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GRAPHS WHOSE SQUARE IS PANCONNECTED

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ศิริรัตน์ สิงหันต์ : กราฟที่กำลังสองมีสมบัติเชื่อมโยงรวม. (GRAPHS WHOSE SQUARE IS PANCONNECTED) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.วนิดา เหมะกุล, Ph.D., อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : Prof. Gek Ling Chia, Ph.D., 40 หน้า.

*กำลังสอง*ของกราฟ G คือ กราฟที่ได้จากกราฟ G โดยการเติมเส้นเชื่อมระหว่างจุดยอดสองจุดใด ๆ ซึ่งมีระยะทางในกราฟ G เท่ากับสอง กราฟมี*สมบัติเชื่อมโยงรวม*ถ้าระหว่างจุดยอดสองจุดใด ๆ ที่ต่างกัน จะมีวิถีแต่ละขนาดตั้งแต่ระยะทางระหว่างจุดยอดสองจุดนั้นขึ้นไป ถ้ากราฟ G เป็นกราฟเชื่อมโยง เรานิยาม*จำนวนไซโคมาติก*ของกราฟ G เท่ากับ $|E(G)| - |V(G)| + 1$ เชียและคณะ [4] ได้แสดงลักษณะกราฟทั้งหมดที่จำนวนไซโคมาติกของกราฟไม่เกินหนึ่งซึ่งกำลังสองของกราฟนั้นมีสมบัติเชื่อมโยงรวม ในวิทยานิพนธ์นี้ เราแสดงลักษณะกราฟทั้งหมดที่จำนวนไซโคมาติกของกราฟนั้นเท่ากับสองซึ่งกำลังสองของกราฟมีสมบัติเชื่อมโยงรวม เราแสดงว่า ถ้า กราฟ G มีจำนวนไซโคมาติกเท่ากับสามและกำลังสองของกราฟ G มีสมบัติเชื่อมโยงรวม แล้ว กราฟ G ต้องเป็นหนึ่งในวงศ์ของกราฟจำนวนแปดวงศ์ เราวางนัยทั่วไปสำหรับวงศ์เหล่านี้ของกราฟจำนวนสามวงศ์ให้เป็นวงศ์ที่ใหญ่กว่าของกราฟ เราตรวจสอบเงื่อนไขจำเป็นและเพียงพอสำหรับวงศ์ที่ใหญ่กว่าของกราฟดังกล่าวจำนวนสามวงศ์ที่กำลังสองของกราฟมีสมบัติเชื่อมโยงรวม

ภาควิชา......คณิตศาสตร์......	ลายมือชื่อนิสิต Sirit Sph.
สาขาวิชา....คณิตศาสตร์.....	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก W. Hematul
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SIRIRAT SINGHUN : GRAPHS WHOSE SQUARE IS PANCONNECTED. THESIS ADVISOR : ASSOC. PROF. WANIDA HEMAKUL, Ph.D., THESIS CO-ADVISOR : PROF. GEK LING CHIA, Ph.D., 40 pp.

The *square* of a graph G is the graph obtained from G by adding edges joining those pairs of vertices whose distance from each other in G is two. A graph is *panconnected* if, between any pair of distinct vertices, it contains a path of each length at least the distance between the two vertices. If G is connected, the *cyclomatic number* of G is defined as $|E(G)| - |V(G)| + 1$. Chia et al. [4] has already characterized all graphs with cyclomatic number no more than one whose square is panconnected. In this thesis, we characterize all graphs with cyclomatic number two whose square is panconnected. We show that if G has cyclomatic number three and the square of G is panconnected, then G is one of the eight families of graphs. Three of these families of graphs are generalized to three larger families of graphs. Necessary and sufficient conditions for these three larger families of graphs to have panconnected square are determined.

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Field of Study	: ....Mathematics....	Advisor's Signature : W. Hemakul
Academic Year : ....2010..........		Co-Advisor's Signature : Chia

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CONTENTS

CHAPTER I

INTRODUCTION

All graphs considered in this thesis are undirected and simple. Let G be a graph. The *square* of G, denoted G^2, is the graph obtained from G by adding edges joining those pairs of vertices whose distance from each other in G is two. Although it is not true in general that the square of a graph is hamiltonian, in 1969, Plummer [9] and Nash-Williams [13] conjectured independently that G^2 is hamiltonian if G contains no cut-vertices. In 1974, Fleischner [7] proved the conjecture in the affirmative.

A graph is *Hamilton-connected* if any two vertices are connected by a Hamilton path. In 1974, Chartrand et al. [3] showed that if G is 2-connected, then G^2 is Hamilton-connected. A graph is *panconnected* if, between any pair of distinct vertices, it contains a path of each length at least the distance between the two vertices. In 1976, Faudree and Schelp [5] showed that if G is 2-connected, then G^2 is panconnected. Clearly, a panconnected graph is Hamilton-connected but not conversely. However, in the square of graphs, Fleischner [8] showed that these two concepts are equivalent in 1976. He proved that for a connected graph G, G^2 is panconnected if and only if G^2 is Hamilton-connected.

Suppose G is connected. Then the number $|E(G)| - |V(G)| + 1$, denoted $c(G)$, is called the *cyclomatic number* of G. Thus, $c(G) = 0$ if and only if G is a tree. Also, $c(G) = 1$ if and only if G is a unicyclic graph, a graph with exactly one cycle. A cut-edge xy of G is termed an *internal cut-edge* if both the degrees of x and y in G are at least 2. In 2009, Chia et al. [4] showed that if G^2 is panconnected, then G has no internal cut-edge. An immediate consequence of this result is that, if G has $n(\geq 3)$ vertices and $c(G) = 0$, then G^2 is panconnected if and only if $G \cong K_{1,n-1}$, the tree with all vertices but one of degree 1.

Chia et al. [4] also characterized all graphs G such that $c(G) = 1$ and G^2 is panconnected. They proved that for a unicyclic graph G, G^2 is panconnected if and only if G is a broken SF graph, a graph consisting of only one cycle together with a set of independent vertices joining to each vertex on the cycle and some set of independent vertices is empty.

Motivated by these, we would like to characterize all graphs G such that $c(G) = k$ for some integer $k \geq 2$ and G^2 is panconnected.

In Chapter II, we give preliminaries and literature reviews. We show in Chapter III that if $c(G) = 2$ and G^2 is panconnected, then G must be a member of the two families of graphs defined in Section 3.1. We then determine all graphs G such that $c(G) = 2$ and G^2 is panconnected. In Chapter IV, we first show that if $c(G) = 3$ and G^2 is panconnected, then G is one of eight families of graphs defined in Section 4.1. Next, three of these eight families of graphs are generalized to larger families of graphs. Finally, necessary and sufficient conditions for these three larger families of graphs to have panconnected square are determined.

CHAPTER II

PRELIMINARIES AND LITERATURE REVIEWS

This chapter gives definitions which will be used in our work and then literature reviews are shown.

2.1 Definitions and examples

All definitions not defined in this thesis can be found in [15].

A *Hamilton path* of a graph G is a path that includes all its vertices. A *Hamilton cycle* of G is a cycle that includes all its vertices. If G obtains a Hamilton cycle, then G is called *hamiltonian*.

Figure 2.1(a) shows a hamiltonian graph with a Hamilton cycle indicated in thick edges and Figure 2.1(b) shows a graph with a Hamilton path indicated in thick edges and this graph is not hamiltonian.

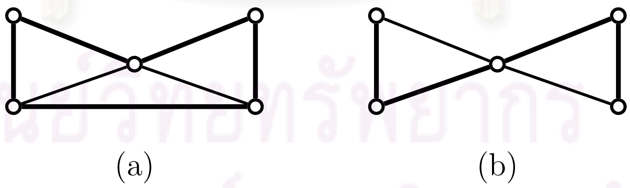
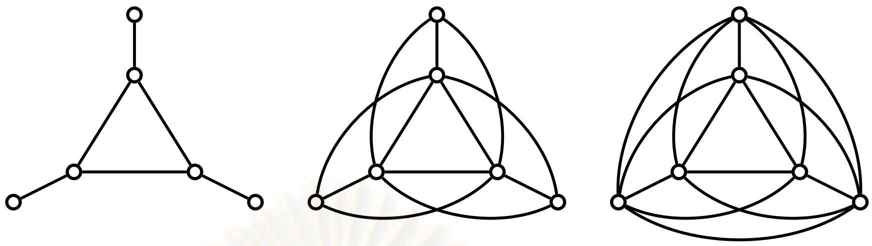


Figure 2.1: (a) A graph with a Hamilton cycle and (b) a graph with a Hamilton path

It is natural to look for graphs with many edges which are hamiltonian.

The *k-power* of G, denoted G^k, is the graph with vertex set $V(G)$ and two vertices u and v are adjacent in G^k if and only if $d(u,v) \leq k$ where $d(u,v)$ is the length of a shortest path from u to v in G.

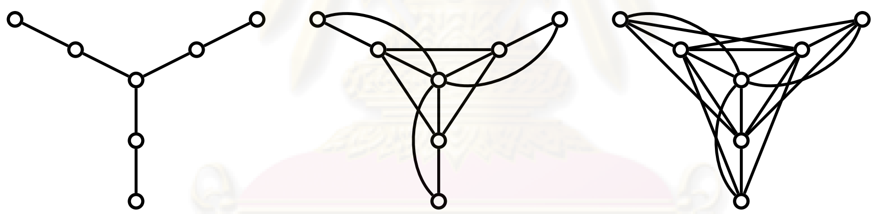
In Figure 2.2, (a) shows a graph G while (b) shows G^2 and (c) shows G^3.



(a) (b) (c)

Figure 2.2: (a) G, (b) G^2 and (c) G^3

It is not true in general that the square of a graph is hamiltonian. Figure 2.3 shows a graph G such that G^2 is not hamiltonian while G^3 is hamiltonian.



