CHAPTER II

FUNDAMENTAL CONSIDERATION

2.1 Theory of Thin, Isotropic Elastic Plates

The thin plate bending theory , in the absence of membrane forces, is based on the assumption that plane sections remain plane during bending and that deflections are small compared with the thickness of the plate. The effect of shear forces on the deflection are also disregarded.

Consider a thin plate element of thickness h the material of which has the modulus of elasticity E and Poisson's ratio ν in Cartesian co-ordinates (x,y) as shown in Fig.1 in which $D = \frac{Eh^3}{12(1-\nu^2)}$ denotes the flexural rigidity. Also the sign convention of stress resultants using the notations and conventions of Timoshenko and Woinowsky-Krieger (10) are shown in Fig.2. The deflection, w, of the middle surface of the plate subjected to a transverse force of intensity q is related to stress resultants, bending and twisting moments and shears per unit length of the section, as follows :

$$M_x = -D\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\} \qquad (1)$$

$$M_y = -D\left\{\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right\} \qquad (2)$$

$$M_{xy} = -M_{yx} = D\,(1-\nu)\,\frac{\partial^2 w}{\partial x\,\partial y} \tag{3}$$

$$Q_x = -D\left\{\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right\} \tag{4}$$

$$Q_y = -D\left\{\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y\,\partial x^2}\right\} \tag{5}$$

If a boundary of a plate is free, it is generally assumed that along this edge there are no bending and twisting moments and also no vertical shears. Kirchhoff proved that three boundary conditions are too many and that two conditions are enough for the complete determination of the solution. He showed that the edge twisting moment can be combined with the shearing force to produce a resultant boundary shear force or called Kirchhoff shear or supplemented shear per unit length, V , as

$$V_x = -D\left\{\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right\} \tag{6}$$

$$V_y = -D\left\{\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y\,\partial x^2}\right\} \tag{7}$$

By incorporating twisting moments into supplemented shears, the corner forces, R , arised from the jump of twisting moments at each corner have to be considered and can be written as

$$R = M_{xy} - M_{yx} = 2M_{xy} \quad . \tag{8}$$

By considering the equilibrium of forces in the z-direction and moments about x- and y-axes, leads to the governing biharmonic equation, called the "plate equation"

$$\nabla^2\nabla^2 w = \nabla^4 w = \frac{q}{D} \tag{9}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

Along boundary of plate, where the boundary integral equations are formed, the transformation of co-ordinates to the system of outward normal and tangential co-ordinates (n,t) (Fig. 3) is neccessary. Using the transformation matrix up to third order given in Appendix A, the above relationships become

$$M_n = -D \left\{ \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right\} \tag{10}$$

$$M_t = -D \left\{ \frac{\partial^2 w}{\partial t^2} + \nu \frac{\partial^2 w}{\partial n^2} \right\} \tag{11}$$

$$M_{nt} = D\,(1-\nu) \frac{\partial^2 w}{\partial n\,\partial t} \tag{12}$$

$$Q_n = -D \left\{ \frac{\partial^3 w}{\partial n^3} + \frac{\partial^3 w}{\partial n\,\partial t^2} \right\} \tag{13}$$

$$V_n = -D \left\{ \frac{\partial^3 w}{\partial n^3} + (2-\nu) \frac{\partial^3 w}{\partial n\,\partial t^2} \right\} . \tag{14}$$

The corner forces can be written as

$$R = M_{n_1 t_1} - M_{n_2 t_2}$$

$$\text{or} \quad R = D(1-\nu)\left\{ \frac{\partial^2 w}{\partial n_1 \partial t_1} - \frac{\partial^2 w}{\partial n_2 \partial t_2} \right\} \quad . \tag{15}$$

In which, the normals and tangents on the first side of the corner are denoted by subscript 1 and the other side by subscript 2 along the path as shown in Fig.4.

2.2 Betti's Reciprocal Theorem

The important energy principle that will be employed later in the boundary integral formulation is the Betti's reciprocal theorem (11). It states that for any two equilibrium states of stresses and compatible displacements, A and B, of a linearly elastic body the total external work done by the forces A during the corresponding displacements caused by the forces B is equal to the total external work done by the forces B during the corresponding displacements caused by the forces A.

