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NEARRING STRUCTURE OF VARIANTS OF SOME TRANSFORMATION SEMIGROUPS

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เนียร์ริงซ้าย [*ขวา*] คือระบบ $(N,+,\cdot)$ ซึ่ง $(N,+)$ เป็นกรุป $(N,\cdot)$ เป็นกึ่งกรุปและตัวดำเนินการ $\cdot$ มีสมบัติการแจกแจงทางซ้าย [ขวา] บนตัวดำเนินการ $+$ สำหรับกึ่งกรุป S ให้ S^0 คือ S ถ้า S มีศูนย์และมีสมาชิกมากกว่าหนึ่งตัว นอกนั้นให้ S^0 คือ กึ่งกรุป S ที่ผนวกด้วยศูนย์ 0 กล่าวว่ากึ่งกรุป S *ให้โครงสร้างเนียร์ริงซ้าย* [*ขวา*] ถ้ามีตัวดำเนินการ $+$ บน S^0 ที่ทำให้ $(S^0,+,\cdot)$ เป็นเนียร์ริงซ้าย [ขวา] สำหรับกึ่งกรุป S และ $a \in S$ นิยามตัวดำเนินการ $*$ บน S โดย $x * y = xay$ สำหรับทุก $x, y \in S$ เรียกกึ่งกรุป $(S,*)$ ว่า *แวเรียนต์* ของ S และเขียนแทน $(S,*)$ ด้วย (S,a)

ให้ X เป็นเซตและ $P(X)$ คือเซตของการแปลงจากสับเซตของ X ไปยัง X ดังนั้น $P(X)$ เป็นกึ่งกรุปภายใต้การประกอบ ให้ V เป็นปริภูมิเวกเตอร์บนริงการหาร R และ $L_R(V)$ คือเซตของการแปลงเชิงเส้นบน V จะได้ว่า $L_R(V)$ เป็นกึ่งกรุปภายใต้การประกอบ เราศึกษากึ่งกรุปย่อยของแวเรียนต์ของ $P(X)$ และกึ่งกรุปย่อยของแวเรียนต์ของ $L_R(V)$ หลากหลายชนิด จุดมุ่งหมายหลักคือการพิจารณาว่าเมื่อใดกึ่งกรุปเหล่านี้ให้โครงสร้างของเนียร์ริงซ้าย [ขวา]

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A *left* [*right*] *nearring* is a system $(N, +, \cdot)$ such that $(N, +)$ is a group, $(N, \cdot)$ is a semigroup and the operation $\cdot$ left [right] distributes over the operation $+$. For a semigroup S, let S^0 be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup S with a zero 0 adjoined. We say that a semigroup S *admits a left* [*right*] *nearring structure* if there exists an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a left [right] nearring. For a semigroup S and $a \in S$, define an operation $*$ on S by $x * y = xay$ for all $x, y \in S$. The semigroup $(S, *)$ is called a *variant* of S and $(S, *)$ is denoted by (S, a).

Let X be a set and $P(X)$ be the set of all transformations from subsets of X into X. Hence $P(X)$ is a semigroup under composition. Let V be a vector space over a division ring R and $L_R(V)$ be the set of all linear transformations on V. Then $L_R(V)$ is a semigroup under composition. Various types of subsemigroups of variants of $P(X)$ and subsemigroups of variants of $L_R(V)$ are studied. Main results are determining when these semigroups admit the structure of a left [right] nearring.

Department : ....Mathematics.... Student's Signature ....
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CONTENTS

CHAPTER I
INTRODUCTION

For a semigroup S, let S^0 be S if S has a zero and $|S| > 1$. Otherwise S^0 means S with adjoined zero which is a new symbol.

Ring Theory is a classical subject in mathematics and had long been studied. It is well-known that the multiplicative structure of a ring is a semigroup with zero. For a semigroup S, if S is isomorphic to the multiplicative structure of some rings, then S is said to *admit a ring structure*. It is equivalent to there exists an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is an operation on S^0. For various studies in this area, see [7], [12], [14], [15] and [16].

By the definition, every left [right] nearring generalizes rings and the multiplicative structure of a left [right] nearring is a semigroup. It is reasonable to ask that for a semigroup S, whether S^0 is isomorphic to the multiplicative structure of some left [right] nearrings. Then a semigroup $(S, \cdot)$ is said to *admit a left* [*right*] *nearring structure* if there exists an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a left [right] nearring. Some research of semigroups admitting the nearring structure can be seen in [5] and [8].

If S is a semigroup and $a \in S$, the semigroup $(S, *)$ defined by $x * y = xay$ for all $x, y \in S$ is called a *variant* of S and it is denoted by (S, a). Variants of abstract semigroups were first studied by J. Hickey [6] in 1983. In fact, variants of concrete semigroups of relations were earlier considered by Magill [11] in 1967.

In this thesis, we generalize some results in [5], [16] and [8]. The interested semigroups are some variants of transformation semigroups and their subsemigroups. For a considered semigroup S, we characterize a transformation θ

which (S, θ) is a semigroup. The main purpose is to determine when these semigroups admit a left [right] nearring.

Examples, basic definitions, some motivations, elementary results and quoted results, regarded in this thesis, are contained in Chapter II.

In Chapter III, some results of [5] and [16] are generalized. We concern some variants of transformation semigroups which shown in [5] and [16].

Chapter IV and Chapter V contain generalizations of the results in [8]. Some linear transformation semigroups which do not admit a left [right] nearring will be shown in Chapter IV.

In the last chapter, we still consider some subsemigroups of variants of linear transformation semigroups. We provide necessary and sufficient conditions for each of these semigroups to admit a left [right] nearring structure.

Notice that some techniques in [5], [8] and [16] will be applied.

CHAPTER II

PRELIMINARIES

For a set X, the cardinality of X will be denoted by $|X|$. The set of all integers and the set of all real numbers will be denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. The following proposition is well-known in set theory.

Proposition 2.1. *Let X be an infinite set. Then there is a partition $\{A, B\}$ of X such that $|X| = |A| = |B|$.*

Let S be a semigroup. A *left* [*right*] *zero* of S is an element $z \in S$ such that $zx = z$ $[xz = z]$ for all $x \in S$. An element 0 of S is called a *zero* if $0x = 0 = x0$ for all $x \in S$. If S has a left zero z_1 and a right zero z_2, then $z_1 = z_2$ which is the zero of S. A semigroup S with zero 0 is said to be a *zero semigroup* if $ab = 0$ for all $a, b \in S$.

A *left* [*right*] *nearring* is a triple $(N, +, \cdot)$ where

(i) $(N, +)$ is a group,

(ii) $(N, \cdot)$ is a semigroup,

(iii) $z \cdot (x + y) = z \cdot x + z \cdot y$ $[(x + y) \cdot z = x \cdot z + y \cdot z]$ for all $x, y, z \in N$.

An element 0 is called a *zero* of left [right] nearring if $0 \cdot x = 0 = x \cdot 0$ for all $x \in N$. Throughout, for every $x, y \in N$, $x \cdot y$ is denoted by xy.

Proposition 2.2. ([4]) *Let $(N, +, \cdot)$ be a left* [*right*] *nearring with the additive identity 0. Then*

(i) $x0 = 0$ $[0x = 0]$ *for all $x \in N$,*

(ii) $x(-y) = -(xy)$ $[(-x)y = -(xy)]$ *for all $x, y \in N$.*

Throughout, the image of a mapping α at x is written by $x\alpha$. The following examples show that every left [right] nearring is a generalization of rings.

Example 2.3. ([4]) Let $(G,+)$ be a group. Define an operation $*$ on G by

$$a * b = a$$

for all $a, b \in G$. It is easy to see that $(G,+,*)$ is a right nearring. If $|G| > 1$, then $(G,+,*)$ is neither a left nearring nor a ring. To see this, let $a, b \in G$ be distinct. Suppose that $(G,+,*)$ is a left nearring. Then

$$a = a * (a+b) = a * a + a * b = a + a,$$
$$b = b * (a+b) = b * a + b * b = b + b.$$

So $a = b = 0$, the identity of $(G,+)$, this is impossible. Hence $(G,+,*)$ is not a left nearring.

Example 2.4. ([4]) Let $(A,+)$ be an abelian group with identity 0 and $M(A)$ be the set of all functions on A. It is clearly that $(M(A),+,\circ)$ is a left nearring where $+$ and $\circ$ are the usual addition and the composition of functions, respectively. By Proposition 2.2(i), $(M(A),+,\circ)$ is neither a right nearring nor a ring where $|A| > 1$. To show this, let $a, b \in A$ be distinct. Suppose that $(M(A),+,\circ)$ is a right nearring. Define $f, g \in M(A)$ by

$$xf = a \text{ and } xg = b$$

for all $x \in A$. Let $\theta \in M(A)$ be such that $x\theta = 0$ for all $x \in A$. Then θ is the additively identity of $M(A)$. By Proposition 2.2 (i),

$$f = \theta \circ f = \theta = \theta \circ g = g$$

Hence $a = b$, a contradiction. Therefore $(M(A),+,\circ)$ is not a right nearring.

Example 2.5. ([5]) Let $M(\mathbb{R})$ be the set of all functions from $\mathbb{R}$ into itself,

$$C(\mathbb{R}) = \{f \in M(\mathbb{R}) \mid f \text{ is continuous on } \mathbb{R}\} \text{ and}$$
$$D(\mathbb{R}) = \{f \in M(\mathbb{R}) \mid f \text{ is differentiable on } \mathbb{R}\}.$$

Then $(M(\mathbb{R}), +, \circ)$, $(C(\mathbb{R}), +, \circ)$ and $(D(\mathbb{R}), +, \circ)$ are left nearrings which are not rings where $+$ and $\circ$ are the usual addition and the composition of functions, respectively. We will show that $(D(\mathbb{R}), +, \circ)$ is not a ring. Suppose in the contrary that it is a ring. Let n be the natural number but not 2. Define $f, g \in M(\mathbb{R})$ by $xf = n^x$ and $xg = n^{2x}$ for all $x \in \mathbb{R}$. Then $f, g \in D(\mathbb{R})$. Consider

$$n^{n^x + n^{2x}} = (x)((f+g) \circ f) = (x)(f \circ f) + (x)(g \circ f) = n^{n^x} + n^{n^{2x}}$$

for all $x \in \mathbb{R}$. But if $x = 0$, then $n^2 = 2n$, a contradiction. Hence $(D(\mathbb{R}), +, \circ)$ is not a ring.

For a semigroup S, let S^0 be S if S has a zero and contains more than one element, otherwise, let S^0 be a semigroup S with a zero 0 adjoined. Notice that if $|S| = 1$, then $S^0 \cong (\mathbb{Z}_2, \cdot)$. A semigroup S is said to *admit a ring structure* if S^0 is isomorphic to the multiplicative structure of some rings. Similarly, a semigroup S is said to *admit a left* [*right*] *nearring structure* if there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a left [right] nearring where $\cdot$ is the operation on S^0. Let $\mathcal{R}$, $\mathcal{LNR}$ and $\mathcal{RNR}$ denote the class of all semigroups admitting a ring structure, the class of all semigroups admitting a left nearring structure and the class of all semigroups admitting a right nearring structure, respectively. Then $\mathcal{R} \subseteq \mathcal{LNR} \cap \mathcal{RNR}$.

Notice that in [16], S. Srichaiyarat showed that every zero semigroup admits a ring structure. Then every zero semigroup always admit a left [right] nearring structure too.

For a semigroup S and $a \in S$, define an operation $*$ on S by $x * y = xay$ for all $x, y \in S$. The semigroup $(S, *)$ is called a *variant* of S and $(S, *)$ is denoted by (S, a). We then have some properties as follows.

Proposition 2.6. *Let $(S, \cdot)$ be a semigroup with identity e, S_1 be a subsemigroup of S containing e and $a \in S$. Then S_1 is a subsemigroup of a variant (S, a) if and only if $a \in S_1$.*

Proof. Assume that $a \notin S_1$. Since $eae = a \notin S_1$, we see that (S_1, a) is not a semigroup. The converse is obvious. $\square$

For a subsemigroup S and $a \in S$, we note that if S_1 is a subsemigroup of S satisfying Proposition 2.6, then the sentence "(S_1, a) is a subsemigroup of (S, a)" means "S_1 is a subsemigroup of (S, a)" which is equivalent to "(S_1, a) is a semigroup".