G G^2 G^3

Figure 2.3: A graph G such that G^2 is not hamiltonian while G^3 is hamiltonian

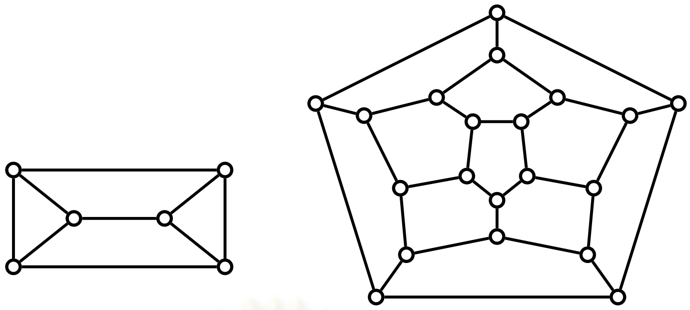
A graph is *pancyclic* if it contains a cycle of each length at least three.

Figure 2.4(a) shows a pancyclic graph. Clearly, a pancyclic graph is a hamiltonian graph. The converse is not true as shown in Figure 2.4(b).

Based on the definition of a pancyclic graph, we have definitions of the specific graphs involving every vertex (respectively edge).

A graph is *vertex-* (respectively *edge-*) *pancyclic* if every of its vertex (respectively edge) is in a cycle of every length.

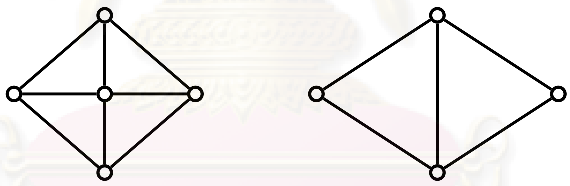
Figure 2.5(a) shows an edge-pancyclic graph. Clearly, an edge-pancyclic graph



(a) (b)

Figure 2.4: (a) A pancyclic graph and (b) a hamiltonian graph which is not pancyclic

is vertex-pancyclic, which of course is pancyclic. The converse is not true. The graph in Figure 2.5(b) is a vertex-pancyclic graph but it is not an edge-pancyclic graph.



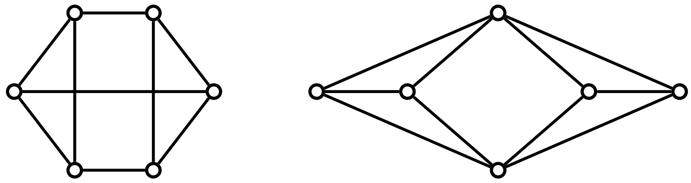
(a) (b)

Figure 2.5: (a) An edge-pancyclic graph and (b) a vertex-pancyclic graph which is not edge-pancyclic

A graph is *Hamilton-connected* if there is a Hamilton path between any pair of distinct vertices.

Figure 2.6(a) shows a Hamilton-connected graph. Clearly, if G is a graph with $|V(G)| > 2$, then a Hamilton-connected graph is necessary hamiltonian, but the converse is not true as shown in Figure 2.6(b).

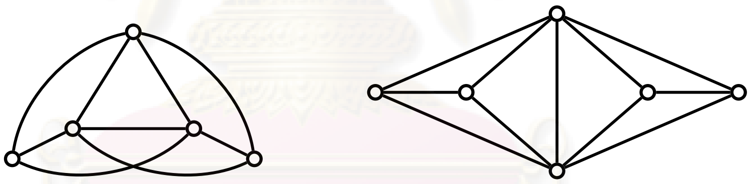
A graph is *panconnected* if, between any pair of distinct vertices, it contains a path of each length at least the distance between the two vertices.



(a) (b)

Figure 2.6: (a) A Hamilton-connected graph and (b) a hamiltonian graph which is not Hamilton-connected

Figure 2.7(a) shows a panconnected graph. Clearly, a panconnected graph is pancyclic and it is Hamilton-connected and hence it is hamiltonian. The converses are not true. Figure 2.7(b) shows a pancyclic graph which is not panconnected and Figure 2.6(a) shows a Hamilton-connected graph which is not panconnected.



(a) (b)

Figure 2.7: (a) A panconnected graph and (b) a pancyclic graph which is not panconnected

2.2 Literature reviews

In 1960, Sekanina [14] and Karaganis [12] obtained a result concerning G^3.

Theorem 2.1. *([14],[12]) If G is a connected graph, then G^3 is Hamilton-connected.*

It is not true in general that G^2 is hamiltonian (see Figure 2.3). In 1969, Plummer [9] and Nash-Williams [13] raised a conjecture independently which is known as the Plummer-Nash-Williams conjecture.

The Plummer-Nash-Williams conjecture *If a graph G contains no cut-vertices, then G^2 is hamiltonian.*

In 1971, Harary and Schwenk [10] characterized trees T such that T^2 is hamiltonian.

Theorem 2.2. *([10]) Let T be a tree on n (≥ 3) vertices. Then T^2 is hamiltonian if and only if T does not contain $S(K_{1,3})$ as a subgraph, where $S(K_{1,3})$ is the graph obtained by subdividing each edge of the complete bipartite graph $K_{1,3}$ exactly once.*

In 1974, Fleischner [7] proved the Plummer-Nash-Williams conjecture in the affirmative.

Theorem 2.3. *([7]) The square of every 2-connected graph is hamiltonian.*

In 1974, under the same condition of the Plummer-Nash-Williams conjecture, Chartrand et al. [3] proved the result involving blocks.

Theorem 2.4. *([3]) The square of a block is Hamilton-connected.*

In 1975, Alavi and Williamson [2] gave a result concerning G^3.

Theorem 2.5. *([2]) If G is a connected graph, then G^3 is panconnected.*

In 1976, Faudree and Schelp [5] obtained a result which is stronger than the Plummer-Nash-Williams conjecture.

Theorem 2.6. *([5]) The square of a block is panconnected.*

Later, Fleischner [8] showed that in the case of square of connected graphs these two properties, Hamilton-connectedness and panconnectedness, are equivalent.

Theorem 2.7. *([8]) Let G be a connected graph. Then*

(i) G^2 is vertex-pancyclic if and only if G^2 is hamiltonian.

(ii) G^2 is panconnected if and only if G^2 is Hamilton-connected.

In 1985, Hendry and Vogler [11] obtained a sufficient condition for a graph which is not a tree such that the square is vertex-pancyclic based on the subgraph $S(K_{1,3})$.

Theorem 2.8. *([11]) Let G be a connected graph on 3 or more vertices which does not contain $S(K_{1,3})$ as a subgraph. Then G^2 is vertex-pancyclic.*

The result of Hendry and Vogler [11] (in Theorem 2.8) motivated Abderrezzak et al. [1] to look for weaker conditions based on the subgraph $S(K_{1,3})$ for which the square of a connected graph remains hamiltonian.

Theorem 2.9. *([1]) If G is a connected graph such that every induced $S(K_{1,3})$ has at least three edges in a block of degree at most 2, then G^2 is hamiltonian.*

In 2009, Chia et al. [4] obtained a sufficient condition for a graph which contains one or more cut-vertices such that the square is panconnected.

Theorem 2.10. *([4]) Let G be a connected graph having only one cut-vertex. Then G^2 is panconnected.*

Theorem 2.11. *([4]) Suppose G is a connected graph with only two cut-vertices. If the block that contains the two cut-vertices is hamiltonian, then G^2 is panconnected.*

Chia et al. [4] also investigated the panconnectedness of graphs having at most one cycle.

Theorem 2.12. *([4]) Let G be a graph. If G^2 is panconnected or edge-pancyclic, then G contains no internal cut-edge.*

Corollary 2.13. *([4]) Let T be a tree on $n(\geq 3)$ vertices. Then the following are equivalent.*

(i) T^2 is panconnected;

(ii) T^2 is edge-pancyclic;

(iii) T is a star $K_{1,n-1}$.

Theorem 2.14. *([4]) Let G be a unicyclic graph. Then G^2 is panconnected if and only if*

(i) G contains no internal cut-edges and

(ii) G contains vertices of degree 2.

Corollary 2.15. *([4]) Let G be a unicyclic graph. Then the following are equivalent.*

(i) G^2 is panconnected;

(ii) G^2 is edge-pancyclic;

(iii) G is a broken SF graph.

Corollaries 2.13 and 2.15 characterize all graphs G with $c(G) = 0$ and $c(G) = 1$ respectively such that G^2 is panconnected.

CHAPTER III

GRAPHS WITH CYCLOMATIC NUMBER TWO HAVING PANCONNECTED SQUARE

In this chapter, we define 2 families of graphs with cyclomatic number 2 and obtain a necessary condition for graphs with cyclomatic number 2 whose square is panconnected. Then, we characterize all graphs with cyclomatic number 2 whose square is panconnected.

3.1 A necessary condition

Let G be a connected graph. The *cyclomatic number* of G, denoted $c(G)$, is defined to be $|E(G)| - |V(G)| + 1$.

Figure 3.1 shows graphs with different cyclomatic numbers.

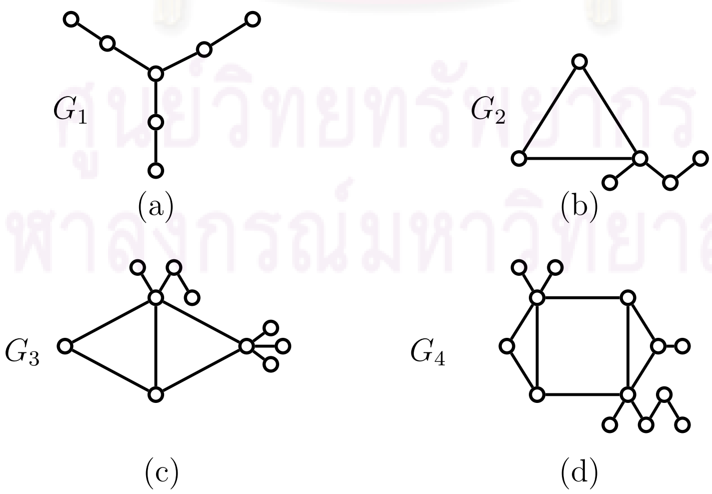


Figure 3.1: (a) $c(G_1) = 0$, (b) $c(G_2) = 1$, (c) $c(G_3) = 2$ and (d) $c(G_4) = 3$

Clearly, $c(G) = 0$ if and only if G is a tree and $c(G) = 1$ if and only if G is a unicyclic graph, a graph with exactly one cycle.

A cut-edge xy of a graph G is termed an *internal cut-edge* if both the degrees of x and y in G are at least 2.

