# CHAPTER II

## PARANORMED AND SEMINORMED SPACES OVER THE QUATERNIONS

Definition 2.1 Let X be a vector space over $\mathbb{H}$. A map $\|.\| : X \to \mathbb{R}$ is said to be a paranorm on X if and only if $\|.\|$ satisfies the following properties :

(PN1) $\|0\| = 0$

(PN2) For all $x \in X$, $\|-x\| = \|x\|$

(PN3) For all $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$

(PN4) If $(t_n)_{n \in \mathbb{N}}$ is a sequence of elements in $\mathbb{H}$ such that $t_n \to t$ and $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements in X such that $\|x_n - x\| \to 0$ then $\|t_n x_n - tx\| \to 0$ (Continuity of multiplication).

A paranorm is called total if and only if we have

(PN5) $\|x\| = 0$ implies that $x = 0$.

Definition 2.2 Let X be a vector space over $\mathbb{H}$. A map $\|.\| : X \to \mathbb{R}$ is said to be a seminorm on X if and only if $\|.\|$ satisfies the following properties :

(SN1) For all $x \in X$ and $t \in \mathbb{H}$, $\|tx\| = |t| \, \|x\|$

(SN2) For all $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$

A seminorm is called a norm if and only if it is total.

Example 2.3 Let $n \in \mathbb{N}$. Define $p_n : \mathbb{H} \to \mathbb{R}$ by $p_n(x) = |x|^{\frac{1}{n}}$. Then $p_n$ is a paranorm on $\mathbb{H}$ for each $n \in \mathbb{N}$ ; but $p_n$ is not a seminorm on $\mathbb{H}$ for $n \geq 2$.

<u>Proof</u> : Let $n \in \mathbb{N}$. It is clear that $p_n$ satisfies (PN1) and (PN2). Let $x, y \in \mathbb{H}$. Then $p_n(x+y) = |x+y|^{\frac{1}{n}} \leq |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}} = p_n(x) + p_n(y)$ Otherwise, $|x+y| > |x| + |y| + \sum_{r=1}^{n-1} \binom{n}{r} (|x|)^{\frac{n-r}{n}} (|y|)^{\frac{r}{n}}$, a contradiction. So $p_n$ satisfies (PN3).

Let $(t_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ such that $t_k \to t$ for some $t \in \mathbb{H}$ and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in X such that $p_n(x_k - x) \to 0$ for some $x \in \mathbb{H}$. Then $p_n(t_k x_k - tx) = |t_k x_k - tx|^{\frac{1}{n}} = |(t_k - t)(x_k - x) + (t_k - t)x + t(x_k - x)|^{\frac{1}{n}} \leq |t_k - t|^{\frac{1}{n}} |x_k - x|^{\frac{1}{n}} + |t_k - t|^{\frac{1}{n}} |x|^{\frac{1}{n}} + |t|^{\frac{1}{n}} |x_k - x|^{\frac{1}{n}} \to 0$ as $n \to \infty$. Hence $p_n$ satisfies (PN4). Thus $p_n$ is a paranorm on $\mathbb{H}$ for each $n \in \mathbb{N}$. It is clear that $p_n$ is not a seminorm on $\mathbb{H}$ for each $n \geq 2$. #

<u>Remark</u> Every seminorm is a paranorm.

<u>Proof</u> : Let $\|.\|$ be a seminorm w.r.t a vector space X over $\mathbb{H}$. Then (PN1), (PN2) and (PN3) are obviously true. Let $x, y \in X$ and let $\varepsilon, t \in \mathbb{H}$. Then $\|(t+\varepsilon)(x+y)-tx\| = \|ty + \varepsilon x + \varepsilon y\| \leq |t| \|y\| + |\varepsilon| \|x\| + |\varepsilon| \|y\|$. If $\varepsilon \to 0$ and $\|y\| \to 0$ then $\|(t+\varepsilon)(x+y)-tx\| \to 0$. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ such that $t_n \to t$ for some $t \in \mathbb{H}$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $\|x_n - x\| \to 0$. Since $t_n - t \to 0$ and $\|x_n - x\| \to 0$ as $n \to \infty$ then $\|t_n x_n - tx\| = \|((t_n - t) + t)((x_n - x) + x) - tx\| \to 0$. Thus $\|.\|$ is a paranorm on X. #

<u>Definition 2.4</u> <u>A paranormed space over $\mathbb{H}$</u> is a pair $(X, \|.\|)$ where X is a vector space over $\mathbb{H}$ and $\|.\|$ is a paranorm on X.

