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TOTAL COLORINGS OF GLUED GRAPHS

Mr. Wongsakorn Charoenpanitseri

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Sciences Program in Mathematics

Department of Mathematics
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วงศกร เจริญพานิชเสรี : การระบายสีแบบรวมของกราฟปะติด (TOTAL COLORINGS OF GLUED GRAPHS) อ.ที่ปรึกษาหลัก : ดร.จริยา อุ่ยยะเสถียร อ.ที่ปรึกษาร่วม : รศ.ดร. วนิดา เหมะกุล 55 หน้า

ข้อความคาดการณ์ของการระบายสีแบบรวมของกราฟ G กล่าวว่า $\chi''(G) \leq \Delta(G)+2$ เมื่อ $\chi''(G)$ แทนรงคเลขแบบรวมของกราฟ G และ $\Delta(G)$ แทนดีกรีสูงสุดในกราฟ G เรากล่าวว่ากราฟ G สอดคล้องสมบัติชนิดที่หนึ่ง ถ้า $\chi''(G) = \Delta(G)+1$ และชนิดที่สอง ถ้า $\chi''(G) = \Delta(G)+2$

ในวิทยานิพนธ์ฉบับนี้เราหาขอบเขตบนของรงคเลขแบบรวมของกราฟปะติดในเทอมของรงคเลขแบบรวมของกราฟเริ่มต้น เราสนใจรงคเลขแบบรวมของกราฟปะติดระหว่างกราฟกลุ่มเดียวกันซึ่งคือ กราฟวง กราฟต้นไม้ กราฟสองส่วน และกราฟบริบูรณ์ และพิสูจน์ว่ากราฟเหล่านี้สอดคล้องข้อความคาดการณ์ของการระบายสีแบบรวมและเสนอเงื่อนไขจำเป็นและเพียงพอสำหรับกราฟปะติดเหล่านี้ยกเว้นกราฟปะติดของกราฟสองส่วน สอดคล้องข้อความคาดการณ์ของการระบายสีแบบรวมและสมบัติชนิดที่หนึ่ง หรือชนิดที่สอง ยิ่งกว่านั้นเราสนใจเงื่อนไขเพียงพอสำหรับกราฟใดๆที่สอดคล้องข้อความคาดการณ์ของการระบายสีแบบรวมและสมบัติชนิดที่หนึ่ง หรือ ชนิดที่สอง เพื่อใช้เงื่อนไขเหล่านี้กับผลลัพธ์ของกราฟปะติดของกราฟใดๆและกราฟต้นไม้ใดๆ

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The Total Coloring Conjecture states that for every graph G, $\chi''(G) \leq \Delta(G) + 2$ when $\chi''(G)$ is the total chromatic number of G and $\Delta(G)$ is the maximum number of degree of vertices of G. We say that a graph G is of *type* 1 if $\chi''(G) = \Delta(G) + 1$ and *type* 2 if $\chi''(G) = \Delta(G) + 2$.

In this thesis, upper bounds of the total chromatic number of glued graphs in terms of the total chromatic number of original graphs are presented. We investigate the total chromatic number of glued graphs of same class where the classes are cycles, trees, bipartite graphs and complete graphs and prove that these glued graphs satisfy the Total Coloring Conjecture and obtain necessary and sufficient conditions for these glued graphs except the glued graph of bipartite graphs to be either of type 1 or type 2. Furthermore, we study sufficient conditions for any graph to satisfy the Total Coloring Conjecture and be either type 1 graph or type 2 graph and use these conditions to obtain the result of glued graphs of any graph and any tree.

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Field of Study :	...Mathematics...	Principle Advisor's Signature ....
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If I did the wrong thing, I do apologize. It does not happen by intention.

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CHAPTER I

INTRODUCTION

1.1 Introduction

The glue operator is a mathematical operator defined by Uiyyasathain[1]. She studies maximal-clique partitions of different sizes whether or not there exists a clique-inseparable graph with n maximal-clique partitions of n different sizes, so the glue operator is defined to solve the problem. Later, Promsakon[2] studies colorability of the glued graphs. Bounds of the chromatic number and the edge-chromatic number of the glued graphs in term of the chromatic number and the edge-chromatic number of the original graphs are obtained in [3] and [4]. This is a motivation for us to study total colorings of glued graphs. In section 1.2, we show literature reviews of vertex colorings, edge colorings and total colorings. In section 1.3, we give examples and also investigate some basic properties of glued graphs. In chapter 2, we analyze the results of total colorings of glued graphs for some classes of graphs such as cycles, bipartite graphs , trees and complete graphs. In chapter 3, we study total colorings of the glued graphs between any graph and any tree. Moreover, there are some necessary conditions of graphs satisfying the Total Coloring Conjecture. In chapter 4, we give conclusions and open problems from Chapter 1, Chapter 2 and Chapter 3.

In this thesis, we consider only a *connected graph without loops and multiple edges.* $V(G)$ and $E(G)$ stand for the vertex set and edge set of a graph G, respectively. The number of elements in $V(G)$ is represented by $n(G)$ and the

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number of elements in $E(G)$ is represented by $e(G)$. We use v_i for a vertex and use e_i for an edge. We also use v_iv_j for the edge whose endpoints v_i and v_j.

1.2 Basic Properties of Colorings, Edge-colorings and Total Colorings

Let $[k]$ represent the set $\{1, 2, ..., k\}$ and we use $\{1, 2, ..., k\}$ as the set of k colors. A *k-coloring* of a graph G is a coloring $f : V(G) \to [k]$. A k-coloring is *proper* if adjacent vertices have different colors. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least positive integer k such that G is k-colorable.

A *k-edge-coloring* of a graph G is a coloring $f : E(G) \to [k]$. A k-edge-coloring is *proper* if incident edges have different colors. A graph is *k-edge-colorable* if it has a proper k-edge-coloring. The *edge-chromatic number* $\chi'(G)$ of a graph G is the least positive integer k such that G is k-edge-colorable.

A *k-total coloring* of a graph G is a coloring $f : V(G) \cup E(G) \to [k]$. A k-total coloring is *proper* if incident edges have different colors, adjacent vertices have different colors, and edges and its endpoints have different colors. A graph is *k-total colorable* if it has a proper k-total coloring. The *total chromatic number* $\chi''(G)$ of a graph G is the least positive integer k such that G is k-total colorable.

Remark 1.2.1. Let G be a graph. Then $\chi''(G) \geq \Delta(G) + 1$.

Proof. Let v be a vertex of a graph G with maximum degree. There are $\Delta(G)$ edges which are incident to v. Since these $\Delta(G)$ edges and v have different colors, we have $\Delta(G) + 1 \leq \chi''(G)$. □

The Total Coloring Conjecture, introduced independently by Behzad[5] and Vizing[6], states that for every graph G, $\chi''(G) \leq \Delta(G) + 2$. It is known that for

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any graph G, $\chi''(G) \geq \Delta(G) + 1$. A graph G is of *type* 1 if $\chi''(G) = \Delta(G) + 1$ and *type* 2 if $\chi''(G) = \Delta(G) + 2$.

Remark 1.2.2. Let G be a graph and H be a subgraph of G. Then

(a) $\chi(H) \leq \chi(G)$,

(b) $\chi'(H) \leq \chi'(G)$,

(c) $\chi''(H) \leq \chi''(G)$.

Proposition 1.2.3. *Let G be a nontrivial graph. Then $\chi''(G) \geq 3$.*

Proof. Since G is a nontrivial graph, there is an edge uv where $u, v \in V(G)$. We need 3 colors to label vertices u, v and edge uv. Thus $\chi''(G) \geq 3$. □

Remark 1.2.4. Let G be a graph. Then

(a) $\chi''(G) \geq \chi(G)$,

(b) $\chi''(G) \geq \chi'(G)$.

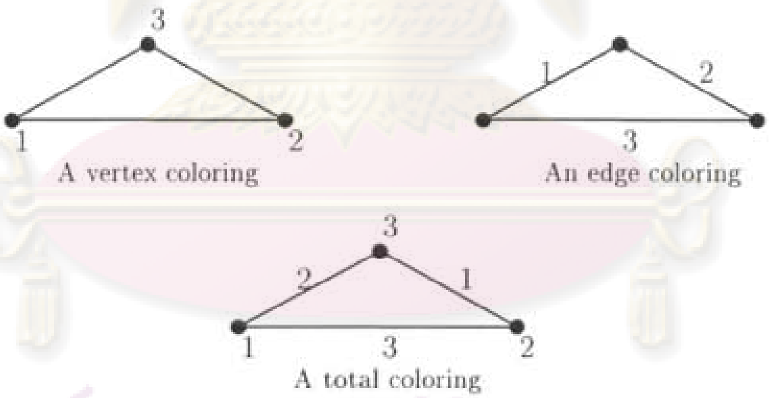

Figure 1.2.1: $\chi(C_3) = \chi'(C_3) = \chi''(C_3)$

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As shown in Figure 1.2.1, we are interested in determining a necessary and sufficient condition for equality of $\chi(G), \chi'(G)$ and $\chi''(G)$.

Remark 1.2.5. $\chi(C_n) = \chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Proposition 1.2.6. [7] $\chi''(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$

Theorem 1.2.7. [8],[9] *For every graph G, $\chi(G) \leq \Delta(G)+1$. The equality holds if and only if G is a complete graph or an odd cycle.*

Remark 1.2.8. $\chi''(C_n) \geq \chi'(C_n) = \chi(C_n)$.

Proposition 1.2.9. $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$ *if and only if* $n \equiv 3 \pmod 6$.

Proof. Sufficiency. Assume that $n \equiv 3 \pmod 6$. Since C_n is an odd cycle, we get $\chi(C_n) = 3$ and $\chi'(C_n) = 3$. By Proposition 1.2.6, we get $\chi''(C_n) = 3$. Therefore, $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$.

Necessity. We will prove by contrapositive. Assume that $n \not\equiv 3 \pmod 6$. By the division algorithm, $n = 6k, 6k+1, 6k+2, 6k+4$ or $6k+5$ for some integer k.

Case 1. $n = 6k, 6k+2$ or $6k+4$.

Since C_n is an even cycle, we get $\chi(C_n) = 2$. However, $\chi''(C_n) \geq \Delta(C_n)+1 = 3$. Then $\chi(C_n) \neq \chi''(C_n)$.

Case 2. $n = 6k+1$ or $n = 6k+5$.

Since n is not divisible by 3, by Proposition 1.2.6, we get $\chi''(C_n) = 4$. By Theorem 1.2.7, $\chi(C_n) \leq \Delta(C_n)+1 = 3$ and $\chi''(C_n) = 4$. Then $\chi(C_n) \neq \chi''(C_n)$.

Therefore, $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$ if and only if $n \equiv 3 \pmod 6$. □

Remark 1.2.10. For every integer n, $\chi(K_n) = n$.

Proposition 1.2.11. [10] $\chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$

Proposition 1.2.12. [11] $\chi''(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$

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Proposition 1.2.13. *If n is odd then $\chi(K_n) = \chi'(K_n) = \chi''(K_n)$. Otherwise, $\chi(K_n) = \chi'(K_n) + 1 = \chi''(K_n) - 1$.*

Proof. *Case 1.* n is odd. By Remark 1.2.10, Proposition 1.2.11 and Proposition 1.2.12, we get $\chi(K_n) = \chi'(K_n) = \chi''(K_n) = n$.

Case 2. n is even

By Proposition 1.2.12, we get $\chi''(K_n) = n + 1$. However, $\chi(K_n) = n$. Thus $\chi(K_n) = \chi''(K_n) - 1$. By Proposition 1.2.11, we get $\chi'(K_n) = n - 1$. Thus $\chi(K_n) = \chi''(K_n) + 1$. □

Theorem 1.2.14. *Let G be a graph. If G is not a complete graph of even degree, then $\chi''(G) \geq \chi'(G) \geq \chi(G)$. Otherwise, $\chi(G) = \chi'(G) - 1 = \chi''(G) + 1$.*

Proof. *Case 1.* G is neither a complete graph nor an odd cycle. By Theorem 1.2.7, $\chi(G) \leq \Delta(G)$. Since $\Delta(G) \leq \chi'(G)$ and $\chi'(G) \leq \chi''(G)$, we get $\chi''(G) \geq \chi'(G) \geq \chi(G)$.

Case 2. G is an odd cycle. By Remark 1.2.8, $\chi''(G) \geq \chi'(G) \geq \chi(G)$.

Case 3. G is a complete graph. If n is odd then $\chi(K_n) = \chi'(K_n) = \chi''(K_n)$ and if n is even then $\chi(K_n) = \chi'(K_n) + 1 = \chi''(K_n) - 1$ by Proposition 1.2.13. □

The following theorem gives necessary and sufficient conditions for the equality of the chromatic number, the edge-chromatic number and the total chromatic number.

Theorem 1.2.15. *Let G be a graph with n vertices. $\chi(G) = \chi'(G) = \chi''(G)$ if and only if G is C_n where $n \equiv 3 \pmod 6$ or K_n where n is odd.*

Proof. *Sufficiency.* $\chi(G) = \chi'(G) = \chi''(G)$ by Proposition 1.2.9 and Proposition 1.2.13.

Necessity. Assume that $\chi(G) = \chi'(G) = \chi''(G)$. By Theorem 1.2.7 and Remark 1.2.1, we get $\chi(G) \leq \Delta(G) + 1 \leq \chi''(G)$. Then $\chi(G) = \Delta(G) + 1 = \chi''(G)$.