Figure 3.2 shows a graph with an internal cut-edge indicated in the thick edge.

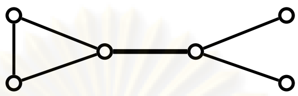


Figure 3.2: A graph with an internal cut-edge indicated in the thick edge

An *SF graph*, denoted $G(m)$ where $m \geq 3$, is a graph obtained from a cycle $u_1u_2 \ldots u_mu_1$ by joining each vertex u_i to a set of independent vertices A_{u_i}. That is, A_{u_i} is the *pendent set* of u_i. If for some $1 \leq i \leq m$, A_{u_i} is an empty set, then we say that the SF graph is *broken*. Each vertex u_i is termed a *c-vertex* of $G(m)$.

Note that a broken SF graph has a vertex of degree 2.

In Figure 3.3, (a) and (b) show some SF graphs $G(3)$ and (c), (d) and (e) show some broken SF graphs $G(4)$.

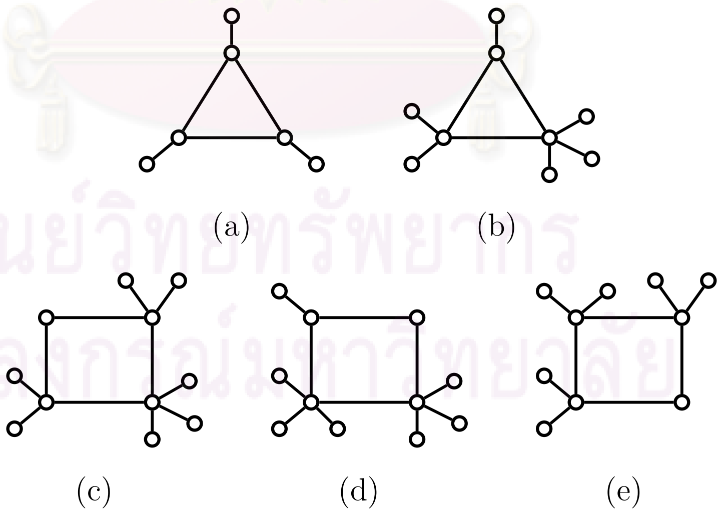
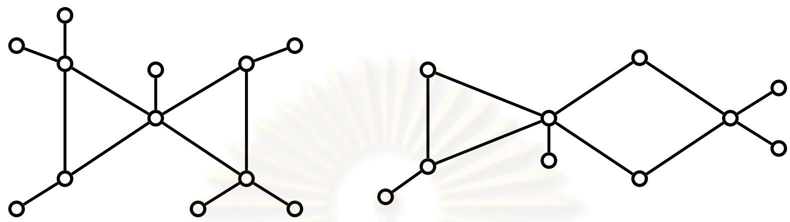


Figure 3.3: (a) and (b) are some $G(3)$ and (c), (d) and (e) are some $G(4)$

Let $G(m)$ and $G(n)$ be two SF graphs whose cycles are $x_1x_2 \ldots x_mx_1$ and $y_1y_2 \ldots y_ny_1$ respectively. Let $G(m,n)$ denote the graph obtained from $G(m)$ and $G(n)$ by identifying the two vertices x_1 and y_1. In this case, we may take

$A_{x_1} = A_{y_1}$. We say that $G(m,n)$ is *broken* if there exist $i, j \geq 2$ such that $A_{x_i} = \varnothing$ and $A_{y_j} = \varnothing$.

Figure 3.4(a) shows a non-broken $G(3,3)$ and Figure 3.4(b) shows a broken $G(3,4)$.

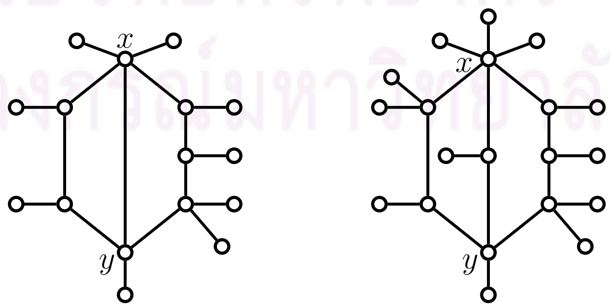


(a) (b)

Figure 3.4: (a) $G(3,3)$ and (b) $G(3,4)$

Let $\mathcal{P}_m = x_1x_2\dots x_m$, $\mathcal{P}_n = y_1y_2\dots y_n$ and $\mathcal{P}_r = z_1z_2\dots z_r$ denote three paths on m, n and r vertices respectively, where $2 \leq m \leq n, r$. Identifying the end vertices of three paths so that $x_1 = y_1 = z_1 = x$ and $x_m = y_n = z_r = y$, we obtain the generalized θ-graph. If $m = 2$, then we require that $n, r \geq 3$. Let $\Theta(m,n,r)$ denote the graph obtained by joining each vertex v of the generalized θ-graph to a new set of independent vertices A_v. That is, A_v is the *pendent set* of v. A vertex v in $\Theta(m,n,r)$ is called a *t-vertex* if $v \in \{x, y\}$.

Figure 3.5(a) shows $\Theta(2,4,5)$ and Figure 3.5(b) shows $\Theta(3,4,5)$.



(a) (b)

Figure 3.5: (a) $\Theta(2,4,5)$ and (b) $\Theta(3,4,5)$

Note that the union of any two paths of $\mathcal{P}_m$, $\mathcal{P}_n$ and $\mathcal{P}_r$ together with all their pendent sets forms an SF graph.

It is routine to check that $G(m,n)$ and $\Theta(m,n,r)$ have cyclomatic number 2.

In [4], Chia et al. gave a necessary condition for graphs whose square is panconnected.

Theorem 3.1. *([4]) Let G be a graph such that G^2 is panconnected. Then G has no internal cut-edge.*

We now obtain a necessary condition for graphs with cyclomatic number 2 whose square is panconnected.

Lemma 3.2. *Let G be a graph with $c(G) = 2$. If G^2 is panconnected, then G is either the graph $G(m,n)$ or else the graph $\Theta(m,n,r)$.*

Proof. Since $c(G) = 2$, it is clear that G is obtained from a unicyclic graph H by adding a new edge uv to two non-adjacent vertices u and v of H. Then either uv creates (i) one or (ii) two extra cycles in $H + uv$.

Since G^2 is panconnected, G has no internal cut-edge (by Theorem 3.1). As such, Case (i) implies that G is the graph $G(m,n)$ while Case (ii) implies that G is the graph $\Theta(m,n,r)$. $\square$

Remark 3.3. *Suppose v is a vertex of a graph G. If A_v, which is a pendent set of vertex v, is not empty, then A_v induces a complete subgraph in G^2. Let P_v denote a Hamilton path in this induced subgraph. In what follows, very often, we shall be dealing with subpath of the form vP_vw or zP_vw in G^2, where z and w are vertices adjacent to v and $z,w \notin A_v$. In the event that A_v is an empty set, then P_v is an empty path and the corresponding subpath of the form vP_vw or zP_vw reduces to the edge vw or zw respectively.*

Suppose u and v are two vertices in a graph G. In what follows, whenever we use $P(u,v) = ua_1a_2\cdots a_{n-1}a_nv$ to denote a path in G from u to v, then by $P(v,u)$ we mean the path $va_na_{n-1}\cdots a_2a_1u$.

3.2 $G(m,n)$

In this section, we obtain a necessary and sufficient condition for the graph $G(m,n)$ to have panconnected square.

In [4], Chia et al. characterized all graphs with cyclomitic number 1 having panconnected square.

Theorem 3.4. *([4]) Let G be a unicyclic graph. Then G^2 is panconnected if and only if G is a broken SF graph.*

Theorem 3.5. *Let G denote the graph $G(m,n)$. Then G^2 is panconnected if and only if G is broken.*

Proof. To verify the necessary condition, suppose that G is not broken and assume that $A_{x_i} \neq \varnothing$ for all $i \geq 2$. We just need to show that there is no Hamilton path in G^2 having x_2 and x_3 as end vertices.

Let H denote the graph obtained from G^2 by deleting the vertices x_2 and x_3 together with all edges incident to them. Notice that, in H, the vertices in A_{x_2} (respectively A_{x_3}) are adjacent only to the vertex x_1 (respectively x_4, or x_1 if $m = 3$). This means that if there is a Hamilton path $P(x_2, x_3)$ in G^2 with x_2 and x_3 as end vertices, then $P(x_2, x_3)$ must contain the subpaths $uP_{x_2}x_1$ and $vP_{x_3}x_4$ where $\{u, v\} = \{x_2, x_3\}$. (Note that, when $m = 3$, $vP_{x_3}x_4 = vP_{x_3}x_1$).

Now, in order that $P(x_2, x_3)$ covers all the vertices in G_1, the subpath $vP_{x_3}x_4$ must be extended to a subpath of the form $vP_{x_3}x_4P_{x_4} \cdots x_{m-1}P_{x_{m-1}}x_mP_{x_m}x_1$. But this is a contradiction.

Next we assume that G is broken. Then there exist $i, j \geq 2$ such that $A_{x_i} = \varnothing$ and $A_{y_j} = \varnothing$. Let u and v be two vertices in G. We shall show that there is a Hamilton path $P(u, v)$ in G^2 having u and v as end vertices.

Case (1): u is in G_1 and v is in G_2.

By Theorem 3.4, there is a Hamilton path P (respectively Q) in G_1^2 (respectively G_2^2) with u and x_1 (respectively y_1 and v) as end vertices. As such PQ is a Hamilton path in G^2.

Case (2): u and v are both in G_1.

Without loss of generality, assume that $u = x_k$ and $v = x_l$ for some $1 \leq k < l \leq m$.

Case (2.1): $2 \leq i \leq k < l \leq m$.

Let $L = x_k P_{x_k} x_{k-1} P_{x_{k-1}} x_{k-2} P_{x_{k-2}} \cdots x_{i+1} P_{x_{i+1}} x_i P_{x_{i-1}} x_{i-1} P_{x_{i-2}} x_{i-2}$
$\cdots x_2 P_{x_1} x_1 P_{x_m} x_m P_{x_{m-1}} x_{m-1} \cdots x_{l+1} P_{x_l} x_{l-1} P_{x_{l-2}} x_{l-3}$.