<u>A seminormed space over $\mathbb{H}$</u> is a pair $(X, \|.\|)$ where X is a vector space over $\mathbb{H}$ and $\|.\|$ is a seminorm on X.

<u>Theorem 2.5</u> Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of paranorms on a vector space X over $\mathbb{H}$. Then there exists a paranorm $\|.\|$ on X such that for any net $(x_\delta)_{\delta \in D}$ in X, $\|x_\delta\| \to 0$ if and only if $p_n(x_\delta) \to 0$ for each $n \in \mathbb{N}$.

<u>Proof</u> : Define $\|.\| : X \to \mathbb{R}$ by $\|x\| = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1+p_n(x)}$, $x \in X$.

It is clear that $\|.\|$ satisfies (PN1) and (PN2).

Let $x, y \in X$. We must show that $\|x+y\| \leq \|x\| + \|y\|$. Consider $f(t) = \frac{1}{1+t}$ $= 1 - \frac{1}{1+t}$, $t \geq 0$. Then $f'(t) = \frac{1}{(1+t)^2} > 0$ for all $t \geq 0$. Then f is an increasing function; so $\frac{p_n(x+y)}{1+p_n(x+y)} \leq \frac{p_n(x)+p_n(y)}{1+p_n(x)+p_n(y)}$ for each $n \in \mathbb{N}$. Since $\frac{p_n(x)+p_n(y)}{1+p_n(x)+p_n(y)} - \frac{p_n(x)}{1+p_n(x)} = \frac{p_n(y)}{(1+p_n(x))(1+p_n(x)+p_n(y))}$ $\leq \frac{p_n(y)}{1+p_n(y)}$ for each $n \in \mathbb{N}$, $\frac{p_n(x)+p_n(y)}{1+p_n(x)+p_n(y)} \leq \frac{p_n(x)}{1+p_n(x)} + \frac{p_n(y)}{1+p_n(y)}$ for each $n \in \mathbb{N}$. Then $\|x+y\| = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x+y)}{1+p_n(x+y)} \leq \sum_{n=1}^{\infty} \frac{p_n(x)+p_n(y)}{1+p_n(x)+p_n(y)} . 2^{-n}$ $\leq \sum_{n=1}^{\infty} \frac{p_n(x)\, 2^{-n}}{1+p_n(x)} + \sum_{n=1}^{\infty} \frac{p_n(y) . 2^{-n}}{1+p_n(y)} = \|x\| + \|y\|$. Hence $\|.\|$ satisfies (PN3).

Claim that for any net $(x_\delta)_{\delta \in D}$ in X, $\|x_\delta\| \to 0$ if and only if $p_n(x_\delta) \to 0$ for each $n \in \mathbb{N}$.

Let $(x_\delta)_{\delta \in D}$ be a net in X. Suppose $\| x_\delta \| \to 0$. Let $n \in \mathbb{N}$ and $\delta \in D$. Then

$\| x_\delta \| = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x_\delta)}{1+p_k(x_\delta)} \geq 2^{-n} \frac{p_n(x_\delta)}{1+p_n(x_\delta)}$, therefore $p_n(x_\delta)$

$\leq \frac{2^n \|x_\delta\|}{1-2^n \|x_\delta\|}$. Since $\| x_\delta \| \to 0$ and $p_n(x_\delta) \leq \frac{2^n \| x_\delta \|}{1-2^n \| x_\delta \|}$ for all $\delta \in D$,

$p_n(x_\delta) \to 0$. Since $n \in \mathbb{N}$ was arbitrary, $p_n(x_\delta) \to 0$ for all $n \in \mathbb{N}$. Conversely, suppose that $p_n(x_\delta) \to 0$ for all $n \in \mathbb{N}$. We must show that $\| x_\delta \| \to 0$ that is for all $\varepsilon > 0$ there exists a $\delta \in D$ such that $\delta > \delta_0$ implies that $\|x_\delta\| < \varepsilon$. Let $\varepsilon > 0$ be given. Since

