CHAPTER I

PRELIMINARIES

Let S be a semigroup.

An element e of S is called an *identity* of S if $xe = ex = x$ for all $x \in S$. If an identity of S exists, then it is unique and usually denoted by 1.

An element x of S is called a *right* [left] *zero* of S if $yx = x$ $[xy = x]$ for all $y \in S$. An element z of S is called a *zero* of S if it is both a right and a left zero of S. A zero of S is unique if it exists, and it is usually denoted by 0.

A semigroup in which every element is a right [left] zero is called a *right* [left] *zero* *semigroup*.

For a nonempty subset A of S, let $\langle A \rangle$ denote the subsemigroup of S generated by A, that is,

$$\langle A \rangle = \{a_1 a_2 \ldots a_n \mid a_i \in A,\ n \in \mathbb{N}\}$$

where $\mathbb{N}$ is the set of all positive integers. For $a \in S$, let $\langle a \rangle$ denote $\langle \{a\} \rangle$, that is,

$$\langle a \rangle = \{a^n \mid n \in \mathbb{N}\} .$$

S is said to be a *cyclic* *semigroup* if $S = \langle a \rangle$ for some $a \in S$. If $S = \langle a \rangle$, $a \in S$, then S is finite if and only if $a^i = a^j$ for some $i, j \in \mathbb{N}$, $i \neq j$.

A subsemigroup I of S is called an <u>ideal</u> of S if $xa, ax \in I$ for all $x \in S$, $a \in I$.

A semigroup S is called an <u>inverse semigroup</u> if for each element $x \in S$, there exists a unique element x^{-1} in S such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$.

Let S and T be semigroups and $\varphi: S \to T$. The map φ is called a <u>homomorphism</u> of S into T if

$$(xy)\varphi = (x\varphi)(y\varphi)$$

for all $x, y \in S$.

Let X be a nonempty set. A nonempty finite sequence $a_1, a_2, \ldots, a_n$, usually written by juxtaposition, $a_1a_2\ldots a_n$, of elements of X is called a <u>word</u> over the alphabet X. The set $\mathcal{F}_X$ of all words with the operation of juxtaposition

$$(a_1a_2\ldots a_m)(b_1b_2\ldots b_n) = a_1a_2\ldots a_m\, b_1b_2\ldots b_n$$

is a semigroup called the <u>free semigroup</u> on the set X.

Let $E = X \times \{1,-1\}$. We shall write the element of E as a^α where $a \in X$ and $\alpha \in \{1,-1\}$. A finite sequence of elements of E is a <u>word</u>, a word may be written in the form $w = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m}$ where $x_i \in X$, $\alpha_i = \pm 1$, $i = 1,2,\ldots, m$. The word w is a <u>reduced word</u> if and only if no symbol x^{+1} is adjacent to x^{-1}. The null set is called the <u>empty word</u>, and denoted by 1. The product of nonempty reduced words is given by juxtaposition, that is,

$$(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m})(y_1^{\alpha_1}y_2^{\alpha_2}\ldots y_n^{\alpha_n}) = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m}y_1^{\alpha_1}y_2^{\alpha_2}\ldots y_n^{\alpha_n}.$$

More precisely, if $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$ and $y_1^{\lambda_1} y_2^{\lambda_2} \ldots y_n^{\lambda_n}$ are nonempty reduced words on X with $m \leqslant n$, let k be the largest integer $(0 \leqslant k \leqslant m)$ such that $x_{m-j}^{\alpha_{m-j}} = y_{j+1}^{-\lambda_{j+1}}$ for $j = 0,1,\ldots, k-1$. Then define

$$(x_1^{\alpha_1} \ldots x_m^{\alpha_m})(y_1^{\lambda_1} \ldots y_n^{\lambda_n}) = \begin{cases} x_1^{\alpha_1} \ldots x_{m-k}^{\alpha_{m-k}} y_{k+1}^{\lambda_{k+1}} \ldots y_n^{\lambda_n} & \text{if } k < m , \\ y_{m+1}^{\lambda_{m+1}} \ldots y_n^{\lambda_n} & \text{if } k = m < n , \\ 1 & \text{if } k = m = n . \end{cases}$$

If $m > n$, the product is defined analogously. The definition insures that the product of reduced words is a reduced word.

Let $\mathcal{G}_X$ be the set of all reduced words on E. Then $\mathcal{G}_X$ is a group under the operation defined above. The group $\mathcal{G}_X$ is called the <u>free group</u> on the set X.

Let X be a set. A <u>partial transformation</u> of X is a map from a subset of X into X. The <u>empty transformation</u> of X is the partial transformation of X with empty domain and it is denoted by 0. For a partial transformation α of X, the domain and the range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let P_X be the set of all partial transformations of X (including 0). For $\alpha, \beta \in P_X$, define the product $\alpha\beta$ as follows. If $\nabla\alpha \cap \Delta\beta = \emptyset$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \emptyset$, let $\alpha\beta = (\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}})(\beta|_{\nabla\alpha \cap \Delta\beta})$ (the composition of the maps $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$) where $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$ denote the restrictions of α and β to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then P_X is a semigroup with identity 1_X (the identity map on X) and zero 0 and it is called the partial transformation semigroup on X.

By a transformation semigroup on X, we mean a subsemigroup of P_X.

Let I_X denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in P_X \mid \alpha \text{ is 1-1.}\}.$$

Then I_X is an inverse subsemigroup of P_X with identity 1_X and zero 0, which is called the 1-1 partial transformation semigroup on X or the symmetric inverse semigroup on X.