If k and l are of different parity, then take M to be the following Hamilton path in G_1^2 with x_k and x_l as end vertices

$$L \cdots x_{k+2} P_{x_{k+1}} x_{k+1} P_{x_{k+2}} x_{k+3} P_{x_{k+4}} x_{k+5} \cdots x_{l-2} P_{x_{l-1}} x_l.$$

If k and l are of the same parity, then take M to be the following Hamilton path in G_1^2 with x_k and x_l as end vertices

$$L \cdots x_{k+3} P_{x_{k+2}} x_{k+1} P_{x_{k+1}} x_{k+2} P_{x_{k+3}} x_{k+4} \cdots x_{l-2} P_{x_{l-1}} x_l.$$

In the event that $l = m$, M reduces to $x_k P_{x_k} x_{k-1} P_{x_{k-1}} x_{k-2} P_{x_{k-2}} \cdots x_{i+1} P_{x_{i+1}} x_i P_{x_{i-1}} x_{i-1} P_{x_{i-2}} x_{i-2} \cdots x_2 P_{x_1} x_1 P_{x_m} x_m$.

Let N denote the following path in G_2^2 with y_2 and y_n as end vertices

$$y_2 P_{y_2} y_3 P_{y_3} \cdots y_{j-1} P_{y_{j-1}} y_j P_{y_{j+1}} y_{j+1} P_{y_{j+2}} \cdots y_{n-1} P_{y_n} y_n. \qquad (*)$$

Let M_1 (respectively M_2) denote the subpath of M with x_k and x_2 (respectively x_1 and x_l) as end vertices. Since $x_2 y_2$ is an edge in G^2, we see that $M_1 N P_{x_1} M_2$ is a suitable Hamilton path $P(u, v)$ in G^2.

Case (2.2): $1 \leq k < i < l \leq m$.

If $l < m$, let

$L = x_{k-3} P_{x_{k-2}} x_{k-1} P_{x_k} x_{k+1} P_{x_{k+1}} x_{k+2} P_{x_{k+2}}$
$\cdots x_{i-1} P_{x_{i-1}} x_i P_{x_{i+1}} x_{i+1} P_{x_{i+2}} x_{i+2} \cdots P_{x_{l-1}} x_{l-1} P_{x_l} x_{l+1} P_{x_{l+2}} x_{l+3}$.

Further, let L_1 denote the following path

$$x_{m-3} P_{x_{m-2}} x_{m-1} P_{x_m} x_m P_{x_{m-1}} x_{m-2} \cdots x_{l+2} P_{x_{l+1}} x_l$$

or the path

$$x_{m-2} P_{x_{m-1}} x_m P_{x_m} x_{m-1} P_{x_{m-2}} x_{m-3} \cdots x_{l+2} P_{x_{l+1}} x_l$$

depending on whether l and m are of the same or different parity.

If $l = m$, let

$$L_2 = x_{k-3}P_{x_{k-2}}x_{k-1}P_{x_k}x_{k+1}P_{x_{k+1}}x_{k+2}P_{x_{k+2}} \cdots x_{i-1}P_{x_{i-1}}x_iP_{x_{i+1}}x_{i+1}P_{x_{i+2}}x_{i+2}\cdots x_{l-1}P_{x_l}x_l.$$

(i) Suppose $k = 1$.

If $l = m$, then we take

$M = x_1P_{x_1}x_2P_{x_2}x_3P_{x_3}\cdots x_{i-1}P_{x_{i-1}}x_iP_{x_{i+1}}x_{i+1}P_{x_{i+2}}x_{i+2}\cdots x_{l-1}P_{x_l}x_l$

to be the Hamilton path in G_1^2 with x_k and x_l as end vertices. If $l < m$, then we take

$M = x_1P_{x_1}x_2P_{x_2}x_3P_{x_3}\cdots x_{i-1}P_{x_{i-1}}x_iP_{x_{i+1}}x_{i+1}P_{x_{i+2}}x_{i+2}\cdots x_{l-1}P_{x_l}x_{l+1}P_{x_{l+2}}x_{l+3}\cdots L_1.$

(ii) Now consider the case $k > 1$.

Suppose k is odd. Then take M to be the Hamilton path in G_1^2 with x_k and x_l as end vertices where

$$M = x_kP_{x_{k-1}}x_{k-2}\cdots x_3P_{x_2}x_1P_{x_1}x_2P_{x_3}\cdots L\cdots L_1 \qquad \text{if } l < m$$

and

$$M = x_kP_{x_{k-1}}x_{k-2}\cdots x_3P_{x_2}x_1P_{x_1}x_2P_{x_3}\cdots L_2 \qquad \text{if } l = m.$$

Suppose k is even. Then take M to be the Hamilton path in G_1^2 with x_k and x_l as end vertices where

$$M = x_kP_{x_{k-1}}x_{k-2}\cdots x_4P_{x_3}x_2P_{x_1}x_1P_{x_2}x_3\cdots L\cdots L_1 \qquad \text{if } l < m$$

and

$$M = x_kP_{x_{k-1}}x_{k-2}\cdots x_4P_{x_3}x_2P_{x_1}x_1P_{x_2}x_3\cdots L_2 \qquad \text{if } l = m.$$

Let N denote the path in G_2^2 with y_2 and y_n as end vertices as defined in (*) (of Case (2.1)).

Suppose $k = 1$. Then let $P(u, v)$ be the Hamilton path obtained from M by replacing x_1 with x_1N.

Suppose $k > 1$.

If k is odd, let M_1 (respectively M_2) denote the subpath of M from x_k to x_1 (respectively P_{x_1} to x_l). Since $y_n P_{x_1} x_2$ is a path in G^2, we see that $M_1 N M_2$ (where $x_1 P_{x_1}$ is replaced by $y_n P_{x_1}$) is a suitable Hamilton path $P(u,v)$ in G^2.

If k is even, let M_1 (respectively M_2) denote the subpath of M from x_k to P_{x_1} (respectively x_1 to x_l). Since $x_2 P_{x_1} y_2$ is a path in G^2, we see that $M_1 N M_2$ (where $P_{x_1} x_1$ is replaced by $P_{x_1} y_2$) is a suitable Hamilton path $P(u,v)$ in G^2.

The case where u or v lies on some pendent set A_{x_i} or A_{y_j} can be easily reduced to the above cases. $\square$

3.3 $\Theta(m,n,r)$

In this section, we obtain a lemma which is a necessary condition for the graph $\Theta(m,n,r)$ to have panconnected square. Then a necessary and sufficient condition for $\Theta(m,n,r)$ to have panconnected square is determined.

We first give a well-known fact which is a necessary condition for the existence of a Hamilton path in a graph.

Theorem 3.6. *([15]) Suppose a graph G has a Hamilton path. Then the deletion of any s vertices from it will result in G with at most $s+1$ components.*

Let A be a subset of the vertex set of a graph G and let $G[A]$ denote the subgraph of G induced by the set A.

Lemma 3.7. *Let G denote the graph $\Theta(m,n,r)$ where $2 \leq m < n, r$. Suppose G has no vertex of degree 2. Then G^2 is not panconnected unless $m = 2$ and G has a t-vertex such that its pendent set is empty.*

Proof. We shall show that there is no Hamilton path in G^2 having x_1 and x_m as end vertices unless $m = 2$ and G has a t-vertex such that its pendent set is empty.

First, assume that there is no t-vertex such that its pendent set is empty in G.

Let H denote the graph obtained from G^2 by deleting the vertices x_1 and x_m together with all edges incident to them.

Let $S = \{x_2, \ldots, x_{m-1}, y_2, \ldots, y_{n-1}, z_2, \ldots, z_{r-1}\}$. Then $|S| = m+n+r-6$ and $H-S$ has $m+n+r-4$ components, $H[A_{x_i}]$, $H[A_{y_j}]$, $H[A_{z_k}]$ where $i = 1, 2, \ldots, m$, $j = 2, \ldots, n-1$ and $k = 2, \ldots, r-1$. This implies that H has no Hamilton path and hence G^2 has no Hamilton path with x_1 and x_m as end vertices unless $A_{x_1} = \varnothing$ or $A_{x_m} = \varnothing$.

Now, assume that $A_{x_1} = \varnothing$ and $m \geq 3$. Suppose there is a Hamilton path $P(x_1, x_m)$ in G^2 having x_1 and x_m as end vertices. Then, without loss of generality, we may assume that $P(x_1, x_m)$ must begin with a subpath of the form

$M_1 = x_1 P_{x_2} x_2 P_{x_3} x_3 \cdots x_{m-2} P_{x_{m-1}} x_{m-1}$, or

$M_2 = x_1 P_{x_2} x_3 x_2 P_{x_3} x_4 P_{x_4} \cdots x_{m-1} P_{x_{m-1}}$, or

$M_3 = x_1 x_3 P_{x_2} x_2 P_{x_3} x_4 P_{x_4} \cdots x_{m-1} P_{x_{m-1}}$, or

$M_4 = x_1 x_2 P_{x_2} x_3 P_{x_3} x_4 P_{x_4} \cdots x_{m-1} P_{x_{m-1}}$

in order that $P(x_1, x_m)$ covers all the vertices in $\mathcal{P}_m$ (except x_m) and all the corresponding pendent sets. Since M_2, M_3 and M_4 each cannot be extended to cover the rest of the vertices in G^2, it follows that $P(x_1, x_m)$ must take the form $M_1 L$, where L is either the subpath L_i or the subpath $P_{x_m} L_i$, for some $i \in \{1, 2\}$. Here $L_1 = y_{n-1} P_{y_{n-1}} y_{n-2} P_{y_{n-2}} \cdots y_2 P_{y_2}$ and $L_2 = z_{r-1} P_{z_{r-1}} z_{r-2} P_{z_{r-2}} \cdots z_2 P_{z_2}$.

Either case is a contradiction because the resulting path cannot be extended to cover the rest of the vertices of G^2 (because the vertices in P_{y_2} are not adjacent to those in P_{z_2} and vice versa). $\square$

We now obtain necessary and sufficient conditions for the graph $\Theta(m, n, r)$ to have panconnected square.