Proposition 2.7. *Let $(S, \cdot)$ be a semigroup, S_1 be a subsemigroup of S and $a \in S$. If $(S_1, \cdot)$ admits a left [right] nearring structure and (S_1, a) is a semigroup, then (S_1, a) also admits a left [right] nearring structure.*

Proof. Since $0ax = 0 = xa0$ for all $x \in S_1$, 0 is also a zero element in (S_1, a). Suppose that $(S_1^0, +, \cdot)$ is a left nearring for some operation $+$ on S_1^0. Claim that $(S_1^0, +, a)$ is a left nearring. It suffices to show that the left distributive law holds. Let $x, y, z \in S_1^0$. Then $xa(y + z) = xay + xaz$. Hence $(S_1, a) \in \mathcal{LNR}$. $\square$

Proposition 2.8. *Let $(S, \cdot)$ be a semigroup, S_1 be a subsemigroup of S and $a \in S$. Assume that (S_1, a) is a semigroup. If $(S_1, \cdot) \cong (S_1, a)$, then $(S_1, \cdot)$ admits a left [right] nearring structure if and only if (S_1, a) admits a left [right] nearring structure.*

Proof. By Proposition 2.7, it suffices to prove the converse. Assume that $(S_1, \cdot) \cong (S_1, a)$ and $(S_1^0, +, a)$ is a left nearring for some operation $+$ on S_1^0. We remark that both of (S_1^0, a) and $(S_1^0, \cdot)$ have the same zero, 0. Let $\varphi : (S_1, \cdot) \to (S_1, a)$ be an isomorphism. Define a mapping $\psi : (S_1^0, \cdot) \to (S_1^0, a)$ by

$$x\psi = \begin{cases} x\varphi & \text{if } x \in S_1 \smallsetminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Then ψ is an isomorphism. Define $\oplus$ on S_1^0 by

$$\begin{aligned} x \oplus y &= (x\psi + y\psi)\psi^{-1} && \text{if } x, y \in S_1 \smallsetminus \{0\}, \\ x \oplus 0 &= x = 0 \oplus x && \text{if } x \in S_1^0. \end{aligned}$$

Clearly, $(S_1^0, \oplus)$ is a binary operation on S_1^0. Let $x, y, z \in S_1^0$. Thus

$$\begin{aligned}(x \oplus y) \oplus z &= ((x\psi + y\psi)\psi^{-1}) \oplus z \\ &= (((x\psi + y\psi)\psi^{-1})\psi + z\psi)\psi^{-1} \\ &= ((x\psi + y\psi) + z\psi)\psi^{-1} \\ &= (x\psi + (y\psi + z\psi))\psi^{-1} \\ &= (x\psi + ((y\psi + z\psi)\psi^{-1})\psi)\psi^{-1} \\ &= x \oplus (y\psi + z\psi)\psi^{-1} \\ &= x \oplus (y \oplus z).\end{aligned}$$

Hence $(S_1^0, \oplus)$ is a semigroup. Since

$$0\psi^{-1} = 0 \oplus (0\psi^{-1}) = (0\psi + (0\psi^{-1})\psi)\psi^{-1} = (0\psi + 0)\psi^{-1} = 0\psi\psi^{-1} = 0,$$

we have $0\psi^{-1} = 0$ is the identity of $(S_1^0, \oplus)$. Let $x \in S_1$ be such that $x \neq 0$. Consider

$$x \oplus (-(x\psi))\psi^{-1} = (x\psi + (-(x\psi))\psi^{-1}\psi)\psi^{-1} = (x\psi + (-(x\psi)))\psi^{-1} = 0\psi^{-1} = 0,$$

where $-(x\psi)$ is an inverse of $x\psi$ under the operation $+$. Then we can see that $(-(x\psi))\psi^{-1}$ is an inverse of x on $(S_1^0, \oplus)$. Hence $(S_1^0, \oplus)$ is a group. Next, we will show that $\cdot$ is left distributive over $\oplus$. Let $x, y, z \in S_1^0$. Then

$$\begin{aligned}x(y \oplus z) &= x(y\psi + z\psi)\psi^{-1} \\ &= x\psi\psi^{-1}(y\psi + z\psi)\psi^{-1} \\ &= ((x\psi)a(y\psi + z\psi))\psi^{-1} \\ &= ((x\psi)a(y\psi) + (x\psi)a(z\psi))\psi^{-1}\end{aligned}$$

$$= (xy\psi + xz\psi)\psi^{-1}$$
$$= xy \oplus xz.$$

Therefore $(S_1^0, \cdot) \in \mathcal{LNR}$. □

Next, let X be a set and

$$P(X) = \{\alpha : A \to X \mid A \subseteq X\}.$$

Let 0 be the empty transformation. For each $\alpha \in P(X)$, the domain of α and range of α are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For $\alpha, \beta \in P(X)$, it is well-known that

$$\alpha\beta = \begin{cases} 0 & \text{if } \operatorname{ran}\alpha \cap \operatorname{dom}\beta = \varnothing, \\ \alpha_{|(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\alpha^{-1}} \circ \beta_{|(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)} & \text{if } \operatorname{ran}\alpha \cap \operatorname{dom}\beta \neq \varnothing. \end{cases}$$

Then $P(X)$ is a semigroup with zero under the composition of all partial transformations on X and the empty transformation is the zero element. The followings are standard transformation subsemigroups of $P(X)$:

$$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is an injection}\},$$
$$C(X) = \{\alpha \in P(X) \mid |\operatorname{ran}\alpha| \leq 1\},$$
$$T(X) = \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\},$$
$$G(X) = \{\alpha \in T(X) \mid \alpha \text{ is a bijection}\}.$$

Then $G(X) \subseteq T(X)$ and $G(X) \subseteq I(X)$. It is easy to see that $I(X)$ and $C(X)$ alway have the zero. In the otherwise, $G(X)$ and $T(X)$ have no zero if $|X| > 1$. For $\varnothing \neq A \subseteq X$ and $x \in X$, let A_x be the constant mapping whose domain and range are A and $\{x\}$, respectively, and the identity map on A will be denoted by 1_A. Observe that $1_X \in G(X)$, but $1_X \in C(X)$ if and only if $|X| = 1$. For distinct $a, b \in X$, the notation $\binom{a}{b}$ means the mapping in $I(X)$ such that the domain and the range are $\{a\}$ and $\{b\}$, respectively.

Proposition 2.9. *Let X be a set and $S(X)$ be any transformation semigroups defined as above. If $X = \varnothing$, then $S(X)$ admits a left [right] nearring structure.*

Proof. Assume that $X = \varnothing$. Then $S(X) = \{0\}$, so $S^0(X) \cong (\mathbb{Z}_2, \cdot) \in \mathcal{R}$. Hence $S(X) \in \mathcal{LNR} \cap \mathcal{RNR}$. □

By Proposition 2.9, it suffices to consider a set X as a nonempty set. The first purpose of this thesis is to determining when variants of transformation semigroups $G(X)$, $T(X)$, $P(X)$, $I(X)$ and $C(X)$ belong to $\mathcal{LNR}$ and $\mathcal{RNR}$. These results are shown in Chapter III.

Next, let V be a vector space over a division ring R and $L_R(V)$ be the semigroup under the composition of all linear transformations on V. Then $L_R(V)$ admits a ring structure under the usual addition of linear transformations. Recall that the image of v under $\alpha \in L_R(V)$ is written by $v\alpha$. For $\alpha \in L_R(V)$, let $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ denote the kernel and the image of α, respectively. For any subspace W of V, $\dim_R W$ means the dimension of W. The identity map on V and the zero map on V will be denoted by 1_V and 0, respectively. The following five propositions are provided in this thesis. They are simple facts of vector spaces and linear transformations which will be used. The proofs are routine and elementary, so then we omitted them.

Proposition 2.10. *Let B be a basis for V. If $u \in B$ and $v \in \langle B \smallsetminus \{u\}\rangle$, then $(B \smallsetminus \{u\}) \cup \{u+v\}$ is also a basis for V.*

Proposition 2.11. *Let B be a basis for V, $A \subseteq B$ and $\varphi : B \smallsetminus A \to V$ a one-to-one mapping such that $(B \smallsetminus A)\varphi$ is a linearly independent subset of V. If $\alpha \in L_R(V)$ is defined by*

$$v\alpha = \begin{cases} 0 & \text{if } v \in A, \\ v\varphi & \text{if } v \in B \smallsetminus A, \end{cases}$$

then $\operatorname{Ker}\alpha = \langle A \rangle$ *and* $\operatorname{Im}\alpha = \langle B \smallsetminus A \rangle \varphi$.

Proposition 2.12. *Let B be a basis for V and $A \subseteq B$. Then*
(i) $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ *is a basis for the quotient space* $V/\langle A \rangle$ *and*
(ii) $\dim_R(V/\langle A \rangle) = |B \smallsetminus A|$.

Proposition 2.13. *Let B be a basis for V and B_1, B_2 and B_3 be disjoint subsets of B. Then $\langle B_1 \cup B_2 \rangle \cap \langle B_1 \cup B_3 \rangle = \langle B_1 \rangle$.*

Proposition 2.14. *Let B be a basis for V, $\alpha \in L_R(V)$ and B_1 be a basis for* $\operatorname{Ker}\alpha$ *such that $B_1 \subseteq B$. Then $(B \smallsetminus B_1)\alpha$ is a basis for* $\operatorname{Im}\alpha$.

Proposition 2.15. ([8]) *Let B be a basis for V and C a nonempty subset of B. Then* $\bigcap\limits_{v \in C} \langle B \smallsetminus \{v\} \rangle = \langle B \smallsetminus C \rangle$.

Let

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $G_R(V)$ is the unit group of $L_R(V)$. The following subsets of $L_R(V)$ are clearly subsemigroups of $L_R(V)$ containing $G_R(V)$,

$$M_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is a monomorphism}\},$$
$$E_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an epimorphism}\}.$$

Observe that $M_R(V) = E_R(V) = G_R(V)$ if and only if $\dim_R V$ is finite. Next, if V is an infinite dimensional vector space, let

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite }\},$$
$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im}\alpha) \text{ is infinite }\}.$$

We knew from [2] that $OM_R(V)$ and $OE_R(V)$ are both subsemigroups of $L_R(V)$ containing the zero mapping. For this case, the semigroups $OM_R(V)$ and $OE_R(V)$

may be referred to respectively as the *opposite semigroup of* $M_R(V)$ and the *opposite semigroup of* $E_R(V)$. For a cardinal number k with $k \leq \dim_R V$, let

$$K_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \geq k\},$$
$$CI_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im} \alpha) \geq k\},$$
$$I_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq k\}.$$

Then the zero mapping belongs to all of the above three subsets of $L_R(V)$. By [1], we can conclude that all of $K_R(V,k)$, $CI_R(V,k)$ and $I_R(V,k)$ are subsemigroups of $L_R(V)$. Observe that if $\dim_R V$ is infinite, the notations $OM_R(V)$ and $OE_R(V)$ defined previously denote $K_R(V,\aleph_0)$ and $CI_R(V,\aleph_0)$, respectively, that is,

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \geq \aleph_0\},$$
$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im} \alpha) \geq \aleph_0\},$$

where $\aleph_0$ is the smallest infinite cardinal number. Notice that if $\dim_R V$ is finite, then for $\alpha \in L_R(V)$, $\dim_R \operatorname{Ker} \alpha = \dim_R (V/\operatorname{Im} \alpha) = \dim_R V - \dim_R \operatorname{Im} \alpha$ since $\dim_R V = \dim_R \operatorname{Ker} \alpha + \dim_R \operatorname{Im} \alpha$ and $\dim_R V = \dim_R (V/\operatorname{Im} \alpha) + \dim_R \operatorname{Im} \alpha$. So we have

Proposition 2.16. ([1]) *If* $\dim_R V$ *is finite, then*

$$K_R(V,k) = CI_R(V,k) = I_R(V, \dim_R V - k)$$

for every cardinal number $k \leq \dim_R V$.

However, these are not generally true if $\dim_R V$ is infinite. This is shown by the following proposition. This proposition also shows that the semigroups $K_R(V,k)$, $CI_R(V,k)$ and $I_R(V,k)$ should be considered independently if $\dim_R V$ is infinite.

Proposition 2.17. ([1]) *Let* V *be an infinite dimensional vector space and a nonzero cardinal number* $k \leq \dim_R V$. *Then the following statements hold.*

(i) $CI_R(V,k) \neq K_R(V,l)$ for every cardinal number $l \leq \dim_R V$.
(ii) If $k < \dim_R V$, then $I_R(V,k) \neq K_R(V,l)$ and $I_R(V,k) \neq CI_R(V,l)$ for every cardinal number $l \leq \dim_R V$.

Next, let $K'_R(V,k)$, $CI'_R(V,k)$ and $I'_R(V,k)$ be subsets of $K_R(V,k)$, $CI_R(V,k)$ and $I_R(V,k)$ respectively as follows:

$$K'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha > k\} \text{ where } k < \dim_R V,$$
$$CI'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im}\alpha) > k\} \text{ where } k < \dim_R V,$$
$$I'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha < k\} \text{ where } 0 < k \leq \dim_R V.$$

Thus $K'_R(V,k)$, $CI'_R(V,k)$ and $I'_R(V,k)$ contain 0, moreover, they are respectively subsemigroups of $L_R(V)$ under the composition. For a cardinal number k, the *successor* of k is the smallest cardinal number greater than k. Note that if $k < \dim_R V$, then $K'_R(V,k) = K_R(V,k')$ and $CI'_R(V,k) = CI_R(V,k')$ where k' is the successor of k. Also, if $0 < k \leq \dim_R V$, k is a finite cardinal number and $\widetilde{k} = k-1$, then $I'_R(V,k) = I_R(V,\widetilde{k})$.

For $\alpha \in L_R(V)$, let

$$F(\alpha) = \{v \in V \mid v\alpha = v\}.$$

Then for $\alpha \in L_R(V)$, $F(\alpha)$ is a subspace of V and α is called an *almost identical linear transformation* on V if $\dim_R(V/F(\alpha))$ is finite. The set of all almost identical linear transformations on V will be denoted by $AI_R(V)$, that is,

$$AI_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/F(\alpha)) < \infty\}.$$

Observe that $1_V \in AI_R(V)$.

Proposition 2.18. ([2]) *$AI_R(V)$ is a subsemigroup of $L_R(V)$.*

Note that if $\dim_R V$ is finite, then $AI_R(V) = L_R(V)$ which admits a ring structure. Moreover, the semigroup $AI_R(V)$ does not contain the zero mapping when

$\dim_R V$ is infinite. We remark that every referred linear transformation semigroups $S(V)$, if $0 \in S(V)$ and $|S(V)| > 1$, the zero mapping is a zero element in $S(V)$.

Since every linear transformation on a vector space V can be defined on a basis for V, for convenience, we may write $\alpha \in L_R(V)$ by using a bracket notation. For examples,

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}$$

means

$$v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ v & \text{if } v \in B \smallsetminus B_1, \end{cases}$$

and

$$\beta = \begin{pmatrix} u & w & v \\ w & 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}$$

is equivalent to

$$v\beta = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v = w, \\ v & \text{if } v \in B \smallsetminus \{u, w\}, \end{cases}$$

where B is a basis for V, $B_1 \subseteq B$ and $u, w \in B$ are distinct.