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Thus $\chi(G) > \Delta(G)$. From Theorem 1.2.7, G is an odd cycle or a complete graph. By Proposition 1.2.9 and Proposition 1.2.13, G is a cycle of length $n \equiv 3 \pmod 6$ or a complete graph of order n when n is odd. □

1.3 Basic Properties of Glued Graphs

In this section, we introduce the glued graph and give some properties of glued graphs. Let G_1 and G_2 be any two vertex-distinct graphs. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f, then *the glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H as its *clone* and refer to G_1 and G_2 as its *original graphs*.

The glued graph of G_1 *and* G_2 *at the clone* H, written $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, means that there exist a subgraph H_1 of G_1 and a subgraph H_2 of G_2 and an isomorphism f between H_1 and H_2 such that $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$ and H is the copy of H_1 and H_2 in the resulting graph.

We denote $G_1 \triangleleft\!\triangleright G_2$ an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

Notation $K_n(v_1, v_2, ..., v_n)$ denotes a complete graph on vertices $v_1, v_2, ..., v_n$, $C_n(v_1, v_2, ..., v_n)$ denotes a cycle on vertices $v_1, v_2, ..., v_n$ and $P_n(v_1, v_2, ..., v_n)$ denotes a path on vertices $v_1, v_2, ..., v_n$.

Example 1.3.1. Let G_1 and G_2 be graphs as shown in Figure 1.3.1.

Let $H_1 \cong K_3(1, 3, 4)$ be a subgraph of G_1 and $H_2 \cong K_3(a, b, c)$ be a subgraph of G_2. Consider three isomorphisms f, g and h between H_1 and H_2, as follows:

$f(1) = a, f(3) = b, f(4) = c,$

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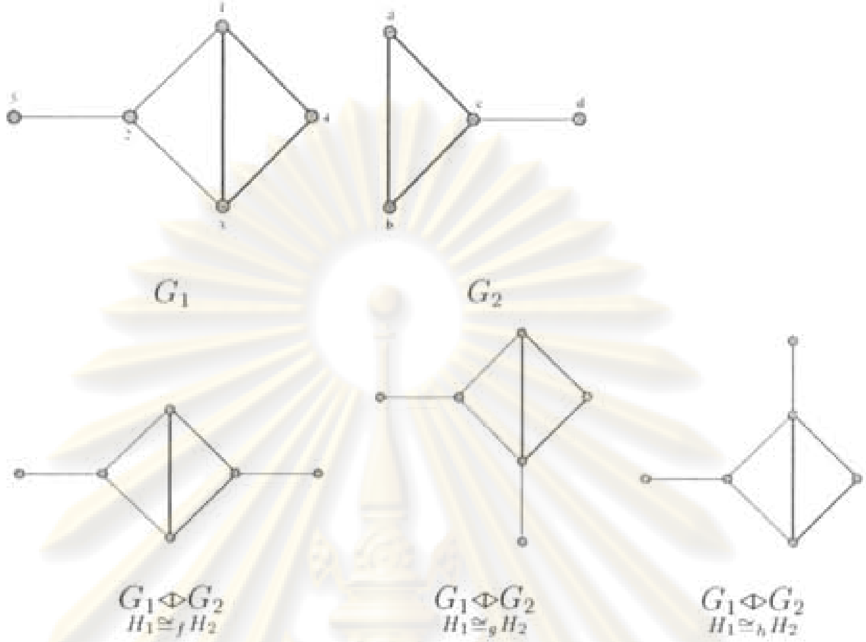

Figure 1.3.1: The results of glued graphs G_1 and G_2 in different isomorphisms

$g(1) = b, g(3) = c, g(4) = a$ and

$h(1) = c, h(3) = a, h(4) = b.$

The glued graphs between G_1 and G_2 with respect to f, g and h are shown in Figure 1.3.1

Example 1.3.1 shows that different isomorphisms can give the different or the same result. However, in some cases it is possible that all isomorphisms give the same result as shown in the next example.

Example 1.3.2. Let G_1 and G_2 be graphs as shown in Figure 1.3.2.

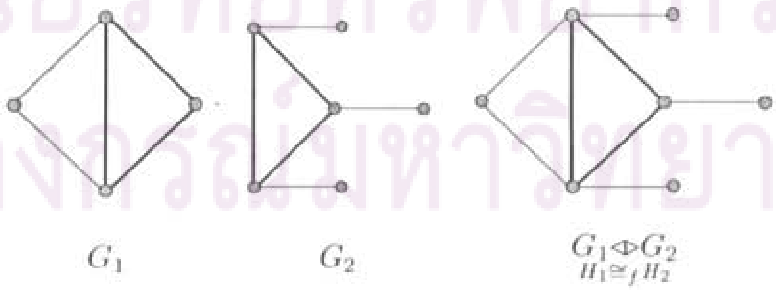

Figure 1.3.2: The results of glued graphs G_1 and G_2 in different isomorphisms

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Let $H_1 \cong K_3(2,3,4)$ be a subgraph of G_1 and $H_2 \cong K_3(a,b,c)$ be a subgraph of G_2. There are six isomorphisms between H_1 and H_2, but all of them give the same result as shown in a Figure 1.3.2 where f is arbitrary isomorphism between H_1 and H_2.

We first observe some basic properties of glued graphs in the following remark.

Remark 1.3.3.

1. The original graphs are subgraphs of their glued graph.
2. The graph gluing does not create or destroy an edge.
3. A glued graph between disconnected graphs is also disconnected and a glued graph between connected graphs is also connected.
4. If $u \in V(G_1) - V(H)$ and $v \in V(G_2) - V(H)$ where G_1 and G_2 are graphs and H is a clone of $G_1 \underset{H}{\oplus} G_2$, then u and v are not adjacent in $G_1 \underset{H}{\oplus} G_2$.

A glued graph could be a simple or not simple graph. Clearly the graph gluing of G_1 and G_2 is not a simple graph if G_1 or G_2 is not a simple graph. If original graphs are simple graphs, it is not necessary that their glued graph is a simple graph. The necessary and sufficiency condition for glued graphs to be simple is given in next theorem. In this thesis, we consider only simple connected glued graphs.

Theorem 1.3.4. [2] *Let G_1 and G_2 be simple graphs and let H be the clone of a glued graph $G_1 \underset{H}{\oplus} G_2$. Then $G_1 \underset{H}{\oplus} G_2$ is a simple graph if and only if there are no vertices u and v in H such that there are edges $e_1 \in E(G_1) - E(H)$ and $e_2 \in E(G_2) - E(H)$ whose endpoints are u and v.*

Remark 1.3.5. [2] Let G_1 and G_2 be nontrivial graphs.
Then $\Delta(G_1 \oplus G_2) \leq \Delta(G_1) + \Delta(G_2) - 1$.

Theorem 1.3.6. *Let G_1 and G_2 be graphs. Then $\chi''(G_1 \oplus G_2) \geq \max\{\chi''(G_1), \chi''(G_2)\}$*

Proof. Since G_1 and G_2 are subgraphs of $G_1 \oplus G_2$, we get $\chi''(G_1) \leq \chi''(G_1 \oplus G_2)$ and $\chi''(G_2) \leq \chi''(G_1 \oplus G_2)$. Then $\chi''(G_1 \oplus G_2) \geq \max\{\chi''(G_1), \chi''(G_2)\}$. □

Theorem 1.3.7 (a) gives an upper bound of the chromatic number of glued graphs in terms of the chromatic number of original graphs and Theorem 1.3.7 (b) shows an upper bound of the edge-chromatic number of glued graphs in terms of the edge-chromatic number of original graphs. Furthermore, Theorem 1.3.8 shows an upper bound of the chromatic number of glued graphs when the clone is an induced subgraph of original graphs in terms of the chromatic number of original graphs.

Theorem 1.3.7. [3],[4] *Let G_1 and G_2 be graphs. Then*

(a) $\chi(G_1 \oplus G_2) \leq \chi(G_1)\chi(G_2)$,

(b) $\chi'(G_1 \oplus G_2) \leq \chi'(G_1) + \chi'(G_2)$.

Theorem 1.3.8. [3] *Let G_1 and G_2 be graphs and $G_1 \underset{H}{\oplus} G_2$ a glued graph with clone H. If H is an induced subgraph, then $\chi(G_1 \underset{H}{\oplus} G_2) \leq \chi(G_1) + \chi(G_2)$.*

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CHAPTER II

TOTAL COLORINGS OF SOME CLASSES OF GLUED GRAPHS

2.1 Upper Bounds of the Total Chromatic Numbers of Glued Graphs

In this section, we investigate the values and bounds of the chromatic numbers, the edge-chromatic numbers and the total chromatic numbers of some classes of graphs and their glued graphs.

Theorem 2.1.1. *Let G_1 and G_2 be graphs. If $\chi''(G_1 \oplus G_2) \leq \Delta(G_1 \oplus G_2) + 2$ then $\chi''(G_1 \oplus G_2) \leq \chi''(G_1) + \chi''(G_2) - 1$.*

Proof. Assume that $\chi''(G_1 \oplus G_2) \leq \Delta(G_1 \oplus G_2) + 2$. Then

$$\begin{aligned} \chi''(G_1 \oplus G_2) &\leq \Delta(G_1 \oplus G_2) + 2 \\ &\leq \Delta(G_1) + \Delta(G_2) - 1 + 2, && \text{by Remark 1.3.5,} \\ &= (\Delta(G_1) + 1) + (\Delta(G_2) + 1) - 1 \\ &\leq \chi''(G_1) + \chi''(G_2) - 1, && \text{by Remark 1.2.1.} \end{aligned}$$

□

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We obtained an upper bound of the total chromatic numbers of glued graphs in terms of the total chromatic number of original graphs. Note that if graphs G_1 and G_2 with $\chi''(G_1 \oplus G_2) \leq \Delta(G_1 \oplus G_2) + 2$ satisfy following conditions

(a) a vertex with maximum degree of G_1 is glued to a vertex with maximum degree of G_2 and the corresponding vertex in the clone has degree 1,

(b) G_1 and G_2 are of type 1,

(c) $G_1 \oplus G_2$ is of type 2.

Then we have $\chi''(G_1 \oplus G_2) = \chi''(G_1) + \chi''(G_2) - 1$.

However, any two conditions among above three conditions yield $\chi''(G_1 \oplus G_2) \leq \chi''(G_1) + \chi''(G_2) - 2$. We conjecture that no graph satisfies all of the above conditions; hence, we have the following conjecture.

Conjecture 2.1.2. Let G_1, G_2 be graphs. Then $\chi''(G_1 \oplus G_2) \leq \chi''(G_1) + \chi''(G_2) - 2$.

We next try to prove the conjecture by first considering some classes of graph such as cycles, bipartite graphs, trees and complete graphs.

2.2 Total Colorings of Glued Graphs of Cycles

In this section, we investigate the values or bounds of the chromatic number, the edge-chromatic numbers and the total chromatic numbers of cycles and their glued graphs. Moreover, we prove that any glued graph of cycles satisfies the Total Coloring Conjecture and give a necessary and sufficient condition to be either of type 1 or of type 2 of glued graphs of cycles.

Proposition 2.2.1. [9] $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Proposition 2.2.2. [9] $\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Proposition 2.2.3. [7] $\chi''(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$

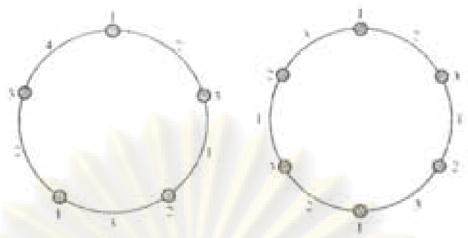

Figure 2.2.1: Total colorings of cycles

A graph is said to be *s-degenerated* for an integer $s \geq 1$ if it can be reduced to a trivial graph by successive removal of vertices with degree at most s.

For example, the graph in Figure 2.2.2 is 2-degenerated and every planar graph is 5-degenerated.

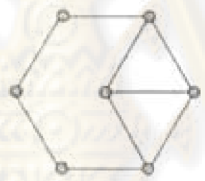

Figure 2.2.2: A 2-degenerated graph

Theorem 2.2.4. [12],[13] *If G is an s-degenerated graph then $\chi(G) \leq s+1$.*

Proposition 2.2.5. [3] *Let G_1 and G_2 be graphs. Then $G_1 \oplus G_2$ is bipartite if and only if G_1 and G_2 are bipartite.*

Theorem 2.2.6. $\chi(C_m \oplus C_n) = \begin{cases} 2 & \textit{if } m \textit{ and } n \textit{ are even,} \\ 3 & \textit{otherwise.} \end{cases}$

Proof. Case 1. m and n are even. Consequently, C_m and C_n are bipartite. By Proposition 2.2.5, $C_m \oplus C_n$ is bipartite. Hence $\chi(C_m \oplus C_n) = 2$.

Case 2. m or n are odd. Then $C_m \oplus C_n$ is not bipartite. Thus $\chi(C_m \oplus C_n) \geq 3$. Since $C_m \oplus C_n$ has at most 2 vertices with degree greater than 2, $C_m \oplus C_n$ is a 2-degenerated graph. By Theorem 2.2.4, $\chi(C_m \oplus C_n) \leq 3$. Thus $\chi(C_m \oplus C_n) = 3$. □

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Let G be a connected graph. The *line graph* $L(G)$ of G is the graph generated from G by $V(L(G)) = E(G)$ and for any two vertices $e, f \in V(L(G))$, vertex e and vertex f are adjacent in $L(G)$ if and only if edge e and edge f share a common vertex in G. If H is the line graph of G, we call G the *root graph* of H.

