CHAPTER II

FUNDAMENTAL CONSIDERATION

Theory of Thin, Isotropic Elastic Plates

The thin plate bending theory, in the absence of membrane forces, is based on the assumption that plane sections remain plane during bending and that the deflection are small comparing with the thickness of the plate. The effect of shear forces on the deflection are also disregarded.

Consider a thin plate element of thickness h the material of which has the modulus of elasticity E and Poisson's ratio ν in Cartesian co-ordinates (x,y) as shown in Fig.1 in which $D = Eh^3/12(1-\nu^2)$ denotes the flexural rigidity. Also the sign convention of stress resultants using the notations and conventions of Timoshenko and Woinowsky-Krieger (10) are shown in Fig.2. The deflection, w, of the middle surface of the plate subjected to transverse force of intensity q is related to stress resultants, bending and twisting moments and shears per unit length of the section, as follows :

$$M_x = -D\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\nu\partial^2 w}{\partial y^2}\right\} \qquad (1)$$

$$M_y = -D\left\{\frac{\partial^2 w}{\partial y^2} + \frac{\nu\partial^2 w}{\partial x^2}\right\} \quad (2)$$

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} \quad (3)$$

$$Q_x = -D\left\{\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\partial y^2}\right\} \quad (4)$$

$$Q_y = -D\left\{\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y\partial x^2}\right\} \quad (5)$$

If a boundary of a plate is free, it is generally assumed that along this edge there are no bending and twisting moments and also no vertical shears. Kirchhoff proved that three boundary conditions are too many and that two conditions are enough for the complete determination of the solution. He showed that the edge twisting moment can be combined with the shearing force to produce a resultant boundary shear force or called Kirchhoff shear or supplemented shear per unit length, V, as

$$V_x = -D\left\{\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}\right\} \quad (6)$$

$$V_y = -D\left\{\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y\partial x^2}\right\} \quad (7)$$

By incorporating twisting moments into supplemented shears, the corner forces , R, arised from the jump of twisting moments at each corner have to be considered and can be written as

$$R = M_{xy} - M_{yx} = 2M_{xy} . \qquad (8)$$

By considering the equilibrium of forces in the z-direction and moments about x- and y-axes, leads to the governing biharmonic equation, called the "plate equation"

$$\nabla^2\nabla^2 w = \nabla^4 w = q/D \qquad (9)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplacian operator.

In terms of normal and tangential co-ordinates of a section (n,t) (Fig.3) the above relationships become

$$M_n = -D\left\{\frac{\partial^2 w}{\partial n^2} + \frac{\nu\partial^2 w}{\partial t^2}\right\} \qquad (10)$$

$$M_t = -D\left\{\frac{\partial^2 w}{\partial t^2} + \frac{\nu\partial^2 w}{\partial n^2}\right\} \qquad (11)$$

$$M_{nt} = D(1-\nu)\frac{\partial^2 w}{\partial n\partial t} \qquad (12)$$

$$Q_n = -D\left\{\frac{\partial^3 w}{\partial n^3} + \frac{\partial^3 w}{\partial n \partial t^2}\right\} \tag{13}$$

$$V_n = -D\left\{\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial t^2}\right\} \tag{14}$$

The corner forces can be written as

$$R = M_{n_1 t_1} - M_{n_2 t_2}$$

$$\text{or} \quad R = D(1-\nu)\left\{\frac{\partial^2 w}{\partial n_1 \partial t_1} - \frac{\partial^2 w}{\partial n_2 \partial t_2}\right\} \tag{15}$$

In which, the normals and tangents on the first side of the corner are denoted by subscript 1 and the other side by subscript 2 along the path as shown in Fig.4. The useful transformation matrices for normal and tangential derivatives up to the third order are also given in the Appendix A.

Betti's Reciprocal Theorem

The important energy principle that will be employed later in the boundary integral formulation is the Betti's reciprocal theorem (11). It states that for any two equilibrium states of stresses and compatible displacements, A and B, of a linearly elastic body the total external work done by the forces A during the corresponding displacements caused by the forces B is equal to the total external work done by the forces B during the corresponding displacements caused by the forces A.