If H and T are respectively subsemigroups of $G_R(V)$ and $AI_R(V)$, we show in Chapter IV that if $\dim_R V$ is infinite, the following systems

$(OM_R(V), \theta), (OE_R(V), \theta)$ where $\theta \in L_R(V)$,

$(OM_R(V) \cup H, \theta), (OE_R(V) \cup H, \theta)$ where $\theta \in H \cup \{1_V\}$,

$(OM_R(V) \cup T, \theta)$ where $\theta \in (E_R(V) \cap T) \cup \{1_V\}$,

$(OE_R(V) \cup T, \theta)$ where $\theta \in (M_R(V) \cap T) \cup \{1_V\}$,

are subsemigroups of $(L_R(V), \theta)$. The main purpose of Chapter IV is to show that if $\dim_R V$ is infinite, then $(OM_R(V), \theta)$ where $\theta \in E_R(V)$, $(OE_R(V), \theta)$ where $\theta \in M_R(V)$ and some subsemigroups of variants $L_R(V)$ containing $OM_R(V)$ and $OE_R(V)$, defined as above, do not admit the structure of both a left nearring and a right nearring.

In the last chapter, we also show that $K_R(V,k)$, $K'_R(V,k)$, $CI_R(V,k)$, $CI'_R(V,k)$, $I_R(V,k)$ and $I'_R(V,k)$ are semigroups of $(L_R(V), \theta)$ where $\theta \in L_R(V)$. We will determine when the following semigroups,

$$(K_R(V,k), \theta), (K'_R(V,k), \theta) \text{ where } \theta \in E_R(V),$$
$$(CI_R(V,k), \theta), (CI'_R(V,k), \theta), (I_R(V,k), \theta), (I'_R(V,k), \theta) \text{ where } \theta \in M_R(V),$$

admit the structure of a left nearring and a right nearring.

CHAPTER III
VARIANTS OF SOME TRANSFORMATION SEMIGROUPS ADMITTING THE NEARRING STRUCTURE

For this chapter, we consider well-known transformation semigroups on a nonempty set X. They are recalled as follows:

$$\begin{aligned} P(X) &= \{\alpha : A \to X \mid A \subseteq X\}, \\ I(X) &= \{\alpha \in P(X) \mid \alpha \text{ is an injection}\}, \\ C(X) &= \{\alpha \in P(X) \mid |\operatorname{ran}\alpha| \leq 1\}, \\ T(X) &= \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\}, \\ G(X) &= \{\alpha \in T(X) \mid \alpha \text{ is a bijection}\}. \end{aligned}$$

We knew that $G(X)$, $T(X)$, $P(X)$ and $I(X)$ contain the identity map on X. So if we consider variants of them, we then confident that they will generalize results in [5]. Since $C(X)$ dose not alway contain the identity mapping on X, then we cannot conclude that variants of $C(X)$ generalizes a semigroup $C(X)$. Since

$$\operatorname{ran}(f\theta g) \subseteq \operatorname{ran} g \text{ and } |\operatorname{ran}(f\theta g)| \leq |\operatorname{ran} g| \leq 1,$$

$f\theta g \in C(X)$ where $f, g \in C(X)$ and $\theta \in P(X)$. Then $C(X)$ is a subsemigroup of a variant $(P(X), \theta)$ where $\theta \in P(X)$.

A question,"if we use $\theta \in P(X)$, are $(G(X), \theta)$, $(T(X), \theta)$ and $(I(X), \theta)$ semigroups?", was answered by Proposition 2.6.

In this chapter, we will consider variants of $G(X)$, $T(X)$, $P(X)$ and $I(X)$ to admit the nearring structure. Moreover, $C(X)$ will be considered as a subsemigroup of all variants of $P(X)$.

3.1 The Variants of $G(X)$ and $T(X)$

Throughout this section, we assume that every left [right] nearring is additively commutative. The following lemma is useful.

Lemma 3.1.1. ([5]) (i) *$G(X)$ admits a left nearring structure if and only if $|X| \leq 2$.*
(ii) *$G(X)$ admits a right nearring structure if and only if $|X| \leq 2$.*

Theorem 3.1.2. (i) *Every variant of $G(X)$ admits a left nearring structure if and only if $|X| \leq 2$.*
(ii) *Every variant of $G(X)$ admits a right nearring structure if and only if $|X| \leq 2$.*

Proof. Since the mapping $\varphi : (G(X), \theta) \to (G(X), \circ)$ defined by $\alpha\varphi = \alpha\theta$ is clearly an isomorphism for all $\alpha, \theta \in G(X)$, we have $(G(X), \theta) \cong G(X)$ for all $\theta \in G(X)$. By Lemma 3.1.1 and Proposition 2.7, the proof is complete. □

Corollary 3.1.3. ([16]) *Every variant of $G(X)$ admits a ring structure if and only if $|X| \leq 2$.*

We can see that Theorem 3.1.2 generalizes Lemma 3.1.1. Next, we characterize when variants of $T(X)$ admit the structure of a left [right] nearring. The quoted result is used.

Lemma 3.1.4. ([5]) (i) *$T(X)$ always admit a left nearring structure.*
(ii) *$T(X)$ admits a right nearring structure if and only if $|X| = 1$.*

Theorem 3.1.5. (i) *Every variant of $T(X)$ always admit a left nearring structure.*
(ii) *Every variant of $T(X)$ admits a right nearring structure if and only if $|X| = 1$.*

Proof. (i) is applied from Lemma 3.1.4 (i) and Proposition 2.7.

(ii) Let $\theta \in T(X)$. Suppose that $|X| > 1$. Let $a, b \in X$ be distinct. Since $fX_a = X_a$ and $fX_b = X_b$ for all $f \in T(X)$, implies that a semigroup $T(X)$ has no left zero. Assume that $(T(X), \theta) \in \mathcal{RNR}$. Then there is an operation $+$ on $T^0(X)$ such that $(T^0(X), +, \theta)$ is a right nearring. Thus $X_a + X_b = f$ for some $f \in T^0(X)$.

Case 1: $f \neq 0$. Thus

$$X_a = f\theta X_a = (X_a + X_b)\theta X_a = X_a\theta X_a + X_b\theta X_a = X_a + X_a.$$

That is, $X_a = 0$, which is a contradiction.

Case 2: $f = 0$. So $0 = (X_a + X_b)\theta X_a = X_a + X_a$. Since $X_a + X_b = 0$, we have $X_a = X_b$, a contradiction. Hence $|X| = 1$.

The converse follows immediately from Lemma 3.1.4 (ii) and Proposition 2.7.

□

As a consequence of Theorem 3.1.5, the following corollary holds and Lemma 3.1.4 is also a speacial case.

Corollary 3.1.6. ([16]) *Every variant of* $T(X)$ *admits a ring structure if and only if* $|X| = 1$.

3.2 The Variants of $P(X)$, $I(X)$ and $C(X)$

Recall that we determine variants of both $P(X)$ and $I(X)$ and $(C(X), \theta)$ to admit a nearring structure where $\theta \in P(X)$.

Lemma 3.2.1. ([5]) (i) $P(X)$ *always admit a left nearring structure.*

(ii) $P(X)$ *admits a right nearring structure if and only if* $|X| = 1$.

Theorem 3.2.2. *For* $\theta \in P(X)$,

(i) *every variant of* $P(X)$ *always admit a left nearring structure,*

(ii) *if* $(P(X), \theta)$ *admits a right nearring structure, then* $\theta \in C(X)$,

(iii) *if* $|X| = 1$ *or* $\theta = 0$, *then* $(P(X), \theta)$ *admits a right nearring structure.*

Proof. (i) follows from Lemma 3.2.1 (i) and Proposition 2.7.

(ii) Assume that there is an operation $+$ such that $(P(X), +, \theta)$ is a right nearring. Suppose on the contrary that $|\operatorname{ran}\theta| > 1$. Let $x, y \in \operatorname{ran}\theta$ be distinct. Then there exist $a, b \in \operatorname{dom}\theta$ such that $a\theta = x$ and $b\theta = y$. Since $(P(X), +, \theta)$ is a right nearring, we have $\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = f$ for some $f \in P(X)$. Then

$$f\theta\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} a \\ x \end{pmatrix} + 0 = \begin{pmatrix} a \\ x \end{pmatrix}.$$

Case 1: $f\theta = 0$. Thus $\begin{pmatrix} a \\ x \end{pmatrix} = 0$, which is a contradiction.

Case 2: $f\theta \neq 0$. Thus

$$f\theta\begin{pmatrix} y \\ x \end{pmatrix} = 0 + \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} a \\ x \end{pmatrix}.$$

So $af\theta = x$ and $af\theta = y$, a contradiction.

(iii) If $\theta = 0$, then $(P(X), \theta)$ is a zero semigroup, which implies that $(P(X), \theta) \in \mathcal{LNR}$. Otherwise, it is proved by Lemma 3.2.1 (ii) and Proposition 2.7. $\square$

If we choose $\theta = 1_V$, then $(P(X), \theta) = P(X)$. In this case, Lemma 3.2.1 and Theorem 3.2.2 have the same results. Moreover, we can conclude that for $\theta \in T(X)$ and θ is surjective or injective, $(P(X), \theta)$ admits a right nearring structure if and only if $|X| = 1$.

Lemma 3.2.3. ([5]) (i) *$I(X)$ admits a left nearring structure if and only if $|X| = 1$.*

(ii) *$I(X)$ admits a right nearring structure if and only if $|X| = 1$.*

Theorem 3.2.4. *For $\theta \in I(X)$,*

1. (i) if $(I(X), \theta)$ admits a left nearring structure, then $\theta \in C(X)$,

(ii) if $|X| = 1$ or $\theta = 0$, then $(I(X), \theta)$ admits a left nearring structure,

2. (i) if $(I(X), \theta)$ admits a right nearring structure, then $\theta \in C(X)$,

(ii) if $|X| = 1$ or $\theta = 0$, then $(I(X), \theta)$ admits a right nearring structure.

Proof. It suffices to show that 1. holds.

(i) Assume $(I(X), \theta) \in \mathcal{LNR}$. Then there is an operation $+$ such that $(I(X), +, \theta)$

is a left nearring. Suppose on the contrary that $|\operatorname{ran}\theta| > 1$. Let $x, y \in \operatorname{ran}\theta$ be distinct. So there are distinct $a, b \in \operatorname{dom}\theta$ such that $a\theta = x$ and $b\theta = y$. Then $\begin{pmatrix} x \\ x \end{pmatrix} + \begin{pmatrix} y \\ x \end{pmatrix} = f$ for some $f \in I(X)$.

Case 1: $f = 0$. Then $0 = \begin{pmatrix} a \\ a \end{pmatrix}\theta f = \begin{pmatrix} a \\ x \end{pmatrix} + 0 = \begin{pmatrix} a \\ x \end{pmatrix}$, which is a contradiction.

Case 2: $f \neq 0$. Then

$$\begin{pmatrix} a \\ a \end{pmatrix}\theta f = \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}\theta f.$$

Thus $a\theta f = x$ and $b\theta f = x$, respectively. Since $\theta f \in I(X)$, we have $a = b$ which is a contradiction.

(ii) If $\theta = 0$, then $(I(X), \theta)$ is a zero semigroup, so that $(I(X), \theta) \in \mathcal{LNR}$. Otherwise, it follows from Lemma 3.2.3 and Proposition 2.7. $\square$

Note that we can see Lemma 3.2.3 as Theorem 3.2.4 where θ is the identity mapping on X. The last main purpose of this section is to determine when $(C(X), \theta)$ admits the structure of a left [right] nearring where $\theta \in P(X)$. The following tool is needed.

Lemma 3.2.5. ([16]) *$C(X)$ admits a ring structure if and only if $|X| = 1$.*

Theorem 3.2.6. *For $\theta \in P(X)$,*

1. (i) if $(C(X), \theta)$ admits a left nearring structure, then $\theta \in C(X)$,

(ii) if $|X| = 1$ or $\theta = 0$, then $(C(X), \theta)$ admits a right nearring structure,

2. (i) if $(C(X), \theta)$ admits a right nearring structure, then $\theta \in C(X)$,

(ii) if $|X| = 1$ or $\theta = 0$, then $(C(X), \theta)$ admits a left nearring structure.

Proof. It suffices to show that 1. holds.

(i) Assume that $(C(X), \theta) \in \mathcal{LNR}$. Then there is an operation $+$ such that $(C(X), +, \theta)$ is a left nearring. Suppose that $\theta \notin C(X)$. Then there exist distinct $x, y \in \operatorname{ran}\theta$. So we have $a, b \in \operatorname{dom}\theta$ such that $a \neq b$, $a\theta = x$ and $b\theta = y$. Thus $\begin{pmatrix} x \\ x \end{pmatrix} + \begin{pmatrix} y \\ y \end{pmatrix} = f$ for some $f \in C(X)$.

Case 1: $f = 0$. Then $0 = \binom{a}{a}\theta f = \binom{a}{x} + 0 = \binom{a}{x}$, which is a contradiction.

Case 2: $f \neq 0$. Then $\binom{a}{a}\theta f = \binom{a}{x}$ and $\binom{a}{b}\theta f = 0 + \binom{a}{y} = \binom{a}{y}$. So distinct $x, y \in \operatorname{ran} f$ contradicts $f \in C(X)$.

(ii) Assume that $|X| = 1$ or $\theta = 0$. By Lemma 3.2.5, $C(X) \in \mathcal{R}$. This implies that $C(X) \in \mathcal{RNR}$. By Proposition 2.7, $(C(X), \theta) \in \mathcal{RNR}$. □

The above theorem yields an immediate corollary.

