CHAPTER II

CERTAIN PROPERTIES OF GENERALIZED TRANSFORMATION SEMIGROUPS

The properties of the generalized transformation semigroups $(\mathcal{T}(X,Y),\theta)$, $(\mathcal{PT}(X,Y),\theta)$ and $(\mathcal{I}(X,Y),\theta)$, presented here, will be used for the next chapter. All of their proofs require only basic knowledge of mappings and cardinalities of sets.

Proposition 2.1. *Let X and Y be sets and $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$. Let $\theta \in \mathcal{S}(Y,X)$ be such that $|\nabla\theta| < \min\{|X|,|Y|\}$.*

(i) *If X or Y is finite, then for every $\alpha \in \mathcal{S}(X,Y)$, $\nabla\theta\alpha = \nabla\alpha$ implies that there exists $\beta \in \mathcal{S}(X,Y)$ such that $|\nabla\beta| > |\nabla\alpha|$ and $\theta\beta = \theta\alpha$.*

(ii) *If X and Y are infinite, then for every $\alpha \in \mathcal{S}(X,Y)$ there exists $\beta \in \mathcal{S}(X,Y)$ such that $|\nabla\beta| = \min\{|X|,|Y|\}$ and $\theta\beta = \theta\alpha$.*

Proof. (i) Since $|\nabla\alpha| = |\nabla\theta\alpha| \leq |\nabla\theta| < \min\{|X|,|Y|\}$, it follows that $\nabla\theta \subsetneq X$ and $\nabla\alpha \subsetneq Y$. From the fact that X or Y is finite, we have that $\nabla\theta$ and $\nabla\alpha$ are both finite. Let $x_0 \in X \smallsetminus \nabla\theta$ and $y_0 \in Y \smallsetminus \nabla\alpha$. Define $\beta\colon \Delta\alpha \cup \{x_0\} \to Y$ by

$$x\beta = \begin{cases} y_0 & \text{if } x = x_0, \\ x\alpha & \text{if } x \in \Delta\alpha \smallsetminus \{x_0\}. \end{cases}$$

It is clear that $\beta \in \mathcal{S}(X,Y)$ for the case that $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$ or $\mathcal{PT}(X,Y)$. And it is also true for the case that $\mathcal{S}(X,Y) = \mathcal{I}(X,Y)$ since $y_0 \notin \nabla\alpha$. The equality $\theta\beta = \theta\alpha$ holds because $x_0 \notin \nabla\theta$. Claim that $\nabla\beta = \nabla\alpha \cup \{y_0\}$, which implies that $|\nabla\beta| > |\nabla\alpha|$ since $\nabla\alpha$ is finite. By the definition of β, $\nabla\beta = (\Delta\alpha \setminus \{x_0\})\,\alpha \cup \{y_0\}$. If $x_0 \notin \Delta\alpha$, we have that $\nabla\beta = \nabla\alpha \cup \{y_0\}$. Assume that $x_0 \in \Delta\alpha$. Then $x_0\alpha \in \nabla\alpha = \nabla\theta\alpha = (\nabla\theta \cap \Delta\alpha)\alpha$, so $x_0\alpha = z\alpha$ for some $z \in \nabla\theta \cap \Delta\alpha$. But $x_0 \notin \nabla\theta$, so $z \in \Delta\alpha \setminus \{x_0\}$. These imply that

$$\begin{aligned}
\nabla\alpha &= (\Delta\alpha)\alpha \\
&= (\Delta\alpha \setminus \{x_0\})\alpha \cup \{x_0\alpha\} \\
&= (\Delta\alpha \setminus \{x_0\})\alpha \cup \{z\alpha\} \quad (\text{since } x_0\alpha = z\alpha) \\
&= ((\Delta\alpha \setminus \{x_0\}) \cup \{z\})\alpha \\
&= ((\Delta\alpha \setminus \{x_0\})\alpha \quad (\text{since } z \in \Delta\alpha \setminus \{x_0\}).
\end{aligned}$$

Hence $\nabla\beta = (\Delta\alpha \setminus \{x_0\})\alpha \cup \{y_0\} = \nabla\alpha \cup \{y_0\}$, so we have the claim.