2.3 Method of Analysis

Consider two distinct systems of compatible deflections and equilibrium states of stresses as shown in Fig.5. One is the real plate, the problem under consideration , which is the rectilinear plate of K sides having L columns and M line supports, each with length ψ_m inside the plate domain, Ω , with free boundary condition loaded by transverse force of intensity $q(\xi,\eta)$. The other is the virtual plate, designated by asterisks , subjected to a unit singular load, the Dirac delta function δ , acting at a point (x,y) , the solution of which satisfying the non-homogeneous equation

$$\nabla^4 w^*(x,y;\xi,\eta) \quad = \quad \frac{\delta(x,y)}{D} \quad . \tag{16}$$

A solution of (16) above may be taken as :

$$w^*(x,y;\xi,\eta) \quad = \quad \frac{r^2 \ \ln(r)}{8\pi D} \tag{17}$$

where $r = \sqrt{(\xi-x)^2 + (\eta-y)^2}$ is the distance between point (x,y) and point (ξ,η). It should be noted that (ξ,η) and (x,y) are the Cartesian co-ordinates. The expressions of slopes and desired stress resultants of the virtual plate can be obtained by appropriate differentiation of the deflection function $w^*(x,y;\xi,\eta)$.

The direct boundary integral methods of analysis are those that make use of the energy principle, i.e. Betti's reciprocal theorem, as mentioned earlier. Imposing the boundary conditions on the real plate where normal bending moment and supplemented shear are prescribed as zero and applying Betti's reciprocal theorem between the two systems in Fig.5, we obtain

$$1^* \cdot w(x,y) + \int_\Gamma \left\{ -M_n^*(x,y;\bar{\xi},\bar{\eta}) \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} + V_n^*(x,y;\bar{\xi},\bar{\eta})\, w(\bar{\xi},\bar{\eta}) \right\} d\Gamma(\bar{\xi},\bar{\eta})$$

$$+ \sum_{k=1,2}^{K} R_k^*(x,y;\bar{\xi}_k,\bar{\eta}_k)\; w(\bar{\xi}_k,\bar{\eta}_k)$$

$$= \int_\Omega q(\xi,\eta)\; w^*(x,y;\xi,\eta)\; d\Omega(\xi,\eta)$$

$$+\sum_{l=1,2}^{L}\left[\left\{-K^{c}_{al}\, w_{cl}(\xi_{cl},\eta_{cl})\right\} w^{*}(x,y;\xi_{cl},\eta_{cl})\right.$$

$$+\left\{-K^{y}_{rl}\,\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi}\right\}\frac{\partial w^{*}(x,y;\xi_{cl},\eta_{cl})}{\partial \xi}$$

$$\left.+\left\{-K^{x}_{rl}\,\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta}\right\}\frac{\partial w^{*}(x,y;\xi_{cl},\eta_{cl})}{\partial \eta}\right]$$

$$+\sum_{m=1,2}^{M}\int_{\psi m}\left[\left\{-K^{w}_{am}\, w_{wm}(\xi_{wm},\eta_{wm})\right\} w^{*}(x,y;\xi_{wm},\eta_{wm})\right.$$

$$+\left\{-K^{t}_{rm}\,\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}}\right\}\frac{\partial w^{*}(x,y;\xi_{wm},\eta_{wm})}{\partial n_{wm}}$$

$$\left.+\left\{-K^{n}_{rm}\,\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}}\right\}\frac{\partial w^{*}(x,y;\xi_{wm},\eta_{wm})}{\partial t_{wm}}\right] d\psi(\xi_{w},\eta_{w})$$

$$,\quad (x,y)\in\Omega \qquad (18)$$

in which

(x,y), (ξ,η) = co-ordinates

(x_c,y_c), (ξ_c,η_c) = co-ordinates of column support

(x_w,y_w), (ξ_w,η_w) = co-ordinates of line support

$(\bar{x},\bar{y})$, $(\bar{\xi},\bar{\eta})$ = co-ordinates on the boundary.

w , $\frac{\partial w}{\partial n}$ = deflection and slope of the real plate with respect to outward normal

$\frac{\partial w}{\partial \xi}$, $\frac{\partial w}{\partial \eta}$ = slope of the real plate with respect to ξ and η respectively

w^* = deflection of the virtual plate

$\frac{\partial w^*}{\partial \xi}$, $\frac{\partial w^*}{\partial \eta}$ = slope of the virtual plate with respect to ξ and η respectively

$\frac{\partial w^*}{\partial n_\omega}$, $\frac{\partial w^*}{\partial t_\omega}$ = slope of the virtual plate with respect to n and t respectively