Theorem 3.8. *Let G denote the graph $\Theta(m, n, r)$.*

(i) Suppose $m = 2$ and $n, r \geq 3$. Then G^2 is panconnected if and only if G has a vertex w such that A_w is an empty set.

(ii) Suppose $m, n, r \geq 3$. Then G^2 is panconnected if and only if G has a vertex of degree 2.

Proof. The necessary part has been established in Lemma 3.7. We now prove the sufficiency.

(i) Suppose $m = 2$. Let H be the graph obtained from G by deleting the edge x_1x_m. Then H is an SF graph. Since G has a vertex w such that A_w is an empty set, we see that H is a broken SF graph and H^2 is panconnected by Theorem 3.4. Consequently, G^2 is also panconnected.

(ii) Suppose $m, n, r \geq 3$. Let u and v be two vertices in G. We shall show that there is a Hamilton path $P(u, v)$ in G^2 having u and v as end vertices. We can just assume that u and v are in $\mathcal{P}_m \cup \mathcal{P}_n \cup \mathcal{P}_r$ (since the other cases can be reduced to this case).

Recall that x and y are the two common vertices in G where all the end vertices of the three paths $\mathcal{P}_m, \mathcal{P}_n$ and $\mathcal{P}_r$ have been identified.

Case (1): u and v are on different paths of G.

In this case, since G has a vertex of degree 2, there exist two paths, say $\mathcal{P}_m$ and $\mathcal{P}_n$ whose union (together with their pendent sets) form a broken SF graph W. There is no loss of generality to assume that u is in W and v is in $\mathcal{P}_r$.

By Theorem 3.4, there is a Hamilton path $P_1(u, x)$ in W^2 with u and x as end vertices. We wish to extend $P_1(u, x)$ to a Hamilton path in G^2.

Suppose $v = z_k$ for some $2 \leq k \leq r - 1$.

If $k = r - 1$, then let $P_2(x, v) = z_1 P_{z_2} z_2 P_{z_3} z_3 \cdots P_{z_{r-2}} z_{r-2} P_{z_{r-1}} z_{r-1}$.

If $k < r - 1$, let

$L_1 = z_1 P_{z_2} z_2 P_{z_3} z_3 \cdots z_{k-2} P_{z_{k-1}} z_{k-1} P_{z_k} z_{k+1}$ and $L_2 = P_{z_{k+3}} z_{k+2} P_{z_{k+1}} z_k$.

Also, let $P_2(x, v)$ denote the following path (which covers all the vertices in $G - W$) with end vertices x and v

$$L_1 P_{z_{k+2}} z_{k+3} \cdots z_{r-3} P_{z_{r-2}} z_{r-1} P_{z_{r-1}} z_{r-2} P_{z_{r-3}} \cdots L_2$$

or

$$L_1 P_{z_{k+2}} z_{k+3} \cdots z_{r-2} P_{z_{r-1}} z_{r-1} P_{z_{r-2}} z_{r-3} \cdots L_2$$

depending on whether k and r have the same or different parity.

We can then take $P_1(u, x) P_2(x, v)$ to be a suitable Hamilton path $P(u, v)$.

Case (2): u and v are on the same path of G.

Suppose u and v are on the path $\mathcal{P}_r$ say $u = z_k$ and $v = z_l$.

Consider the case $2 \leq k < l \leq r-1$ first.

Case (2.1): $\mathcal{P}_r$ has no vertex of degree 2.

Let $P_1(y,v)$ denote the subpath $z_rP_{z_{r-1}}z_{r-1}P_{z_{r-2}}\cdots z_{l-1}P_{z_l}z_l$. Further, let $L_1 = z_kP_{z_{k+1}}z_{k+2}P_{z_{k+3}}z_{k+4}$ and $L_2 = z_{k+1}P_{z_k}z_{k-1}P_{z_{k-1}}z_{k-2}P_{z_{k-2}}\cdots z_2P_{z_2}z_1$.

Now let $P_2(u,x)$ denote the subpath

$$L_1\cdots z_{l-4}P_{z_{l-3}}z_{l-2}P_{z_{l-1}}z_{l-1}P_{z_{l-2}}z_{l-3}P_{z_{l-4}}\cdots L_2$$

or the subpath

$$L_1\cdots z_{l-3}P_{z_{l-2}}z_{l-1}P_{z_{l-1}}z_{l-2}P_{z_{l-3}}z_{l-4}P_{z_{l-5}}\cdots L_2$$

depending on whether k and l are of the same or different parity.

Let W be the subgraph obtained from G by deleting all vertice of $\mathcal{P}_r - \{x,y\}$ together with their pendent sets. Then W is a broken SF graph (because $\mathcal{P}_r$ has no vertex of degree 2). Hence there is a Hamilton path $P_3(z_1,z_r)$ in W^2 with z_1 and z_r as end vertices. Then $P_2(u,x)P_3(z_1,z_r)P_1(y,v) = P(u,v)$ is a suitable Hamilton path in G^2.

Case (2.2): $\mathcal{P}_r$ has some vertices of degree 2.

Suppose $A_{z_i} = \varnothing$ with $i \notin \{1,r\}$.

(i) Suppose $1 < i < k$. Then let $M_1 = P_{z_{r-1}}z_{r-1}P_{z_{r-2}}z_{r-2}\cdots P_{z_{l+1}}z_{l+1}P_{z_l}z_l$ and $L_1 = z_{k+1}P_{z_k}z_{k-1}P_{z_{k-1}}z_{k-2}P_{z_{k-2}}\cdots P_{z_{i+1}}z_iP_{z_{i-1}}z_{i-1}P_{z_{i-2}}\cdots z_2P_{z_1}$.

Further let M_2 denote the following path

$$z_kP_{z_{k+1}}z_{k+2}\cdots P_{z_{l-3}}z_{l-2}P_{z_{l-1}}z_{l-1}P_{z_{l-2}}z_{l-3}P_{z_{l-4}}z_{l-5}\cdots L_1$$

or the path

$$z_kP_{z_{k+1}}z_{k+2}\cdots P_{z_{l-4}}z_{l-3}P_{z_{l-2}}z_{l-1}P_{z_{l-1}}z_{l-2}P_{z_{l-3}}z_{l-4}\cdots L_1$$

depending on whether k and l have the same or different parity.

In the event that $k = l-1$, M_2 reduces to $z_kP_{z_k}z_{k-1}P_{z_{k-1}}\cdots P_{z_{i+1}}z_iP_{z_{i-1}}z_{i-1}$ $P_{z_{i-2}}\cdots z_2P_{z_1}$.

Then we see that $G-(M_1 \cup M_2) = W$ is a broken SF graph. By Theorem 3.4, W^2 has a Hamilton path $P_3(z_1, z_r)$ with z_1 and z_r as end vertices. As such $M_2P_3(z_1, z_r)M_1$ is a suitable Hamilton path $P(u, v)$.

(ii) Suppose $i = k$. Then let $M_1 = P_{z_1}z_2P_{z_2}z_3 \cdots z_{k-1}P_{z_{k-1}}z_k$ and

$L_1 = z_{l-3}P_{z_{l-2}}z_{l-1}P_{z_l}z_{l+1}P_{z_{l+1}}z_{l+2}P_{z_{l+2}} \cdots z_{r-1}P_{z_{r-1}}$.

Further let M_2 denote the following path

$$z_lP_{z_{l-1}}z_{l-2}P_{z_{l-3}} \cdots z_{k+2}P_{z_{k+1}}z_{k+1}P_{z_{k+2}}z_{k+3}P_{z_{k+4}}z_{k+5} \cdots L_1$$

or

$$z_lP_{z_{l-1}}z_{l-2}P_{z_{l-3}} \cdots z_{k+3}P_{z_{k+2}}z_{k+1}P_{z_{k+1}}z_{k+2}P_{z_{k+3}}z_{k+4} \cdots L_1$$

depending on whether k and l have the same or different parity.

In the event that $k = l-1$, M_2 reduces to $z_lP_{z_l}z_{l+1}P_{z_{l+1}} \cdots z_{r-1}P_{z_{r-1}}$.

Then $G-(M_1 \cup M_2) = W$ is a broken SF graph. By Theorem 3.4, W^2 has a Hamilton path $P_3(z_1, z_r)$ with z_1 and z_r as end vertices. As such $M_2P_3(z_r, z_1)M_1$ is a suitable Hamilton path $P(v, u)$.

(iii) Suppose $2 \le k < i < l \le r-1$. Then let

$L_1 = z_{k-3}P_{z_{k-2}}z_{k-1}P_{z_k}z_{k+1}P_{z_{k+1}}z_{k+2} \cdots z_{i-1}P_{z_{i-1}}z_iP_{z_{i+1}}z_{i+1}P_{z_{i+2}}z_{i+2}$. Also, let M_1 denote the following path

$$z_kP_{z_{k-1}}z_{k-2}P_{z_{k-3}}z_{k-4} \cdots z_5P_{z_4}z_3P_{z_2}z_2P_{z_3}z_4P_{z_5} \cdots L_1$$

or

$$z_kP_{z_{k-1}}z_{k-2}P_{z_{k-3}}z_{k-4} \cdots P_{z_5}z_4P_{z_3}z_2P_{z_2}z_3P_{z_4}z_5 \cdots L_1$$

depending on whether k is odd or even.

Now if l and r are of the same parity, then let

$L_2 = P_{z_{r-1}}z_{r-2}P_{z_{r-3}}z_{r-4} \cdots z_{l+2}P_{z_{l+1}}z_l$ and $L_3 = z_{l-2}P_{z_{l-1}}z_{l-1}P_{z_l}z_{l+1} \cdots z_{r-1}P_{z_r}$

otherwise let

$L_2 = P_{z_r}z_{r-1}P_{z_{r-2}}z_{r-3} \cdots z_{l+2}P_{z_{l+1}}z_l$ and $L_3 = z_{l-2}P_{z_{l-1}}z_{l-1}P_{z_l}z_{l+1} \cdots z_{r-2}P_{z_{r-1}}$.