(ii) We have by the assumption that $|\nabla\theta| < |X|$ and $|\nabla\theta| < |Y|$. Since $|\nabla\theta\alpha| \leq |\nabla\theta|$, we have that $|\nabla\theta\alpha| < |Y|$. Since X and Y are infinite, it follows that $|X \setminus \nabla\theta| = |X|$ and $|Y \setminus \nabla\theta\alpha| = |Y|$.

Case 1: $|X| \leq |Y|$. Then $|X \setminus \nabla\theta| \leq |Y \setminus \nabla\theta\alpha|$, and so there exists $\lambda \in \mathcal{I}(X,Y)$ with $\Delta\lambda = X \setminus \nabla\theta$ and $\nabla\lambda \subseteq Y \setminus \nabla\theta\alpha$. Then $|\Delta\lambda| = |\nabla\lambda|$. Define $\beta\colon \Delta\lambda \cup (\nabla\theta \cap \Delta\alpha) \to Y$ by $\beta|_{\Delta\lambda} = \lambda$ and $\beta|_{\nabla\theta\cap\Delta\alpha} = \alpha|_{\nabla\theta\cap\Delta\alpha}$. Then $\beta \in \mathcal{S}(X,Y)$ for the case that $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$ or $\mathcal{PT}(X,Y)$. Since $(\Delta\lambda)\beta = (\Delta\lambda)\lambda = \nabla\lambda \subseteq Y \setminus \nabla\theta\alpha$ and $(\nabla\theta \cap \Delta\alpha)\beta = (\nabla\theta \cap \Delta\alpha)\alpha = \nabla\theta\alpha$, we have $(\Delta\lambda)\beta \cap (\nabla\theta \cap \Delta\alpha)\beta = \varnothing$. This implies that $\beta \in \mathcal{S}(X,Y)$ for the case that $\mathcal{S}(X,Y) = \mathcal{I}(X,Y)$. Also $|\nabla\beta| \geq |\nabla\lambda| = |\Delta\lambda| = |X \setminus \nabla\theta| = |X|$. But $|\nabla\beta| \leq |X|$, so $|\nabla\beta| = |X| = \min\{|X|, |Y|\}$. We have that $\theta\beta = \theta\alpha$ from the following equalities:

$$\begin{aligned}
\Delta\theta\beta &= (\nabla\theta \cap \Delta\beta)\theta^{-1} \\
&= [\nabla\theta \cap (\Delta\lambda \cup (\nabla\theta \cap \Delta\alpha))]\,\theta^{-1}
\end{aligned}$$

$= [(\nabla\theta \cap \Delta\lambda) \cup (\nabla\theta \cap \Delta\alpha)]\,\theta^{-1}$

$= (\nabla\theta \cap \Delta\alpha)\theta^{-1} \quad (\text{since } \Delta\lambda = X \smallsetminus \nabla\theta)$

$= \Delta\theta\alpha$

and for $x \in \Delta\theta\beta$ $(= \Delta\theta\alpha)$,

$x\theta\beta = (x\theta)\beta$

$= (x\theta)\alpha \quad (\text{since } x\theta \in \nabla\theta \cap \Delta\alpha \text{ and } \beta|_{\nabla\theta\cap\Delta\alpha} = \alpha|_{\nabla\theta\cap\Delta\alpha})$

$= x\theta\alpha.$

Case 2: $|X| > |Y|$. Then $|X \smallsetminus \nabla\theta| > |Y \smallsetminus \nabla\theta\alpha|$. Thus there exists $\lambda \in \mathcal{I}(X, Y)$ with $\Delta\lambda \subseteq X \smallsetminus \nabla\theta$ and $\nabla\lambda = Y \smallsetminus \nabla\theta\alpha$.

Subcase 2.1: $\mathcal{S}(X, Y) = \mathcal{T}(X, Y)$ or $\mathcal{PT}(X, Y)$. Define $\beta\colon \Delta\lambda \cup ((X \smallsetminus \Delta\lambda) \cap \Delta\alpha) \to Y$ by $\beta|_{\Delta\lambda} = \lambda$ and $\beta|_{(X\smallsetminus\Delta\lambda)\cap\Delta\alpha} = \alpha|_{(X\smallsetminus\Delta\lambda)\cap\Delta\alpha}$. Thus $\beta \in \mathcal{S}(X, Y)$. Since $|\nabla\beta| \geq |(\Delta\lambda)\beta| = |(\Delta\lambda)\lambda| = |\nabla\lambda| = |Y \smallsetminus \nabla\theta\alpha| = |Y|$, it follows that $|\nabla\beta| = \min\{|X|, |Y|\}$. The following equalities yield $\theta\beta = \theta\alpha$:

$\Delta\theta\beta = (\nabla\theta \cap \Delta\beta)\theta^{-1}$

$= [\nabla\theta \cap (\Delta\lambda \cup ((X \smallsetminus \Delta\lambda) \cap \Delta\alpha))]\,\theta^{-1}$

$= [(\nabla\theta \cap \Delta\lambda) \cup (\nabla\theta \cap (X \smallsetminus \Delta\lambda) \cap \Delta\alpha)]\,\theta^{-1}$

$= [\nabla\theta \cap (X \smallsetminus \Delta\lambda) \cap \Delta\alpha)]\,\theta^{-1} \quad (\text{since } \Delta\lambda \subseteq X \smallsetminus \nabla\theta)$

$= (\nabla\theta \cap \Delta\alpha)\theta^{-1} \quad (\text{since } \Delta\lambda \subseteq X \smallsetminus \nabla\theta \text{ implies} \nabla\theta \subseteq X \smallsetminus \Delta\lambda)$

$= \Delta\theta\alpha$

and for $x \in \Delta\theta\beta$ $(= \Delta\theta\alpha)$,

$x\theta\beta = (x\theta)\beta$

$= (x\theta)\alpha \quad (\text{ since } x\theta \in \nabla\theta \cap \Delta\alpha \subseteq (X \smallsetminus \Delta\lambda) \cap \Delta\alpha$

$\text{and } \beta|_{(X\smallsetminus\Delta\lambda)\cap\Delta\alpha} = \alpha|_{(X\smallsetminus\Delta\lambda)\cap\Delta\alpha})$

$= x\theta\alpha.$

Subcase 2.2: $\mathcal{S}(X,Y)=\mathcal{I}(X,Y)$. Since $\Delta\lambda \subseteq X \setminus \nabla\theta$, we have that $\Delta\lambda \cap (\nabla\theta \cap \Delta\alpha) = \varnothing$. Define $\beta\colon \Delta\lambda \cup (\nabla\theta \cap \Delta\alpha) \to Y$ by $\beta|_{\Delta\lambda} = \lambda$ and $\beta|_{\nabla\theta\cap\Delta\alpha} = \alpha|_{\nabla\theta\cap\Delta\alpha}$. Since $\lambda, \alpha \in \mathcal{I}(X,Y)$ and $(\Delta\lambda)\beta \cap (\nabla\theta \cap \Delta\alpha)\beta = (\Delta\lambda)\lambda \cap (\nabla\theta \cap \Delta\alpha)\alpha = \nabla\lambda \cap \nabla\theta\alpha = (Y \setminus \nabla\theta\alpha) \cap \nabla\theta\alpha = \varnothing$, it follows that $\beta \in \mathcal{I}(X,Y)$. Since $\nabla\lambda \subseteq \nabla\beta$ and $|\nabla\lambda| = |Y \setminus \nabla\theta\alpha| = |Y|$, we have that $|\nabla\beta| = |Y| = \min\{|X|,|Y|\}$. We get that $\theta\beta = \theta\alpha$ because of the following equalities:

$$\begin{aligned}
\Delta\theta\beta &= (\nabla\theta \cap \Delta\beta)\theta^{-1} \\
&= [\nabla\theta \cap (\Delta\lambda \cup (\nabla\theta \cap \Delta\alpha))]\,\theta^{-1} \\
&= [(\nabla\theta \cap \Delta\lambda) \cup (\nabla\theta \cap \Delta\alpha)]\,\theta^{-1} \\
&= (\nabla\theta \cap \Delta\alpha)\theta^{-1} \quad (\text{since } \Delta\lambda \subseteq X \setminus \nabla\theta) \\
&= \Delta\theta\alpha
\end{aligned}$$

and for $x \in \Delta\theta\beta$ $(= \Delta\theta\alpha)$,

$$\begin{aligned}
x\theta\beta &= (x\theta)\beta \\
&= (x\theta)\alpha \quad (\text{since } x\theta \in \nabla\theta \cap \Delta\alpha \text{ and } \beta|_{\nabla\theta\cap\Delta\alpha} = \alpha|_{\nabla\theta\cap\Delta\alpha}) \\
&= x\theta\alpha.
\end{aligned}$$