$\sum_{n=m}^{\infty} \frac{1}{2^n} \to 0$ as $m \to \infty$, there exists an $M \in \mathbb{N}$ such that $m > M$ implies that

$\sum_{n=m}^{\infty} \frac{1}{2^n} < \varepsilon/_2$. Since $p_n(x_\delta) \to 0$ for all $n \in \mathbb{N}$, there exists a $\delta_i \in D$ such that $\delta > \delta_i$ implies that $p_i(x_\delta) < \frac{\varepsilon}{2M}$, $i = 1,2,\ldots,M$. Let $\delta_0 \geq$

$\max\{\delta_1,\delta_2,\ldots,\delta_M\}$. Let $\delta > \delta_0$. Then $\|x_\delta\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x_\delta)}{1+p_n(x_\delta)}$

$= \sum_{n=1}^{M} \frac{1}{2^n} \frac{p_n(x_\delta)}{1+p_n(x_\delta)} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} \frac{p_n(x_\delta)}{1+p_n(x_\delta)} < \frac{\varepsilon/_{2M} + \varepsilon/_{2M} + \ldots + \varepsilon/_{2M}}{M \text{ times}}$

$+ \varepsilon/_2 = \varepsilon/_2 + \varepsilon/_2 = \varepsilon$. Hence $\|x_\delta\| \to 0$. So we have the claim. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{H}$ such that $t_n \to t$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $\|x_n - x\| \to 0$. We must show that $\|t_n x_n - tx\| \to 0$. Since $\|x_n - x\| \to 0$, by the claim, $p_k(x_n - x) \to 0$ for each $k \in \mathbb{N}$. Since $p_k$ is a paranorm for each $k \in \mathbb{N}$, $t_n \to t$ and $p_k(x_n - x) \to 0$ for each $k \in \mathbb{N}$, we get that $p_k(t_n x_n - tx) \to 0$ for each $k \in \mathbb{N}$. Hence $\|t_n x_n - tx\|$

$= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(t_n x_n - tx)}{1+p_k(t_n x_n - tx)} \to 0$. So $\|.\|$ satisfies (PN4). Thus $\|.\|$ is a paranorm on X such that for any net $(x_\delta)_{\delta \in D}$ in X, $\|x_\delta\| \to 0$ if and only if $p_n(x_\delta) \to 0$ for each $n \in \mathbb{N}$.

<u>Theorem 2.6</u> Let A be a balanced convex absorbing set in a vector space X over $\mathbb{H}$. Then there exists a unique seminorm $\|.\|$ on X such that $\{x \mid \|x\| < 1\} \subseteq A \subseteq \{x \mid \|x\| \leq 1\}$. The seminorm is a norm if and only is A does not contain a one dimensional vector subspace.

<u>Proof</u> : Define $\|.\| : X \to \mathbb{R}$ by $\|x\| = \inf\{t > 0 \mid x \in tA\}$. Since A is absorbing, for each $x \in X$, there exists a $t_0 > 0$ such that $x \in t_0A$; so $\{t > 0 \mid x \in tA\} \neq \emptyset$. Hence $\|.\|$ is well-defined. We must show that $\|.\|$ is a seminorm on X. Let $x \in X$ and $t \in \mathbb{H}$. Must show that $\|tx\| = |t|\,\|x\|$.

<u>Case 1</u> $t = 0$. The result is obvious.