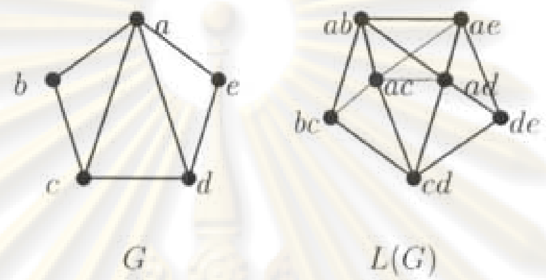

Figure 2.2.3: A graph G and its line graph $L(G)$

Since colorings of the line graph of a graph G are edge-colorings of G, it follows that the chromatic number of the line graph of G is equal to the edge-chromatic number of G.

Theorem 2.2.7. *For a glued graph $C_m \Phi C_n$, $\chi'(C_m \Phi C_n) \leq 3$.*

Proof. If $C_m \Phi C_n$ is a cycle, we are done. Assume that $C_m \Phi C_n$ is not a cycle.

Case 1. The clone of $C_m \Phi C_n$ is not P_2. Then every vertex in the line graph of $C_m \Phi C_n$ has degree at most 3. Hence $\Delta(L(C_m \Phi C_n)) \leq 3$. Since $L(C_m \Phi C_n)$ is neither an odd cycle nor a complete graph, by Theorem 1.2.7, $\chi(L(C_m \Phi C_n)) \leq \Delta(L(C_m \Phi C_n)) \leq 3$. Thus $\chi'(C_m \Phi C_n) \leq 3$.

Case 2. The clone of $C_m \Phi C_n$ is P_2. Let C_m be a cycle with a vertex set $\{u_1, u_2, \ldots, u_m\}$ and an edge set $\{e_1, e_2, \ldots, e_m\}$ where $e_i = u_i u_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $e_m = u_m u_1$. Let C_n be a cycle with a vertex set $\{v_1, v_2, \ldots, v_n\}$ and an edge set $\{f_1, f_2, \ldots, f_n\}$ where $f_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $f_n = v_n v_1$. Since the clone of $C_m \underset{P_2}{\Phi} C_n$ is P_2, without loss of generality, assume that we glue u_1 to v_1 and u_2 to v_2. Let $f : E(C_m \underset{P_2}{\Phi} C_n) \to [3]$ be an edge-coloring of $C_m \underset{P_2}{\oplus} C_n$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = e_k \text{ where } k \text{ is odd and } k < m, \\ 3 & \text{if } x = e_k \text{ where } k \text{ is even and } k < m, \\ 2 & \text{if } x = e_m, \\ 1 & \text{if } x = f_k \text{ where } k \text{ is odd and } k < n, \\ 2 & \text{if } x = f_k \text{ where } k \text{ is even and } k < n, \\ 3 & \text{if } x = f_n. \end{cases}$$

Then f is a proper edge-coloring from $E(C_m \underset{P_2}{\oplus} C_n)$ to [3]. Thus $\chi'(C_m \oplus C_n) \leq 3$. □

Theorem 2.2.8. $\chi'(C_m \oplus C_n) = \begin{cases} 2 & \textit{if } C_m \oplus C_n \textit{ is an even cycle,} \\ 3 & \textit{otherwise.} \end{cases}$

Proof. Case 1. $C_m \oplus C_n$ is a cycle. If $C_m \oplus C_n$ is an even cycle then $\chi'(C_m \oplus C_n) = 2$. If $C_m \oplus C_n$ is an odd cycle then $\chi'(C_m \oplus C_n) = 3$.

Case 2. $C_m \oplus C_n$ is not a cycle. Then $\Delta(C_m \oplus C_n) = 3$, hence, $\chi'(C_m \oplus C_n) \geq \Delta(C_m \oplus C_n) = 3$. By Theorem 2.2.7, we get $\chi'(C_m \oplus C_n) \leq 3$. Consequently, $\chi'(C_m \oplus C_n) = 3$. □

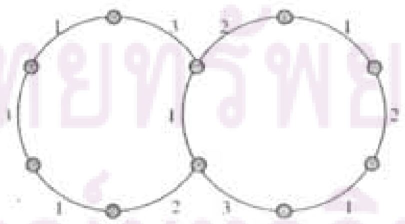

Figure 2.2.4: An edge-coloring of $C_6 \oplus C_6$ when its clone is P_2.

Theorem 2.2.9. [14] *Let G be a graph. Then* $\chi''(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$.

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Theorem 2.2.10. *For a glued graph* $C_m \oplus C_n$, $\chi''(C_m \oplus C_n) \leq 4$.

Proof. By Theorem 2.2.9, $\chi''(C_m \oplus C_n) \leq \lfloor \frac{3}{2}\Delta(C_m \oplus C_n) \rfloor = \lfloor \frac{3}{2} \times 3 \rfloor = 4$. □

Theorem 2.2.11. *For a glued graph* $C_m \oplus C_n$,

$$\chi''(C_m \oplus C_n) = \begin{cases} 3 & \text{if } C_m \oplus C_n \text{ is a cycle and } m = n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$$

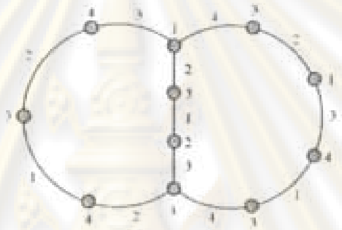

Figure 2.2.5: A total coloring of $C_7 \oplus C_8$ when its clone is P_4

Proof. *Case 1.* $C_m \oplus C_n$ is a cycle. Then $m = n$ and $C_m \oplus C_n \cong C_m \cong C_n$. If $m = n \equiv 0 \pmod 3$, by Theorem 2.2.3, $\chi''(C_m \oplus C_n) = \chi''(C_m) = 3$. If $m = n \equiv 1, 2 \pmod 3$, by Theorem 2.2.3, $\chi''(C_m \oplus C_n) = \chi''(C_m) = 4$.

Case 2. $C_m \oplus C_n$ is not a cycle. Then $\Delta(C_m \oplus C_n) = 3$. Thus $\chi''(C_m \oplus C_n) \geq \Delta(C_m \oplus C_n) + 1 = 4$. By Theorem 2.2.10, $\chi''(C_m \oplus C_n) \leq 4$. Hence $\chi''(C_m \oplus C_n) = 4$. □

Theorem 2.2.12. *Any glued graph of cycles satisfies the Total Coloring Conjecture.*

Proof. By Theorem 2.2.10, we get $\chi''(C_m \oplus C_n) \leq 4$. Since $\Delta(C_m \oplus C_n) + 2 \geq 4$, we get $\chi''(C_m \oplus C_n) \leq \Delta(C_m \oplus C_n) + 2$. □

Theorem 2.2.13. *If the glued graph* $C_m \oplus C_n$ *is a cycle and* $m = n \equiv 1, 2 \pmod 3$ *then* $C_m \oplus C_n$ *is of type* 2. *Otherwise,* $C_m \oplus C_n$ *is of type* 1.

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Proof. *Case 1.* $C_m \oplus C_n$ is not a cycle. Then $\Delta(C_m \oplus C_n) = 3$. By Theorem 2.2.10, $\chi''(C_m \oplus C_n) = 4 = \Delta(C_m \oplus C_n) + 1$. Hence, $C_m \oplus C_n$ is of type 1.

Case 2. $C_m \oplus C_n$ is a cycle. Then $m = n$ and $C_m \oplus C_n \cong C_m \cong C_n$. If $C_m \oplus C_n$ is a cycle and $m = n \equiv 0 \pmod 3$. By Theorem 2.2.3, we get $\chi''(C_m \oplus C_n) = 3 = \Delta(C_m \oplus C_n) + 1$. Thus $C_m \oplus C_n$ is of type 1. If $C_m \oplus C_n$ is a cycle and $m = n \equiv 1, 2 \pmod 3$, by Theorem 2.2.3, $\chi''(C_m \oplus C_n) = 4 = \Delta(C_m \oplus C_n) + 2$. □

Corollary 2.2.14.

(a) $C_m \oplus C_n$ *is of type* 1 *if and only if* $C_m \oplus C_n$ *is not a cycle or* $m \neq n$ *or* $m = n \equiv 0 \pmod 3$,

(b) $C_m \oplus C_n$ *is of type* 2 *if and only if* $C_m \oplus C_n$ *is a cycle and* $m = n \equiv 1, 2 \pmod 3$.

Proof. It follows from Theorem 2.2.12 and Theorm 2.2.13. □

Theorem 2.2.15. $\chi''(C_m \oplus C_n) \leq \chi''(C_m) + \chi''(C_n) - 2$. *The equality holds if* $m, n \equiv 0 \pmod 3$ *and* $\chi''(C_m \oplus C_n) = 4$.

Proof. By Theorem 2.2.10, we get $\chi''(C_m \oplus C_n) \leq 4$. Since $\chi''(C_m), \chi''(C_n) \geq 3$, we get $\chi''(C_m) + \chi''(C_n) - 2 \geq 4$. Then $\chi''(C_m \oplus C_n) \leq \chi''(C_m) + \chi''(C_n) - 2$. If $m, n \equiv 0 \pmod 3$, by Proposition 2.2.3, $\chi''(C_m) = \chi''(C_n) = 3$. Thus $\chi''(C_m) + \chi''(C_n) - 2 = 4$. Since $\chi''(C_m \oplus C_n) = 4$, we get $\chi''(C_m \oplus C_n) = 4 = \chi''(C_m) + \chi''(C_n) - 2$. □

2.3 Total Colorings of Glued Graphs of Bipartite Graphs

In this section, we investigate the values or bounds of the chromatic numbers, the edge-chromatic numbers and the total chromatic numbers of bipartite graphs and their glued graphs. Moreover, we prove that any glued graph of bipartite graphs satisfy the Total Coloring Conjecture.

Proposition 2.3.1. [9] *Let G be a nontrivial bipartite graph. Then* $\chi(G) = 2$.

Proof. It follows from the definition of a bipartite graph. □

Theorem 2.3.2. (König [15]) *For every bipartite graph G,* $\chi'(G) = \Delta(G)$.

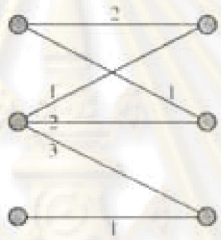

Figure 2.3.1: An edge-coloring of a bipartite graph

Proposition 2.3.3. *Let G be a bipartite graph. Then* $\chi''(G) \leq \Delta(G) + 2$.

Proof. Let G be a bipartite graph. It is easy to see that $\chi''(G) \leq \chi'(G) + \chi(G)$ By Proposition 2.3.1 and Theorem 2.3.2, $\chi(G) = 2$ and $\chi'(G) = \Delta(G)$. Then $\chi''(G) \leq \Delta(G) + 2$. □

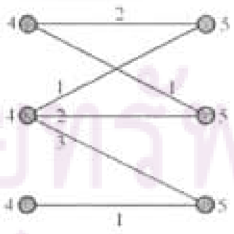

Figure 2.3.2: A total coloring of a bipartite graph

Theorem 2.3.4. *Let G_1 and G_2 be nontrivial graphs.*
Then $\chi(G_1 \oplus G_2) = 2$ *if and only if* $\chi(G_1) = 2$ *and* $\chi(G_2) = 2$.

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Proof. Sufficiency. Assume that $\chi(G_1 \Phi G_2) = 2$. Since $\chi(G_1) \leq \chi(G_1 \Phi G_2) = 2$ and G_1 is nontrivial, we get $\chi(G_1) = 2$. Similarly, $\chi(G_2) = 2$.
Necessity. Assume that $\chi(G_1) = 2$ and $\chi(G_2) = 2$. Then G_1 and G_2 are bipartite. By Proposition 2.2.5, $G_1 \Phi G_2$ is also bipartite.
Hence $\chi(G_1 \Phi G_2) = 2$. □

Remark 2.3.5. Let G_1 and G_2 be nontrivial bipartite graphs. Then $\chi(G_1 \Phi G_2) = 2$.