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Lemma 2.2. *Let X and Y be sets and (S,θ) a generalized transformation semigroup of X into Y. Assume that $A \subseteq \nabla\theta$ and for each $a \in A$, let $y_a \in a\theta^{-1}$. Let $\mathbf{c}$ be an infinite cardinal number and let*

$$U = \{\alpha \in S \mid |A\alpha \cap (Y \setminus \{y_a \mid a \in A\})| < \mathbf{c}\}.$$

Then for all $\alpha, \beta \in U$, $\alpha\theta\beta \in U$, that is, if $U \neq \varnothing$, then U is a subsemigroup of (S,θ).

Proof. First we note that $\{y_a \mid a \in A\}\theta = A$. Let $\alpha, \beta \in U$. Then $|A\alpha \cap (Y \setminus \{y_a \mid a \in A\})\,| < \mathbf{c}$ and $|A\beta \cap (Y \setminus \{y_a \mid a \in A\})\,| < \mathbf{c}$. To prove that $\alpha\theta\beta \in U$, we consider the following equalities and inclusions:

$A(\alpha\theta\beta) \cap (Y \setminus \{y_a \mid a \in A\})$

$= (A\alpha)\theta\beta \cap (Y \setminus \{y_a \mid a \in A\})$

$= [(A\alpha \cap (Y \setminus \{y_a \mid a \in A\})) \cup (A\alpha \cap \{y_a \mid a \in A\})]\,\theta\beta \cap (Y \setminus \{y_a \mid a \in A\})$

$\subseteq [(A\alpha \cap (Y \setminus \{y_a \mid a \in A\})) \cup \{y_a \mid a \in A\}]\,\theta\beta \cap (Y \setminus \{y_a \mid a \in A\})$

$\subseteq [A\alpha \cap (Y \setminus \{y_a \mid a \in A\})]\,\theta\beta \cup [\{y_a \mid a \in A\}\theta\beta \cap (Y \setminus \{y_a \mid a \in A\})]$

$= [A\alpha \cap (Y \setminus \{y_a \mid a \in A\})]\,\theta\beta \cup [A\beta \cap (Y \setminus \{y_a \mid a \in A\})]\,.$

Since $|A\alpha \cap (Y \setminus \{y_a \mid a \in A\})\,| < \mathbf{c}$ and $|A\beta \cap (Y \setminus \{y_a \mid a \in A\})\,| < \mathbf{c}$, it follows that $|A(\alpha\theta\beta) \cap (Y \setminus \{y_a \mid a \in A\})\,| < \mathbf{c}$. Hence $\alpha\theta\beta \in U$, as required. #

Proposition 2.3. *Let X and Y be infinite sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and let $\theta \in \mathcal{S}(Y,X)$ be such that $\nabla\theta$ is infinite. Assume that $A \subseteq \nabla\theta$ such that A is infinite and $|\nabla\theta \setminus A| = |\nabla\theta|$. For each $a \in A$, let $y_a \in a\theta^{-1}$. Let*

$$U = \left\{\alpha \in \mathcal{S}(X,Y) \;\middle|\; |A\alpha \cap (Y \setminus \{y_a \mid a \in A\})\,| < |A|\right\}.$$

Then U is a proper subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

Proof. By Lemma 2.2, it suffices to show that $U \neq \varnothing$ and $\mathcal{S}(X,Y) \setminus U \neq \varnothing$. We have by the definition of U that U contains every $\alpha \in \mathcal{S}(X,Y)$ with $|\nabla\alpha| < \infty$ since A is infinite. Then $U \neq \varnothing$. It follows from the assumption that $|A| \leq |\nabla\theta| = |\nabla\theta \setminus A|$. But $|\nabla\theta \setminus A| \leq |(\nabla\theta \setminus A)\theta^{-1}| = |\Delta\theta \setminus A\theta^{-1}| \leq |Y \setminus \{y_a | a \in A\}|$ since $\Delta\theta \subseteq Y$ and $\{y_a \mid a \in A\} \subseteq A\theta^{-1}$, so $|A| \leq |Y \setminus \{y_a \mid a \in A\}|$. Then there exists $\lambda \in \mathcal{I}(X,Y)$ such that $\Delta\lambda = A$ and $\nabla\lambda \subseteq Y \setminus \{y_a \mid a \in A\}$, and thus