<u>Case 2</u> $t \neq 0$. Let $s > 0$ be such that $\|x\| < s$. Then there exists a $t_0 \in \{t > 0 \mid x \in tA\}$ such that $\|x\| < t_0 < s$. Since $\frac{x}{t_0} \in A$, $\frac{t_0}{s} < 1$ and A is balanced, $\frac{x}{s} = (\frac{t_0}{s})(\frac{x}{t_0}) \in A$. Then $x \in sA$. So $tx \in stA$. Claim that $\ell A = |\ell| A$. If $\ell = 0$. The result is obvious. Assume $\ell \neq 0$. Let $z \in \ell A$. Since A is balanced and $|\frac{\ell}{|\ell|}| = 1$, $\frac{z}{|\ell|} = (\frac{\ell}{|\ell|})(\frac{z}{\ell}) \in A$. Hence $z \in |\ell| A$. So $\ell A \subseteq |\ell| A$. Similarly, we can show that $|\ell| A \subseteq \ell A$, so we have the claim. Thus $tx \in stA = s|t|A$. So $|tx| \leq s|t|$ for all $s > 0$ such that $\|x\| < s$. Hence $\|tx\| \leq |t|\,\|x\|$. Suppose that $\|tx\| < |t|\,\|x\|$. Then there exists an $s > 0$ such that $\|tx\| < s < |t|\,\|x\|$. As a result, $tx \in sA$, hence $x \in \frac{s}{t} A$. Since $s < |t|\,\|x\|$, $\frac{s}{|t|} < \|x\|$, so $x \notin \frac{s}{|t|} A$, a contradiction. Hence $\|tx\| = |t|\,\|x\|$. So $\|.\|$ satisfies (SN1).

Let $x, y \in X$. We must show that $\|x + y\| \leq \|x\| + \|y\|$. Let $s, t \in \mathbb{R}^+$ be such that $\|x\| < s$ and $\|y\| < t$. Then $x \in sA$ and $y \in tA$. So $x + y \in sA + tA$. Claim that $sA + tA = (s + t)A$. Clearly $(s+t)A \subseteq sA + tA$. Then $z = sa_1 + ta_2$ for some $a_1, a_2 \in A$.

But $z = sa_1 + ta_2 = (\frac{s}{s+t})(s+t)a_1 + (\frac{t}{s+t})(s+t)a_2$

$$= (s+t)[\frac{sa_1}{s+t} + \frac{ta_2}{s+t}]$$

Since A is convex, $\frac{sa_1}{s+t} + \frac{ta_2}{s+t} \in A$. Thus $z \in (s+t)A$. So we have the claim. Thus $x+y \in sA + tA = (s+t)A$, so $\|x+y\| \leq s+t$ for all s, $t \in \mathbb{R}^+$ such that $\|x\| < s$ and $\|y\| < t$. Hence $\|x+y\| \leq \|x\| + \|y\|$ and so $\|\cdot\|$ satisfies (SN2), therefore $\|.\|$ is a seminorm on X.

We shall now show that $\{x \mid \|x\| < 1\} \subseteq A \subseteq \{x \mid \|x\| \leq 1\}$. Let $x \in \{x \mid \|x\| < 1\}$. Then $\|x\| < 1$. We must show that $x \in A$. Since $\|x\| < 1$, $\inf\{t > 0 \mid x \in tA\} < 1$. Then there exists a $t_0 > 0$ such that $\|x\| < t_0 < 1$. and $x \in t_0A$. As a result $\frac{x}{t_0} \in A$. Since $|t_0| = t_0 < 1$ and A is balanced, $x = t_0(\frac{x}{t_0}) \in A$. Hence $\{x \mid \|x\| \leq 1\}$ $\subseteq A$. Let $x \in A$. Then $x \in 1.A$, so $1 \in \{t > 0 \mid x \in tA\}$. Hence $\|x\| \leq 1$. So $A \subseteq \{x \mid \|x\| \leq 1\}$.

To show uniqueness, suppose that p is a seminorm on X such that

$$\{x \mid p(x) < 1\} \subseteq A \subseteq \{x \mid p(x) \leq 1\} .$$

We must show that $p = \|.\|$ on X. Suppose that there is an $x_0 \in X$ such that $p(x_0) \neq \|x_0\|$.

<u>Case 1</u> $p(x_0) > \|x_0\|$. Then there exists a $t_0 > 0$ such that $p(x_0) > t > \|x_0\|$. Let $y = \frac{x_0}{t}$. Then $p(\frac{x_0}{t}) = \frac{1}{t}p(x_0) > 1$ and $\|\frac{x_0}{t}\| = \frac{1}{t}\|x_0\| < 1$, a contradiction. (because $\|\frac{x_0}{t}\| < 1$, so $\frac{x_0}{t} \in \{x \mid \|x\| < 1\}$. So $\frac{x_0}{t} \in A$. Hence $\frac{x_0}{t} \in \{x \mid p(x) \leq 1\}$, therefore $p(\frac{x_0}{t}) \leq 1$, a contradiction)

<u>Case 2</u> $p(x_0) < \|x_0\|$. The proof is similar to the proof of case 1.