Method of Analysis

Consider two distinct systems of compatible deflections and equilibrium states of stresses as shown in Fig.5. One is the real plate, the problem under consideration, which is the rectilinear plate of K sides having L columns inside the plate domain, Ω , with free boundary condition loaded by transverse force of intensity $q(\xi,\eta)$. The other is the virtual plate, designated by asterisks, subjected to a unit singular load, the Dirac delta function δ, acting at a point (x,y), the solution of which satisfying the non-homogeneous equation

$$\nabla^4 w^*(x,y;\xi,\eta) \quad = \quad \delta(x,y)/D \quad . \qquad (16)$$

A solution of (16) above may be taken as :

$$w^*(x,y;\xi,\eta) \quad = \quad \frac{r^2 \ln r}{8\pi D} \qquad (17)$$

where $r = \sqrt{(\xi-x)^2 + (\eta-y)^2}$ is the distance between point (x,y) and point (ξ,η). It should be noted that (ξ,η) and (x,y) are the Cartesian co-ordinates. The expressions of slopes and desired stress resultants of the virtual plate can be obtained by appropriate differentiation of the deflection function $w^*(x,y;\xi,\eta)$.

The direct boundary integral methods of analysis are those that make use of the energy principle, i.e. Betti's reciprocal theorem, as mentioned earlier. Imposing the boundary conditions on the real plate where normal bending moment and supplemented shear are prescribed as zero and applying Betti's reciprocal theorem between the two systems in Fig.5, we obtain

$$w(x,y) + \int_{\Gamma} [-\overset{*}{M}_n(x,y;\bar{\xi},\bar{\eta})\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} + \overset{*}{V}_n(x,y;\bar{\xi},\bar{\eta})w(\bar{\xi},\bar{\eta})]d\Gamma(\bar{\xi},\bar{\eta})$$

$$+ \sum_{k=1,2}^{K} \overset{*}{R}_k(x,y;\bar{\xi}_k,\bar{\eta}_k)w(\bar{\xi}_k,\bar{\eta}_k) = \int_{\Omega} q(\xi,\eta)\overset{*}{w}(x,y;\xi,\eta)d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} [- K_{al}w_{cl}(\xi_{cl},\eta_{cl})\overset{*}{w}(x,y;\xi_{cl},\eta_{cl})$$

$$- K^{y}_{rl}\frac{\partial w_{cl}}{\partial \xi}(\xi_{cl},\eta_{cl})\frac{\partial \overset{*}{w}}{\partial \xi}(x,y;\xi_{cl},\eta_{cl})$$

$$- K^{x}_{rl}\frac{\partial w_{cl}}{\partial \eta}(\xi_{cl},\eta_{cl})\frac{\partial \overset{*}{w}}{\partial \eta}(x,y;\xi_{cl},\eta_{cl})] , \quad (x,y) \in \Omega \qquad (18)$$

in which

$\overset{*}{w}$ = deflection of the virtual plate

$\frac{\partial \overset{*}{w}}{\partial \xi}$, $\frac{\partial \overset{*}{w}}{\partial \eta}$ = slope with respect to ξ and η respectively of the virtual plate

$\overset{*}{V}_n$, $\overset{*}{M}_n$, $\overset{*}{R}$ = supplemented shear, normal bending moment and corner force of the virtual plate

w , $\frac{\partial w}{\partial n}$ = deflection and slope with respect to outward normal of the real plate

$\frac{\partial w}{\partial \xi}$, $\frac{\partial w}{\partial \eta}$ = slope with respect to ξ and η respectively of the real plate

K_a , K_r^x , K_r^y = axial stiffness, rotational stiffness about x-axis and y-axis of the interior support

Γ = boundary of plate

Ω = domain of plate

(x,y), (ξ,η) = co-ordinates

(x_c,y_c),(ξ_c,η_c) = co-ordinates of interior support

$(\bar{x},\bar{y})$, $(\bar{\xi},\bar{\eta})$ = co-ordinates on the boundary.