Corollary 3.2.7. (i) $C(X)$ *admits a left nearring structure if and only if* $|X| = 1$.
(ii) $C(X)$ *admits a right nearring structure if and only if* $|X| = 1$.

Finally, this section is ended by giving

Remark 3.2.8. Let $\theta \in T(X)$ be surjective or injective and $S(X)$ be $I(X)$ or $C(X)$. Then the following statements hold.
(i) $(S(X), \theta)$ admits a left nearring structure if and only if $|X| = 1$.
(ii) $(S(X), \theta)$ admits a right nearring structure if and only if $|X| = 1$.

CHAPTER IV
SUBSEMIGROUPS OF VARIANTS OF $L_R(V)$ WHICH DO NOT ADMIT THE NEARRING STRUCTURE

These linear transformation semigroups on V, an infinite dimensional vector space over a division ring R, given in Chapter II are recalled as follows:

$$\begin{aligned}
L_R(V) &= \{\alpha : V \to V \mid \alpha \text{ is a linear transformation}\}\,,\\
G_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism }\}\,,\\
M_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is a monomorphism }\}\,,\\
E_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is an epimorphism }\}\,,\\
OM_R(V) &= \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite }\}\,,\\
OE_R(V) &= \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) \text{ is infinite }\}\,,\\
AI_R(V) &= \{\alpha \in L_R(V) \mid \dim_R (V/F(\alpha)) < \infty\}\\
&\qquad \text{where } \; F(\alpha) = \{v \in V \mid v\alpha = v\}\,,\\
&= \{\alpha \in L_R(V) \mid \alpha \text{ is almost identical }\}\,.
\end{aligned}$$

The chapter is concerned with subsemigroups of variants of $L_R(V)$ defined from these linear transformation semigroups.

4.1 Generalizations of the Semigroups $OM_R(V)$ and $OE_R(V)$

We begin this section by recalling that 0, the zero mapping on V, belongs to both $OM_R(V)$ and $OE_R(V)$, but the identity mapping 1_V dose not contain in both $OM_R(V)$ and $OE_R(V)$. First, we will show that $(OM_R(V), \theta)$ and $(OE_R(V), \theta)$

are semigroups where $\theta \in L_R(V)$.

Lemma 4.1.1. ([2]) (i) *$OM_R(V)$ is a right ideal of $L_R(V)$.*
(ii) *$OE_R(V)$ is a left ideal of $L_R(V)$.*

Next proposition follows immediately from Lemma 4.1.1.

Proposition 4.1.2. *Let $\theta \in L_R(V)$. Then following statements hold.*
(i) *$(OM_R(V), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*
(ii) *$(OE_R(V), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*

In this thesis, we will show that semigroups

$$(OM_R(V), \theta) \text{ where } \theta \in E_R(V),$$
$$(OE_R(V), \theta) \text{ where } \theta \in M_R(V)$$

do not admit both a left nearring structure and a right nearring structure.

Theorem 4.1.3. *Let $\theta \in E_R(V)$. Then the following statements hold.*
(i) *$(OM_R(V), \theta)$ does not admit a left nearring structure.*
(ii) *$(OM_R(V), \theta)$ does not admit a right nearring structure.*

Proof. Let B be a basis for V. Since B is infinite, there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. For each $v \in B$, we choose $u_v \in V$ such that $u_v\theta = v$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ u_v & 0 \end{pmatrix}_{v \in B_1} \text{ and } \beta = \begin{pmatrix} B_1 & v \\ 0 & u_v \end{pmatrix}_{v \in B_2}.$$

Then $\operatorname{Ker} \alpha = \langle B_2 \rangle$ and $\operatorname{Ker} \beta = \langle B_1 \rangle$. That is, $\alpha, \beta \in OM_R(V)$. Observe that $\alpha\theta\alpha = \alpha$, $\beta\theta\beta = \beta$ and $\alpha\theta\beta = 0 = \beta\theta\alpha$.
(i) Suppose that $(OM_R(V), \oplus, \theta)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in OM_R(V)$. Then $\alpha\theta\lambda = \alpha$ and $\beta\theta\lambda = \beta$. These show that

$$\text{for every } v \in B_1, v\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta = v,$$
$$\text{for every } v \in B_2, v\lambda\theta = v\beta\theta\lambda\theta = v\beta\theta = v.$$

Then we have $\lambda\theta = 1_V \notin OM_R(V)$, which is a contradiction since $OM_R(V)$ is a right ideal of $L_R(V)$. Hence $(OM_R(V), \theta) \notin \mathcal{LNR}$.

(ii) Suppose that $(OM_R(V), \oplus, \theta)$ is a right nearring. Let $\lambda = \alpha \oplus \beta \in OM_R(V)$. Then $\lambda\theta\alpha = \alpha$ and $\lambda\theta\beta = \beta$. Consequently,

$$\text{for every } v \in B_1, v\lambda\theta\beta = v\beta = 0,$$
$$\text{for every } v \in B_2, v\lambda\theta\alpha = v\alpha = 0.$$

Thus

$$\text{for every } v \in B_1, v\lambda\theta \in \operatorname{Ker}\beta = \langle B_1 \rangle,$$
$$\text{for every } v \in B_2, v\lambda\theta \in \operatorname{Ker}\alpha = \langle B_2 \rangle.$$

So

$$\text{for every } v \in B_1, v\lambda\theta\alpha\theta = v\lambda\theta,$$
$$\text{for every } v \in B_2, v\lambda\theta\beta\theta = v\lambda\theta.$$

Since $\lambda\theta\alpha = \alpha$ and $\lambda\theta\beta = \beta$, we have

$$\text{for every } v \in B_1, v\lambda\theta\alpha\theta = v\alpha\theta = v \text{ and}$$
$$\text{for every } v \in B_2, v\lambda\theta\beta\theta = v\beta\theta = v$$

which imply that $v\lambda\theta = v$ for all $v \in B$. This shows that $\lambda\theta = 1_V \notin OM_R(V)$ contradicts the right ideal property of $OM_R(V)$. Hence $(OM_R(V), \theta) \notin \mathcal{RNR}$. □

Theorem 4.1.3 generalizes the next corollary when $\theta = 1_V$.

Corollary 4.1.4. ([8]) *$OM_R(V)$ does not admit both a left nearring structure and a right nearring structure.*

We need the following fact for our next main theorem.

Lemma 4.1.5. $OE_R(V)M_R(V) \subseteq OE_R(V)$.

Proof. Let $\alpha \in OE_R(V)$ and $\beta \in M_R(V)$. Define $\varphi : V/\operatorname{Im}\alpha \to V/\operatorname{Im}\alpha\beta$ by

$$(v + \operatorname{Im}\alpha)\varphi = v\beta + \operatorname{Im}\alpha\beta \text{ for all } v \in V.$$

Since β is a monomorphism on V, we have φ is also a monomorphism. Let B be a basis for $V/\operatorname{Im}\alpha$. By Proposition 2.14 and φ is an injection, $B\varphi$ is a basis for $\operatorname{Im}\varphi$. Extend $B\varphi$ to a basis C for $V/\operatorname{Im}\alpha\beta$. Thus $\dim_R(V/\operatorname{Im}\alpha) = |B| = |B\varphi| \leq |C| = \dim_R(V/\operatorname{Im}\alpha\beta)$. Since $\alpha \in OE_R(V)$, $\dim_R(V/\operatorname{Im}\alpha)$ is infinite and we obtain that $\dim_R(V/\operatorname{Im}\alpha\beta)$ is also infinite. Therefore $\alpha\beta \in OE_R(V)$. □

Our next target is showing that $(OE_R(V), \theta)$ does not admit both a left nearring structure and a right nearring structure where $\theta \in M_R(V)$.

Theorem 4.1.6. *Let $\theta \in M_R(V)$. Then the following statements hold.*
(i) $(OE_R(V), \theta)$ does not admit a left nearring structure.
(ii) $(OE_R(V), \theta)$ does not admit a right nearring structure.

Proof. Let B be a basis for V. Then there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Since θ is injective, by Proposition 2.14, we have $B\theta$ is a basis for $\operatorname{Im}\theta$ and $\{B_1\theta, B_2\theta\}$ is also a partition of $B\theta$ such that $|B\theta| = |B_1\theta| = |B_2\theta| = |B|$. Extend $B\theta$ to a basis C for V. Set $B_3 = C \setminus B\theta$. Then $C = B\theta \cup B_3$. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_1\theta}, \quad \beta = \begin{pmatrix} v & B_1\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_2\theta}$$

and

$$\gamma = \begin{pmatrix} v & B\theta \\ v & 0 \end{pmatrix}_{v \in B_3}.$$

Then $\dim_R(V/\operatorname{Im}\alpha) = |B \setminus B_1| = |B_2|$, $\dim_R(V/\operatorname{Im}\beta) = |B \setminus B_2| = |B_1|$ and $\dim_R(V/\operatorname{Im}\gamma) = |C \setminus B_3| = |B\theta|$. This implies that $\alpha, \beta, \gamma \in OE_R(V)$. It is easy to see that $\alpha\theta\alpha = \alpha$, $\beta\theta\beta = \beta$ and $\alpha\theta\beta = \beta\theta\alpha = \alpha\theta\gamma = \beta\theta\gamma = 0$.

(i) Suppose that $(OE_R(V), \oplus, \theta)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in OE_R(V)$. Then $\alpha\theta\lambda = \alpha$ and $\beta\theta\lambda = \beta$. Hence

$$\text{for every } v \in B_1\theta, v\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta = v,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta = v\beta\theta\lambda\theta = v\beta\theta = v.$$

So, for every $v \in B\theta$, $v\lambda\theta = v$. Consequently, $(B\theta)\lambda\theta = B\theta$. This implies that $B(\theta\lambda) = (B\theta\lambda)\theta\theta^{-1} = (B\theta\lambda\theta)\theta^{-1} = B\theta\theta^{-1} = B$ since θ is an injection. Thus $B \subseteq \operatorname{Im}\theta\lambda \subseteq \operatorname{Im}\lambda$, hence λ is onto, which contradicts $\lambda \in OE_R(V)$. Hence $(OE_R(V), \theta) \notin \mathcal{LNR}$, as desired.

(ii) Suppose that $(OE_R(V), \oplus, \theta)$ is a right nearring. Let $\lambda = \alpha \oplus \beta \in OE_R(V)$. Then $\lambda\theta\alpha = \alpha$, $\lambda\theta\beta = \beta$ and $\lambda\theta\gamma = 0$. Thus

$$\text{for every } v \in B_1\theta, v\lambda\theta\beta = v\beta = 0,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta\alpha = v\alpha = 0.$$

So we have

$$\text{for every } v \in B_1\theta, v\lambda\theta \in \operatorname{Ker}\beta = \langle B_1\theta \cup B_3 \rangle,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta \in \operatorname{Ker}\alpha = \langle B_2\theta \cup B_3 \rangle.$$

Claim that for every $v \in B_1\theta$, $v\lambda\theta \in \langle B_1\theta \rangle$ and for every $v \in B_2\theta$, $v\lambda\theta \in \langle B_2\theta \rangle$. Let $v \in B_1\theta$. Then $v\lambda\theta = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j w_j$ for some $v_i \in B_1\theta$, $w_j \in B_3$ and $a_i, b_j \in R$ where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Thus

$$0 = (v\lambda\theta)\gamma = (\sum_{i=1}^{n} a_i v_i)\gamma + (\sum_{j=1}^{m} b_j w_j)\gamma = 0 + \sum_{j=1}^{m} b_j w_j = \sum_{j=1}^{m} b_j w_j.$$

Consequently, $b_j = 0$ for all $j = 1, 2, \ldots, m$. So $v\lambda\theta = \sum_{i=1}^{n} a_i v_i \in \langle B_1\theta\rangle$. Similarly, $v\lambda\theta \in \langle B_2\theta\rangle$ for all $v \in B_2\theta$, and then the claim is proved. By the claim, we have

$$\text{for every } v \in B_1\theta, v\lambda\theta = v\lambda\theta\alpha\theta = v,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta = v\lambda\theta\beta\theta = v.$$

Hence

$$\text{for every } v \in B_1\theta, v\lambda\theta\alpha\theta = v\alpha\theta = v \text{ and}$$
$$\text{for every } v \in B_2\theta, v\lambda\theta\beta\theta = v\beta\theta = v$$

we obtain that $v\lambda\theta = v$ for all $v \in B\theta$. Similary (i), we have a contradiction. Therefore the proof is complete. □

Finally, the next corollary follows from the above Theorem.

Corollary 4.1.7. ([8]) *$OE_R(V)$ does not admit both a left nearring structure and a right nearring structure.*

4.2 Generalizations of any Semigroups Containing $OM_R(V)$ and Semigroups Containing $OE_R(V)$

In this section, let H be a subsemigroup of $G_R(V)$ and T be a subsemigroup of $AI_R(V)$.

Remark 4.2.1. Let H be a proper subsemigroup of $G_R(V)$ containing 1_V and $\theta \in G_R(V) \smallsetminus H$. Since $1_V\theta 1_V = \theta \notin H$, it follows that both $(OM_R(V) \cup H, \theta)$ and $(OE_R(V) \cup H, \theta)$ are not semigroups.

To fulfill the above remark, we will show that there exist many proper subsemigroups of $G_R(V)$ containing the identity map 1_V.

Example 4.2.2. Let B be a basis for V and for distinct $u, w \in B$, let $\alpha_{u,w} \in G_R(V)$ be defined by

$$\alpha_{u,w} = \begin{pmatrix} u & w & v \\ w & u & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}.$$

Then $\{1_V, \alpha_{u,w}\}$ is a proper subsemigroup of $G_R(V)$ contaning 1_V.