By a transformation of a set X, we mean a map of X into itself. Then an element $\alpha \in P_X$ is a transformation of X if and only if $\Delta\alpha = X$. Let T_X denote the set of all transformations of X, that is,

$$T_X = \{\alpha \in P_X \mid \Delta\alpha = X\} .$$

Then T_X is a subsemigroup of P_X with identity 1_X, which is called the full transformation semigroup on X.

Let $M_X = \{\alpha \in T_X \mid \alpha \text{ is 1-1.}\}$ (the set of all 1-1 transformations of X),

$O_X = \{\alpha \in T_X \mid \alpha \text{ is onto X.}\}$ (the set of all onto transformations of X),

$CP_X = \{\alpha \in P_X \mid \alpha \text{ is a constant map.}\}$ (the set of all constant partial transformations of X (including the empty transformation)) and $CT_X = \{\alpha \in T_X \mid \alpha$ is a constant map.$\}$ (the set of all constant transformations of X).

Then M_X, O_X, CP_X and CT_X are subsemigroups of P_X.

Let $(F,+,\cdot)$ be a field and n a positive integer. Let $M_n(F)$ be the set of all $n \times n$ matrices over F. Then $M_n(F)$ is a semigroup under usual matrix multiplication.

By a <u>matrix</u> <u>semigroup</u> over F, we mean a subsemigroup of $M_k(F)$ under usual matrix multiplication for some positive integer k.

Let $G_n(F)$ = the matrix group of all $n \times n$ nonsingular matrices over F,

$U_n(F)[L_n(F)]$ = the matrix semigroup of all $n \times n$ upper [lower] triangular matrices over F and

$D_n(F)$ = the matrix semigroup of all $n \times n$ diagonal matrices over F.

The following statements are well-known :

(i) For $A, B \in M_n(F)$, $\text{rank}(A) = \text{rank}(B)$ if and only if $A = PBQ$ for some $P, Q \in G_n(F)$.

(ii) $\{\det A \mid A \in M_n(F)\} = F$ and $\{\det A \mid A \in G_n(F)\} = F \smallsetminus \{0\}$.

A subset A of a semigroup S is said to be <u>dense</u>* in S if for any semigroup T, for any homomorphisms α, $\beta : S \to T$, $\alpha|_A = \beta|_A$ implies $\alpha = \beta$.

Let S be a semigroup and U a subsemigroup of S. For any element d of S, d is said to be <u>dominated</u> by U or U <u>dominates</u> d if for any semigroup T and for any homomorphisms α, $\beta : S \to T$, $\alpha|_U = \beta|_U$ implies $d\alpha = d\beta$. The set of all elements of S which are dominated by U is called the <u>dominion</u> of U in S and it is denoted by Dom(U,S).

Hence U is dense in S if and only if Dom(U,S) = S.

The following statements clearly hold :

(i) $U \subseteq$ Dom(U,S).

(ii) Dom(U,S) is a subsemigroup of S.

(iii) If V is a subsemigroup of S such that $U \subseteq V$, then Dom(U,S) $\subseteq$ Dom(V,S).

Let U be a subsemigroup of a semigroup S. A <u>zigzag</u> of <u>length</u> m ($m \in \mathbb{N}$) in U over S with <u>value</u> $d \in S$ is a system of equalities

$$d = u_0 y_1,\ u_0 = x_1 u_1$$

$$x_i u_{2i} = x_{i+1} u_{2i+1},$$

$$u_{2i-1} y_i = u_{2i} y_{i+1} \quad (i=1,2,\ldots,m-1),$$

$$u_{2m-1} y_m = u_{2m},\ x_m u_{2m} = d\ ,$$

---

* In Topology, it is known that for a metric space X and $D \subseteq X$, D is dense in X if and only if for any metric space Y, for any continuous mappings $f, g : X \to Y$, $f|_D = g|_D$ implies $f = g$.

with $u_0, u_1, \ldots, u_{2m}$ in U and $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ in S, that is,

$$\begin{aligned} d &= u_0y_1\ , \quad u_0 \in U\ , y_1 \in S\ , \\ &= x_1u_1y_1\ , \quad u_1 \in U\ , x_1 \in S\ , u_0 = x_1u_1\ , \\ &= x_1u_2y_2\ , \quad u_2 \in U\ , y_2 \in S\ , u_1y_1 = u_2y_2\ , \\ &= x_2u_3y_2\ , \quad u_3 \in U\ , x_2 \in S\ , x_1u_2 = x_2u_3\ , \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &= x_mu_{2m-1}y_m\ , \quad u_{2m-1} \in U,\ x_m \in S,\ x_{m-1}u_{2m-2} = x_mu_{2m-1}\ , \\ &= x_mu_{2m}\ , \quad u_{2m} \in U\ , u_{2m-1}\, y_m = u_{2m}. \end{aligned}$$

The following known results will be used in the thesis :

<u>Theorem 1.1</u> <u>Isbell's</u> <u>Zigzag</u> <u>Theorem</u> ([3] or [4]). Let U be a subsemigroup of a semigroup S. Then d ε Dom(U,S) if and only if d ε U or there exists a zigzag in U over S with value d.

It follows easily from Theorem 1.1 that if I is an ideal of a semigroup S, then Dom(I,S) = I.

<u>Theorem 1.2</u> ([4]). If U is an inverse subsemigroup of a semigroup S, then Dom(U,S) = U.

The following theorem is known :

<u>Theorem 1.3</u> (Higgins [2]). Let X be a set and let S denote any one of T_X, P_X or I_X. Then S has a proper dense subsemigroup if and only if X is infinite.