Proof. Let G_1 and G_2 be nontrivial bipartite graphs. By Proposition 2.2.5, $G_1 \Phi G_2$ is bipartie. Then $\chi(G_1 \Phi G_2) = 2$. □

Theorem 2.3.6. [11] *$K_{m,n}$ is of type 2 if and only if $m = n$.*

Theorem 2.3.7. *If G_1 and G_2 are bipartite graphs then $\chi'(G_1 \Phi G_2) = \Delta(G_1 \Phi G_2)$.*

Proof. Assume that G_1 and G_2 are bipartite graphs. By Proposition 2.2.5, $G_1 \Phi G_2$ is also bipartite. By Proposition 2.3.2, $\chi'(G_1 \Phi G_2) = \Delta(G_1 \Phi G_2)$. □

Theorem 2.3.8. *Any glued graph of bipartite graphs satisfies the Total Coloring Conjecture.*

Proof. Let G_1 and G_2 be bipartite graphs. By Proposition 2.2.5, $G_1 \Phi G_2$ is bipartite. By Proposition 2.3.3, $\chi''(G_1 \Phi G_2) \leq \Delta(G_1 \Phi G_2) + 2$. □

Theorem 2.3.9. *Let G_1 and G_2 be bipartite graphs, $\chi''(G_1 \Phi G_2) \leq \chi''(G_1) + \chi''(G_2) - 1$.*

Proof. By Theorem 2.3.8, we get $\chi''(G_1 \Phi G_2) \leq \Delta(G_1 \Phi G_2) + 2$. Then this theorem holds by Theorem 2.1.1. □

Example 2.3.10. There are bipartite graphs G_1 and G_2 such that $\chi''(G_1 \Phi G_2) = \chi''(G_1) + \chi''(G_2) - 2$. We consider C_m, C_n where $m, n \equiv 0 \pmod 6$ and the clone is an edge of them. Since $m, n \equiv 0 \pmod 3$, we get $\chi''(C_m) = \chi''(C_n) = 3$. By Theorem 2.2.10, since $C_m \Phi C_n$ is not a cycle, we get $\chi''(C_m \Phi C_n) = 4$. Hence $\chi''(C_m \Phi C_n) = 4 = 3 + 3 - 2 = \chi''(C_m) + \chi''(C_n) - 2$.

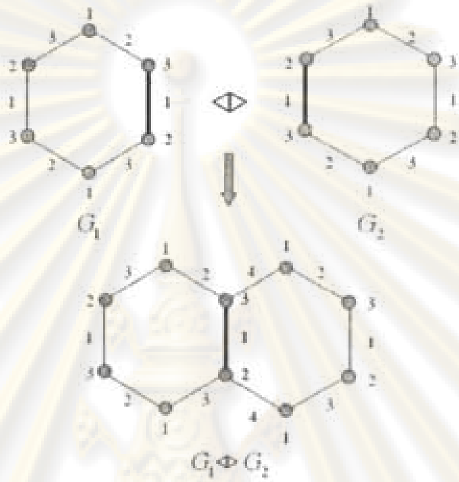

Figure 2.3.3: C_m, C_n and $C_m \Phi C_n$ are bipartite graphs with $\chi''(C_m \Phi C_n) = \chi''(C_m) + \chi''(C_n) - 2$

Figure 2.3.3 is an example of bipartite graphs with $\chi''(G_1 \Phi G_2) = \chi''(G_1) + \chi''(G_2) - 2$. Moreover, G_1, G_2 and $G_1 \Phi G_2$ are of type 1. Example 2.3.11 shows a glued graph of type 2 such that original graphs are of type 1. Furthermore, when original graphs are of type 2, a glued graph can be either of type 1 or of type 2 as shown in Example 2.3.12 and Example 2.3.13.

Example 2.3.11. According to the proper total coloring shown in Figure 2.3.4, $\chi''(G_1) \leq 3$ and $\chi''(G_2) \leq 4$. By Remark 1.2.1, $\chi''(G_1) \geq \Delta(G_1) + 1 = 3$ and $\chi''(G_2) \geq \Delta(G_2) + 1 = 4$. Hence G_1 and G_2 are of type 1. By Theorem 2.3.6, $G_1 \Phi G_2$ is of type 2.

When G_1 and G_2 are of type 2, $G_1 \Phi G_2$ can be both of type 1 and type 2 as shown in Example 2.3.12 and Example 2.3.13.

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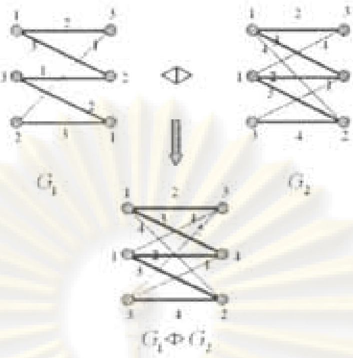

Figure 2.3.4: Both G_1 and G_2 are of type 1 while $G_1 \oplus G_2$ is of type 2

Example 2.3.12. In Figure 2.3.5, G_1 and G_2 are $K_{2,2}$. By Theorem 2.3.6, G_1 and G_2 are of type 2. Since $G_1 \oplus G_2$ is $K_{3,2}$, by Theorem 2.3.6, $G_1 \oplus G_2$ is of type 1.

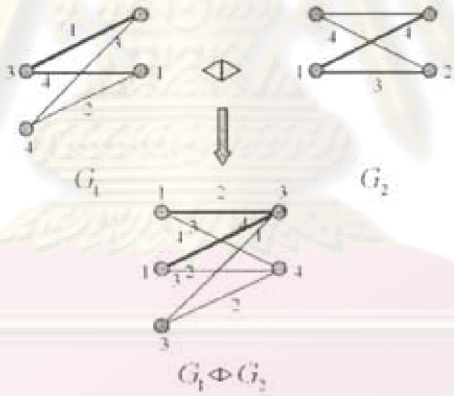

Figure 2.3.5: Both G_1 and G_2 are of type 2 while $G_1 \oplus G_2$ is of type 1

Example 2.3.13. In Figure 2.3.6, G_1 is $K_{2,2}$, G_2 is $K_{3,3}$ and $G_1 \oplus G_2$ is $K_{3,3}$. By Theorem 2.3.6, G_1, G_2 and $G_1 \oplus G_2$ are of type 2.

We show that any glued graph of bipartite graphs satisfies the Total Coloring Conjecture. It is an open problem to find a necessary and sufficient condition of the glued graph of bipartite graphs be either of type 1 or of type 2.

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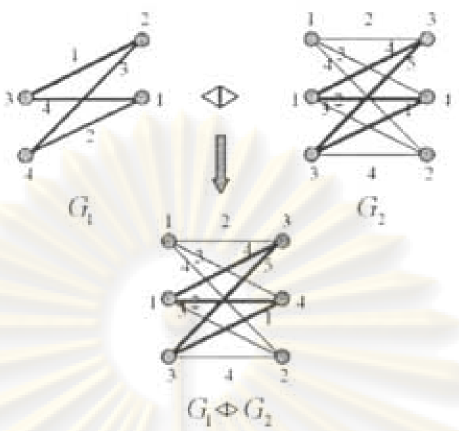

Figure 2.3.6: G_1, G_2 and $G_1 \Diamond G_2$ are of type 2

2.4 Total Colorings of Glued Graphs of Trees

In this section, we investigate the values or bounds of the chromatic numbers, the edge-chromatic numbers and the total chromatic numbers of trees and their glued graphs. Moreover, we prove that any glued graph of trees satisfies the Total Coloring Conjecture and give a necessary and sufficient condition to be either of type 1 or of type 2 of glued graphs of trees.

Throughout this thesis, $G - \{v_1, v_2, ..., v_k\}$ is the induced subgraph on $V(G) - \{v_1, v_2, ..., v_k\}$. We write $G - v$ instead of $G - \{v\}$.

Proposition 2.4.1. *Let T be a nontrivial tree. Then*

(a) $\chi(T) = 2$,

(b) $\chi'(T) = \Delta(T)$,

(c) $\chi''(T) = \begin{cases} \Delta(T) + 2 & \text{if } T \text{ is } P_2, \\ \Delta(T) + 1 & \text{otherwise.} \end{cases}$

Proof. (a) T is nontrivial bipartite. Then $\chi(T) = 2$.

(b) By Proposition 2.3.2, $\chi'(T) = \Delta(T)$.

(c) If T has only one vertex, then $\chi''(T) = 1 = \Delta(T) + 1$. If T is P_2, then we have $\chi''(T) = 3 = \Delta(T) + 2$. Assume that T is a tree with n vertices, where

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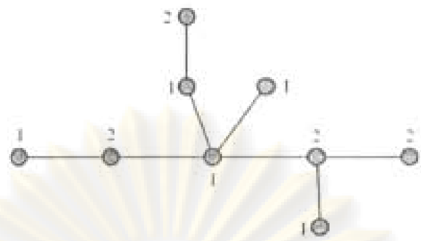

Figure 2.4.1: A coloring of a tree

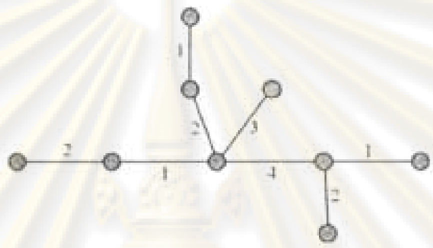

Figure 2.4.2: An edge-coloring of a tree

$n \geq 3$. Thus $\Delta(T) \geq 2$.

When $n = 3$, we get $T \cong P_3$. It is easy to see that $\chi''(T) = 3 = \Delta(T) + 1$.

Assume that $\chi''(T) = \Delta(T) + 1$ for all T with k vertices where $k \geq 3$. Let T be a tree with $k + 1$ vertices where $k \geq 3$ and $m = \Delta(T) + 1$. It suffices to show that there is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Since T is a tree, T has a vertex with degree 1, say v. Let u be a vertex which is adjacent to v.

Case 1. u is a vertex with maximum degree. Then $\Delta(T - v) + 1 = \Delta(T) = m - 1$. Since $T - v$ is a tree with k vertices where $k \geq 3$, by induction hypothesis, $\chi''(T - v) \leq \Delta(T - v) + 1 = m - 1$. Then there is a proper total coloring $f : V(T - v) \cup E(T - v) \to \{1, 2, ..., m - 1\}$. Since $m - 1 = \Delta(T) \geq 2$, there is a color r which differs from $f(u)$. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of T defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T-v) \cup E(T-v), \\ k & \text{if } x = uv, \\ r & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$.

Case 2. u is not a vertex with maximum degree in $T - v$. Then $\Delta(T-v)+1 = \Delta(T)+1 = m$. Since $T - v$ is a tree with k vertices where $k \geq 3$, by induction hypothesis, $\chi''(T-v) \leq \Delta(T-v)+1 = m$. Thus there is a proper total coloring $f : V(T-v) \cup E(T-v) \to \{1, 2, ..., m\}$. Since $d_{T-v}(u)+1 \leq \Delta(T-v) = \Delta(T) = m-1$, at most $m-1$ colors are used to color u and edges incident to u in $T-v$. There is a remaining color in $\{1, 2, ..., m\}$, say r. Since $m = \Delta(T)+1 \geq 2+1 = 3$, there is a color which differs from $f(u)$ and r, say r'. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of T defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T-v) \cup E(T-v), \\ r & \text{if } x = uv, \\ r' & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Hence $\chi''(T) \leq m = \Delta(T)+1$. Since $\chi''(T) \geq \Delta(T)+1$, we get $\chi''(T) = \Delta(T)+1$. □

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Example 2.4.2. Figure 2.4.3 shows an example of a total coloring of tree.

Proposition 2.4.3. [3] *Any glued graph of trees is a tree.*

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Theorem 2.4.4. *Let T_1 and T_2 be nontrivial trees. Then*

(a) $\chi(T_1 \Diamond T_2) = 2$,

(b) $\chi'(T_1 \Diamond T_2) = \Delta(T_1 \Diamond T_2)$,

(c) $T_1 \Diamond T_2$ *is of type* 1 *unless* $T_1 \cong T_2 \cong P_2$.

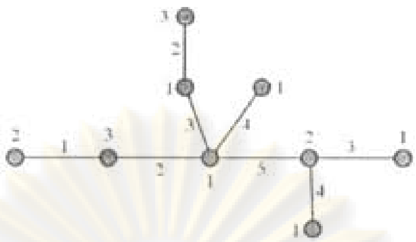

Figure 2.4.3: A total coloring of a tree

Proof. Proposition 2.4.3 states that any glued graph of trees is a tree. Then this theorem follows from Theorem 2.4.1. □

Remark 2.4.5. Any glued graph of trees satisfies the Total Coloring Conjecture.

Theorem 2.4.6. *Let T_1 and T_2 be nontrivial trees. Then $\chi'(T_1 \lhd\!\rhd T_2) \leq \chi'(T_1) + \chi'(T_2) - 1$. The equality holds if and only if $\Delta(T_1 \lhd\!\rhd T_2) = \Delta(T_1) + \Delta(T_2) - 1$.*

Proof. Let T_1 and T_2 be trees.

$$\begin{aligned} \chi'(T_1 \lhd\!\rhd T_2) &= \Delta(T_1 \lhd\!\rhd T_2), && \text{by Proposition 2.4.1 (b)}, \\ &\leq \Delta(T_1) + \Delta(T_2) - 1, && \text{by Remark 1.3.5}, \\ &= \chi'(T_1) + \chi'(T_2) - 1, && \text{by Proposition 2.4.1 (b)}. \end{aligned}$$

As shown above, if $\chi'(T_1 \lhd\!\rhd T_2) = \chi'(T_1) + \chi'(T_2) - 1$ if and only if $\Delta(T_1 \lhd\!\rhd T_2) = \Delta(T_1) + \Delta(T_2) - 1$. □

Theorem 2.4.7. *Let T_1 and T_2 be nontrivial trees. Then $\chi''(T_1 \lhd\!\rhd T_2) \leq \chi''(T_1) + \chi''(T_2) - 2$. The equality holds if and only if $\Delta(T_1 \lhd\!\rhd T_2) = \Delta(T_1) + \Delta(T_2) - 1$ and $T_1 \lhd\!\rhd T_2 \ncong P_2$.*

Proof. If $T_1 \lhd\!\rhd T_2 \cong P_2$ then $T_1 \cong T_2 \cong P_2$. Since $\chi''(P_2) = 3$, we get $\chi''(T_1) = \chi''(T_2) = \chi''(T_1 \lhd\!\rhd T_2) = 3$. Then $\chi''(T_1) + \chi''(T_2) - 2 = 4 > \chi''(T_1 \lhd\!\rhd T_2)$. Assume

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that $T_1 \Diamond T_2$ is not P_2. Thus

$$
\begin{aligned}
\chi''(T_1 \Diamond T_2) &= \Delta(T_1 \Diamond T_2) + 1, && \text{by Proposition 2.4.1,} \\
&\leq (\Delta(T_1) + \Delta(T_2) - 1) + 1, && \text{by Remark 1.3.5,} \\
&= \Delta(T_1) + \Delta(T_2) \\
&= \chi''(T_1) + \chi''(T_2) - 2, && \text{by Proposition 2.4.1.}
\end{aligned}
$$

As proof above, $\chi''(T_1 \Diamond T_2) = \chi''(T_1) + \chi''(T_2) - 2$ if and only if $\Delta(T_1 \Diamond T_2) = \Delta(T_1) + \Delta(T_2) - 1$ and $T_1 \Diamond T_2$ is not P_2. $\square$

Example 2.4.8. Figure 2.4.4 shows examples of trees and their glued graph making the equality in Theorem 2.4.7 holds. By Proposition 2.4.1 (c), we get $\chi''(T_1) = 4, \chi''(T_2) = 3$ and $\chi''(T_1 \Diamond T_2) = 5$. Hence $\chi''(T_1 \Diamond T_2) = \chi''(T_1) + \chi''(T_2) - 2$.

