |f_{n,c}^{-1}(w)|$. Hence $|(f_{n,c}^{-2}(z))^n| < |(f_{n,c}^{-1}(w))^n|$ and $\arg((f_{n,c}^{-2}(z))^n) = n\theta = \arg((f_{n,c}^{-1}(w))^n)$. Therefore, for a fixed c, we have $(f_{n,c}^{-2}(z))^n + c \in C_1$, $(f_{n,c}^{-1}(w))^n + c \in C_0$, and $|(f_{n,c}^{-2}(z))^n + c| < |(f_{n,c}^{-1}(w))^n + c|$, which contradicts the above result. Thus $|f_{n,c}^{-2}(z)| \geqslant |f_{n,c}^{-1}(w)|$ for all $z, w \in C_0$. This implies C_2 is surrounding C_1. By Lemma 4.2., C_2 is a loop surrounding C_1. Using the same argument, we have that C_k is a loop surrounding C_{k-1} where $k = 1, 2, 3, \ldots$ $\square$

Lemma 4.4. *Let* c *be a complex number such that* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$ *and* C_0 *be the circle* $|z| = \dfrac{1}{\sqrt[n-1]{n}}$. *Then, for each* $k > 1$, $\left|f_{n,c}^{-k}(z) - f_{n,c}^{-k+1}(z)\right| < \alpha\gamma^n$ *for some constants* α *and* $\gamma > 1$.

Proof. Since $f'_{n,c}(z_0) = n(z_0)^{n-1}$ and C_0 is the circle $|z| = \dfrac{1}{\sqrt[n-1]{n}}$, there is a positive number $r > 1$ such that $|f'_{n,c}(z)| > r$ for all z outside C_0. Thus, $|(f_{n,c}^{-1}(z))'| \leqslant$

$\dfrac{1}{|(f_{n,c})'(z)|} < \dfrac{1}{r}$ for all z outside C. For each two points z_1, z_2 outside C, let $\beta : [0,1] \to \mathbb{C} \setminus C$ be the straight line joining z_1 to z_2.

Then
$$\begin{aligned}
|f_{n,c}^{-1}(z_2) - f_{n,c}^{-1}(z_1)| &= \left| \int_\beta (f_{n,c}^{-1}(z))' dz \right| \\
&\leqslant \int_{z_1}^{z_2} |(f_{n,c}^{-1}(\beta(t)))'||d\beta(t)| \\
&< \frac{1}{r} \int_{z_1}^{z_2} |d\beta(t)| \\
&= \frac{1}{r}|z_1 - z_2|.
\end{aligned}$$

By direct calculation, for each $z \in C_0$ and $k \in \mathbb{N}$, $f_{n,c}^{-k}(z)$ is outside C_0. Applying the above inequality, we get that

$$\begin{aligned}
\left|f_{n,c}^{-k}(z) - f_{n,c}^{-k+1}(z)\right| &< \left(\frac{1}{r}\right) \left|f_{n,c}^{-k+1}(z) - f_{n,c}^{-k+2}(z)\right| \\
&< \left(\frac{1}{r}\right)^2 |f_{n,c}^{-k+2}(z) - f_{n,c}^{-k+3}(z)| \\
&\vdots \\
&< \left(\frac{1}{r}\right)^{k-2} |f_{n,c}^{-2}(z) - f_{n,c}^{-1}(z)|.
\end{aligned}$$

Hence, for each $k > 1$ and $z \in C_0$, $|f_{n,c}^{-k}(z) - f_c^{-k+1}(z)| < \alpha\gamma^k$ where $\gamma = \left(\dfrac{1}{r}\right)$ and $\alpha = r^2|f_{n,c}^{-2}(z) - f_{n,c}^{-1}(z)|$, as required. □

Lemma 4.5. *Let $\{\psi_k(\theta)\}_{k=0}^{\infty}$ be a sequence of continuous functions on an open domain U such that there is a positive number $\gamma < 1$ such that for each $n \in \mathbb{N}$,*

$$|\psi_k(\theta) - \psi_{k-1}(\theta)| < (\gamma)^k.$$

Then $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as $k \to \infty$.

Proof. Let $g_k(\theta) = \psi_k(\theta) - \psi_{k-1}(\theta)$. Then for each $k \in \mathbb{N}$, $|g_k(\theta)| < (\gamma)^k$. By Weierstrass M-test, $\psi_k(\theta) - \psi_0(\theta)$ converges uniformly to a continuous function $\phi(\theta)$ as k$\to \infty$. Let $\psi(\theta) = \phi(\theta) + \psi_0(\theta)$. Then $\psi(\theta)$ is also a continuous function. Hence $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as k $\to \infty$. □

Theorem 4.6. *Let c be a complex number such that $|c| < \dfrac{n-1}{n \sqrt[n-1]{n}}$ where $n = 2, 3, 4, \ldots$ Then $J(f_{n,c})$ is a simple closed curve.*

Proof. Let C_0 be the circle $|z| = \dfrac{1}{\sqrt[n-1]{n}}$. By Lemma 4.1., c and the attractive fixed point of $f_{n,c}$ are inside C_0. By Lemma 4.2. and 4.3., the inverse image $C_1 = f_{n,c}^{-1}(C_0)$ is a loop surrounding C_0. Let A_1 be the annular region between C_0 and C_1. We fill A_1 by a continuum of curves, called trajectories, which leave C_0 and reach C_1 perpendicularly. For each θ, let $\psi_1(\theta)$ be the point on C_1 at the end of the trajectory leaving C_0 at $\psi_0(\theta) = \dfrac{1}{\sqrt[n-1]{n}}\ e^{i\theta}$. Let A_2 be the inverse image $f_{n,c}^{-1}(A_1)$ which is the annular region with outer boundary the loop $C_2 = f_{n,c}^{-1}(C_1)$ and inner boundary C_1, with $f_{n,c}$ mapping A_2 to A_1 in a n-to-1 manner. Then the inverse image of the trajectories joining C_0 to C_1 provides a family of trajectories joining C_1 to C_2. Let $\psi_2(\theta)$ be the point on C_2 at the end of the trajectory leaving C_1 at $\psi_1(\theta)$. Continuing this process, we get a sequence of loops C_k, each surrounding its predecessor, and families of trajectories joining the points $\psi_k(\theta)$ on C_k to $\psi_{k+1}(\theta)$ on C_{k+1} for each k.