Remark 4.2.3. Let T be a proper subsemigroup of $AI_R(V)$ containing 1_V and $\theta \in G_R(V) \smallsetminus T$. Since $1_V \theta 1_V = \theta \notin (OM_R(V) \cup T) \cup (OE_R(V) \cup T)$, it follows that both $(OM_R(V) \cup T, \theta)$ and $(OE_R(V) \cup T, \theta)$ are not semigroups.

Next, we give examples for a proper subsemigroup of $AI_R(V)$ containing 1_V.

Example 4.2.4. Let B be a basis for V and $u \in B$. Define $\alpha_u \in AI_R(V)$ by

$$\alpha_u = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

Then $\{1_V, \alpha_u\}$ is a proper subsemigroup of $AI_R(V)$ containing 1_V.

By Remark 4.2.1 and Remark 4.2.3, it is valid to consider the followings:

$(OM_R(V) \cup H, \theta)$ and $(OE_R(V) \cup H, \theta)$ where $\theta \in H \cup \{1_V\}$,

$(OM_R(V) \cup T, \theta)$ where $\theta \in (E_R(V) \cap T) \cup \{1_V\}$,

$(OE_R(V) \cup T, \theta)$ where $\theta \in (M_R(V) \cap T) \cup \{1_V\}$.

We will show that they are semigroups which do not admit the structure of nearring.

Lemma 4.2.5. ([2]) $G_R(V) OM_R(V) \subseteq OM_R(V)$.

The following proposition is a direct consequence of Lemma 4.1.1, Lemma 4.1.5 and Lemma 4.2.5.

Proposition 4.2.6. *Let $\theta \in H \cup \{1_V\}$. Then the following statements hold.*
(i) *$(OM_R(V) \cup H, \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*
(ii) *$(OE_R(V) \cup H, \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*

Lemma 4.2.7. ([2]) (i) $AI_R(V)OM_R(V) \subseteq OM_R(V)$.
(ii) $OE_R(V)AI_R(V) \subseteq OE_R(V)$.

Proposition 4.2.8. (i) *If $\theta \in (E_R(V) \cap T) \cup \{1_V\}$, then $(OM_R(V) \cup T, \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*
(ii) *If $\theta \in (M_R(V) \cap T) \cup \{1_V\}$, then $(OE_R(V) \cup T, \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*

Proof. (i) Let $\alpha, \beta \in OM_R(V) \cup T$. If either $\alpha \in OM_R(V)$ or $\beta \in OM_R(V)$, we then have $\alpha\theta\beta \in OM_R(V)$ or $\beta\theta\alpha \in OM_R(V)$ by the right ideal property of $OM_R(V)$ and Lemma 4.2.7, respectively. If $\alpha, \beta \in T$, then it is clear that $\alpha\theta\beta$, $\beta\theta\alpha \in OM_R(V) \cup T$.
(ii) is similar to (i). □

The following theorems are our main purposes.

Theorem 4.2.9. *Let $\theta \in H \cup \{1_V\}$ and $S(V)$ be the semigroup $OM_R(V)$ or $OE_R(V)$. Then the following statements hold.*
(i) *$(S(V) \cup H, \theta)$ does not admit a left nearring structure.*
(ii) *$(S(V) \cup H, \theta)$ does not admit a right nearring structure.*

Proof. Let B be a basis for V and $u \in B$ be a fixed element. Since $B \smallsetminus \{u\}$ is infinite, $B \smallsetminus \{u\}$ has a partition $\{B_1, B_2\}$ such that $|B \smallsetminus \{u\}| = |B_1| = |B_2|$. Then $B = B_1 \cup B_2 \cup \{u\}$ and these three sets are pairwise disjoint. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \cup \{u\} \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_1}, \ \beta = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\theta^{-1} \end{pmatrix}_{v \in B_2}$$

and

$$\gamma = \begin{pmatrix} u & B_1 \cup B_2 \\ u\theta^{-1} & 0 \end{pmatrix}.$$

So $\operatorname{Ker}\alpha = \langle B_2 \cup \{u\}\rangle$, $\operatorname{Ker}\beta = \langle B_1 \cup \{u\}\rangle$, $\operatorname{Ker}\gamma = \langle B_1 \cup B_2\rangle$,

$$\dim_R(V/\operatorname{Im}\alpha) = \dim_R(V/\left\langle B_1\theta^{-1}\right\rangle) = |B\theta^{-1} \smallsetminus B_1\theta^{-1}| = |B \smallsetminus B_1|,$$

$$\dim_R(V/\operatorname{Im}\beta) = \dim_R(V/\left\langle B_2\theta^{-1}\right\rangle) = |B\theta^{-1} \smallsetminus B_2\theta^{-1}| = |B \smallsetminus B_2|,$$

$$\dim_R(V/\operatorname{Im}\gamma) = \dim_R(V/\left\langle u\theta^{-1}\right\rangle) = |B\theta^{-1} \smallsetminus \left\{u\theta^{-1}\right\}| = |B \smallsetminus \{u\}|.$$

Thus $\alpha, \beta, \gamma \in S(V)$. Obviously,

$$\alpha\theta\alpha = \alpha,\ \beta\theta\beta = \beta,\ \alpha\theta\beta = \beta\theta\alpha = \gamma\theta\alpha = \alpha\theta\gamma = \gamma\theta\beta = \beta\theta\gamma = 0.$$

(i) Suppose that $(S(V) \cup H, \oplus, \theta)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in S(V) \cup H$. Clearly, $\alpha\theta\lambda = \alpha$, $\beta\theta\lambda = \beta$ and $\gamma\theta\lambda = 0$. Thus

$$\text{for every } v \in B_1, v\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta = v,$$

$$\text{for every } v \in B_2, v\lambda\theta = v\beta\theta\lambda\theta = v\beta\theta = v,$$

$$u\lambda\theta = u\gamma\theta\lambda\theta = 0.$$

That is,

$$\lambda\theta = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B_1 \cup B_2}.$$

Thus $\operatorname{Ker}\lambda\theta = \langle u\rangle$ and $\dim_R(V/\operatorname{Im}\lambda\theta) = \dim_R(V/\langle B_1 \cup B_2\rangle) = |\{u\}| = 1$. Hence $\lambda\theta \notin S(V) \cup H$ which contradicts Lemma 4.1.1 and Lemma 4.1.5, respectively. Therefore $(S(V) \cup H, \theta) \notin \mathcal{LNR}$.

(ii) Suppose that $(S(V) \cup H, \oplus, \theta)$ is a right nearring. Let $\lambda = \alpha \oplus \beta \in S(V) \cup H$.

Clearly, $\lambda\theta\alpha = \alpha$, $\lambda\theta\beta = \beta$ and $\lambda\theta\gamma = 0$. Then

$$\text{for every } v \in B_1, v\lambda\theta\beta = v\beta = 0,$$
$$\text{for every } v \in B_2, v\lambda\theta\alpha = v\alpha = 0.$$

It follows that

$$\text{for every } v \in B_1, v\lambda\theta \in \operatorname{Ker}\beta = \langle B_1 \cup \{u\}\rangle\,,$$
$$\text{for every } v \in B_2, v\lambda\theta \in \operatorname{Ker}\alpha = \langle B_2 \cup \{u\}\rangle\,.$$

Claim that $v\lambda\theta \in \langle B_1\rangle$ for all $v \in B_1$ and $v\lambda\theta \in \langle B_2\rangle$ for all $v \in B_2$. Let $v \in B_1$. Then $v\lambda\theta = \sum_{i=1}^{n} a_i v_i + au$ for some $v_1, v_2, \ldots, v_n \in B_1$ and $a_1, a_2, \ldots, a_n, a \in R$. Since $0 = v\lambda\theta\gamma = \sum_{i=1}^{n} a_i v_i\gamma + au\gamma = au$, we have $v\lambda\theta \in \langle B_1\rangle$. Similarly, $v\lambda\theta \in \langle B_2\rangle$ for all $v \in B_2$. By the claim,

$$\text{for every } v \in B_1, v\lambda\theta\alpha\theta = v\lambda\theta,$$
$$\text{for every } v \in B_2, v\lambda\theta\beta\theta = v\lambda\theta.$$

Since

$$\text{for every } v \in B_1, v\lambda\theta\alpha\theta = v\alpha\theta = v,$$
$$\text{for every } v \in B_2, v\lambda\theta\beta\theta = v\beta\theta = v,$$
$$\text{for every } v \in V, v\lambda\theta\gamma = 0,$$

then we have that $v\lambda\theta = v$ for all $v \in \langle B_1 \cup B_2\rangle$ and $u\lambda\theta \in \operatorname{Ker}\gamma = \langle B_1 \cup B_2\rangle$. So $(u - u\lambda\theta)\lambda\theta = u\lambda\theta - u\lambda\theta\lambda\theta = u\lambda\theta - u\lambda\theta = 0$. Since $B_1 \cup B_2 \cup \{u\}$ is a basis for V and $u\lambda\theta \in \langle B_1 \cup B_2\rangle$, by Proposition 2.10, $B_1 \cup B_2 \cup \{u - u\lambda\theta\}$ is a basis for V.

Hence

$$\lambda\theta = \begin{pmatrix} u - u\lambda\theta & v \\ 0 & v \end{pmatrix}_{v \in B_1 \cup B_2}.$$

Thus $\operatorname{Ker} \lambda\theta = \langle u - u\lambda\theta \rangle$ and

$$\dim_R(V/\operatorname{Im} \lambda\theta) = \dim_R(V/\langle B_1 \cup B_2 \rangle) = |\{u - u\lambda\theta\}| = 1.$$

Then $\lambda\theta \notin S(V) \cup H$ contradicts Lemma 4.1.1 and Lemma 4.1.5, respectively. Therefore $(S(V) \cup H, \theta) \notin \mathcal{RNR}$. $\square$

By the preceding theorem, we have the following corollary if $\theta = 1_V$.

Corollary 4.2.10. ([8]) *Let $S(V)$ be $OM_R(V)$ or $OE_R(V)$. Then $S(V) \cup H$ does not admit both a left nearring structure and a right nearring structure.*

Theorem 4.2.11. *Let $\theta \in (E_R(V) \cap T) \cup \{1_V\}$. Then the following statements hold.*
(i) $(OM_R(V) \cup T, \theta)$ does not admit a left nearring structure.
(ii) $(OM_R(V) \cup T, \theta)$ does not admit a right nearring structure.

Proof. Let B be a basis for V. Then there is a partition $\{B_1, B_2\}$ such that $|B| = |B_1| = |B_2|$. Since $|B_1| = |B_2|$, there exists a bijection $\varphi : B_1 \to B_2$. It follows from $\theta \in E_R(V)$ that we can choose $u_v \in V$ such that $u_v\theta = v$ for all $v \in V$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ u_{v\varphi} & 0 \end{pmatrix}_{v \in B_1} \text{ and } \beta = \begin{pmatrix} B_1 & v \\ 0 & u_{v\varphi^{-1}} \end{pmatrix}_{v \in B_2}.$$

Thus $\operatorname{Ker} \alpha = \langle B_2 \rangle$ and $\operatorname{Ker} \beta = \langle B_1 \rangle$. Hence we have $\alpha, \beta \in OM_R(V) \cup T$. It is easy to see that $\alpha\theta\alpha = \beta\theta\beta = 0$,

$$\alpha\theta\beta = \begin{pmatrix} v & B_2 \\ u_v & 0 \end{pmatrix}_{v \in B_1} \text{ and } \beta\theta\alpha = \begin{pmatrix} B_1 & v \\ 0 & u_v \end{pmatrix}_{v \in B_2}.$$

(i) Suppose that $(OM_R(V) \cup T, \oplus, \theta)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in OM_R(V) \cup T$. Then $\alpha\theta\lambda = \alpha\theta\beta$ and $\beta\theta\lambda = \beta\theta\alpha$. We then have

$$\text{for every } v \in B_1, (v\varphi)\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta\beta\theta = v = (v\varphi)\varphi^{-1},$$
$$\text{for every } v \in B_2, (v\varphi^{-1})\lambda\theta = v\beta\theta\lambda\theta = v\beta\theta\alpha\theta = v = (v\varphi^{-1})\varphi.$$

Thus

$$\begin{aligned} (\lambda\theta)_{|B_2} &= \varphi^{-1} : B_2 \to B_1 \text{ is a bijection,} \\ (\lambda\theta)_{|B_1} &= \varphi : B_1 \to B_2 \text{ is a bijection.} \end{aligned} \quad (*)$$

Since $\{B_1, B_2\}$ is a partition of B, we have $(\lambda\theta)_{|B} : B \to B$ is a bijection. So $\lambda\theta \in G_R(V)$. That is, $\lambda\theta \notin OM_R(V)$. Since $OM_R(V)$ is a right ideal of $L_R(V)$, $\lambda \notin OM_R(V)$. Claim that $\lambda\theta \notin AI_R(V)$. By (*), $B_1 \cap F(\lambda\theta) = \varnothing$ which is equivalent to $v + F(\lambda\theta) \neq F(\lambda\theta)$ for all $v \in B_1$. Let $v_1, v_2, \ldots, v_n \in B_1$ be distinct and $a_1, a_2, \ldots, a_n \in R$ such that $\sum_{i=1}^{n} a_i(v_i + F(\lambda\theta)) = F(\lambda\theta)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\lambda\theta)$, and we have

$$\sum_{i=1}^{n} a_i v_i = (\sum_{i=1}^{n} a_i v_i)\lambda\theta \in \langle B_2 \rangle .$$

Consequently, $\sum_{i=1}^{n} a_i v_i \in \langle B_1 \rangle \cap \langle B_2 \rangle = \{0\}$ and $a_i = 0$ for all $i = 1, 2, \ldots, n$. This implies that $\{v + F(\lambda\theta) \mid v \in B_1\}$ is a linearly independent set and $v + F(\lambda\theta) \neq w + F(\lambda\theta)$ for distinct $v, w \in B_1$. It follows that $\dim_R(V/F(\lambda\theta)) \geq |B_1|$, so $\lambda\theta \notin AI_R(V)$ and the claim is proved. If $\lambda \in T$, then $\lambda\theta \in T \subseteq AI_R(V)$ which is impossible. Then $\lambda \notin T$, a contradiction. Hence $(OM_R(V) \cup T, \theta) \notin \mathcal{LNR}$.