Hence $p = \|\cdot\|$ on X. We must show that $\|\cdot\|$ is a norm if and only if A dose not contain a one-dimensional vector subspace. Suppose that $\|\cdot\|$ is not a norm. Then there is an $x_0 \in X$ such that $x_0 \neq 0$ and $\| x_0 \| = 0$ Hence $\langle x_0 \rangle$ is a vector subspace of A. Suppose that A has a one dimensional subspace, say B. Let $\{ x_0 \}$ be a base of B. Then $tx_0 \in A$ for all $t \in \mathbb{H}$. Now $|t| \, \|x_0\| = \|tx_0\| \leq 1$ for all $t \in \mathbb{H}$. Hence $\|x_0\| \leq \frac{1}{|t|}$ for all $t \neq 0$, so $\| x_0 \| = 0$, therefore $\|\cdot\|$ is not a norm.

Notation : Let X be a vector space over $\mathbb{H}$ and $A \subseteq X$. Then $A^{\#}$ denote the set of all linear functionals on A.

Theorem 2.7 (Hahn-Banach theorem) Let (X, p) be a seminormed space over $\mathbb{H}$. Let f be a linear functional defined only on a vector subspace S of X and such that $| f(x) | \leq p(x)$ for all $x \in S$. Then f can be extended to $F \in X^{\#}$ with $| F(x) | \leq p(x)$ for all $x \in X$.

Proof : First we shall consider X as a vector space over the field $\mathbb{R}$, S is then a real vector subspace of X and f is a real linear functional defined on S. Let $P = \{ g \mid g \in T_g^{\#}$ for some vector subspace of X such that $T_g \supseteq S$, $g|_S = f$ and $| g(x) | \leq p(x)$ for all $x \in T_g \}$. Since $f \in P$, $P \neq \emptyset$. Partially order P by the relation that $g_1 \leq g_2$ if and only if $g_2$ is an extension of $g_1$.

Let $\{ g_\alpha \}_{\alpha \in I}$ be a chain in P. Then $g_\alpha \in T_{g_\alpha}^{\#}$ for all $\alpha \in I$, where $T_{g_\alpha}$ is a vector subspace of X, $T_{g_\alpha} \supseteq S$ for all $\alpha \in I$, $g_\alpha|_S = f$ and $|g_\alpha(x)| \leq p(x)$ for all $x \in T_{g_\alpha}$. Define $g : \bigcup_{\alpha \in I} T_{g_\alpha} \to \mathbb{R}$ as follows : $x \in \bigcup_{\alpha \in I} T_{g_\alpha}$. Then $x \in T_{g_\alpha}$ for some $\alpha \in I$, let $g(x) = g_\alpha(x)$. Then g is well-defined and $g_\alpha \leq g$ for all $\alpha \in I$. Hence the chain has an upperbound. By Zorn'lemma, $P$ contains a maximal chain. Let $M$ be a maximal chain in $P$. Let $D = \bigcup_{g \in M} T_g$. We must show that $D$ is a vector subspace of $X$. Let $x, y \in D$ and $\alpha, \beta \in \mathbb{R}$. Then $x \in T_g$, $y \in T_{g'}$ for some $g, g' \in M$. Since $M$ is a chain, $g \leq g'$ or $g' \leq g$, say $g' \leq g$. Then $x, y \in T_g$. Since $T_g$ is a vector subspace of $X$, $\alpha x + \beta y \in T_g \subseteq D$. Define $F : D \to \mathbb{R}$ as follows : Let $x \in D$. Then $x \in T_g$ for some $g \in M$. Define $F(x) = g(x)$. We must show that $F$ is $\mathbb{R}$ - linear. Let $x, y \in D$, $\alpha, \beta \in \mathbb{R}$. Since $M$ is chain, $x$, $y$ and $\alpha x + \beta y \in T_g$ for some $g \in M$. Hence $F(\alpha x + \beta y) = g(\alpha x + \beta y) = \alpha g(x) + \beta g(y) = \alpha F(x) + \beta F(y)$, therefore $F$ is $\mathbb{R}$ - linear. Let $x \in D$. Then $x \in T_g$ for some $g \in M$. Then $|F(x)| \leq |g(x)| \leq p(x)$. So we have that $F \in D^{\#}$, $F_{|S} = g_{|S} = f$ and $|F(x)| \leq p(x)$ for all $x \in D$. We must show that $D = X$. Let $y_0 \in X \setminus D$. Let $u = \sup \{ F(x) - p(y_0 - x) \mid x \in D \}$ and $v = \inf \{ F(x) + p(y_0 - x) \mid x \in D \}$.