V_n^* , M_n^* , R^* = supplemented shear, normal bending moment and corner force of the virtual plate

K_a^c , K_r^x , K_r^y = axial stiffness , rotational stiffness about x-axis and y-axis of the column support

K_a^w , K_r^n , K_r^t = axial stiffness , rotational stiffness about n-axis and t-axis of the line support

Ω = domain of plate

Γ = boundary of plate

ψ = range of line support

The above equation gives the deflection of any point, $w(x,y)$, within the plate domain in terms of

- boundary values ; $\frac{\partial w(\xi,\eta)}{\partial n}$, $w(\xi,\eta)$ and $w_k(\xi_k,\eta_k)$ for $k = 1,2,\ldots,K$
- column supports values ; $w_{cl}(\xi_{cl},\eta_{cl})$, $\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi}$ and $\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta}$ for $l = 1,2,\ldots,L$
- each line supports values ; $w_{wm}(\xi_{wm},\eta_{wm})$, $\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n(\xi_m,\eta_m)}$ and $\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t(\xi_m,\eta_m)}$ for $m = 1,2,\ldots,M$.

Using equation (18) as the basic boundary equation, the system of simultaneous equations are created by moving (x,y) to the defined unknown parts of the plate such as at boundary $(\bar{x},\bar{y})$, corners $(\bar{x}_k,\bar{y}_k)$, column supports (x_{cl},y_{cl}) and along line supports (x_{wm},y_{wm}).

Now rearranging equation (18) and then moving (x,y) to the boundary $(\bar{x},\bar{y})$, we obtain

$$
\begin{aligned}
\left(\frac{\emptyset}{2\pi}\right) 1^{*}\cdot w(\bar{x},\bar{y}) = & \int_{\Gamma}\left\{ M_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})\ \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - V_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})\ w(\bar{\xi},\bar{\eta})\right\} d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} R_k^{*}(\bar{x},\bar{y};\bar{\xi}_k,\bar{\eta}_k)\ w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta)\ w^{*}(\bar{x},\bar{y};\xi,\eta)\ d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L}\left[\left\{-K_{al}^{c}\ w_{cl}(\xi_{cl},\eta_{cl})\right\} w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})\right. \\
& \qquad + \left\{-K_{rl}^{y}\ \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi}\right\} \frac{\partial w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial \xi} \\
& \qquad \left. + \left\{-K_{rl}^{x}\ \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta}\right\} \frac{\partial w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial \eta}\right] \\
& + \sum_{m=1,2}^{M}\int_{\psi m}\left[\left\{-K_{am}^{w}\ w_{wm}(\xi_{wm},\eta_{wm})\right\} \partial w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})\right. \\
& \qquad + \left\{-K_{rm}^{t}\ \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}}\right\} \frac{w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})}{\partial n_{wm}} \\
& \qquad \left. + \left\{-K_{rm}^{n}\ \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}}\right\} \frac{\partial w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})}{\partial t_{wm}}\right] d\psi(\xi_w,\eta_w)
\end{aligned}
$$

$$, \quad (\bar{x},\bar{y}) \in \Gamma \qquad (19) $$

where $\emptyset$ is the included angle (for smooth boundary point, $\emptyset = \pi$) at the boundary point.

Differentiating equation (19) with respect to the outward normal direction, $n(\bar{x},\bar{y})$, we obtain

$$\frac{\emptyset}{2\pi}\frac{\partial w(\bar{x},\bar{y})}{\partial n(\bar{x},\bar{y})} = \int_{\Gamma}\left\{\frac{\partial M_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})}{\partial n(\bar{x},\bar{y})}\frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - \frac{\partial V_n^{*}(\bar{x},\bar{y};\bar{\xi},\bar{\eta})}{\partial n(\bar{x},\bar{y})} w(\bar{\xi},\bar{\eta})\right\} d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} \frac{\partial R_k^{*}(\bar{x},\bar{y};\bar{\xi}_k,\bar{\eta}_k)}{\partial n(\bar{x},\bar{y})} w(\bar{\xi}_k,\bar{\eta}_k)$$

$$+ \int_{\Omega} q(\xi,\eta)\frac{\partial w^{*}(\bar{x},\bar{y};\xi,\eta)}{\partial n(\bar{x},\bar{y})} d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L}\left[\left\{-K_{al}^{c}\, w_{cl}(\xi_{cl},\eta_{cl})\right\}\frac{\partial w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial n(\bar{x},\bar{y})}\right.$$