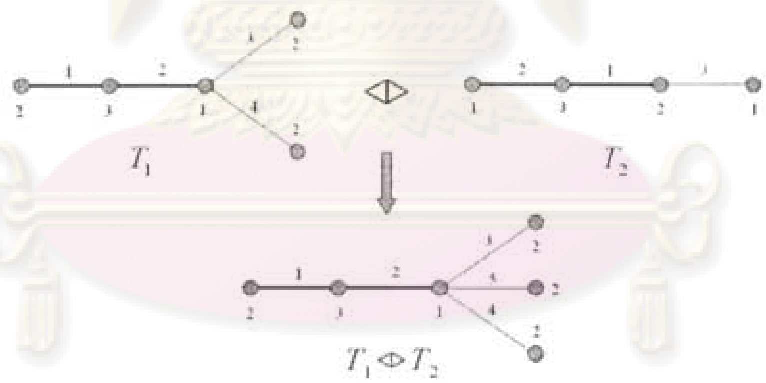

Figure 2.4.4: Total colorings of T_1, T_2 and $T_1 \Diamond T_2$

2.5 Total Colorings of Glued Graphs of Complete Graphs

In this section, we investigate the values or bounds of the chromatic numbers, the edge-chromatic numbers and the total chromatic numbers of complete graphs and their glued graphs. Moreover, we prove that any glued graph of complete

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graphs satisfies the Total Coloring Conjecture and give a necessary and sufficient condition to be either of type 1 or of type 2 of glued graphs of complete graphs.

Proposition 2.5.1. $\chi(K_n) = n$.

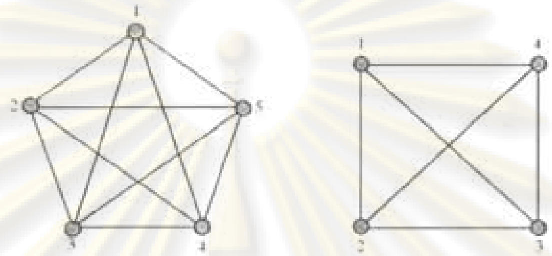

Figure 2.5.1: Colorings of complete graphs with 5 and 4 vertices

Proof. Proposition holds since each vertex is adjacent to all remaining vertices. □

Proposition 2.5.2. [10] $\chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$

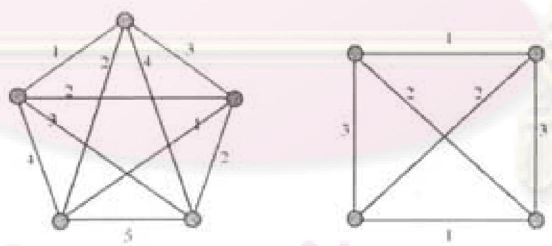

Figure 2.5.2: Edge-colorings of complete graphs with 5 and 4 vertices

Proposition 2.5.3. [11] $\chi''(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$

Lemma 2.5.4. *If a glued graph* $K_m \oplus K_n$ *is a simple graph, then* $\Delta(K_m \oplus K_n) = n(K_m \oplus K_n) - 1$.

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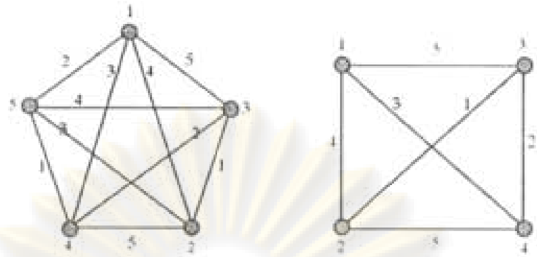

Figure 2.5.3: Total colorings of complete graphs with 5 and 4 vertices

Proof. Assume that $K_m \oplus K_n$ is a simple graph. Then the clone of $K_m \oplus K_n$ is a complete graph, say K_r. Each vertex in the clone of $K_m \underset{K_r}{\oplus} K_n$ gives the maximum degree. Hence $\Delta(K_m \underset{K_r}{\oplus} K_n) = (m-1)+(n-1)-(r-1) = m+n-r-1$. Besides, $n(K_m \underset{K_r}{\oplus} K_n) = n(K_m) + n(K_n) - n(K_r) = m+n-r$. Therefore, $\Delta(K_m \underset{K_r}{\oplus} K_n) = n(K_m \underset{K_r}{\oplus} K_n) - 1$. □

Theorem 2.5.5. *Any glued graph of complete graphs satisfies the Total Coloring Conjecture.*

Proof. Let K_m and K_n be complete graphs of order m and n, respectively. Let $k = n(K_m \oplus K_n)$. Then

$$\begin{aligned}
\chi''(K_m \oplus K_n) &\leq \chi''(K_k) && \text{since } K_m \oplus K_n \text{ is a subgraph of } K_k, \\
&\leq \Delta(K_k) + 2, && \text{by Proposition 2.5.3,} \\
&= n(K_m \oplus K_n) - 1 + 2 \\
&= \Delta(K_m \oplus K_n) + 2, && \text{by Lemma 2.5.4.}
\end{aligned}$$

□

We have already proved that any glued graph of complete graphs satisfies the Total Coloring Conjecture. Next, Theorem 2.5.8 gives a necessary and sufficient condition to be either of type 1 or of type 2 for a glued graph of complete graphs by using Theorem 2.5.6 and Lemma 2.5.7.

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A *matching* in a graph G is a set of edges with no shared endpoints. The maximum size of matching of a graph G is denoted by $\alpha'(G)$

Theorem 2.5.6. [16] *Suppose that G is a graph of order $2n$ and $\Delta(G) = 2n-1$, then $\chi''(G) = 2n$ if and only if $e(\overline{G}) + \alpha'(\overline{G}) \geq n$.*

Lemma 2.5.7. *For $m, n, r \in \mathbb{R}$,*
$m < r + \frac{2r-n}{2n-2r-1}$ if and only if $(m-r)(n-r) + (n-r) < \frac{m+n-r}{2}$.

Proof.

$$
\begin{aligned}
m < r + \frac{2r-n}{2n-2r-1} &\Leftrightarrow m < \frac{r(2n-2r-1)+(2r-n)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{2r(n-r)-(n-r)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{n-r}{2n-2r-1}(2r-1) \\
&\Leftrightarrow \frac{2n-2r-1}{n-r}m < 2r-1 \\
&\Leftrightarrow 2m - \frac{m}{n-r} < 2r-1 \\
&\Leftrightarrow 2m-2r+1 < \frac{1}{n-r}m \\
&\Leftrightarrow (n-r)(2m-2r+1) < m \\
&\Leftrightarrow (n-r)(2m-2r)+(n-r) < m \\
&\Leftrightarrow 2(n-r)(m-r)+2(n-r) < m+n-r \\
&\Leftrightarrow (n-r)(m-r)+(n-r) < \frac{m+n-r}{2}
\end{aligned}
$$

□

Theorem 2.5.8. *Let $m \geq n$. If $m+n-r$ is even and $m < r + \frac{2r-n}{2n-2r-1}$ then $K_m \underset{K_r}{\oplus} K_n$ is of type 2. Otherwise, $K_m \underset{K_r}{\oplus} K_n$ is of type 1.*

Proof. Let $m \geq n$ and $G = K_m \underset{K_r}{\oplus} K_n$.
Case 1. $m+n-r$ is odd. By Proposition 2.5.3, $\chi''(K_{m+n-r}) = m+n-r =$

$\Delta(K_{m+n-r})+1$. Since G is a subgraph of K_{m+n-r} and $\Delta(G)=\Delta(K_{m+n-r})$, we get $\chi''(G)\le\chi''(K_{m+n-r})=\Delta(K_{m+n-r})+1=\Delta(G)+1$. Thus G is of type 1.

Case 2. $m+n-r$ is even. By Lemma 2.5.4, $\Delta(G)=n(G)-1=m+n-r-1$. The complement of G, $\overline{K_m \underset{K_r}{\oplus} K_n}$ has only one nontrivial component, $K_{m-r,n-r}$. Then $e(\overline{G})=(m-r)(n-r)$. Since $m\ge n$, we get $\alpha'(\overline{G})=n-r$. Thus $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)$. If $m\ge r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.5.7, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)\ge\frac{m+n-r}{2}$. Consequently, by Theorem 2.5.6, G is of type 1. If $m<r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.5.7, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)<\frac{m+n-r}{2}$. Hence, by Theorem 2.5.6, $\chi''(G)\ne n+m-r$. Since $n+m-r=n(G)=\Delta(G)+1$, we have $\chi''(G)\ne\Delta(G)+1$. By Theorem 2.5.5, $\chi''(G)\le\Delta(G)+2$. Therefor, $\chi''(G)=\Delta(G)+2$; hence, G is of type 2. □

Corollary 2.5.9. *Let $m\ge n$. Then*

(a) $K_m \underset{K_r}{\oplus} K_n$ *is of type 1 if and only if $m+n-r$ is odd or* $m\ge r+\dfrac{2r-n}{2n-2r-1}$,

(b) $K_m \underset{K_r}{\oplus} K_n$ *is of type 2 if and only if $m+n-r$ is even and* $m<r+\dfrac{2r-n}{2n-2r-1}$.

Proof. They follow immediately from Theorem 2.5.5 and Theorem 2.5.8. □

Theorem 2.5.10. $\chi''(K_m \oplus K_n)\le\chi''(K_m)+\chi''(K_n)-2$.

Proof. Since the clone of $K_m \oplus K_n$ must be nontrivial, we get $m,n\ge 2$. If $m=2$, we get the clone of $K_m \oplus K_n$ is K_2 and $K_m \oplus K_n=K_n$. Since $\chi''(K_2)=3$, we get $\chi''(K_m \oplus K_n)<\chi''(K_n)+1=\chi''(K_m)+\chi''(K_n)-2$. If $n=2$, similarly, $\chi''(K_m \oplus K_n)<\chi''(K_m)+\chi''(K_n)-2$. Assume that $m,n\ge 3$.

$$\begin{aligned}\chi''(K_m \oplus K_n) &\le \Delta(K_m \oplus K_n)+2, && \text{by Theorem 2.5.5,}\\ &\le \Delta(K_m)+\Delta(K_n)-1+2, && \text{by Lemma 1.3.5,}\\ &\le \chi''(K_m)+\chi''(K_n)-1, && \text{by Remark 1.2.1.}\end{aligned}$$

Note that $\chi''(K_m \oplus K_n)=\chi''(K_m)+\chi''(K_n)-1$ if a vertex with maximum degree of K_m is glued to a vertex with maximum degree of K_n and the corre-

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sponding vertex in clone has degree 1, K_m and K_n are of type 1 and $K_m \oplus K_n$ is of type 2.

Assume that a vertex with maximum degree of K_m is glued to a vertex with maximum degree of K_n and the corresponding vertex in clone has degree 1 and K_m and K_n are of type 1. Since the clone must be a complete graph, the clone is K_2. Without loss of generality, assume that $m \geq n$. Since $m \geq 3$ and $n \geq 3$, we get $(n-2)(2m-2(2)+1) = (n-2)(2m-3) \geq m$. By Theorem 2.5.8, $K_m \underset{K_r}{\oplus} K_n$ is of type 1. Thus $\chi''(K_m \oplus K_n) \neq \chi''(K_m) + \chi''(K_n) - 1$. Hence $\chi''(K_m \oplus K_n) \leq \chi''(K_m) + \chi''(K_n) - 2$. □

Theorem 2.5.11. *$\chi''(K_m \oplus K_n) = \chi''(K_m) + \chi''(K_n) - 2$ if and only if m, n are odd and the clone of $K_m \oplus K_n$ is K_2.*

Proof. Since we are interested in simple glued graphs, the clone is a complete graph, say K_r. By Theorem 2.5.5, $\chi''(K_m \underset{K_r}{\oplus} K_n) \leq \Delta(K_m \underset{K_r}{\oplus} K_n)+2$. By Lemma 2.5.4, we have $\Delta(K_m \underset{K_r}{\oplus} K_n) = n(K_m \underset{K_r}{\oplus} K_n) - 1$. Then $\chi''(K_m \underset{K_r}{\oplus} K_n) = \Delta(K_m \underset{K_r}{\oplus} K_n) + 1 = n(K_m \underset{K_r}{\oplus} K_n) = m + n - r$ or $\chi''(K_m \underset{K_r}{\oplus} K_n) = \Delta(K_m \underset{K_r}{\oplus} K_n) + 2 = n(K_m \underset{K_r}{\oplus} K_n) + 1 = m + n - r + 1$.