The above equation gives the deflection of any interior point (x,y), $w(x,y)$, in terms of boundary values, $\partial w(\bar{\xi},\bar{\eta})/\partial n$, $w(\bar{\xi},\bar{\eta})$ and $w(\bar{\xi}_k,\bar{\eta}_k)$, $w_{ci}(\xi_{ci},\eta_{ci})$, $\partial w_{ci}(\xi_{ci},\eta_{ci})/\partial\xi$ and $\partial w_{ci}(\xi_{ci},\eta_{ci})/\partial\eta$, for $k = 1,2,\ldots,K$, $i = 1,2,\ldots,L$.

Approaching (x,y) to the boundary $(\bar{x},\bar{y})$ and rearranging, equation (18) becomes

$$\frac{\phi w(\bar{x},\bar{y})}{2\pi} = \int_{\Gamma} [\, M^{*}_{n}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - V^{*}_{n}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})w(\bar{\xi},\bar{\eta})]d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} R^{*}_{k}(\bar{x},\bar{y};\bar{\xi}_k,\bar{\eta}_k)w(\bar{\xi}_k,\bar{\eta}_k) + \int_{\Omega} q(\xi,\eta)w^{*}(\bar{x},\bar{y};\xi,\eta)d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} [\, - K_{al}w_{cl}(\xi_{cl},\eta_{cl})w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})$$

$$- K^{y}_{rl}\frac{\partial w_{cl}}{\partial \xi}(\xi_{cl},\eta_{cl})\frac{\partial w^{*}}{\partial \xi}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})$$

$$- K^{x}_{rl}\frac{\partial w_{cl}}{\partial \eta}(\xi_{cl},\eta_{cl})\frac{\partial w^{*}}{\partial \eta}(x,y;\xi_{cl},\eta_{cl})] \;,\; (\bar{x},\bar{y}) \in \Gamma \qquad (19)$$

where ϕ is the included angle (for smooth boundary point, $\phi = \pi$) at the boundary point.

Differentiating equation (19) with respect to the outward normal direction, $n(\bar{x},\bar{y})$, to obtain

$$\frac{\phi\,\partial w(\bar{x},\bar{y})}{2\pi\,\partial n(\bar{x},\bar{y})} = \int_{\Gamma} \left[\frac{\partial M_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})}{\partial n(\bar{x},\bar{y})}\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - \frac{\partial V_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})}{\partial n(\bar{x},\bar{y})} w(\bar{\xi},\bar{\eta})\right] d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} \frac{\partial R_k^{*}(\bar{x},\bar{y};\xi_k,\eta_k)}{\partial n(\bar{x},\bar{y})} w(\xi_k,\eta_k) + \int_{\Omega} q(\xi,\eta)\frac{\partial w^{*}(\bar{x},\bar{y};\xi,\eta)}{\partial n(\bar{x},\bar{y})} d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} \left[- K_{al} w_{cl}(\xi_{cl},\eta_{cl})\frac{\partial w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial n(\bar{x},\bar{y})}\right.$$

$$- K_{rl}^{y}\frac{\partial w_{cl}}{\partial \xi}(\xi_{cl},\eta_{cl})\frac{\partial^{2} w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial n(\bar{x},\bar{y})\partial \xi}$$

$$\left.- K_{rl}^{x}\frac{\partial w_{cl}}{\partial \eta}(\xi_{cl},\eta_{cl})\frac{\partial^{2} w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial n(\bar{x},\bar{y})\partial \eta}\right] , \quad (\bar{x},\bar{y}) \in \Gamma \qquad (20)$$

Now, approach (x,y) to the location of each supports :