As k $\to \infty$, the curve C_k approach the boundary of the basin of attraction fixed point of $f_{n,c}$, that is, they approach the Julia set $J(f_{n,c})$. By Lemma 4.4., we get the length of the trajectory joining $\psi_k(\theta)$ to $\psi_{k+1}(\theta)$ converges to 0 at a geometric rate as k $\to \infty$. By Lemma 4.5., $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as k$\to \infty$. Hence $J(f_{n,c})$ is the closed curve given by $\psi(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$.

Now, it remains to show that $\psi(\theta)$ represents a simple curve. Assume that $\psi(\theta_1) = \psi(\theta_2)$. Let D be the region bounded by C_0 and the two trajectories joining $\psi_0(\theta_1)$ and $\psi_0(\theta_2)$ to this common point. The boundary of D remains bounded under iterates of $f_{n,c}$, so by the maximum modulus Theorem (See Appendix), D remains bounded under iterates of $f_{n,c}$. By Lemma 2.5., we have the interior of

D cannot contain any points of $J(f_{n,c})$. Hence $\psi(\theta_1) = \psi(\theta) = \psi(\theta_2)$ for all θ between θ_1 and θ_2. It follows that $\psi(\theta)$ has no point self-intersection. Therefore $J(f_{n,c})$ is a simple closed curve. $\square$

Proposition 4.7. *Suppose c is a complex number which is not real and $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$. Then $J(f_{n,c})$ is a simple nowhere differentiable closed curve.*

Proof. By Theorem 4.6., $J(f_{n,c})$ is a simple closed curve. Let z_1 be a repelling fixed point of $f_{n,c}$. It is easy to check that $f'_{n,c}(z_1)$ is a complex number which is not real. We will show that z_1 does not lie in a smooth arc in $\psi(\theta)$. Suppose not. Since $J(f_{n,c})$ is invariant under $f_{n,c}$, the image of $\psi(\theta)$ would also be a smooth arc in $J(f_{n,c})$ passing through z_1. Since $arg(f'_{n,c}(z_1)) \neq 0$ and $\neq \pi$, the tangents to these two curves would not be parallel. Hence, $\psi(\theta)$ would not be simple at z_1, which ia a contradiction. This implies z_1 does not lie in a smooth arc in $\psi(\theta)$. By Proposition 2.6.(b), the preimages of z_1 are dense in $J(f_{n,c})$. It follows that $J(f_{n,c})$ contains no smooth arcs. $\square$

APPENDIX

Suppose f is holomorphic in $D(a;r)$ for some r. The point a is said to be a *zero* of f if $f(a) = 0$. The zero a is *isolated* if there exists ε such that $D(a;\varepsilon) \setminus \{a\}$ contains no zeros of f. A function f has an *isolated singularity* at a point $z = a$ if there is a number $R > 0$ such that f is holomorphic on $D(a;R) \setminus \{a\}$ and is not holomorphic at point a. The point a is called a *removable singularity* if there is holomorphic function $g : D(a;R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z-a| < R$. If $\lim_{z\to a} |f(z)| = \infty$, then a is a *pole* of f. If f has a pole at $z = a$ and m is the smallest positive integer such that $f(z)(z-a)^m$ has removable singularity at $z = a$ then f has a *pole of order* m at $z = a$. If G is open and f is a function defined and holomorphic in G except for poles, then f is a *meromorphic function* on G.

Theorem (Rouché's Theorem) *Suppose f and g are meromorphic in the region G, an open connected subset of the complex plane, and $\overline{D}(a;R) \subset G$. If f and g have no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$ and $|f(z)-g(z)| < |g(z)|$ for z on γ then $Z_f - P_f = Z_g - P_g$ where $Z_f, Z_g(P_f, P_g)$ are the number of zeros (poles) of f and g inside $|z| = R$ counted according to multiplicity.*

Theorem (Motel's Theorem) *Let $\{g_k\}$ be a family of complex analytic functions on an open domain U. If $\{g_k\}$ is not a normal family, then for all $w \in \mathbb{C}$ with at most one exception we have $g_k(z) = w$ for some $z \in U$ and some k.*

Theorem (Maximum Modulus Theorem) *Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\overline{G}$ which is holomorphic in G. Then $max\{|f(z)| : z \in \overline{G}\} = max\{|f(z)| : z \in \partial G\}$.*

Theorem (Weierstrass M-Test) *Let $u_n : X \to \mathbb{C}$ be a such that $|u_n(x)| \leqslant M_n$ for every x in X and suppose the constants satisfy $\displaystyle\sum_{n=1}^{\infty} M_n < \infty$.*

Then $\sum_{n=1}^{\infty} u_n$ *is uniformly convergent.*

Theorem *Let* (X, d) *and* (Ω, ρ) *be metric spaces. Suppose* $f_n : (X, d) \to (\Omega, \rho)$ *is continuous for each* n *and that a sequence* $\{f_n\}$ *converges uniformly to* f*. Then* f *is continuous.*

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