Finally, let $P_1(z_k) = M_1 \cdots L_3$. Then we see that $G-(P_1(z_k) \cup L_2) = W$ is a broken SF graph. So, by Theorem 3.4, W^2 has a Hamilton path $P(z_r, z)$ with

z_r and z as end vertices , where z is a vertex in W and z is adjacent to z_r. In this case, $P_1(z_k)P(z, z_r)L_2$ or $P_1(z_k)P(z_r, z)L_2$ (depending on whether l and r are of the same or different parity) is a suitable Hamilton path $P(u, v)$ (because z is adjacent to a vertex of P_{z_r} and z_r is adjacent to a vertex of $P_{z_{r-1}}$).

We now consider the remaining case where $k = 1$ or $l = r$.

Suppose $l = r$ and $k \geq 2$. Let $P_1(z_k)$ and $P(z, z_r)$ be as defined in Case(2.2)(iii). Then $P_1(z_k)P(z, z_r)$ is a Hamilton path in G^2 with z_k and z_r as end vertices .

Suppose $l = r$ and $k = 1$. Since A_{z_i} is an empty set where $2 \leq i \leq r - 1$, we may take $P(u, v)$ to be the following Hamilton path

$$z_1 P_{y_2} y_2 P_{y_3} y_3 \cdots y_{n-2} P_{y_{n-1}} y_{n-1} P_{y_n} z_{r-1} P_{z_{r-1}} z_{r-2} P_{z_{r-2}} \cdots$$
$$z_{i+1} P_{z_{i+1}} z_i P_{z_{i-1}} z_{i-1} \cdots z_3 P_{z_2} z_2 P_{z_1} x_2 P_{x_2} x_3 P_{x_3} \cdots P_{x_{m-2}} x_{m-1} P_{x_{m-1}} z_r.$$

This finishes the proof. $\square$

CHAPTER IV

GRAPHS WITH CYCLOMATIC NUMBER THREE HAVING PANCONNECTED SQUARE

In this chapter, we present 8 families of graphs denoted $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_8$ each with cyclomatic number 3 having no internal cut-edges. It turns out that these are the only such families of graphs whose square could be panconnected (see Proposition 4.1). Furthermore, we define three larger families of graphs $G(m_1, m_2, \ldots, m_r)$, $H(r, s, t)$ and $\Theta(m_1, m_2, \ldots, m_r)$ which contain $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_5$ respectively as subfamilies. We then determine necessary and sufficient conditions for each of these larger families of graphs to have panconnected square.

4.1 A necessary condition

Figure 4.1 depicts a list of 8 graphs each with cyclomatic number 3 having no internal cut-edge.

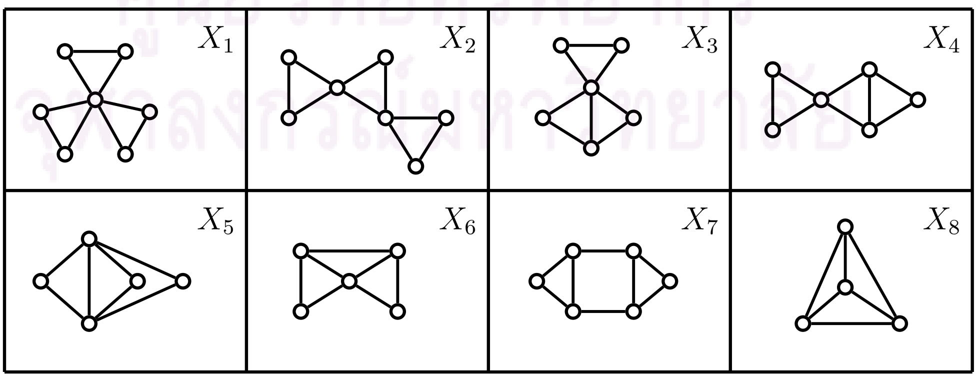


Figure 4.1: A list of 8 graphs with cyclomatic number 3.

It is routine to check that these are the only smallest simple graphs with cyclomatic number 3 having neither internal cut-edges nor vertices of degree 1. To each of these graphs we do the following operations:

(i) Subdivide any edge an arbitrary number of times. This is equivalent to replacing any edge by a path of arbitrary length. This operation yields many graphs with cyclomatic number 3.

(ii) To each resulting graph G obtained in (i), and to each vertex v of G we join a new set of independent vertices A_v, that is the pendent set of v, (which may be empty).

Let $\mathcal{C}_3$ be the set of all graphs obtained by performing the operations (i) and (ii) above to every graph in Figure 4.1.

We now obtain a necessary condition for graphs with cyclomatic number 3 whose square is panconnected.

Proposition 4.1. *Suppose G is a graph with $c(G) = 3$. If G^2 is panconnected, then $G \in \mathcal{C}_3$.*

Proof. Clearly a graph has cyclomatic number 0 if and only if it is a tree. Hence graphs with cyclomatic number 1 are those that are obtained from the trees by adding a new edge which are unicyclic graphs. Likewise, graphs with cyclomatic number 2 are those that are obtained from unicyclic graphs by adding a new edge.

Since $c(G) = 3$, G is obtained from a graph with cyclomatic number 2 by adding a new edge to two non-adjacent vertices. Since G^2 is panconnected, G has no internal cut-edge by Theorem 3.1. If we delete all vertices of degree 1 and then contract those edges that are incident to vertices of degree 2 in the resulting graph until we get a graph H with neither multiple edges nor loops, then H must be one of those graphs shown in Figure 4.1. This shows that $G \in \mathcal{C}_3$. $\square$

For each $i = 1, 2, \ldots, 8$, let $\mathcal{F}_i$ denote the set of all graphs obtained from the graph X_i (in Figure 4.1) by applying the operations (i) and (ii) described above. Then clearly, $\mathcal{C}_3 = \cup_{i=1}^{8} \mathcal{F}_i$.

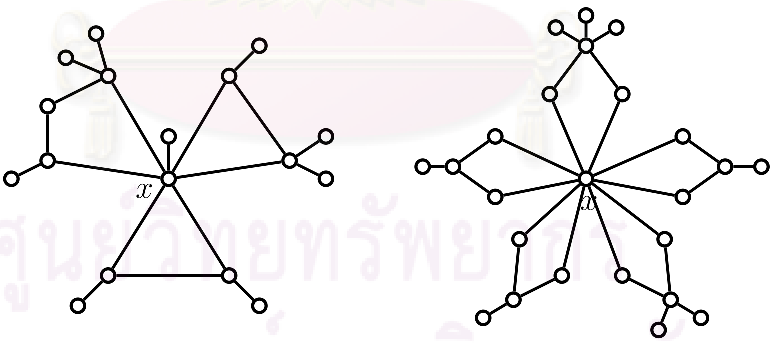
We focus our attention on 3 families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_5$ of graphs. We determine necessary and sufficient conditions for these families of graphs to have panconnected square.

4.2 $G(m_1, m_2, \ldots, m_r)$

Suppose $r \geq 2$. Let $G(m_1), G(m_2), \ldots, G(m_r)$ be SF graphs. For each $i = 1, 2, \ldots, r$, let z_i be a c-vertex in $G(m_i)$. Let $G(m_1, m_2, \ldots, m_r)$ denote the graph obtained by identifying all the c-vertices $z_1, z_2, \ldots, z_r$ into a single vertex x. We call $G(m_1, m_2, \ldots, m_r)$ a *bouquet* of r SF graphs. The graph $G(m_1, m_2, \ldots, m_r)$ is said to be *broken* if for every $i = 1, 2, \ldots, r$, there exists a vertex z in $G(m_i)$ where $z \neq x$ and $A_z = \varnothing$.

Note that each of r SF graphs $G(m_1), G(m_2), \ldots, G(m_r)$ of a broken $G(m_1, m_2, \ldots, m_r)$ has a vertex of degree 2.

Figure 4.2(a) shows $G(3, 3, 4)$ and Figure 4.2(b) shows $G(4, 4, 4, 4, 4)$ which is broken.



(a) (b)

Figure 4.2: (a) $G(3, 3, 4)$ and (b) $G(4, 4, 4, 4, 4)$

Clearly, $\mathcal{F}_1$ is the set of all bouquet of 3 SF graphs. Bouquet of 2 SF graphs having panconnected square are completely characterized in Theorem 3.5.

The following proposition will be needed for the necessary part of the proof of main results (Theorems 4.5 and 4.6).

Proposition 4.2. *Let $G(m)$ denote an SF graph with cycle $x_1x_2\ldots x_mx_1$. Let H_1 and H_2 be two graphs with $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. Let G be a graph obtained by*

(i) identifying any vertex of H_1 with the vertex x_1 and

(ii) identifying any vertex of H_2 with x_j for some $2 \leq j \leq m$ of $G(m)$.

If $A_{x_i} \neq \varnothing$, for every $i \notin \{1, j\}$, then G^2 is not panconnected.

Proof. Since we can relabel the vertices (in the reverse order if necessary), we may assume without loss of generality that $2 \leq j \leq m-1$.

We shall show that there is no Hamilton path in G^2 having x_1 and x_m as end vertices. Let Q be the graph obtained from G^2 by deleting x_1 and x_m.

Let Q_1 denote the graph obtained from Q by deleting $m-2$ vertices $x_2, x_3, \ldots, x_{m-1}$. Then there are at least m components in Q_1, namely $Q_1[A_{x_2}], \ldots, Q_1[A_{x_{j-1}}]$, $Q_1[(A_{x_j} \cup H_2) - x_j], Q_1[A_{x_{j+1}}], \ldots, Q_1[A_{x_m}]$ and $Q_1[(A_{x_1} \cup H_1) - x_1]$. Thus Q_1 has no Hamilton path and hence G^2 has no Hamilton path having x_1 and x_m as end vertices. $\square$

Suppose x is a vertex of a graph G, we let $N(x)$ denote the set of vertices adjacent to x in G.

The next 2 lemmas will be needed for the sufficient part of the proof of main results (Theorems 4.5 and 4.6).

Lemma 4.3. *(i) Suppose G is an SF graph with cycle $x_1x_2\ldots x_mx_1$. Suppose $A_{x_i} = \varnothing$ for some $2 \leq i \leq m$. Then, for any $2 \leq k < i$, there is a Hamilton path in $G^2 - (\{x_1\} \cup A_{x_1})$ having x_k and x_m as end vertices.*

(ii) Suppose $G(m_1, m_2, \ldots, m_r)$ is a broken bouquet of r SF graphs. Let J_r denote the graph obtained from $G(m_1, m_2, \ldots, m_r)^2$ by deleting all the vertices in $\{x\} \cup A_x$. Suppose $z \in N(x) \cap V(G(m_1))$ and $w \in N(x) \cap V(G(m_r))$. Then there is a Hamilton path $P(z, w)$ in J_r with z and w as end vertices.