(ii) Suppose that $(OM_R(V) \cup T, \oplus, \theta)$ is a right nearring. Let $\lambda = \alpha \oplus \beta \in OM_R(V) \cup T$. Then $\lambda\theta\alpha = \beta\theta\alpha$ and $\lambda\theta\beta = \alpha\theta\beta$. We can conclude that

$$\text{for every } v \in B_1, v(\lambda\theta\alpha)\theta\beta = v(\beta\theta\alpha)\theta\beta = 0,$$
$$\text{for every } v \in B_2, v(\lambda\theta\beta)\theta\alpha = v(\alpha\theta\beta)\theta\alpha = 0.$$

Consequently,

$$\text{for every } v \in B_1, v\lambda\theta \in \operatorname{Ker}(\alpha\theta\beta) = \langle B_2 \rangle\,,$$
$$\text{for every } v \in B_2, v\lambda\theta \in \operatorname{Ker}(\beta\theta\alpha) = \langle B_1 \rangle\,.$$

Since

$$\text{for every } v \in B_1, v(\lambda\theta\beta)\theta = v(\alpha\theta\beta)\theta = v = (v\varphi)\beta\theta,$$
$$\text{for every } v \in B_2, v(\lambda\theta\alpha)\theta = v(\beta\theta\alpha)\theta = v = (v\varphi^{-1})\alpha\theta,$$

$(\beta\theta)_{|_{\langle B_2 \rangle}}$ and $(\alpha\theta)_{|_{\langle B_1 \rangle}}$ are monomorphisms, we have $v\lambda\theta = v\varphi$ for all $v \in B_1$ and $v\lambda\theta = v\varphi^{-1}$ for all $v \in B_2$. Thus

$$(\lambda\theta)_{|_{B_1}} = \varphi : B_1 \to B_2 \text{ is a bijection},$$
$$(\lambda\theta)_{|_{B_2}} = \varphi^{-1} : B_2 \to B_1 \text{ is a bijection}.$$

Since $\{B_1, B_2\}$ is a partition of B, $(\lambda\theta)_{|_B} : B \to B$ is a bijection. So $\lambda\theta \in G_R(V)$. That is $\lambda\theta \notin OM_R(V)$. Similarly to (i), $\lambda \notin OM_R(V) \cup T$, a contradiction. Hence $(OM_R(V) \cup T, \theta) \notin \mathcal{RNR}$. □

Corollary 4.2.12. ([8]) *$OM_R(V) \cup T$ does not admit both a left nearring structure and a right nearring structure.*

Theorem 4.2.13. *Let $\theta \in (M_R(V) \cap T) \cup \{1_V\}$. Then the following statements hold.*
(i) $(OE_R(V) \cup T, \theta)$ does not admit a left nearring structure.
(ii) $(OE_R(V) \cup T, \theta)$ does not admit a right nearring structure.

Proof. Let B be a basis for V. Then there is a partition $\{B_1, B_2\}$ such that $|B| = |B_1| = |B_2|$. Since θ is injective, by Proposition 2.14, we then have $B\theta$ is a basis for $\operatorname{Im}\theta$ such that $\{B_1\theta, B_2\theta\}$ is a partition of $B\theta$ and $|B\theta| = |B_1\theta| = |B_2\theta|$. Extend $B\theta$ to a basis C for V. Set $B_3 = C \smallsetminus B\theta$. Let $\varphi : B_1\theta \to B_2\theta$ be a bijection.

Notice that $\theta^{-1} : V\theta \to V$ exists. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2\theta \cup B_3 \\ v\varphi\theta^{-1} & 0 \end{pmatrix}_{v \in B_1\theta}, \quad \beta = \begin{pmatrix} v & B_1\theta \cup B_3 \\ v\varphi^{-1}\theta^{-1} & 0 \end{pmatrix}_{v \in B_2\theta}$$

and

$$\gamma = \begin{pmatrix} v & B\theta \\ v & 0 \end{pmatrix}_{v \in B_3}.$$

Then $\dim_R(V/\operatorname{Im}\alpha) = |B \smallsetminus B_2| = |B_1|$, $\dim_R(V/\operatorname{Im}\beta) = |B \smallsetminus B_1| = |B_2|$ and $\dim_R(V/\operatorname{Im}\gamma) = |C \smallsetminus B_3| = |B\theta|$. It follows that $\alpha, \beta, \gamma \in OE_R(V)$. Observe that $\alpha\theta\alpha = \beta\theta\beta = \alpha\theta\gamma = \beta\theta\gamma = 0$,

$$\alpha\theta\beta = \begin{pmatrix} v & B_2\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_1\theta} \text{ and } \beta\theta\alpha = \begin{pmatrix} v & B_1\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_2\theta}.$$

(i) Suppose that $(OE_R(V) \cup T, \oplus, \theta)$ is a left nearring. Let $\lambda = \alpha \oplus \beta \in OE_R(V) \cup T$. Then $\alpha\theta\lambda = \alpha\theta\beta$ and $\beta\theta\lambda = \beta\theta\alpha$. We have that

for every $v \in B_1\theta, (v\varphi)\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta\beta\theta = v = (v\varphi)\varphi^{-1}$,

for every $v \in B_2\theta, (v\varphi^{-1})\lambda\theta = v\beta\theta\lambda\theta = v\beta\theta\alpha\theta = v = (v\varphi^{-1})\varphi$.

Then

$$\begin{aligned} (\lambda\theta)_{|B_2\theta} &= \varphi^{-1} : B_2\theta \to B_1\theta \text{ is a bijection,} \\ (\lambda\theta)_{|B_1\theta} &= \varphi : B_1\theta \to B_2\theta \text{ is a bijection.} \end{aligned} \tag{*}$$

Since $\{B_1\theta, B_2\theta\}$ is a partition of $B\theta$, $(\lambda\theta)_{|B\theta} : B\theta \to B\theta$ is also a bijection. So $(B\theta)\lambda = (B\theta)\lambda\theta\theta^{-1} = (B\theta)\theta^{-1} = B$. This implies that $B \subseteq \operatorname{Im}\lambda$. Consequently, λ is onto. Hence $\lambda \notin OE_R(V)$. Next, we claim that $\lambda \notin T$. By (*), $B_1\theta \cap F(\lambda\theta) = \varnothing$. That is, $v + F(\lambda\theta) \neq F(\lambda\theta)$ for all $v \in B_1\theta$. Let $v_1, v_2, \ldots, v_n \in B_1\theta$ be distinct and $a_1, a_2, \ldots, a_n \in R$ such that $\sum_{i=1}^{n} a_i(v_i + F(\lambda\theta)) = F(\lambda\theta)$. Then

$\sum_{i=1}^{n} a_i v_i \in F(\lambda\theta)$. It follows that

$$\sum_{i=1}^{n} a_i v_i = (\sum_{i=1}^{n} a_i v_i)\lambda\theta \in \langle B_2\theta \rangle .$$

So $\sum_{i=1}^{n} a_i v_i \in \langle B_1\theta \rangle \cap \langle B_2\theta \rangle = \{0\}$. Then $a_i = 0$ for all $i = 1, 2, \ldots, n$. Thus $\{v + F(\lambda\theta) \mid v \in B_1\theta\}$ is linearly independent and $v + F(\lambda\theta) \neq w + F(\lambda\theta)$ for different $v, w \in B_1\theta$. Hence $\dim_R(V/F(\lambda\theta)) \geq |B_1\theta|$, which implies that $\lambda\theta \notin AI_R(V)$. If $\lambda \in T$, then $\lambda\theta \in T \subseteq AI_R(V)$, which is a contradiction. Thus $\lambda \notin T$. Hence $\lambda \notin OE_R(V) \cup T$, which is impossible. Therefore $(OE_R(V) \cup T, \theta) \notin \mathcal{LNR}$.
(ii) Suppose that $(OE_R(V) \cup T, \oplus, \theta)$ is a right nearring. Let $\lambda = \alpha \oplus \beta \in OE_R(V) \cup T$. Then $\lambda\theta\alpha = \beta\theta\alpha$, $\lambda\theta\beta = \alpha\theta\beta$ and $\lambda\theta\gamma = 0$. Thus

$$\text{for every } v \in B_1\theta, v\lambda\theta\alpha\theta\beta = v\beta\theta\alpha\theta\beta = 0,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta\beta\theta\alpha = v\alpha\theta\beta\theta\alpha = 0.$$

Consequently,

$$\text{for every } v \in B_1\theta, v\lambda\theta \in \operatorname{Ker}(\alpha\theta\beta) = \langle B_2\theta \cup B_3 \rangle ,$$
$$\text{for every } v \in B_2\theta, v\lambda\theta \in \operatorname{Ker}(\beta\theta\alpha) = \langle B_1\theta \cup B_3 \rangle .$$

Claim that $v\lambda\theta \in \langle B_2\theta \rangle$ for all $v \in B_1\theta$ and $v\lambda\theta \in \langle B_1\theta \rangle$ for all $v \in B_2\theta$. Let $v \in B_1\theta$. Then $v\lambda\theta = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j w_j$ for some $v_i \in B_2\theta$, $w_j \in B_3$ and $a_i, b_j \in R$ where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then

$$0 = (v\lambda\theta)\gamma = (\sum_{i=1}^{n} a_i v_i)\gamma + (\sum_{j=1}^{m} b_j w_j)\gamma = 0 + \sum_{j=1}^{m} b_j w_j = \sum_{j=1}^{m} b_j w_j.$$

This implies that $b_j = 0$ for all $j = 1, 2, \ldots, m$. So $v\lambda\theta = \sum_{i=1}^{n} a_i v_i \in \langle B_2\theta \rangle$.

Similarly, $v\lambda\theta \in \langle B_1\theta \rangle$ for all $v \in B_2\theta$. The claim is complete. Since

$$\text{for every } v \in B_1\theta, v(\lambda\theta\beta)\theta = v(\alpha\theta\beta)\theta = v = (v\varphi)\beta\theta,$$
$$\text{for every } v \in B_2\theta, v(\lambda\theta\alpha)\theta = v(\beta\theta\alpha)\theta = v = (v\varphi^{-1})\alpha\theta,$$

$(\beta\theta)_{|_{\langle B_2\theta\rangle}}$ and $(\alpha\theta)_{|_{\langle B_1\theta\rangle}}$ are monomorphisms, we have $v\lambda\theta = v\varphi$ for all $v \in B_1\theta$ and $v\lambda\theta = v\varphi^{-1}$ for all $v \in B_2\theta$, respectively. That is,

$$(\lambda\theta)_{|_{B_1\theta}} = \varphi : B_1\theta \to B_2\theta \text{ is a bijection},$$
$$(\lambda\theta)_{|_{B_2\theta}} = \varphi^{-1} : B_2\theta \to B_1\theta \text{ is a bijection}.$$

Since $\{B_1\theta, B_2\theta\}$ is a partition of $B\theta$, $(\lambda\theta)_{|_{B\theta}} : B\theta \to B\theta$ is a bijection and $(B\theta)\lambda\theta = B\theta$. It follows that $B(\theta\lambda) = B$ since θ is injective. So $B \subseteq \operatorname{Im}\theta\lambda \subseteq \operatorname{Im}\lambda$ Thus λ is onto. Consequently, $\lambda \notin OE_R(V)$. It is similar to (i), we have $\lambda \notin T$. That is $\lambda \notin OE_R(V) \cup T$, a contradiction. Hence $(OE_R(V) \cup T, \theta) \notin \mathcal{RNR}$.

Therefore the theorem is proved. □

Corollary 4.2.14. ([8]) *$OE_R(V) \cup T$ does not admit both a left nearring structure and a right nearring structure.*

CHAPTER V

SUBSEMIGROUPS OF VARIANTS OF $L_R(V)$ ADMITTING THE NEARRING STRUCTURE

Throughout this section, let V be a vector space over a division ring R and k be a cardinal number. We recall that the followings are linear transformation semigroups on V.

$$K_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \geq k\} \text{ where } k \leq \dim_R V,$$
$$K'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha > k\} \text{ where } k < \dim_R V,$$
$$CI_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im}\alpha) \geq k\} \text{ where } k \leq \dim_R V,$$
$$CI'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im}\alpha) > k\} \text{ where } k < \dim_R V,$$
$$I_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha \leq k\} \text{ where } k \leq \dim_R V,$$
$$I'_R(V,k) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha < k\} \text{ where } 0 < k \leq \dim_R V.$$

Note that these semigroups contain 0, the zero map on V.

5.1 Generalizations of the Semigroups $K_R(V,k)$ and $K'_R(V,k)$

We begin this section by showing both $(K_R(V,k),\theta)$ and $(K'_R(V,k),\theta)$ are subsemigroups of $(L_R(V),\theta)$ where $\theta \in L_R(V)$.

Lemma 5.1.1. (i) *For* $k \leq \dim_R V$, $K_R(V,k)$ *is a right ideal of* $L_R(V)$.
(ii) *For* $k < \dim_R V$, $K'_R(V,k)$ *is a right ideal of* $L_R(V)$.

Proof. The results are obtained immediately from $\operatorname{Ker}\alpha\beta \supseteq \operatorname{Ker}\alpha$ for all $\alpha,\beta \in L_R(V)$. □

The following proposition is a directly consequence from Lemma 5.1.1.