Claim that $u \leq v$. Let $a, b \in D$. Then $F(b) - F(a) = F(b - a) \leq p(b - a) \leq p(b - y_0) + p(y_0 - a)$. Thus $F(b) - p(y_0 - b) \leq F(a) + p(y_0 - a)$. Since $a$, $b$ were arbitrary, $\sup \{ F(b) - p(y_0 - b) \mid b \in D \}$ is a lower bound of $\{ F(a) + p(y_0 - a) \mid a \in D \}$. Hence $u = \sup \{ F(b) - p(y_0 - b) \mid b \in D \} \leq \inf \{ F(a) + p(y_0 - a) \mid a \in D \} = v$, so we have the claim. Let $D_1 = D + \langle y_0 \rangle$. Define $F_1 \in D_1^{\#}$ as follows : Let $x \in D_1$. Then $x = d + ty_0$ for some $d \in D$ and $t \in \mathbb{R}$, define $F_1(d + ty_0) = F(d) + tu$. We must show that $F_1 \in P$. Since $F_1|_D = F$, $f \leq F \leq F_1$. Let $x \in D_1$. Then $x = d + ty_0$ for some $d \in D$ and $t \in \mathbb{R}$.

<u>Case 1</u> $t = 0$. Then $|F_1(x)| = |F(d)| \leq p(d) = p(x)$.

Case 2 $t \neq 0$. Since $d \in D$ and $D$ a vector subspace of $X$ and $t \neq 0$ then $\frac{-d}{t} \in D$. Hence $F(\frac{-d}{t}) - p(y_0 + \frac{d}{t}) \leq u \leq v \leq F(\frac{-d}{t}) + p(y_0 + \frac{d}{t})$ (*). If $t > 0$, multiply the inequality (*) by $t$. Then we have the inequality $-F(d) - p(ty_0 + d) \leq tu \leq -F(d) + p(ty_0 + d)$. Then $-p(x) = -p(ty_0 + d) \leq F(d) + tu = F_1(x) \leq p(x)$, so $|F_1(x)| \leq p(x)$ in the case where $t > 0$. If $t < 0$, multiply the inequality (*) by $-t$. Then we get that $F(d) - p(-ty_0 - d) \leq -tu \leq F(d) + p(-ty_0 - d)$. Hence $-p(-x) \leq -F(d) - tu = -F_1(x) \leq p(-x)$. Since $p$ is a seminorm on $X$, $-p(x) = -p(-x) \leq -F_1(x) \leq p(-x) = p(x)$. Hence $|F_1(x)| \leq p(x)$ in the case where $t < 0$. As a result, we have that $|F_1(x)| \leq p(x)$ for all $x \in D_1$, so $F_1 \in P$. Hence $F_1 \notin M$, so $M \cup \{F_1\} \supsetneq M$, a contradiction (because $M$ is maximal). Hence $D = X$. So we have that $F \in X^{\#}$ is such that $F|_S = f$ and $|F(x)| \leq p(x)$ for all $x \in X$. We now consider $X$ as a vector space over $\mathbb{H}$, $S$ an $\mathbb{H}$ - vector subspace of $X$ and $f$ an $\mathbb{H}$ - linear functional defined on $S$. Let $g$ be the real part of $f$. Since $|g(x)| \leq |f(x)| \leq p(x)$ for all $x \in S$, we can extend $g$ to an $\mathbb{R}$ - linear functional $G$ on $X$ with $|G(x)| \leq p(x)$ for all $x \in X$.