Proof. (i) Let L denote the following path

$$x_5P_{x_4}x_3P_{x_2}x_2P_{x_3}x_4P_{x_5}x_6 \quad \text{or} \quad x_6P_{x_5}x_4P_{x_3}x_2P_{x_2}x_3P_{x_4}x_5$$

depending on whether k is odd or even.

If $k > 2$, then a suitable Hamilton path is given by

$x_kP_{x_{k-1}}x_{k-2}\cdots L\cdots x_{k-1}P_{x_k}x_{k+1}P_{x_{k+1}}\cdots x_{i-1}P_{x_{i-1}}x_iP_{x_{i+1}}x_{i+1}P_{x_{i+2}}\cdots P_{x_m}x_m$

If $k = 2$, then a suitable Hamilton path is given by

$x_2P_{x_2}x_3P_{x_3}\cdots P_{x_{i-1}}x_iP_{x_{i+1}}\cdots P_{x_m}x_m$

(ii) For each $i = 1, 2, \ldots, r$, let z_i and w_i be two vertices in $N(x)\cap V(G(m_i))$. From (i) (with $k = 2$), we know that there is a Hamilton path $P_i(z_i, w_i)$ in $G(m_i)^2-(\{x\}\cup A_x)$ with z_i and w_i as end vertices. Then $P(z, w) = P_1(z_1, w_1)P_2(z_2, w_2)\cdots P_r(z_r, w_r)$ where $z_1 = z$ and $w_r = w$ is a suitable Hamilton path in J_r. □

Lemma 4.4. *Let $G(m)$ denote a broken SF graph with cycle $x_1x_2\ldots x_mx_1$. Suppose W is a non-empty subset of $\{1, 2, \ldots, m\}$ such that $A_{x_k} \neq \varnothing$ whenever $k \in W$. For each $k \in W$, let H_k denote a graph with the following properties:*

(i) H_k contains vertices u_k, v_k, w_k such that u_k is adjacent to both v_k and w_k.

(ii) $H_k^2 - u_k$ has a Hamilton path having v_k and w_k as end vertices.

Let G be the graph obtained from $G(m)$ by first deleting A_{x_k} and then identifying u_k of H_k with x_k for each $k \in W$. Then for any two vertices u and v in $G(m)$, there is a Hamilton path in G^2 having u and v as end vertices.

Proof. Since $G(m)$ is a broken SF graph, for any two vertices u and v, there is a Hamilton path $P(u, v)$ in $G(m)^2$ with u and v as end vertices by Theorem 3.4.

Using the Hamilton path $P(u, v)$ in $G(m)^2$, we shall construct a Hamilton path $P^*(u, v)$ in G^2 with u and v as end vertices in the following way.

First, if $k \notin W$, then any subpath of $P(u, v)$ involving x_k or A_{x_k} in $G(m)^2$ is taken to be a subpath of $P^*(u, v)$ in G^2.

Next, suppose $k \in W$. Let $P_k(v_k, w_k)$ denote a Hamilton path in $H_k^2 - u_k$ with v_k and w_k as end vertices. (i) If $P(u, v)$ contains a subpath of the form $x_jP_{x_k}x_k$ for some $j \in \{k-1, k+1\}$, then in G^2, we take $x_jP_k(v_k, w_k)x_k$ to be a subpath of

$P^*(u,v)$. (ii) If $P(u,v)$ contains a subpath of the form $x_{k-1}P_{x_k}x_{k+1}$, then in G^2, we take $x_{k-1}P_k(v_k,w_k)x_{k+1}$ to be a subpath of $P^*(u,v)$. □

We now obtain a necessary and sufficient condition for the graph $G(m_1,m_2,\dots,m_r)$ to have panconnected square.

Theorem 4.5. *Suppose $r \geq 2$. Let G denote the graph $G(m_1,m_2,\dots,m_r)$. Then G^2 is panconnected if and only if G is broken.*

Proof. To prove the necessity, suppose some SF graph, say $G(m_r)$ of G is not broken. If we take H_1 to be the graph $G(m_1,\dots,m_{r-1})$ and H_2 to be the subgraph of $G(m_r)$ induced by some c-vertex and its pendent set. Then the resulting graph as constructed in Proposition 4.2 is isomorphic to the graph G. By Proposition 4.2, G^2 is not panconnected.

Next, we shall prove the sufficiency by induction on r.

For $r = 2$, G is the graph $G(m_1,m_2)$. Since G is broken, by Theorem 3.5, G^2 is panconnected.

Let $r \geq 3$ and assume that the result holds for any broken bouquet of k SF graphs for $k < r$. Let G be the graph $G(m_1,m_2,\dots,m_r)$ which is broken and let u and v be two vertices in G. We shall show that there is a Hamilton path $P(u,v)$ in G^2 with u and v as end vertices.

For each $i \in \{1,2,\dots,m\}$, let B_i denote the graph obtained from $G(m_1,m_2,\dots,m_r)$ by deleting all the vertices in $G(m_i)$ except those in $\{x\} \cup A_x$. Then B_i is a bouquet of $r-1$ SF graphs.

Suppose u and v are on different SF graphs of G.

Without loss of generality, assume that u is in $G(m_1)$ and v is in $G(m_r)$. Since B_r is the graph $G(m_1,m_2,\dots,m_{r-1})$ which is broken, by the induction hypothesis, B_r^2 is panconnected. So there is a Hamilton path $P_1(u,x)$ in B_r^2 with u and x as end vertices. Since G is broken, $G(m_r)$ is also broken. By Theorem 3.4, $G(m_r)^2$ is panconnected. So there is a Hamilton path $P_2(x,v)$ in $G(m_r)^2$ with x and v as end vertices. Then $P_1(u,x)P_2(x,v)$ is a suitable Hamilton path $P(u,v)$.

Hence we assume that u and v are both on the same SF graph of G.

Without loss of generality, assume that u and v are both in $G(m_r)$ whose cycle is $x_1x_2\ldots x_{m_r}x_1$. Suppose $x = x_1$ and $A_{x_i} = \varnothing$ for some $2 \le i \le m_r$.

Recall that B_r is the graph $G(m_1, m_2, \ldots, m_{r-1})$. Now if we take $z \in N(x) \cap V(G(m_1))$ and $w \in N(x) \cap V(G(m_{r-1}))$. Then, by Lemma 4.3(ii), there is a Hamilton path in $B_r^2 - (\{x\} \cup A_x)$ having z and w as end vertices.

By Lemma 4.4, there is a Hamilton path in G^2 having u and v as end vertices. This completes the proof. $\square$

4.3 $H(r, s, t)$

Suppose r, s and t are integers such that $r, s \ge 1$ and $t \ge 3$. Let $G(t)$ be an SF graph with cycle $z_1z_2\ldots z_tz_1$. Let $H(r, s, t)$ denote any graph obtained from $G(t)$ by identifying a c-vertex of each of the r SF graphs $G(m_1), G(m_2), \ldots, G(m_r)$ with z_1 and identifying a c-vertex of each of the s SF graphs $G(n_1), G(n_2), \ldots, G(n_s)$ with z_m where $2 \le m \le t$. As such, the graph $H(r, s, t)$ contains $r + s + 1$ SF graphs $G(m_1), \ldots, G(m_r), G(n_1), \ldots, G(n_s)$ and $G(t)$ as subgraphs. If each of these SF graphs has a vertex of degree 2, we say that $H(r, s, t)$ is *broken*.

Clearly, $\mathcal{F}_2$ is the set of all the graphs $H(1, 1, t)$.

Let G_1 (respectively G_2) denote the subgraph of $H(r, s, t)$ induced by $G(m_1)$, $G(m_2), \ldots, G(m_r)$ (respectively $G(n_1), G(n_2), \ldots, G(n_s)$). Then G_1 is the graph $G(m_1, m_2, \ldots, m_r)$ and G_2 is the graph $G(n_1, n_2, \ldots, n_s)$. Further $G_1 \cup G(t)$ is the graph $G(m_1, m_2, \ldots, m_r, t)$ and $G_2 \cup G(t)$ is the graph $G(n_1, n_2, \ldots, n_s, t)$.

In Figure 4.3, (a) shows $H(1, 1, 3)$ with the subgraphs G_1 and G_2 and (b) shows a broken $H(1, 2, 4)$ with the subgraphs G_1 and G_2.

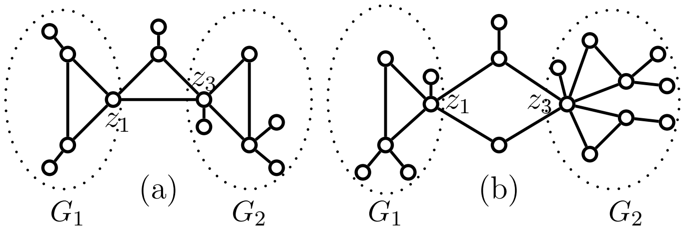


Figure 4.3: (a) $H(1, 1, 3)$ and (b) $H(1, 2, 4)$

We now obtain a necessary and sufficient condition for the graph $H(r,s,t)$ to have panconnected square.

Theorem 4.6. *Let G denote the graph $H(r,s,t)$. Then G^2 is panconnected if and only if G is broken.*

Proof. Suppose G is not broken. We shall show that G^2 is not panconnected by using Proposition 4.2. Let J be an SF subgraph of G which is not broken. If J is some subgraph of the type $G(m_i)$ or $G(n_j)$, we take H_2 to be the subgraph of J induced by some c-vertex and its pendent set, and H_1 to be the subgraph induced by the the rest of the SF subgraphs of G. If J is the SF subgraph $G(t)$, then we take H_1 to be the subgraph G_1 of G and H_2 to be the subgraph G_2 of G. This proves the necessity.