Proposition 5.1.2. *Let $\theta \in L_R(V)$. Then the following statements hold.*
(i) *For $k \leq \dim_R V$, $(K_R(V,k),\theta)$ is a subsemigroup of $(L_R(V),\theta)$.*
(ii) *For $k < \dim_R V$, $(K'_R(V,k),\theta)$ is a subsemigroup of $(L_R(V),\theta)$.*

In this thesis, we shall choose $\theta \in E_R(V)$ to determine when the semigroup $(K_R(V,k),\theta)$ admits the structure of a left [right] nearring. These facts are helpful for our main theorems.

Lemma 5.1.3. *Let $k \leq \dim_R V$ and $\theta \in G_R(V)$. Then $(K_R(V,k),\theta) \cong K_R(V,k)$.*

Proof. Define a map $\varphi : (K_R(V,k),\theta) \to K_R(V,k)$ by $\alpha\varphi = \alpha\theta$ for all $\alpha \in K_R(V,k)$. Since $K_R(V,k)$ is a right ideal, φ is well-defined. Moreover, φ is also one-to-one. By the right ideal property of $K_R(V,k)$, $\alpha\theta^{-1} \in K_R(V,k)$ and then $\alpha = \alpha\theta^{-1}\theta = (\alpha\theta^{-1})\varphi$ for all $\alpha \in K_R(V,k)$, so φ is surjective. For any $\alpha,\beta \in K_R(V,k)$, we have $\alpha\theta\beta \in K_R(V,k)$ and $(\alpha\theta\beta)\varphi = \alpha\theta\beta\theta = \alpha\varphi\beta\varphi$. Hence φ is an isomorphism. □

Lemma 5.1.4. ([8]) *Let $k \leq \dim_R V$.*

1. *$K_R(V,k)$ admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

2. *$K_R(V,k)$ admits the structure of a right nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

By Lemma 5.1.3, Lemma 5.1.4 and Proposition 2.8, the following theorem holds where $\theta \in G_R(V)$. Then we will generalize these results by choosing $\theta \in E_R(V)$.

Theorem 5.1.5. *Let $k \le \dim_R V$ and $\theta \in E_R(V)$.*

1. $(K_R(V,k),\theta)$ *admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

2. $(K_R(V,k),\theta)$ *admits the structure of a right nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

Proof. Assume that (i) or (ii) holds. By Lemma 5.1.4, $K_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. By Proposition 2.7, we know that $(K_R(V,k),\theta) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $(K_R(V,k),\oplus,\theta)$ is a left nearring or a right nearring. Suppose (i) and (ii) are all false. Then either $k > 0$ and $\dim_R V$ is infinite or $0 < k < \dim_R V < \infty$.

Case 1: $k > 0$ and $\dim_R V$ is infinite. Let B be a basis for V. Then there is a partition $\{B_1, B_2\}$ such that $|B| = |B_1| = |B_2|$. For each $v \in B$, we can choose $u_v \in V$ such that $u_v\theta = v$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ u_v & 0 \end{pmatrix}_{v \in B_1} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & u_v \end{pmatrix}_{v \in B_2}.$$

Then $\operatorname{Ker}\alpha = \langle B_2\rangle$ and $\operatorname{Ker}\beta = \langle B_1\rangle$, so $\dim_R \operatorname{Ker}\alpha = |B_2| = |B| \ge k$ and $\dim_R \operatorname{Ker}\beta = |B_1| = |B| \ge k$. Thus $\alpha, \beta \in K_R(V,k)$. Obviously, $\alpha\theta\alpha = \alpha$, $\beta\theta\beta = \beta$ and $\alpha\theta\beta = 0 = \beta\theta\alpha$. Similar to the proof of Theorem 4.1.3, we have $\lambda\theta = 1_V$. Thus $\dim_R \operatorname{Ker}\lambda\theta = 0 < k$, this implies that $\lambda\theta \notin K_R(V,k)$, which contradicts the right ideal property of $K_R(V,k)$. Hence $(K_R(V),\theta) \notin \mathcal{LNR} \cup \mathcal{RNR}$.

Case 2: $0 < k < \dim_R V < \infty$. Then $\theta \in G_R(V)$ and Lemma 5.1.4 shows that $K_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$. By Lemma 5.1.3 and Proposition 2.8, it follows that

$(K_R(V,k),\theta) \cong K_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$.

Therefore the proof is complete. □

Since $K'_R(V,k) = K_R(V,k')$ where k' is the successor of k, the following theorem will be proved. Moreover, this lemma will be used.

Lemma 5.1.6. ([8]) *Let* $k < \dim_R V$.
(i) $K'_R(V,k)$ *admits the structure of a left nearring if and only if* $\dim_R V$ *is finite and* $k = \dim_R V - 1$.
(ii) $K'_R(V,k)$ *admits the structure of a right nearring if and only if* $\dim_R V$ *is finite and* $k = \dim_R V - 1$.

Theorem 5.1.7. *Let* $k < \dim_R V$ *and* $\theta \in E_R(V)$.
(i) $(K'_R(V,k),\theta)$ *admits the structure of a left nearring if and only if* $\dim_R V$ *is finite and* $k = \dim_R V - 1$.
(ii) $(K'_R(V,k),\theta)$ *admits the structure of a right nearring if and only if* $\dim_R V$ *is finite and* $k = \dim_R V - 1$.

Proof. Let k' be the successor of k. Then $k' > 0$ and $K'_R(V,k) = K_R(V,k')$. Suppose that $(K'_R(V,k),\theta) \in \mathcal{LNR} \cup \mathcal{RNR}$. By Theorem 5.1.5, $\dim_R V < \infty$ and $k' = \dim_R V$. So $k = \dim_R V - 1$.

Conversely, assume that $\dim_R V < \infty$ and $k = \dim_R V - 1$. Then by Lemma 5.1.6, $K'_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. Hence $(K'_R(V,k),\theta) \in \mathcal{LNR} \cap \mathcal{RNR}$ is obtained from Proposition 2.7. □

Notice that we can generalize Lemma 5.1.4 and Lemma 5.1.6 by choosing $\theta = 1_V$.

5.2 Genaralizations of the Semigroups $CI_R(V,k)$ and $CI'_R(V,k)$

First, we will show that $(CI_R(V,k),\theta)$ and $(CI'_R(V,k),\theta)$ are subsemigroups of $(L_R(V),\theta)$ where $\theta \in L_R(V)$.

Lemma 5.2.1. *(i) For $k \leq \dim_R V$, $CI_R(V,k)$ is a left ideal of $L_R(V)$.*
(ii) For $k < \dim_R V$, $CI'_R(V,k)$ is a left ideal of $L_R(V)$.

Proof. The results are obtained directly from the fact that $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$ for all $\alpha, \beta \in L_R(V)$ □

The following proposition is a direct consequence of Lemma 5.2.1.

Proposition 5.2.2. *Let $\theta \in L_R(V)$. Then the following statements hold.*
(i) For $k \leq \dim_R V$, $(CI_R(V,k), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.
(ii) For $k < \dim_R V$, $(CI'_R(V,k), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.

In this thesis, we will choose $\theta \in M_R(V)$ to determine when the semigroup $(CI_R(V,k), \theta)$ admits the structure of a left [right] nearring. The following lemmas are needed for our main theorems.

Lemma 5.2.3. *Let $k \leq \dim_R V$. Then $CI_R(V,k)M_R(V) \subseteq CI_R(V,k)$.*

Proof. Let $\alpha \in CI_R(V,k)$ and $\beta \in M_R(V)$. Define $\varphi : V/\operatorname{Im}\alpha \to V/\operatorname{Im}\alpha\beta$ by

$$(v + \operatorname{Im}\alpha)\varphi = v\beta + \operatorname{Im}\alpha\beta \text{ for all } v \in V.$$

It can be seen from the proof of Lemma 4.1.5 and $\alpha \in CI_R(V,k)$ that

$$k \leq \dim_R(V/\operatorname{Im}\alpha) \leq \dim_R(V/\operatorname{Im}\alpha\beta).$$

Therefore $\alpha\beta \in CI_R(V,k)$. □

Lemma 5.2.4. *Let $k \leq \dim_R V$ and $\theta \in G_R(V)$. Then $(CI_R(V,k), \theta) \cong CI_R(V,k)$.*

Proof. Define a map $\varphi : (CI_R(V,k), \theta) \to CI_R(V,k)$ by $\alpha\varphi = \alpha\theta$ for all $\alpha \in CI_R(V,k)$. By Lemma 5.2.3, φ is well-defined. Since θ is one-to-one, so is φ. Lemma 5.2.3 implies that $\alpha\theta^{-1} \in CI_R(V,k)$ and $\alpha = \alpha\theta^{-1}\theta = (\alpha\theta^{-1})\varphi$ for all $\alpha \in CI_R(V,k)$. Hence φ is surjective. For any $\alpha, \beta \in CI_R(V,k)$, we then have $\alpha\theta\beta \in CI_R(V,k)$ so $(\alpha\theta\beta)\varphi = \alpha\theta\beta\theta = \alpha\varphi\beta\varphi$. Hence φ is an isomorphism. □

The following quoted result is useful.

Lemma 5.2.5. ([8]) *Let* $k \leq \dim_R V$.

1. $CI_R(V,k)$ *admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

2. $CI_R(V,k)$ *admits the structure of a right nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

The next theorem is obtained immediately from Lemma 5.2.4, Lemma 5.2.5 and Proposition 2.8 when $\theta \in G_R(V)$. We will generalize this result by considering $\theta \in M_R(V)$.

Theorem 5.2.6. *Let* $k \leq \dim_R V$ *and* $\theta \in M_R(V)$.

1. $(CI_R(V,k),\theta)$ *admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

2. $(CI_R(V,k),\theta)$ *admits the structure of a right nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $\dim_R V$ *is finite and* $k = \dim_R V$.

Proof. Suppose that (i) or (ii) holds. It is direct from Lemma 5.2.5 that $CI_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. By Proposition 2.7, $(CI_R(V,k),\theta) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $(CI_R(V,k),\oplus,\theta)$ is a left nearring or a right nearring. Suppose (i) and (ii) are false. Then either $k > 0$ and $\dim_R V$ is infinite or

$0 < k < \dim_R V < \infty$.

Case 1: $k > 0$ and $\dim_R V$ is infinite. Let B be a basis for V. Then there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Since θ is injective, by Proposition 2.14, we have $B\theta$ is a basis for $\operatorname{Im}\theta$ and $\{B_1\theta, B_2\theta\}$ is also a partition of $B\theta$ with $|B\theta| = |B_1\theta| = |B_2\theta| = |B|$. Let C be a basis for V containing $B\theta$ and $B_3 = C \smallsetminus B\theta$. Then $C = B\theta \cup B_3$. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_1\theta}, \quad \beta = \begin{pmatrix} v & B_1\theta \cup B_3 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_2\theta}$$

and

$$\gamma = \begin{pmatrix} v & B\theta \\ v & 0 \end{pmatrix}_{v \in B_3}.$$

Then $\dim_R(V/\operatorname{Im}\alpha) = |B \smallsetminus B_1| = |B_2| = |B| \geq k$, $\dim_R(V/\operatorname{Im}\beta) = |B \smallsetminus B_2| = |B_1| = |B| \geq k$ and $\dim_R(V/\operatorname{Im}\gamma) = |C \smallsetminus B_3| = |B\theta| = |B| \geq k$. Hence $\alpha, \beta, \gamma \in CI_R(V,k)$. It is easy to see that $\alpha\theta\alpha = \alpha$, $\beta\theta\beta = \beta$ and $\alpha\theta\beta = \beta\theta\alpha = \alpha\theta\gamma = \beta\theta\gamma = 0$. Let $\lambda = \alpha \oplus \beta \in CI_R(V)$. The proof of Theorem 4.1.6 shows that λ is onto, which contradicts $\lambda \in CI_R(V,k)$. Therefore $(CI_R(V,k), \theta) \notin \mathcal{LNR} \cup \mathcal{RNR}$.

Case 2: $0 < k < \dim_R V < \infty$. Then $\theta \in G_R(V)$. By Lemma 5.2.5, we have $CI_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$. It follows from Lemma 5.2.4 and Proposition 2.8 that $(CI_R(V,k), \theta) \cong CI_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$. $\square$

Since $CI'_R(V,k) = CI_R(V,k')$ where k' is the successor of k, the neccessary and sufficient conditions for $(CI'_R(V,k), \theta)$ admitting the nearring structure are also obtained by Theorem 5.2.6 where $\theta \in M_R(V)$. The following quoted result is helpful.

Lemma 5.2.7. ([8]) *Let* $k < \dim_R V$.

(i) $CI'_R(V,k)$ *admits the structure of a left nearring if and only if* $\dim_R V$ *is finite and* $k = \dim_R V - 1$.

(ii) *$CI'_R(V,k)$ admits the structure of a right nearring if and only if $\dim_R V$ is finite and $k = \dim_R V - 1$.*

Theorem 5.2.8. *Let $k < \dim_R V$ and $\theta \in M_R(V)$.*
(i) *$(CI'_R(V,k), \theta)$ admits the structure of a left nearring if and only if $\dim_R V$ is finite and $k = \dim_R V - 1$.*
(ii) *$(CI'_R(V,k), \theta)$ admits the structure of a right nearring if and only if $\dim_R V$ is finite and $k = \dim_R V - 1$.*

Proof. Assume that $\dim_R V$ is finite and $k = \dim_R V - 1$. By Lemma 5.2.7, $CI'_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. Finally, Proposition 2.7 shows that $(CI'_R(V,k), \theta) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, assume that $(CI'_R(V,k), \theta) \in \mathcal{LNR} \cup \mathcal{RNR}$. Let k' be the successor of k. Then $k' > 0$ and $CI'_R(V,k) = CI_R(V,k')$. By Theorem 5.2.6, we have $\dim_R V < \infty$ and $k' = \dim_R V$. Hence $\dim_R V < \infty$ and $k = \dim_R V - 1$. □

Therefore quoted results, Lemma 5.2.5 and Lemma 5.2.7, are special cases of Theorem 5.2.6 and Theorem 5.2.8 where $\theta = 1_V$.