$$+\left\{-K_{rl}^{y}\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi}\right\}\frac{\partial^2 w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial \xi\, \partial n(\bar{x},\bar{y})}$$

$$\left.+\left\{-K_{rl}^{x}\frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta}\right\}\frac{\partial^2 w^{*}(\bar{x},\bar{y};\xi_{cl},\eta_{cl})}{\partial \eta\, \partial n(\bar{x},\bar{y})}\right]$$

$$+ \sum_{m=1,2}^{M}\int_{\psi m}\left[\left\{-K_{am}^{w}\, w_{wm}(\xi_{wm},\eta_{wm})\right\}\frac{\partial w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})}{\partial n(\bar{x},\bar{y})}\right.$$

$$+\left\{-K_{rm}^{t}\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}}\right\}\frac{\partial^2 w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})}{\partial n_{wm}\, \partial n(\bar{x},\bar{y})}$$

$$\left.+\left\{-K_{rm}^{n}\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}}\right\}\frac{\partial^2 w^{*}(\bar{x},\bar{y};\xi_{wm},\eta_{wm})}{\partial t_{wm}\, \partial n(\bar{x},\bar{y})}\right] d\psi(\xi_w,\eta_w)$$

$$, \quad (\bar{x},\bar{y}) \in \Gamma \qquad (20)$$

Now move (x,y) to the location of each column supports ; differentiate with respect to x ; differentiate with respect to y to obtain equations (21), (22) and (23) as shown below

$$
\begin{aligned}
w(x_{ci},y_{ci}) = & \int_{\Gamma} \left\{ M_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta}) \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} \right. \\
& \left. - V_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta}) \, w(\bar{\xi},\bar{\eta}) \right\} d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} R_k^*(x_{ci},y_{ci};\bar{\xi}_k,\bar{\eta}_k) \, w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta) \, w^*(x_{ci},y_{ci};\xi,\eta) \, d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \left[\left\{ -K_{al}^{c} \, w_{cl}(\xi_{cl},\eta_{cl}) \right\} w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl}) \right. \\
& + \left\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \xi} \\
& \left. + \left\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \eta} \right] \\
& + \sum_{m=1,2}^{M} \int_{\psi m} \left[\left\{ -K_{am}^{w} \, w_{wm}(\xi_{wm},\eta_{wm}) \right\} w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm}) \right. \\
& + \left\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial n_{wm}} \\
& \left. + \left\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial t_{wm}} \right] d\psi(\xi_w,\eta_w)
\end{aligned}
$$

$$, \; i = 1,2,\ldots,L \qquad (21)$$

$$
\begin{aligned}
\frac{\partial w(x_{ci},y_{ci})}{\partial x} = & \int_{\Gamma} \left\{ \frac{\partial M_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial x} \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} \right. \\
& \left. - \frac{\partial V_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial x} w(\bar{\xi},\bar{\eta}) \right\} d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} \frac{\partial R_k^*(x_{ci},y_{ci};\bar{\xi}_k,\bar{\eta}_k)}{\partial x} w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta) \frac{\partial w^*(x_{ci},y_{ci};\xi,\eta)}{\partial x} d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \left[\left\{ -K_{al}^{c} w_{cl}(\xi_{cl},\eta_{cl}) \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial x} \right. \\
& + \left\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \xi \, \partial x} \\
& \left. + \left\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \eta \, \partial x} \right] \\
& + \sum_{m=1,2}^{M} \int_{\Psi m} \left[\left\{ -K_{am}^{w} w_{wm}(\xi_{wm},\eta_{wm}) \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial x} \right. \\
& + \left\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial n_{wm} \, \partial x} \\
& \left. + \left\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial t_{wm} \, \partial x} \right] d\Psi(\xi_w,\eta_w)
\end{aligned}
$$

$$, \quad i = 1,2,\ldots,L \qquad (22)$$

$$\frac{\partial w(x_{ci},y_{ci})}{\partial y} = \int_{\Gamma} \left\{ \frac{\partial M_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial y} \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - \frac{\partial V_n^*(x_{ci},y_{ci};\bar{\xi},\bar{\eta})}{\partial y} w(\bar{\xi},\bar{\eta}) \right\} d\Gamma(\bar{\xi},\bar{\eta})$$