$|A\lambda| = |A|$. Let $\lambda' \in \mathcal{T}(X,Y)$ be an extension of λ. Then $A\lambda' = A\lambda = \nabla\lambda \subseteq Y \setminus \{y_a | a \in A\}$. Therefore $|A\lambda' \cap (Y \setminus \{y_a | a \in A\})| = |A\lambda \cap (Y \setminus \{y_a \mid a \in A\})| = |A\lambda| = |A|$. Hence $\lambda' \in \mathcal{S}(X,Y) \setminus U$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$, and for the case that $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, we have $\lambda \in \mathcal{S}(X,Y) \setminus U$. This proves that $\mathcal{T}(X,Y) \setminus U \neq \varnothing$. #

Proposition 2.4. *Let X and Y be sets and $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$. Then the following statements hold:*

(i) *If $\alpha \in \mathcal{S}(X,Y)$ and $\beta \in \mathcal{S}(X,X)$ are such that $\Delta\alpha \subseteq \Delta\beta$ and for every $x \in \Delta\alpha$, $(x\alpha)\alpha^{-1} = (x\beta)\beta^{-1} \cap \Delta\alpha$, then there exists $\gamma \in \mathcal{S}(X,Y)$ such that $(\beta\gamma)|_{\Delta\alpha} = \alpha$.*

(ii) *If $\alpha \in \mathcal{S}(X,Y)$ and $\beta \in \mathcal{S}(Y,Y)$ are such that $\nabla\alpha \subseteq \nabla\beta$, then there exists $\gamma \in \mathcal{S}(X,Y)$ such that $\gamma\beta = \alpha$.*

Proof. (i) Define $\gamma_1 \colon (\Delta\alpha)\beta \to \nabla\alpha$ by $(x\beta)\gamma_1 = x\alpha$ for all $x \in \Delta\alpha$. Since $(x\alpha)\alpha^{-1} = (x\beta)\beta^{-1} \cap \Delta\alpha$ for all $x \in \Delta\alpha$, γ_1 is well-defined. Let $\gamma_2 \colon X \to Y$ be an extension of γ_1 and let $\gamma = \gamma_1$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\gamma = \gamma_2$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$. Then $\gamma \in \mathcal{S}(X,Y)$ and for each $x \in \Delta\alpha$, $x\alpha = (x\beta)\gamma_1 = (x\beta)\gamma = x\beta\gamma$. Hence $(\beta\gamma)|_{\Delta\alpha} = \alpha$.

(ii) Since $\nabla\alpha \subseteq \nabla\beta$, we have that $y\beta^{-1} \neq \varnothing$ for all $y \in \nabla\alpha$. From the fact that $\Delta\alpha = \bigcup\limits_{y \in \nabla\alpha} y\alpha^{-1}$, $y_1\alpha^{-1} \cap y_2\alpha^{-1} = \varnothing$ and $y_1\beta^{-1} \cap y_2\beta^{-1} = \varnothing$ for all $y_1, y_2 \in \nabla\alpha, y_1 \neq y_2$, it follows that there exists $\gamma \colon \Delta\alpha \to Y$ such that $(y\alpha^{-1})\gamma \subseteq y\beta^{-1}$ for all $y \in \nabla\alpha$. Then $\Delta\gamma = \Delta\alpha, \nabla\gamma \subseteq \Delta\beta$ and $((x\alpha)\alpha^{-1})\gamma \subseteq (x\alpha)\beta^{-1}$ for all $x \in \Delta\alpha$. It is clear that $\gamma \in \mathcal{S}(X,Y)$ for every case of $\mathcal{S}(X,Y)$. We have that $\gamma\beta = \alpha$ because of the following equalities and inclusions: $\Delta\gamma\beta = (\nabla\gamma \cap \Delta\beta)\gamma^{-1} = (\nabla\gamma)\gamma^{-1} = \Delta\gamma = \Delta\alpha$ and for every $x \in \Delta\alpha, x(\gamma\beta) \in \left((x\alpha)\alpha^{-1}\right)(\gamma\beta) = \left(\left((x\alpha)\alpha^{-1}\right)\gamma\right)\beta \subseteq \left((x\alpha)\beta^{-1}\right)\beta = \{x\alpha\}$. #