Case 1. m and n are odd. By Proposition 2.5.3, $\chi''(K_m) + \chi''(K_n) - 2 = m + n - 2$. Then $\chi''(K_m \underset{K_r}{\oplus} K_n) = \chi''(K_m) + \chi''(K_n) - 2$ if and only if $r = 2$.

Case 2. Either m or n is odd. Then $\chi''(K_m) + \chi''(K_n) - 2 = m + n - 1$. If $r \geq 3$ then $\chi''(K_m \underset{K_r}{\oplus} K_n) \leq m + n - r + 1 \leq m + n - 2 < \chi''(K_m) + \chi''(K_n) - 2$. Assume that $r = 2$. Then $m + n - r$ is odd. By Corollary 2.5.9, $\chi''(K_m \underset{K_2}{\oplus} K_n) = \Delta(K_m \underset{K_2}{\oplus} K_n) + 1 = m + n - 2 < \chi''(K_m) + \chi''(K_n) - 2$.

Case 3. m and n are even. By Proposition 2.5.3, $\chi''(K_m) + \chi''(K_n) - 2 = m + n > \chi''(K_m \underset{K_r}{\oplus} K_n)$ because $r \geq 2$. □

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CHAPTER III

TRIMMED GRAPHS VS GLUED GRAPHS

3.1 Total Colorings of t-trimmed Graphs

A graph H is a *t-trimmed graph* of a graph G if G can be reduced to a graph H by successive removal of vertices with degree at most t. Among t-trimmed graphs of G, the smallest t-trimmed graph of G is the one with the minimum number of vertices.

Example 3.1.1. A graph G have a lot of 2-trimmed graphs but only one smallest 2-trimmed graph.

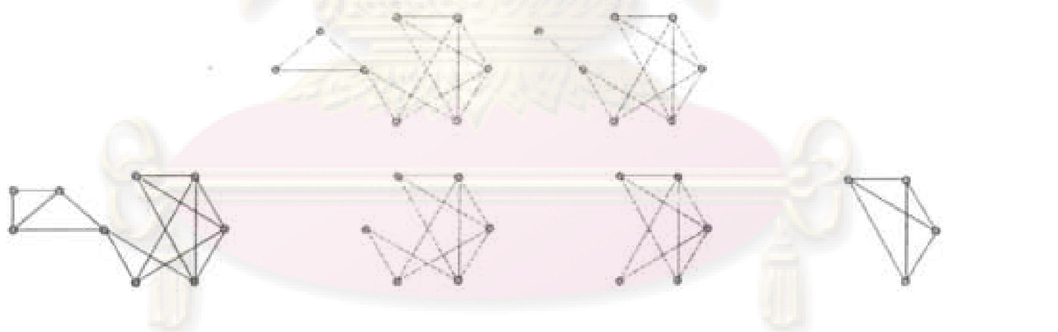

Figure 3.1.1: A graph G with its 2-trimmed graphs and its smallest 2-trimmed graph

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Theorem 3.1.2. *The is the only one smallest t-trimmed graph of a graph G, unless the smallest t-trimmed graph has one vertex.*

Proof. If the smallest trimmed graph of a graph G has only one vertex, then it is unique up to isomorphism. Assume that the smallest trimmed graph of a graph G has more than one vertex. Let H_1 and H_2 be the smallest t-trimmed graphs of G. Let H_1 be obtained from successive removal vertices $v_1, v_2, ..., v_k$, respectively. Assume that $H_1 \neq H_2$. By the definition of t-trimmed graph, a t-trimmed graph of G is an induced subgraph of G. Then $V(H_1) \neq V(H_2)$. Since H_2 is the smallest t-trimmed graph of G, $V(H_2) \not\subseteq V(H_1)$. Let $j \in [k]$ be the smallest number such that $v_j \in V(H_2) - V(H_1)$. If $j = 1$, let $K = G$. Then H_2 is a subgraph of K. If $j \geq 2$, let $K = G - \{v_1, v_2, ..., v_{j-1}\}$. Since $v_1, v_2, ..., v_{j-1} \notin V(H_2)$, H_2 is a subgraph of K. Both cases, H_2 is a subgraph of K. Since $d_K(v_j) \leq t$, we get $d_{H_2}(v_j) \leq t$. Thus $H_2 - v_j$ is a t-trimmed graph of G. It is a contradiction because H_2 is the smallest t-trimmed graph. Hence $H_1 = H_2$. □

Lemma 3.1.3. *Let G be a graph with $\Delta(G) \geq 2$ and contain a vertex v with degree 1. If $\chi''(G - v) \leq \Delta(G - v) + 2$ and $\Delta(G) = \Delta(G - v) + 1$, then G is of type 1.*

Proof. Since v has degree 1, let u be a vertex of G which is adjacent to v. Assume that $\chi''(G - v) \leq \Delta(G - v) + 2$ and $\Delta(G) = \Delta(G - v) + 1$. Since $\Delta(G) = \Delta(G - v) + 1$, u is a vertex with maximum degree in $G - v$. Let $k = \Delta(G) + 1$. It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

We get $\Delta(G-v)+2 = (\Delta(G)-1)+2 = k$. Since $\chi''(G-v) \leq \Delta(G-v)+2$, there is a proper total coloring $f : V(G - v) \cup E(G - v) \to [k]$. Since $d_{G-v}(u) + 1 \leq \Delta(G - v) + 1 = \Delta(G) = k - 1$, we use at most $k - 1$ colors to color u and edges incident to u in $G - v$, there is a remaining color in $[k]$, say r. Since $k = \Delta(G) + 1 \geq 3$, there is a color s which differs from $f(u)$ and r. Let

$f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = uv, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$. Hence $\chi''(G) = \Delta(G) + 1$ and G is of type 1. □

Lemma 3.1.4. *Let v be a vertex with degree 1 of a graph G. If $\chi''(G-v) \leq \Delta(G-v) + 2$ then $\chi''(G) \leq \Delta(G) + 2$.*

Proof. Since v is a vertex with degree 1, let u be the vertex which is adjacent to v. Assume that $\chi''(G-v) \leq \Delta(G-v) + 2$. Let $k = \Delta(G) + 2$. It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Case 1. u is a vertex with maximum degree in $G - v$.

Then $\Delta(G) = \Delta(G-v) + 1$. By Lemma 3.1.3, $\chi''(G) = \Delta(G) + 1 < \Delta(G) + 2$.

Case 2. u is not a vertex with maximum degree in $G - v$.

Hence $\Delta(G-v) + 2 = \Delta(G) + 2 = k$. Since $\chi''(G-v) \leq \Delta(G-v) + 2$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [k]$. Since $d_{G-v}(u) + 1 \leq \Delta(G-v) + 1 = \Delta(G) + 1 = k - 1$, we use at most $k - 1$ colors to color u and edges incident to u in $G - v$, there is a remaining color in $[k]$, say r. Since $k = \Delta(G) + 2 \geq d(v) + 2 = 3$, there is a color which differs from $f(u)$ and r, say s. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = uv, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$. Hence $\chi''(G) \leq k = \Delta(G) + 2$. □

Lemma 3.1.5. *Let v be a vertex with degree 2 of a graph G. If $\chi''(G-v) \leq \Delta(G-v)+2$ then $\chi''(G) \leq \Delta(G)+2$.*

Proof. Since v is a vertex with degree 2, let u_1 and u_2 be vertices which are adjacent to v. Assume that $\chi''(G-v) \leq \Delta(G-v)+2$. If $\Delta(G) \leq 2$, then G is a path or a cycle. So $\chi''(G) \leq \Delta(G)+2$. Assume that $\Delta(G) \geq 3$. Let $k = \Delta(G)+2$. It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Case 1. u_1 or u_2 is a vertex with maximum degree in $G-v$. Without loss of generality, assume that u_1 is a vertex with maximum degree in $G-v$. Then $\Delta(G-v)+2 = (\Delta(G)-1)+2 = k-1$. Since $\chi''(G-v) \leq \Delta(G-v)+2$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [k-1]$. Since $d_{G-v}(u_2)+1 \leq \Delta(G-v)+1 = \Delta(G) = k-2$, we use at most $k-2$ colors to color u_2 and edges incident to u_2 in $G-v$, there is a remaining color in $[k]$, say r. Since $\Delta(G) \geq 3$, we get $k \geq 5$. Let s be a color which differs from $f(u_1), f(u_2), r$ and k. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ k & \text{if } x = u_1 v, \\ r & \text{if } x = u_2 v, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Case 2. u_1 and u_2 are not vertices with maximum degree in $G-v$. Then $\Delta(G-v)+2 = \Delta(G)+2 = k$. Since $\chi''(G-v) \leq \Delta(G-v)+2$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [k]$. Since $d_{G-v}(u_1)+1 \leq \Delta(G-v) = \Delta(G) = k-2$, we use at most $k-2$ colors to color u_1 and edges incident to u_1 in $G-v$. Then there are 2 remaining unused colors. Let one be r. Similarly for

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u_2, there are 2 remaining colors. Pick the one which differs from r, say r'. Since $\Delta(G) \geq 3$, we get $k \geq 5$. Let s be a color which differs from $f(u_1), f(u_2), r$ and r'. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = u_1 v, \\ r' & \text{if } x = u_2 v, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$. Hence $\chi''(G) \leq k = \Delta(G) + 2$. □

Theorem 3.1.6. *If a graph G has a 2-trimmed graph H such that $\chi''(H) \leq \Delta(H) + 2$ then $\chi''(G) \leq \Delta(G) + 2$. In particular, if a graph G has a 1-trimmed graph K such that $\chi''(K) \leq \Delta(K) + 2$ then $\chi''(G) \leq \Delta(G) + 2$.*

Proof. Assume that a graph G has a 2-trimmed graph H such that $\chi''(H) \leq \Delta(H) + 2$. Without loss of generality, let H be obtained from G by successive removal vertices $v_k, v_{k-1}, \ldots, v_1$, respectively. Let $H_0 = H$ and $H_i = G[V(H) \cup \{v_1, v_2, \ldots, v_i\}]$. Let $P(n)$ be the statement that $\chi''(H_n) \leq \Delta(H_n) + 2$.
Basic Step. By the assumption, $\chi''(H_0) \leq \Delta(H_0) + 2$.
Inductive Step. Assume that $\chi''(H_{i-1}) \leq \Delta(H_{i-1}) + 2$ when $i \leq k$. By the definition of 2-trimmed graph, $d_{H_i}(v_i) \leq 2$. If $d_{H_i}(v_i) = 1$, by Lemma 3.1.4 and the induction hypothesis, $\chi''(H_i) \leq \Delta(H_i) + 2$. If $d_{H_i}(v_i) = 2$, by Lemma 3.1.5 and the induction hypothesis, $\chi''(H_i) \leq \Delta(H_i) + 2$. By mathematical induction, we get $\chi''(G) \leq \Delta(G) + 2$. □

An *outerplanar graph* is a graph with an imbedding in the plane such that every vertex appears on the boundary of the exterior face.

Theorem 3.1.8 shows that every outerplanar graph satisfies the Total Coloring Conjecture.

Proposition 3.1.7. [9] *Every outerplanar graph has a vertex of degree at most* 2.

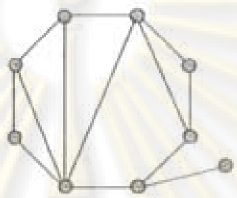

Figure 3.1.2: An outerplanar graph

Theorem 3.1.8. *For every outerplanar graph* G, *we have* $\chi''(G) \leq \Delta(G) + 2$.

Proof. By the fact that a subgraph of outerplanar graph is an outerplanar graph and Proposition 3.1.7, the smallest 2-trimmed graph of every outerplanar graph is a trivial graph with one vertex. Since a trivial graph satisfies the Total Coloring Conjecture, by Theorem 3.1.6, every outerplanar graph satisfies the Total Coloring Conjecture. □

Lemma 3.1.9. *Let* G *be a graph with* $\Delta(G) \geq 2$ *and containing a vertex with degree* 1, *say* v. *If* $\chi''(G - v) = \Delta(G - v) + 1$ *then* $\chi''(G) = \Delta(G) + 1$.