Define $F : X \to \mathbb{H}$ as follows : For $x \in X$, let $F(x) = G(x) + iI(x) + jJ(x) + kK(x)$ where $I(x) = -G(ix)$, $J(x) = -G(jx)$, $K(x) = -G(kx)$. We must show that $F$ is $\mathbb{H}$ - linear. Clearly, $F(x+y) = F(x) + F(y)$ for all $x, y \in X$. Let $x \in X$ and $\lambda \in \mathbb{H}$. We must show that $F(\lambda x) = \lambda F(x)$. Since $\lambda \in \mathbb{H}$, there exist $a,b,c,d \in \mathbb{R}$ such that $\lambda = a + bi + cj + dk$.

$$\begin{aligned} \text{Then } F(\lambda x) &= F((a+bi+cj+dk)x) \\ &= F(ax+bix+cjx+dkx) \\ &= G(ax+bix+cjx+dkx) + iI(ax+bix+cjx+dkx) \\ &\quad + jJ(ax+bix+cjx+dkx) + kK(ax+bix+cjx+dkx) \end{aligned}$$

$$= aG(x)+bG(ix)+cG(jx)+dG(kx)+i(-G(i(ax+bix+cjx+dkx)))+j(-G(j(ax+bix+cjx+dkx)))+k(-G(k(ax+bix+cjx+dkx))).$$

$$= aG(x)+bG(ix)+cG(jx)+dG(kx)-aiG(ix)+ibG(x)-ciG(kx)+diG(jx)+ja(-G(jx)+bjG(kx)+cjG(x)+dj(-G(ix)+ak(-G(kx))+bk(-G(jx))+ckG(ix)+dkG(x).$$

$$= (a+bi+cj+dk)G(x)+(a+bi+cj+dk)i(-G(ix))+(a+bi+cj+dk)j(-G(jx))+(a+bi+cj+dk)k(-G(kx)).$$

$$= (a+bi+cj+dk)(G(x)+iI(x)+jJ(x)+kK(x))$$

$$= \lambda F(x).$$

Hence $F$ is $\mathbb{H}$- linear on $X$.

We now show that $F$ is unique determined by its real part. Suppose there exists on $F' \in X^{\#}$ such that the real part of $F' = G$. Suppose $F' = G + iI' + jJ' + kK'$. Since $F'$ is $\mathbb{H}$ - linear, $G(ix)+iI'(ix)+jJ'(ix)+kK'(ix) = F'(ix) = iG(x) - I'(x) + kJ'(x) - jK'(x)$. Then $-I'(x) = G(ix)$ ; so $I'(x) = -G(ix)$. Similarly, $J'(x) = -G(jx)$, $K'(x) = -G(kx)$.

Then
$$\begin{aligned} F'(x) &= G(x)\, i I'(x) + jJ'(x) + kK'(x) \\ &= G(x)+i(-G(ix))+j(-G(jx))+k(-G(kx)). \\ &= G(x)+iI(x)+jJ(x)+kK(x) \\ &= F(x) . \end{aligned}$$

On $S$, the real part of $F = G = g =$ the real part of $f$. Since $F$ is uniquely determined by its real part and the real part of $F =$ the real part of $f$ (on $S$) then $F_{|S} = f$. We must show that $|F(x)| \leq p(x)$ for all $x \in X$.

Let $x \in X$. Since $F(x) \in \mathbb{H}$. there exists a $u_x \in \mathbb{H}$ such that $|u_x| = 1$ and $F(x) = |F(x)|\, u_x$. Since $\{ x \in \mathbb{H} \mid |x| = 1 \}$ is a group, $u_x^{-1}$ exist and $|u_x^{-1}| = 1$.

Let $y = u_x^{-1} x$. Then $F(y) = F(u_x^{-1}x) = u_x^{-1}F(x) = u_x^{-1}|F(x)|u_x = F(x)$.
So $|F(x)| = F(y)$ = Real part of $F(y) = G(y) = |G(y)| \leq p(y) = p(u_x^{-1}x)$ $= |u_x^{-1}|p(x) = 1 \cdot p(x) = p(x)$. Hence $|F(x)| \leq p(x)$ for all $x \in X$.