$$- \sum_{k=1,2}^{K} \frac{\partial R_k^*(x_{ci},y_{ci};\bar{\xi}_k,\bar{\eta}_k)}{\partial y} w(\bar{\xi}_k,\bar{\eta}_k)$$

$$+ \int_{\Omega} q(\xi,\eta) \frac{\partial w^*(x_{ci},y_{ci};\xi,\eta)}{\partial y} d\Omega(\xi,\eta)$$

$$+ \sum_{l=1,2}^{L} \left[\left\{ -K_{al}^{c} \, w_{cl}(\xi_{cl},\eta_{cl}) \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial y} + \left\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \xi \, \partial y} + \left\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{cl},\eta_{cl})}{\partial \eta \, \partial y} \right]$$

$$+ \sum_{m=1,2}^{M} \int_{\Psi m} \left[\left\{ -K_{am}^{w} \, w_{wm}(\xi_{wm},\eta_{wm}) \right\} \frac{\partial w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial y} + \left\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial n_{wm} \, \partial y} + \left\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \right\} \frac{\partial^2 w^*(x_{ci},y_{ci};\xi_{wm},\eta_{wm})}{\partial t_{wm} \, \partial y} \right] d\Psi(\xi_w,\eta_w)$$

$$, \quad i = 1,2,\ldots,L \;. \qquad (23)$$

Moving (x,y) to the location of line support (x_{wm},y_{wm}) ; differentiating with respect to normal direction , n_w , and differentiating with respect to tangential direction , t_w , equations (24), (25) and (26) are obtained.

$$
\begin{aligned}
w(x_{wi},y_{wi}) = \int_{\Gamma} \Bigg\{ & M_n^*(x_{wi},y_{wi};\bar{\xi},\bar{\eta}) \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} \\
& - V_n^*(x_{wi},y_{wi};\bar{\xi},\bar{\eta})\, w(\bar{\xi},\bar{\eta}) \Bigg\}\, d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} R_k^*(x_{wi},y_{wi};\bar{\xi}_k,\bar{\eta}_k)\, w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta)\, w^*(x_{wi},y_{wi};\xi,\eta)\, d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \Bigg[\Big\{ -K_{al}^{c}\, w_{cl}(\xi_{cl},\eta_{cl}) \Big\}\, \partial w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl}) \\
& \quad + \Big\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \Big\} \frac{w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial \xi} \\
& \quad + \Big\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \Big\} \frac{\partial w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial \eta} \Bigg] \\
& + \sum_{m=1,2}^{M} \int_{\psi m} \Bigg[\Big\{ -K_{am}^{w}\, w_{wm}(\xi_{wm},\eta_{wm}) \Big\}\, w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm}) \\
& \quad + \Big\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \Big\} \frac{\partial w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial n_{wm}} \\
& \quad + \Big\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \Big\} \frac{\partial w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial t_{wm}} \Bigg]\, d\psi(\xi_w,\eta_w)
\end{aligned}
$$

$$
, \quad i = 1,2,\ldots,M \text{ and } (x_{wi},y_{wi}) \in \psi \qquad (24)
$$

$$
\begin{aligned}
\frac{\partial w(x_{wi},y_{wi})}{\partial n_{wi}} = \int_{\Gamma} \Bigg\{ & \frac{\partial M_n^{*}(x_{wi},y_{wi};\bar{\xi},\bar{\eta})}{\partial n_{wi}} \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} \\
& - \frac{\partial V_n^{*}(x_{wi},y_{wi};\bar{\xi},\bar{\eta})}{\partial n_{wi}} w(\bar{\xi},\bar{\eta}) \Bigg\} d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} \frac{\partial R_k^{*}(x_{wi},y_{wi};\bar{\xi}_k,\bar{\eta}_k)}{\partial n_{wi}} w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta) \frac{\partial w^{*}(x_{wi},y_{wi};\xi,\eta)}{\partial n_{wi}} d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \Bigg[\Big\{ -K_{al}^{c} \, w_{cl}(\xi_{cl},\eta_{cl}) \Big\} \frac{\partial w^{*}(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial n_{wi}} \\
& + \Big\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \Big\} \frac{\partial^2 w^{*}(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial \xi \, \partial n_{wi}} \\
& + \Big\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \Big\} \frac{\partial^2 w^{*}(x_{wm},y_{wm};\xi_{cl},\eta_{cl})}{\partial \eta \, \partial n_{wi}} \Bigg] \\
& + \sum_{m=1,2}^{M} \int_{\Psi m} \Bigg[\Big\{ -K_{am}^{w} \, w_{wm}(\xi_{wm},\eta_{wm}) \Big\} \frac{\partial w^{*}(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial n_{wi}} \\
& + \Big\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \Big\} \frac{\partial^2 w^{*}(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial n_{wm} \, \partial n_{wi}} \\
& + \Big\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \Big\} \frac{\partial^2 w^{*}(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial t_{wm} \, \partial n_{wi}} \Bigg] d\Psi(\xi_w,\eta_w)
\end{aligned}
$$