Proof. Let G be a graph with $\Delta(G) \geq 2$. Since v is a vertex with degree 1, let u be the vertex which is adjacent to v. Assume that $\chi''(G - v) \leq \Delta(G - v) + 1$. Let $k = \Delta(G) + 1$. It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Case 1. u is a vertex with maximum degree in $G - v$. We get $\Delta(G - v) = \Delta(G) - 1$. Then $\Delta(G - v) + 1 = \Delta(G) = k - 1$. Since $\chi''(G - v) \leq \Delta(G - v) + 1$, there is a proper total coloring $f : V(G - v) \cup E(G - v) \to [k - 1]$. Since $k - 1 = \Delta(G) \geq 2$.

there is a color s which differs from $f(u)$. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ k & \text{if } x = uv, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Case 2. u is not a vertex with maximum degree in G_v. We get $\Delta(G-v) = \Delta(G)$. Then $\Delta(G-v)+1 = \Delta(G)+1 = k$. Since $\chi''(G-v) \leq \Delta(G-v)+1$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [k]$. Since $d_{G-v}(u)+1 \leq \Delta(G-v) = \Delta(G) = k-1$, we use at most $k-1$ colors to color u and edges incident to u in $G-v$, there is a remaining color in $[k]$, say r. Since $k = \Delta(G)+1 \geq 3$, there is a color which differs from $f(u)$ and r, say s. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = uv, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$. Hence $\chi''(G) \leq k = \Delta(G)+1$. □

$\Delta(G) \geq 2$ is a sufficient condition in Lemma 3.1.9. Since G is connected, when $\Delta(G) = 1$, G is K_2. Then $\chi''(G) = 3 = \Delta(G)+2$. However, for any vertex v of G, we get $\chi''(G-v) = 1 = \Delta(G-v)+1$.

Theorem 3.1.10. *If a graph G has a 1-trimmed graph K such that K is of type 1 then G is of type 1.*

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Proof. Assume that a graph G has a 1-trimmed graph K such that $\chi''(K) = \Delta(K) + 1$. Without loss of generality, let K be obtained from G by successive removal vertices $v_k, v_{k-1}, \ldots, v_1$, respectively. Let $K_0 = K$ and $K_i = G[V(K) \cup \{v_1, v_2, \ldots, v_i\}]$. Let $P(n)$ be the statement that $\chi''(K_n) = \Delta(K_n) + 1$.

Basic Step. By the assumption, $\chi''(K_0) = \Delta(K_0) + 1$.

Inductive Step. Assume that $\chi''(K_{i-1}) = \Delta(K_{i-1}) + 1$ when $i \leq k$. By the definition of 1-trimmed graph, $d_{K_i}(v_i) = 1$. From Lemma 3.1.9 and the induction hypothesis, $\chi''(K_i) = \Delta(K_i) + 1$. By mathemaical induction, we get $\chi''(G) = \Delta(G) + 1$. □

In other words, let K be a 1-trimmed graph of G. If K is of type 1, so is G. However, if K is of type 2, G can be both of type 1 and type 2 as illustrated in Example 3.1.13 and Example 3.1.14.

Lemma 3.1.11. *Let G be a graph with $\Delta(G) \geq 2$ and v be a vertex with degree 1. If $\chi''(G - v) \leq \Delta(G - v) + t$ then $\chi''(G) \leq \Delta(G) + t$ for each positive integer t.*

Proof. Let G be a graph with $\Delta(G) \geq 2$ and v be a vertex with degree 1 of G. Since v is a vertex with degree 1, let u be the vertex which is adjacent to v. Assume that $\chi''(G - v) \leq \Delta(G - v) + t$. If $t = 1$, by Lemma 3.1.4, $\chi''(G) \leq \Delta(G) + 1$. If $t = 2$, by Lemma 3.1.5, $\chi''(G) \leq \Delta(G) + 2$. Assume that $t \geq 3$. Let $k = \Delta(G) + t$. It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[k]$.

Then $\Delta(G - v) + t \leq \Delta(G) + t = k$. Since $\chi''(G - v) \leq \Delta(G - v) + t$, there is a proper total coloring $f : V(G - v) \cup E(G - v) \to [k]$. Since $d_{G-v}(u) + 1 \leq \Delta(G - v) + 1 \leq \Delta(G) + 1 \leq k - 1$, we use at most $k - 1$ colors to color u and edges incident to u in $G - v$, there is a remaining color in $[k]$, say r. Since

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$k = \Delta(G) + t \geq 3$, there is a color which differs from $f(u)$ and r, say s. Let $f' : V(G) \cup E(G) \to [k]$ be a total coloring of a graph G defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = uv, \\ s & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[k]$. Hence $\chi''(G) \leq k = \Delta(G) + t$. □

Theorem 3.1.12. *If a graph G has a 1-trimmed graph K such that $\chi''(K) \leq \Delta(K) + t$ then $\chi''(G) \leq \Delta(G) + t$.*

Proof. Assume that a graph G has a 1-trimmed grpah K such that $\chi''(K) \leq \Delta(K) + t$. Without loss of generality, let K be obtained from G by successive removal vertices $v_k, v_{k-1}, \ldots, v_1$, respectively. Let $K_0 = K$ and $K_i = G[V(K) \cup \{v_1, v_2, \ldots, v_i\}]$. Let $P(n)$ be the statement that $\chi''(K_n) = \Delta(K_n) + t$.
Basic Step. By the assumption, $\chi''(K_0) = \Delta(K_0) + t$.
Inductive Step. Assume that $\chi''(K_{i-1}) = \Delta(K_{i-1}) + t$ when $i \leq k$. By the definition of 1-trimmed graph, $d_{K_i}(v_i) = 1$. By Lemma 3.1.11 and the induction hypothesis, $\chi''(K_i) = \Delta(K_i) + t$. By mathematical induction, we get $\chi''(G) = \Delta(G) + t$. □

Example 3.1.13. Let G be a graph as in Figure 3.1.3. As the given proper total coloring shown in the figure, $\chi''(G) = \Delta(G) + 1$. Hence G is of type 1. Moreover, K_4 is a 1-trimmed graph of G. Since $\chi''(K_4) = 5$, K_4 is of type 2.

Cycles whose length are not divisible by 3 and complete graphs with even vertices are of type 2 [7][11]. Fews other type 2 graphs are found. In 1992, Bor-Liang Chen and Hung-Lin Fu found nonregular type 2 graphs [17]. Their results

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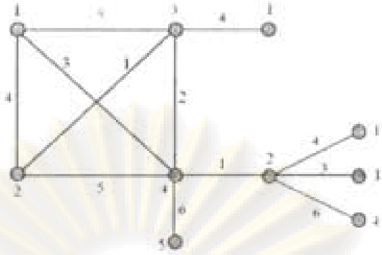

Figure 3.1.3: A type 1 graph having a 1-trimmed graph of type 2

and Theorem 3.1.6 yield a construction of type 2 graphs whose 1-trimmed graphs are of type 2.

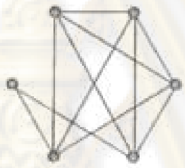

Figure 3.1.4: A type 2 graph which is constructed by Chen and Fu[17]

Example 3.1.14. Let K be a graph in Figure 3.1.4 and let G be a graph whose the smallest 1-trimmed graph K and $\Delta(G) = \Delta(K)$. Figure 3.1.5 shows an example of such a graph.

Since K is of type 2 and K is a subgraph of a graph G, $\chi''(G) \geq \chi''(K) = \Delta(K) + 2 = \Delta(G) + 2$. Since the smallest 1-trimmed graph of a graph G in Figure 3.1.5 is K and $\chi''(K) = \Delta(K) + 2$, by Theorem 3.1.6, we get $\chi''(G) \leq \Delta(G) + 2$. Then we get $\chi''(G) = \Delta(G) + 2$. Hence G is of type 2.

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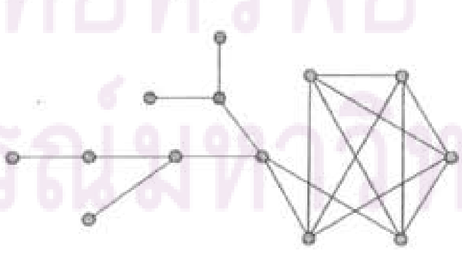

Figure 3.1.5: A type 2 graph which has a 1-trimmed graph of type 2

Theorem 3.1.15. *Let G be a graph with $\Delta(G) \geq 2$. Let K be a 1-trimmed graph of type 2 of G. Then G is of type 1 if and only if $\Delta(G) > \Delta(K)$.*

Proof. Let G be a graph with $\Delta(G) \geq 2$ and K a 1-trimmed graph of G. Assume that K is of type 2.

Necessity. Assume that $\Delta(G) \leq \Delta(K)$. Since K is a subgraph of G, $\Delta(K) = \Delta(G)$. Since K is of type 2, we get $\chi''(K) = \Delta(K) + 2$. Also $\chi''(G) \geq \chi''(K) = \Delta(K) + 2 = \Delta(G) + 2$. Hence G is not of type 1.

Sufficiency. Assume that $\Delta(G) > \Delta(K)$. Let K be obtained from successive removal vertices $v_1, v_2, ..., v_k$, respectively. Let $j \in [k]$ be the smallest number such that $\Delta(G - \{v_1, v_2, ..., v_j\}) < \Delta(G)$. If $j = 1$, by Lemma 3.1.3, $\chi''(G) \leq \Delta(G) + 1$. Assume that $j \geq 2$. Let $K_1 = G - \{v_1, v_2, ..., v_j\}$ and $K_2 = G - \{v_1, v_2, ..., v_{j-1}\}$. Since $\chi''(K) = \Delta(K) + 2$ and K is a 1-trimmed graph of K_1, by Theorem 3.1.6, we get $\chi''(K_1) \leq \Delta(K_1) + 2$. Since $\Delta(K_1) = \Delta(K_2) + 1$, by Lemma 3.1.3, K_2 is of type 1. Since K_2 is a 1-trimmed graph of G, by Theorem 3.1.10, G is of type 1. □

Proposition 3.1.16. *If a graph G has a regular 1-trimmed graph K such that $\chi''(K) \leq \Delta(K) + 2$ and $K \neq G$, then G is of type 1.*

Proof. Assume that a graph G has a regular 1-trimmed graph K such that $\chi''(K) \leq \Delta(K) + 2$ and $K \neq G$. If K is of type 1, by Theorem 3.1.6, G is of type 1. If K is of type 2, since K is regular, we get $\Delta(G) > \Delta(K)$. By Theorem 3.1.15, G is of type 1. □

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3.2 Trimmed Graphs and Glued Graphs

Remark 3.2.1. Let K be a 1-trimmed graph of a connected graph G. Then there is a tree T such that $K \triangleleft\!\triangleright T = G$.

Proof. Let a graph K be a 1-trimmed graph of a connected graph G. Since G is connected, G has a spanning tree T. Thus $K \triangleleft\!\triangleright T = G$. □

Remark 3.2.2. Let G be a graph and T be a tree. Then G is a 1-trimmed graph of $G \triangleleft\!\triangleright T$.

Theorem 3.2.3. *Let G be a graph and T be a tree. If G is of type 1, so is $G \triangleleft\!\triangleright T$.*

Proof. Let G be a graph and T be a tree. Assume that G is of type 1. Since G is a 1-trimmed graph of $G \triangleleft\!\triangleright T$, by Theorem 3.1.10, $G \triangleleft\!\triangleright T$ is of type 1. □

Theorem 3.2.4. *Let G be a graph and T be a tree, if G satisfies the Total Coloring conjecture, so is $G \triangleleft\!\triangleright T$.*

Proof. Let G be a graph and T be a tree. Assume that $\chi''(G) \leq \Delta(G) + 2$. Since G is a 1-trimmed graph of $G \triangleleft\!\triangleright T$, by Theorem 3.1.6, $\chi''(G \triangleleft\!\triangleright T) \leq \Delta(G \triangleleft\!\triangleright T) + 2$. □

Theorem 3.2.5. *Let G be a type 2 graph and T be a tree, $G \triangleleft\!\triangleright T$ is of type 1 if and only if $\Delta(G \triangleleft\!\triangleright T) > \Delta(G)$*

Proof. Let G be a type 2 graph and T be a tree. Since G is a 1-trimmed graph of $G \triangleleft\!\triangleright T$, by Theorem 3.1.15, this theorem holds. □

Theorem 3.2.6. *Let T be a tree and G a regular graph. If $G \neq G \triangleleft\!\triangleright T$ and $\chi''(G) \leq \Delta(G) + 2$ then $G \triangleleft\!\triangleright T$ is of type 1.*

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Proof. This theorem follows from the fact that G is a 1-trimmed graph of $G \oplus T$ and Proposition 3.1.16. □

Theorem 3.2.7. *Let G be a graph and T be a tree. If $\chi''(G) \leq \Delta(G) + t$ then $\chi''(G \oplus T) \leq \Delta(G \oplus T) + t$.*

Proof. This theorem follows from the fact that G is a 1-trimmed graph of $G \oplus T$ and Theorem 3.1.12. □

Recall that for any graph T if $T \ncong P_2$ then $\chi''(T) = \Delta(T) + 1$.