<u>Definition 2.8</u> Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be seminormed spaces over $\mathbb{H}$. Let T be a mapping between X and Y. Define $\|T\| = \sup \{ \|T(x)\|_2 | \|x\|_1 \leq 1 \}$

<u>Theorem 2.9</u> Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be seminormed spacs over $\mathbb{H}$ and let $T : X \to Y$ be a linear map. Then the following are equivalent :

(a) T is continuous at some $a \in X$.

(b) T is continuous on X.

(c) T is bounded on the unit disc, i.e $\|T\| < \infty$.

(d) There exists an $M \in \mathbb{R}$ such that $\|T(x)\|_2 \leq M \|x\|_1$ for all $x \in X$.

Proof : (a)$\Longrightarrow$(b). Assume that T is continuous at a point $a \in X$. We must show that T is continuous on X. Let $x \in X$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \to x$. Then $a + x_n - x \to a$. Since T is continuous at a, $T(a + x_n - x) \to T(a)$. Since T is $\mathbb{H}$ - linear, $T(a) + T(x_n) - T(x) = T(a + x_n - x) \to T(a) \to T(a)$. Thus $T(x_n) \to T(x)$, so T is continuous at the point x. But x was arbitrary, so T is continuous on X.

(b)$\Longrightarrow$(c) Claim that if $a, b \in \mathbb{R}^+$ are such that $\|x\|_1 < a$ implies that $\|T(x)\|_2 < b$, then $a\|T(x)\|_2 \leq b \|x\|_1$ for all $x \in X$. To prove this suppose not. Then there exists an $x_0 \in X$ such that $a\|T(x_0)\|_2 > b\|x_0\|_1$. Now there is a $t > 0$ such that $a\|T(x_0)\|_2 > t > b\|x_0\|_1$. Let $x = (\frac{ab}{t})x_0$. Then $\|x\|_1 = \|(\frac{ab}{t})x_0\|_1 = a\left(\frac{b\|x_0\|_1}{t}\right) < a.1 = a$

But $\| T(x)\|_2 = \| T(\frac{ab}{t})x_0\|_2 = \frac{ab}{t} \| T(x_0)\|_2 = b \frac{a\| t(x_0)\|_2}{t} > 1.b = b$, a contradicti

So we have the claim. Assume that T is continuous on X. We must show that $\| T \| < \infty$. By assumption, T is continuous at x = 0. Thus $T^{-1}(B(0, 1)) = \{x \mid \|Tx\|_2 < 1\}$ is a neighborhood of 0. Then there exists an $\varepsilon > 0$ and open set $U = \{ x \mid \|x\|_1 < \varepsilon\}$ such that $U \subseteq \{ x \mid \|Tx\|_2 < 1\}$. By the claim $\varepsilon \|Tx\|_2 \leq \|x\|_1$ for all $x \in X$. Hence $\|Tx\|_2 \leq \frac{1}{\varepsilon} \|x\|_1$ for all $x \in X$. So $\|T\| = \sup \{ \|Tx\|_2 \mid \|x\| \leq 1 \} \leq \frac{1}{\varepsilon}$, there fore $\|T\| < \infty$.

(c) $\Longrightarrow$ (d). Assume that $\|T\| < \infty$. Choose $M = \|T\|$. Must show that $\| Tx \| \leq M \| x \|$ for all $x \in X$. Let $x \in X$. If $\| x \| < 1$ then $\| Tx \| \leq \|T\| = M$. By the claim, $\| Tx \| \leq M \| x \|$, for all $x \in X$ (consider a = 1 and $b = \| T \|$).

(d) $\Longrightarrow$ (a) Assume that there is an $M \in \mathbb{R}$ such that $\| Tx \| \leq M \| x \|$ for all $x \in X$. Must show that T is continuous at x = 0. Let $\varepsilon > 0$ be given. By assumption,. There exists an $M \in \mathbb{R}$ such that $\|Tx\| \leq M \|x\|$. Choose $\delta = \frac{\varepsilon}{1+|M|}$. Then $\delta > 0$. Let $x \in B(0, \delta)$. Then $\| T(x) - T(0)\| = \|T(x)\| \leq M \|x\| < M \frac{\varepsilon}{1+|M|} \leq \varepsilon$, so T is continuous at x = 0.