5.3 Generalizations of the Semigroups $I_R(V,k)$ and $I'_R(V,k)$

In the last section, we will show that both of $I_R(V,k)$ and $I'_R(V,k)$ are subsemigroups of $(L_R(V), \theta)$ where $\theta \in L_R(V)$.

Lemma 5.3.1. (i) *For $k \le \dim_R V$, $I_R(V,k)$ is a left ideal of $L_R(V)$.*
(ii) *For $0 < k \le \dim_R V$, $I'_R(V,k)$ is a left ideal of $L_R(V)$.*

Proof. The results are obtained directly from the fact that $\operatorname{Im} \alpha\beta \subseteq \operatorname{Im} \beta$ for all $\alpha, \beta \in L_R(V)$. □

Proposition 5.3.2. *Let $\theta \in L_R(V)$. Then the following statements hold.*
(i) *For $k \le \dim_R V$, $(I_R(V,k), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*
(ii) *For $0 < k \le \dim_R V$, $(I'_R(V,k), \theta)$ is a subsemigroup of $(L_R(V), \theta)$.*

For $I_R(V,k)$ and $I'_R(V,k)$, we will choose $\theta \in M_R(V)$ to determine whether or not $(I_R(V,k),\theta)$ and $(I'_R(V,k),\theta)$ admit the structure of a left [right] nearring. These two lemmas are useful facts for our main theorems.

Lemma 5.3.3. *For $k \leq \dim_R V$, $I_R(V,k)M_R(V) \subseteq I_R(V,k)$.*

Proof. Let $\alpha \in I_R(V,k)$ and $\beta \in M_R(V)$. Let B be a basis for $\operatorname{Im}\alpha$. By Proposition 2.14, we have $B\beta$ is a basis for $(\operatorname{Im}\alpha)\beta = \operatorname{Im}\alpha\beta$. So $\dim_R \operatorname{Im}\alpha\beta = |B\beta| = |B| \leq k$ since β is injective and $\alpha \in I_R(V,k)$. Hence $\alpha\beta \in I_R(V,k)$. $\square$

Lemma 5.3.4. ([8]) *Let $k \leq \dim_R V$.*

1. *$I_R(V,k)$ admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $k = \dim_R V$.
   (iii) *k is an infinite cardinal number.*

2. *$I_R(V,k)$ admits the structure of a right nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $k = \dim_R V$.
   (iii) *k is an infinite cardinal number.*

Next, we begin the first main Theorem.

Theorem 5.3.5. *Let $k \leq \dim_R V$ and $\theta \in M_R(V)$.*

1. *$(I_R(V,k),\theta)$ admits the structure of a left nearring if and only if one of the following statements holds.*
   (i) $k = 0$.
   (ii) $k = \dim_R V$.
   (iii) *k is an infinite cardinal number.*

2. $(I_R(V,k),\theta)$ *admits the structure of a right nearring if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = \dim_R V$.

(iii) k *is an infinite cardinal number.*

Proof. First, assume that (i), (ii) or (iii) holds. By Lemma 5.3.4, $I_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. It follows from Proposition 2.7 that $(I_R(V,k),\theta) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, suppose that (i), (ii) and (iii) are all false. That is, $0 < k < \dim_R V$ and k is finite. Let B be a basis for V. Then there exists B_1, a proper subset of B such that $|B_1| = k$. Since θ is injective, by Proposition 2.14, $B\theta$ is a basis for $\operatorname{Im}\theta$. Moreover, $|B_1\theta| = |B_1| = k$ and $|B\theta| = |B|$. Extend $B\theta$ to C, a basis for V. Set $B_2 = C \setminus B\theta$. Notice that $C = B\theta \cup B_2 = B_1\theta \cup (B \setminus B_1)\theta \cup B_2$. Let $u \in (B \setminus B_1)\theta$ be fixed. Then we have $u_0 \in B \setminus B_1$ such that $u_0\theta = u$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & (B \setminus B_1)\theta \cup B_2 \\ v\theta^{-1} & 0 \end{pmatrix}_{v \in B_1\theta} \quad \text{and} \quad \beta = \begin{pmatrix} u & C \setminus \{u\} \\ u_0 & 0 \end{pmatrix}.$$

Then $\operatorname{Im}\alpha = \langle B_1 \rangle$ and $\operatorname{Im}\beta = \langle u_0 \rangle$. It follows that $\dim_R \operatorname{Im}\alpha = |B_1| = k$ and $\dim_R \operatorname{Im}\beta = 1 \leq k$, so we have $\alpha, \beta \in I_R(V,k)$. It is easy to see that $\alpha\theta\alpha = \alpha$, $\beta\theta\beta = \beta$ and $\alpha\theta\beta = \beta\theta\alpha = 0$.

1. If $(I_R(V,k),\oplus,\theta)$ is a left nearring, let $\lambda = \alpha \oplus \beta \in I_R(V,k)$. Then $\alpha\theta\lambda = \alpha$ and $\beta\theta\lambda = \beta$. So we have

$$\text{for every } v \in B_1\theta,\ v\lambda\theta = v\alpha\theta\lambda\theta = v\alpha\theta = v,$$
$$u\lambda\theta = u\beta\theta\lambda\theta = u\beta\theta = u.$$

This implies that $\langle B_1\theta \cup \{u\} \rangle \subseteq \operatorname{Im}\lambda\theta$. Thus $\dim_R \operatorname{Im}\lambda\theta \geq |B_1\theta \cup \{u\}| = k+1 > k$ since k is finite. Then $\lambda\theta \notin I_R(V,k)$, which contradicts to Lemma 5.3.3. Hence

$(I_R(V,k),\theta) \notin \mathcal{LNR}$.

2. Suppose that $(I_R(V,k),\oplus,\theta)$ is a right nearring, let $\lambda = \alpha \oplus \beta \in I_R(V,k)$. Then $\lambda\theta\alpha = \alpha$ and $\lambda\theta\beta = \beta$. Thus

$$\text{for every } v \in B_1\theta,\ v\lambda\theta\beta = v\beta = 0,$$
$$u\lambda\theta\alpha = u\alpha = 0.$$

Consequently,

$$\text{for every } v \in B_1\theta, v\lambda\theta \in \operatorname{Ker}\beta = \langle C \setminus \{u\}\rangle\,,$$
$$u\lambda\theta \in \operatorname{Ker}\alpha = \langle C \setminus B_1\theta\rangle\,. \qquad (*)$$

Claim that $v\lambda\theta \in \langle B_1\theta\rangle$ for all $v \in B_1\theta$ and $u\lambda\theta \in \langle u\rangle$.

Case 1: $C \setminus B_1\theta = \{u\}$. Then

$$\text{for every } v \in B_1\theta, v\lambda\theta \in \langle C \setminus \{u\}\rangle = \langle B_1\theta\rangle\,,$$
$$u\lambda\theta \in \langle C \setminus B_1\theta\rangle = \langle u\rangle\,.$$

Case 2: $(C \setminus B_1\theta) \setminus \{u\} \neq \varnothing$. That is, $\big((B \setminus B_1)\theta \cup B_2\big) \setminus \{u\} \neq \varnothing$. For each $w \in \big((B \setminus B_1)\theta \cup B_2\big) \setminus \{u\}$, define $\gamma_w \in L_R(V)$ by

$$\gamma_w = \begin{cases} \begin{pmatrix} w & C \setminus \{w\} \\ w\theta^{-1} & 0 \end{pmatrix} & \text{if } w \in (B \setminus B_1)\theta \setminus \{u\}\,, \\[2ex] \begin{pmatrix} w & C \setminus \{w\} \\ w & 0 \end{pmatrix} & \text{if } w \in B_2. \end{cases}$$

If either $w \in (B \setminus B_1)\theta \setminus \{u\}$ or $w \in B_2$, we then have $\operatorname{Im}\gamma_w = \langle w\theta^{-1}\rangle$ or $\operatorname{Im}\gamma_w = \langle w\rangle$, respectively. So $\dim_R \operatorname{Im}\gamma_w = 1 \leq k$. Thus $\gamma_w \in I_R(V,k)$ and

$$\alpha\theta\gamma_w = \beta\theta\gamma_w = 0 \text{ for all } w \in \big((B \smallsetminus B_1)\theta \cup B_2\big) \smallsetminus \{u\}.$$

Since $(I_R(V,k), \oplus, \theta)$ is a right nearring, it follows that $\lambda\theta\gamma_w = 0$ for all $w \in \big((B \smallsetminus B_1)\theta \cup B_2\big) \smallsetminus \{u\}$. From (*), we have for every $v \in B_1\theta$,

$$\begin{aligned} v\lambda\theta &\in \langle C \smallsetminus \{u\}\rangle \cap \Big(\bigcap_{w\in((B\smallsetminus B_1)\theta\cup B_2)\smallsetminus\{u\}} \operatorname{Ker}\gamma_w\Big) \\ &= \langle C \smallsetminus \{u\}\rangle \cap \Big(\bigcap_{w\in((B\smallsetminus B_1)\theta\cup B_2)\smallsetminus\{u\}} \langle C \smallsetminus \{w\}\rangle\Big) \\ &= \langle C \smallsetminus \{u\}\rangle \cap \langle B_1\theta \cup \{u\}\rangle \\ &= \langle B_1\theta\rangle \end{aligned}$$

and

$$\begin{aligned} u\lambda\theta &\in \langle C \smallsetminus B_1\theta\rangle \cap \Big(\bigcap_{w\in((B\smallsetminus B_1)\theta\cup B_2)\smallsetminus\{u\}} \operatorname{Ker}\gamma_w\Big) \\ &= \langle C \smallsetminus B_1\theta\rangle \cap \langle B_1\theta \cup \{u\}\rangle \\ &= \langle u\rangle\,. \end{aligned}$$

Hence the claim is proved, and then we have

$$\begin{aligned} \text{for every } v \in B_1\theta,\ v\lambda\theta &= v\lambda\theta\alpha\theta = v\alpha\theta = v, \\ u\lambda\theta &= u\lambda\theta\beta\theta = u\beta\theta = u, \end{aligned}$$

since $\lambda\theta\alpha = \alpha$ and $\lambda\theta\beta = \beta$. Thus $\langle B_1\theta \cup \{u\}\rangle \subseteq \operatorname{Im}\lambda\theta$. Since k is finite, $\dim_R \operatorname{Im}\lambda\theta \geq |B_1\theta \cup \{u\}| = k + 1 > k$. Hence $\lambda\theta \notin I_R(V,k)$ which contradicts Lemma 5.3.3. Therefore $(I_R(V,k),\theta) \notin \mathcal{RNR}$. □

The following quoted result will be used.

Lemma 5.3.6. ([8]) *Let* $0 < k \leq \dim_R V$.

(i) $I'_R(V,k)$ *admits the structure of a left nearring if and only if either* $k = 1$ *or* k *is an* *infinite cardinal number.*

(ii) $I'_R(V,k)$ *admits the structure of a right nearring if and only if either* $k = 1$ *or* k *is an infinite cardinal number.*

Theorem 5.3.7. *Let* $0 < k \leq \dim_R V$ *and* $\theta \in M_R(V)$.

(i) $(I'_R(V,k),\theta)$ *admits the structure of a left nearring if and only if either* $k = 1$ *or* k *is an infinite cardinal number.*

(ii) $(I'_R(V,k),\theta)$ *admits the structure of a right nearring if and only if either* $k = 1$ *or* k *is an infinite cardinal number.*

Proof. Assume that $k = 1$ or k is an infinite cardinal number. Then, by Lemma 5.3.6, $I'_R(V,k) \in \mathcal{LNR} \cap \mathcal{RNR}$. Hence Proposition 2.7 shows that $(I'_R(V,k),\theta) \in \mathcal{LNR} \cap \mathcal{RNR}$.

Conversely, suppose that $1 < k$ and k is finite. Thus $I'_R(V,k) = I_R(V,k-1)$, $0 < k-1 < \dim_R V$ and $k-1$ is finite. By Theorem 5.3.5, $(I_R(V,k-1),\theta) \notin \mathcal{LNR} \cup \mathcal{RNR}$. Hence $I'_R(V,k) \notin \mathcal{LNR} \cup \mathcal{RNR}$. $\square$

By Theorem 5.3.5 and Theorem 5.3.7, we have generalized Lemma 5.3.4 and Lemma 5.3.6.

From Chapter IV and the current chapter, we can conclude that $(OM_R(V),\theta)$, $(OE_R(V),\theta)$, $(K_R(V,k),\theta)$, $(K'_R(V,k),\theta)$, $(CI_R(V,k),\theta)$, $(CI'_R(V,k),\theta)$, $(I_R(V,k,\theta)$ and $(I'_R(V,k),\theta)$ are semigroups where $\theta \in L_R(V)$. In this thesis, we determine when these semigroups admit the structure of a nearring where $\theta \in E_R(V)$ or $\theta \in M_R(V)$. So we can continue this research by determining when these semigroups with the other linear transformations θ to admit the nearring structure. Moreover, if we extend semigroups containing $OM_R(V)$ and $OE_R(V)$ to sets $OM_R(V) \cup H$ and $OE_R(V) \cup H$ where H is a subsemigroup of $L_R(V)$, the research works are finding necessary and sufficient conditions for $(OM_R(V) \cup H,\theta)$ and $(OE_R(V) \cup H,\theta)$ to be semigroups where $\theta \in L_R(V)$. Then these semigroups can be characterized whether or not they admit the nearring structure.

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