Theorem 3.2.8. *Let G be a graph and T be a tree. $\chi''(G \oplus T) \leq \chi''(G) + \chi''(T) - 2$.*

Proof. Let G be a graph and T be a tree. If $T \cong P_2$ then $\chi''(T) = 3$ and $G \oplus T = G$. Hence $\chi''(G \oplus T) = \chi''(G) < \chi''(G) + \chi''(T) - 2$. Assume that T is not P_2. Let k be an integer such that $\chi''(G) = \Delta(G) + k$.

$$\begin{aligned}
\chi''(G \oplus T) &\leq \Delta(G \oplus T) + k, && \text{by Theorem 3.2.7,} \\
&\leq \Delta(G) + \Delta(T) - 1 + k, && \text{by Remark 1.3.5,} \\
&= (\Delta(G) + k) + (\Delta(T) + 1) - 2 && \\
&= \chi''(G) + \chi''(T) - 2, && \text{by Remark 1.2.1.}
\end{aligned}$$

□

Theorem 3.2.9. *Let G be a graph and T be a tree. Then $\chi''(G \oplus T) = \chi''(G) + \chi''(T) - 2$ if and only if $\chi''(G \oplus T) - \Delta(\oplus T) = \chi''(G) - \Delta(T)$ and $\Delta(G \oplus T) = \Delta(G) + \Delta(T) - 1$.*

Proof. This theorem follows from the proof in Theorem 3.2.8. □

Figure 3.2.1 and Figure 3.2.2 show a graph G and a tree T such that $\chi''(G \oplus T) = \chi''(G) + \chi''(T) - 2$.

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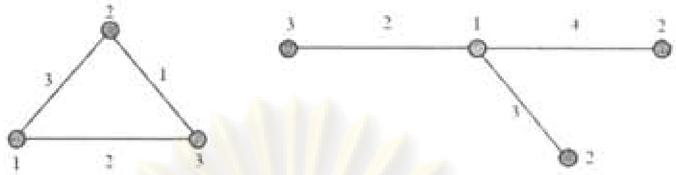

Figure 3.2.1: Total colorings of K_3 and $K_{1,3}$

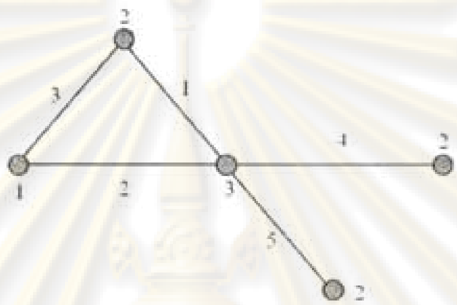

Figure 3.2.2: A total coloring of $K_3 \Diamond K_{1,3}$ when its clone is P_2

As the proper total coloring shown in Figure 3.2.1, we get $\chi''(K_3) \leq 3$. Since $\chi''(K_3) \geq \Delta(K_3) + 1 = 3$, $\chi''(K_3) = 3$. Similarly, $\chi''(K_{1,3}) = 4$ and $\chi''(K_3 \Diamond K_{1,3}) = 5$. Hence we get $\chi''(K_3 \Diamond K_{1,3}) = \chi''(K_3) + \chi''(K_{1,3}) - 2$.

Theorem 3.2.10. *Let G be a graph. If $\chi''(G) \leq \Delta(G)+2$ and $n(G \Diamond C_n) > n(G)$. Then $\chi''(G \Diamond C_n) \leq \Delta(G \Diamond C_n) + 2$.*

Proof. Assume that $\chi''(G) \leq \Delta(G) + 2$ and $n(G \Diamond C) > n(G)$ then G is a 2-trimmed graph of $G \Diamond C$. By Theorem 3.1.6, we get $\chi''(G \Diamond C) \leq \Delta(G \Diamond C) + 2$. □

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CHAPTER IV

CONCLUSIONS AND OPEN PROBLEMS

4.1 Conclusions

In this section, we conclude main results in this thesis.

Equality of the chromatic number, the edge-chromatic number and the total chromatic number

Let G be a graph with n vertices. Then $\chi(G) = \chi'(G) = \chi''(G)$ if and only if G is C_n where $n \equiv 3 \pmod 6$ or K_n where n is odd.

Upper bounds of the total chromatic numbers of glued graphs

1. Let G_1 and G_2 be graphs. If $\chi''(G_1 \Phi G_2) \leq \Delta(G_1 \Phi G_2)+2$ then $\chi''(G_1 \Phi G_2) \leq \chi''(G_1) + \chi''(G_2) - 1$.

2. Let G be a graph and T be a tree. Then $\chi''(G \Phi T) \leq \chi''(G) + \chi''(T) - 2$.

3. $\chi''(C_m \Phi C_n) \leq \chi''(C_m) + \chi''(C_n) - 2$.

4. $\chi''(K_m \Phi K_n) \leq \chi''(K_m) + \chi''(K_n) - 2$.

Colorability of the glued graphs of cycles

1. $\chi(C_m \Phi C_n) = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even,} \\ 3 & \text{otherwise.} \end{cases}$

2. $\chi'(C_m \Phi C_n) = \begin{cases} 2 & \text{if } C_m \Phi C_n \text{ is an even cycle,} \\ 3 & \text{otherwise.} \end{cases}$

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3. $\chi''(C_m \diamond C_n) = \begin{cases} 3 & \text{if } C_m \diamond C_n \text{ is a cycle and } \quad m = n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$

4. Any glued graph of cycles satisfies the Total Coloring Conjecture.

5.
   - $C_m \diamond C_n$ is of type 2 if $C_m \diamond C_n$ is a cycle and $m = n \equiv 1, 2 \pmod 3$. Otherwise, $C_m \diamond C_n$ is of type 1,
   - $C_m \diamond C_n$ is of type 1 if and only if $C_m \diamond C_n$ is not a cycle or $m = n \equiv 0 \pmod 3$,
   - $C_m \diamond C_n$ is of type 2 if and only if $C_m \diamond C_n$ is a cycle and $m = n \equiv 1, 2 \pmod 3$.

Colorability of the glued graphs of nontrivial bipartite graphs G_1 and G_2

1. $\chi(G_1 \diamond G_2) = 2$.

2. $\chi'(G_1 \diamond G_2) = \Delta(G_1 \diamond G_2)$.

3. The glued graph of bipartite graphs satisfies the Total Coloring Conjecture.

4. $\chi''(G_1 \diamond G_2) \leq \chi''(G_1) + \chi''(G_2) - 1$.

Colorability of the glued graphs of nontrivial trees T_1 and T_2

1. $\chi(T_1 \diamond T_2) = 2$.

2. $\chi'(T_1 \diamond T_2) = \Delta(T_1 \diamond T_2)$.

3. $T_1 \diamond T_2$ is of type 1 unless $T_1 \cong T_2 \cong P_2$.

4. Any glued graph of trees satisfies the Total Coloring Conjecture.

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5. $\chi'(T_1 \Phi T_2) \leq \chi'(T_1)+\chi'(T_2)-1$. The equality holds if and only if $\Delta(T_1 \Phi T_2) = \Delta(T_1) + \Delta(T_2) - 1$.

6. $\chi''(T_1 \Phi T_2) \leq \chi''(T_1) + \chi''(T_2) - 2$. The equality holds if and only if $\Delta(T_1 \Phi T_2) = \Delta(T_1) + \Delta(T_2) - 1$ and $T_1 \Phi T_2 \ncong P_2$.

Colorability of the glued graphs of complete graphs

1. Any glued graph of complete graphs satisfies the Total Coloring Conjecture.
   - Let $m \geq n$. If $m+n-r$ is even and $m < r + \frac{2r-n}{2n-2r-1}$ then $K_m \underset{K_r}{\Phi} K_n$ is of type 2. Otherwise, $K_m \underset{K_r}{\Phi} K_n$ is of type 1,
   - $K_m \underset{K_r}{\Phi} K_n$ is of type 1 if and only if $m+n-r$ is odd or $m \geq r+\frac{2r-n}{2n-2r-1}$,
   - $K_m \underset{K_r}{\Phi} K_n$ is of type 2 if and only if $m+n-r$ is even and $m < r+\frac{2r-n}{2n-2r-1}$.

2. $\chi''(K_m \Phi K_n) \leq \chi''(K_m) + \chi''(K_n) - 2$.

3. $\chi''(K_m \Phi K_n) = \chi''(K_m) + \chi''(K_n) - 2$ if and only if m, n are odd and the clone of $K_m \Phi K_n$ is K_2.

Colorability of the glued graphs of a graph G and a tree T

1. $\chi''(G \Phi T) \leq \chi''(G) + \chi''(T) - 2$.

2. $\chi''(G \Phi T) = \chi''(G) + \chi''(T) - 2$ if and only if $\chi''(G \Phi T) - \Delta(G \Phi T) = \chi''(G) - \Delta(T)$ and $\Delta(G \Phi T) = \Delta(G) + \Delta(T) - 1$.

3. If G is of type 1, so is $G \Phi T$.

4. If G satisfies the Total Colorings Conjecture, so is $G \Phi T$.

5. If G is a type 2 graph, then $G \Phi T$ is of type 1 if and only if $\Delta(G \Phi T) > \Delta(G)$.

6. If G is a regular graph such that $\chi''(G) \leq \Delta(G) + 2$ and $G \diamond T \neq G$, then $G \diamond T$ is of type 1.

Colorability of the glued graph of a graph G and a cycle C_n

If $\chi''(G) \leq \Delta(G) + 2$ and $n(G \diamond C_n) > n(G)$, then $\chi''(G \diamond C_n) \leq \Delta(G \diamond C_n) + 2$.

Applications of 2-trimmed graphs

1. If a graph G has a 2-trimmed graph H such that $\chi''(H) \leq \Delta(H) + 2$ then $\chi''(G) \leq \Delta(G) + 2$.

2. Every outerplanar graph G, $\chi''(G) \leq \Delta(G) + 2$.

Applications of 1-trimmed graphs

1. If a graph G has a 1-trimmed graph K such that $\chi''(K) \leq \Delta(K) + 2$ then $\chi''(G) \leq \Delta(G) + 2$.

2. If a graph G has a 1-trimmed graph K such that K is of type 1 then G is of type 1.

3. Let G be a graph with $\Delta(G) \geq 2$. Let K a 1-trimmed graph of type 2 of G. Then G is of type 1 if and only if $\Delta(G) > \Delta(K)$.

4. If a graph G has a regular 1-trimmed graph K such that $\chi''(K) \leq \Delta(K)+2$, then G is of type 1.

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4.2 Open Problems

We propose some open problems in this thesis for further research as follows.

1. In Chapter 2, we obtained an upper bound of the total chromatic numbers of glued graphs in terms of the total chromatic number of original graphs. Theorem 2.1.1 states that for any graphs G_1 and G_2, if $\chi''(G_1 \diamond G_2) \leq \Delta(G_1 \diamond G_2) + 2$,

then $\chi''(G_1 \Phi G_2) \leq \chi''(G_1) + \chi''(G_2) - 1$. Note that if graphs G_1 and G_2 with $\chi''(G_1 \Phi G_2) \leq \Delta(G_1 \Phi G_2) + 2$ satisfy following conditions

(a) a vertex with maximum degree of G_1 is glued to a vertex with maximum degree of G_2 and the corresponding vertex in the clone has degree 1,

(b) G_1 and G_2 are of type 1,

(c) $G_1 \Phi G_2$ is of type 2.

Then we have $\chi''(G_1 \Phi G_2) = \chi''(G_1) + \chi''(G_2) - 1$.

However, any two conditions among above three conditions yield $\chi''(G_1 \Phi G_2) \leq \chi''(G_1) + \chi''(G_2) - 2$. We conjecture that no graph satisfies all of the above conditions; hence, we have the following conjecture.

$$\text{For graphs } G_1, G_2,\ \chi''(G_1 \Phi G_2) \leq \chi''(G_1) + \chi''(G_2) - 2.$$

2. we have already investigated necessary and sufficient conditions to be either of type 1 or of type 2 of glued graphs of trees, cycles and complete graphs. It is an open problem to find a necessary and sufficient condition of the glued graph of bipartite graphs to be either of type 1 or of type 2.

3. In Chapter 3, we have already proved that if there is a 2-trimmed graph H of a graph G such that $\chi''(H) \leq \Delta(H) + 2$ then $\chi''(G) \leq \Delta(G) + 2$. It is interested to prove that for each positive integer $t \geq 3$, if there is a t-trimmed graph H of a graph G such that $\chi''(H) \leq \Delta(H) + 2$ then $\chi''(G) \leq \Delta(G) + 2$. This conjecture has some advantages. For example, the conjecture for case $t = 5$ yields that every planar graph satisfies the Total Coloring Conjecture.

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APPENDIX

ศูนย์วิทยทรัพยากร
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A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessary to be distinct) called its *endpoints*. The number of elements in $V(G)$ is represented by $n(G)$ and the number of elements in $E(G)$ is represented by $e(G)$.

The *degree of a vertex* v in a graph G is the number of edges incident with v and is denoted by $d_G(v)$ or simply by $d(v)$ if the graph G is clear from the context. The *maximum degree* of G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$; the *minimum degree* of G is denoted by $\delta(G)$.

The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possible empty) independent sets called *partite set* of G.

A component graph is *trivial* if it has no edge; otherwise it is *nontrivial*. An *isolated vertex* is a vertex of degree 0.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjecent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutive along the circle.

The (unlabeled) path and cycle with n vertices are denoted P_n and C_n respectively; an n-*cycle* is a cycle with n-vertices. A *complete graph* is a simple graph whose vertices are pairwise adjacent; the (unlabeled) complete graph with n vertices is denoted by K_n. A *complete bipartite graph* or *biclique* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the set has size r and s, the (unlabeled) biclique is denoted $K_{r,s}$.

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A graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph.

An *outerplanar graph* is a graph with an imbedding in the plane such that every vertex appears on the boundary of the exterior face.

The *line graph* of G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.

An *induced subgraph* is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G) - T$; this is the subgraph of G *induced by* T.

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