$$
\begin{aligned}
w(x_{ci},y_{ci}) = & \int_{\Gamma} \Big[M^{*}_{n}(x_{ci},y_{ci};\bar{\xi},\bar{\eta}) \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} \\
& - V^{*}_{n}(x_{ci},y_{ci};\bar{\xi},\bar{\eta}) w(\bar{\xi},\bar{\eta}) \Big] \, d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} R^{*}_{k}(x_{ci},y_{ci};\bar{\xi}_{k},\bar{\eta}_{k}) w(\bar{\xi}_{k},\bar{\eta}_{k}) + \int_{\Omega} q(\xi,\eta) w^{*}(x_{ci},y_{ci};\xi,\eta) \, d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \Big[- K_{al} w_{cl}(\xi_{cl},\eta_{cl}) w^{*}(x_{ci},y_{ci};\xi_{cl},\eta_{cl}) \\
& - K^{y}_{rl} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \frac{\partial w^{*}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \xi} \\
& - K^{x}_{rl} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \frac{\partial w^{*}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \eta} \Big] , \quad i = 1,2,..,L \qquad (21)
\end{aligned}
$$

$$\frac{\partial w(x_{ci},y_{ci})}{\partial x} = \int_{\Gamma} \Big[\frac{\partial M^{*}_{n}}{\partial x}(x_{ci},y_{ci};\bar{\xi},\bar{\eta})\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n}$$

$$- \frac{\partial V^{*}_{n}}{\partial x}(x_{ci},y_{ci};\bar{\xi},\bar{\eta})w(\bar{\xi},\bar{\eta}) \Big] d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} \frac{\partial R^{*}_{k}}{\partial x}(x_{ci},y_{ci};\bar{\xi}_{k},\bar{\eta}_{k})w(\bar{\xi}_{k},\bar{\eta}_{k}) + \int_{\Omega} q(\xi,\eta)\frac{\partial w^{*}}{\partial x}(x_{ci},y_{ci};\xi,\eta)d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} \Big[- K_{al}w_{cl}(\xi_{cl},\eta_{cl})\frac{\partial w^{*}}{\partial x}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})$$

$$- K^{y}_{rl}\frac{\partial w_{cl}}{\partial \xi}(\xi_{cl},\eta_{cl})\frac{\partial^{2} w^{*}}{\partial x \partial \xi}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})$$

$$- K^{x}_{rl}\frac{\partial w_{cl}}{\partial \eta}(\xi_{cl},\eta_{cl})\frac{\partial^{2} w^{*}}{\partial x \partial \eta}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})\Big] ,i = 1,2,..,L \quad (22)$$

$$\frac{\partial w(x_{ci},y_{ci})}{\partial y} = \int_{\Gamma} \Big[\frac{\partial \overset{*}{M}_n(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial y}\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n}$$

$$- \frac{\partial \overset{*}{V}_n(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial y} w(\bar{\xi},\bar{\eta}) \Big] d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} \frac{\partial \overset{*}{R}_k(x_{ci},y_{ci};\bar{\xi}_k,\bar{\eta}_k)}{\partial y} w(\bar{\xi}_k,\bar{\eta}_k) + \int_{\Omega} q(\xi,\eta)\frac{\partial \overset{*}{w}}{\partial y}(x_{ci},y_{ci};\xi,\eta)\, d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} \Big[- K_{al} w_{cl}(\xi_{cl},\eta_{cl}) \frac{\partial \overset{*}{w}}{\partial y}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})$$

$$- K^{y}_{rl} \frac{\partial w_{cl}}{\partial \xi}(\xi_{cl},\eta_{cl}) \frac{\partial^2 \overset{*}{w}}{\partial y \partial \xi}(x_{ci},y_{ci};\xi_{cl},\eta_{cl})$$

$$- K^{x}_{rl} \frac{\partial w_{cl}}{\partial \eta}(\xi_{cl},\eta_{cl}) \frac{\partial^2 \overset{*}{w}}{\partial y \partial \eta}(x_{ci},y_{ci};\xi_{cl},\eta_{cl}) \Big] \quad ,i = 1,2,..,L \qquad (23)$$

Equations (19),(20),(21),(22) and (23) constitute (2+3L) Fredholm integral equations of the second kind in two unknown functions w , $\partial w/\partial n$ which are continuous functions throughout the boundary, K unknown values of w_k, $k = 1,2,3,\ldots,K$ at each corners and 3L unknown values of w_{cl}, $\partial w_{cl}/\partial\xi$, $\partial w_{cl}/\partial\eta$, $l = 1,2,3,\ldots,L$ inside the domain which will be solved numerically as to be elaborated in the next chapter.