$$, \quad i = 1,2,\ldots,M \quad \text{and} \quad (x_{wi},y_{wi}) \in \Psi \qquad (25)$$

$$
\begin{aligned}
\frac{\partial w(x_{wi},y_{wi})}{\partial t_{wi}} = & \int_{\Gamma} \left\{ \frac{\partial M_n^*(x_{wi},y_{wi};\bar{\xi},\bar{\eta})}{\partial t_{wi}} \frac{\partial w(\bar{\xi},\bar{\eta})}{\partial n} - \frac{\partial V_n^*(x_{wi},y_{wi};\bar{\xi},\bar{\eta})}{\partial t_{wi}} w(\bar{\xi},\bar{\eta}) \right\} d\Gamma(\bar{\xi},\bar{\eta}) \\
& - \sum_{k=1,2}^{K} \frac{\partial R_k^*(x_{wi},y_{wi};\bar{\xi}_k,\bar{\eta}_k)}{\partial t_{wi}} w(\bar{\xi}_k,\bar{\eta}_k) \\
& + \int_{\Omega} q(\xi,\eta) \frac{\partial w^*(x_{wi},y_{wi};\xi,\eta)}{\partial t_{wi}} d\Omega(\xi,\eta) \\
& + \sum_{l=1,2}^{L} \left[\left\{ -K_{al}^{c}\, w_{cl}(\xi_{cl},\eta_{cl}) \right\} \frac{\partial w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial t_{wi}} \right. \\
& \qquad + \left\{ -K_{rl}^{y} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \xi} \right\} \frac{\partial^2 w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial \xi\, \partial t_{wi}} \\
& \qquad \left. + \left\{ -K_{rl}^{x} \frac{\partial w_{cl}(\xi_{cl},\eta_{cl})}{\partial \eta} \right\} \frac{\partial^2 w^*(x_{wi},y_{wi};\xi_{cl},\eta_{cl})}{\partial \eta\, \partial t_{wi}} \right] \\
& + \sum_{m=1,2}^{M} \int_{\psi m} \left[\left\{ -K_{am}^{w}\, w_{wm}(\xi_{wm},\eta_{wm}) \right\} \frac{\partial w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial t_{wi}} \right. \\
& \qquad + \left\{ -K_{rm}^{t} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}} \right\} \frac{\partial^2 w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial n_{wm}\, \partial t_{wi}} \\
& \qquad \left. + \left\{ -K_{rm}^{n} \frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}} \right\} \frac{\partial^2 w^*(x_{wi},y_{wi};\xi_{wm},\eta_{wm})}{\partial t_{wm}\, \partial t_{wi}} \right] d\psi(\xi_w,\eta_w)
\end{aligned}
$$

$$, \quad i = 1,2,\ldots,M \text{ and } (x_{wi},y_{wi}) \in \psi . \qquad (26)$$

The influence functions of w^* ,only the additional from previous study which appeared in the last term of equations (19) through (26), are stated in Appendix B.

Equations (19) through (26) constitute (2+3L+3M) integral equations in

- two unknown functions w, $\frac{\partial w}{\partial n}$ which are continuous functions throughout the boundary

- K unknown values of w_k , k=1,2,...,K at each corner

- 3L unknown values of w_{c1}, $\frac{\partial w_{c1}}{\partial \xi}$ and $\frac{\partial w_{c1}}{\partial \eta}$, l = 1,2,...,L at each column supports inside the plate domain

- 3M unknown values of w_{wm} , $\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial n_{wm}}$ and $\frac{\partial w_{wm}(\xi_{wm},\eta_{wm})}{\partial t_{wm}}$, m = 1,2,...,M which are continuous function along line support also within the plate domain.

These equations will be solved numerically as to be elaborated in the next chapter.