Next, we prove the sufficiency. We shall show that, for any two vertices u and v in G, there is a Hamilton path $P(u,v)$ in G^2 with u and v as end vertices. Throughout, assume that z_i is a vertex of degree 2 where $2 \leq i \leq m$ (since otherwise we can relabel the vertices of the cycle in $G(t)$ in reverse order).

Case (1): $u, v \in V(G_i)$ for some $i \in \{1,2\}$.

We may just assume that $u, v \in V(G_1)$.

(i) Suppose $u \in V(G(m_1))$, $v \in V(G(m_2))$ and $u, v \neq z_1$. By Theorem 3.4, there is a Hamilton path $P_1(u, z_1)$ in $G(m_1)^2$ with u and z_1 as end vertices.

Let Q_r denote the subgraph of G induced by $G(m_3), \ldots, G(m_r)$ if $r \geq 3$. Let $w_1 \in N(z_1) \cap V(G(m_3))$ and $w_2 \in N(z_1) \cap V(G(m_r))$. Then by Lemma 4.3, there is a Hamilton path $P_2(w_1, w_2)$ in $Q_r^2 - (\{z_1\} \cup A_{z_1})$ with w_1 and w_2 as end vertices. If $r \ngeq 3$, then $P_2(w_1, w_2)$ is an empty path.

We can assume that the cycle $y_1 y_2 \ldots y_n y_1$ in $G(m_2)$ is such that $y_1 = z_1$, $v = y_k$, $A_{y_i} = \varnothing$ with $2 \leq k \leq i \leq n$.

Suppose $2 \leq k < i \leq n$. By Theorem 4.5, there is a Hamilton path $P_3(z_1, z_t)$ in $(G(t) \cup G_2)^2$ from z_1 to z_t. By Lemma 4.3(i), there is a Hamilton path $P_4(y_k, y_n)$ in $G(m_2)^2 - (\{y_1\} \cup A_{y_1})$ from y_k to y_n.

Then $P(u,v) = P_1(u,z_1)P_3(z_1,z_t)P_2(w_1,w_2)P_4(y_n,v)$ is a suitable Hamilton path in G^2.

Suppose $2 < k = i \leq n$. Let $P_3(z_1,z_2)$ denote the following path

$$z_1P_{y_2}y_3P_{y_4}\cdots y_{k-4}P_{y_{k-3}}y_{k-2}P_{y_{k-1}}y_{k-1}P_{y_{k-2}}y_{k-3}\cdots P_{y_3}y_2z_2$$

or

$$z_1P_{y_2}y_3P_{y_4}\cdots y_{k-3}P_{y_{k-2}}y_{k-1}P_{y_{k-1}}y_{k-2}P_{y_{k-3}}y_{k-4}\cdots P_{y_3}y_2z_2$$

depending on whether k is odd or even.

Let $P_4(y_n,y_k)$ denote the path $y_nP_{y_n}y_{n-1}P_{y_{n-1}}\cdots y_{k+1}P_{y_{k+1}}y_k$. In the event that $k = i = n$, take $P_4(y_n,y_k) = y_k$.

We shall construct a Hamilton path $P_5(z_2,z_t)$ in $(G(t)\cup G_2)^2 - (\{z_1\}\cup A_{z_1})$. Let $z \in N(z_m)\cap V(G_2)$. Then by Theorem 4.5, there is a Hamilton path $P_6(z,z_m)$ in G_2^2 from z to z_m. Let

$P_5(z_2,z_t) = z_2P_{z_2}\cdots z_{i-1}P_{z_{i-1}}z_iP_{z_{i+1}}\cdots P_{z_{m-1}}z_{m-1}P_6(z,z_m)P_{z_{m+1}}z_{m+1}\cdots P_{z_t}z_t$.

Then $P(u,v) = P_1(u,z_1)P_3(z_1,z_2)P_5(z_2,z_t)P_2(w_1,w_2)P_4(y_n,v)$ is a suitable Hamilton path in G^2.

(ii) Hence assume that $u,v \in V(G(m_1))$.

Let H denote the subgraph $G-(G(m_1)-(\{z_1\}\cup A_{z_1}))$. We shall show that for some vertex $w \in V(H)-(\{z_1,z_2\}\cup A_{z_1})$, there is a Hamilton path $Q(z_2,w)$ in $H^2-(\{z_1\}\cup A_{z_1})$ with z_2 and w as end vertices. Then by Lemma 4.4, we have a Hamilton path in G^2 having u and v as end vertices.

To see this, let $z \in N(z_m)\cap V(G_2)$. Then by Theorem 4.5, there is a Hamilton path $P_1(z,z_m)$ in G_2^2 with z and z_m as end vertices. Let

$P_2(z_2,z_t) = z_2P_{z_2}\cdots z_{i-1}P_{z_{i-1}}z_iP_{z_{i+1}}\cdots P_{z_{m-1}}z_{m-1}P_1(z,z_m)P_{z_{m+1}}z_{m+1}\cdots P_{z_t}z_t$.

If $r = 1$, then take $w = z_t$ and $Q(z_2,w) = P_2(z_2,z_t)$.

If $r \geq 2$, let H_1 denote the subgraph of G induced by $G(m_2),\ldots,G(m_r)$. Also, let $w_1 \in V(G(m_2)), w \in V(G(m_r))$. Then by Lemma 4.3(ii), there is a Hamilton path $P_3(w_1,w)$ in $H_1^2-(\{z_1\}\cup A_{z_1})$ with w_1 and w as end vertices. As such, $P_2(z_2,z_t)P_3(w_1,w)$ is a suitable Hamilton path $Q(z_2,w)$.

Case (2): $u, v \in V(G(t))$.

By Lemma 4.3, we can find two vertices $u_1, u_2 \in N(z_1) \cap V(G_1)$ such that there is a Hamilton path in $G_1^2 - (\{z_1\} \cup A_{z_1})$ having u_1 and u_2 as end vertices. Likewise, we can find two vertices $v_1, v_2 \in N(z_m) \cap V(G_2)$ such that there is a Hamilton path in $G_2^2 - (\{z_m\} \cup A_{z_m})$ having v_1 and v_2 as end vertices.

Since the subgraph $G(t)$ has a vertex of degree 2, by Lemma 4.4, G^2 has a Hamilton path having u and v as end vertices.

Case (3): $u \in V(G_1)$, $v \in V(G_2) \cup V(G(t))$ *and* $\{u, v\} \neq \{z_1, z_m\}$.

Suppose $u \neq z_1$. By Case (1), we can assume that $v \neq z_1$. As such there is a Hamilton path $P_1(u, z_1)$ in G_1^2 with u and z_1 as end vertices by Theorem 4.5 (or Theorem 3.4 depending on the value of r). Also, by Theorem 4.5 again, there is a Hamilton path $P_2(z_1, v)$ in $(G(t) \cup G_2)^2$ with z_1 and v as end vertices. Then $P(u, v) = P_1(u, z_1)P_2(z_1, v)$ is a suitable Hamilton path in G^2.

Hence assume that $u = z_1$. By Case (2), we may assume that $v \in V(G_2)$ and $v \neq z_m$. By Theorem 4.5, there is a Hamilton path $P_1(u, z_m)$ in $(G(t) \cup G_1)^2$ with u and z_m as end vertices. By Theorem 4.5 (or Theorem 3.4 depending on the value of r), there is a Hamilton path $P_2(z_m, v)$ in G_2^2 with z_n and v as end vertices. Then $P(u, v) = P_1(u, z_m)P_2(z_m, v)$ is a suitable Hamilton path in G^2.

The proof is complete. $\square$

4.4 $\Theta(m_1, m_2, \ldots, m_r)$

Suppose $r \geq 3$ is an integer. Let θ_r be a multigraph with 2 vertices, say x and y, together with r multiple edges. Suppose $m_i \geq 2$ is an integer for each $i = 1, 2, \ldots, r$. Let $\theta(m_1, m_2, \ldots, m_r)$ denote the graph obtained by replacing the edges of θ_r with paths $\mathcal{P}_{m_1}, \mathcal{P}_{m_2}, \ldots, \mathcal{P}_{m_r}$ on $m_1, m_2, \ldots, m_r$ vertices respectively. Note that if $m_1 = 2$, then we require that $m_2, m_3, \ldots, m_r \geq 3$. Let $\Theta(m_1, m_2, \ldots, m_r)$ denote any graph obtained by joining each vertex v of $\theta(m_1, m_2, \ldots, m_r)$ to a new set of independent vertices A_v. That is, A_v is the pendent set of v. We call $\Theta(m_1, m_2, \ldots, m_r)$ an *r-stripe cactus graph.*

$\Theta(2,3,3,4)$ is depicted in Figure 4.4(a) and $\Theta(3,4,4,5,5)$ is depicted in Figure 4.4(b).

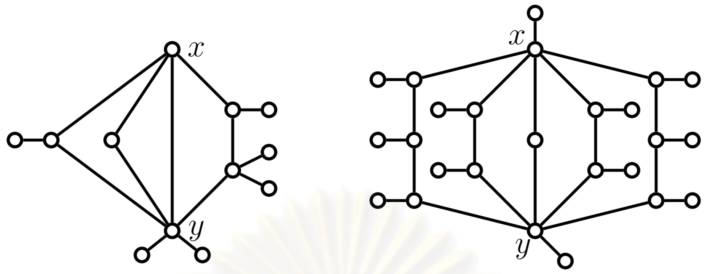


Figure 4.4: (a) $\Theta(2,3,3,4)$ and (b) $\Theta(3,4,4,5,5)$

Clearly, $\mathcal{F}_5$ is the set of all 4-stripe cactus graphs. 3-stripe cactus graphs having panconnected square have been completely characterized in Theorem 3.8.

The next propositon forms the necessary part of the proof of main results (Theorems 4.8 and 4.9).

Proposition 4.7. *Suppose $\Theta(m,n,r)$ has no vertex of degree 2 and $m,n,r \geq 3$. Let H be any graph with $|V(H)| \geq 3$. Let u and v be any two vertices in H. Let G denote any graph obtained from $\Theta(m,n,r)$ and H by identifying x with u and y with v respectively. Then G^2 is not panconnected.*

Proof. We shall show that there is no Hamilton path in G^2 having x and y as end vertices. Suppose on the contrary that there is a Hamilton path $P(x,y)$ in G^2 with x and y as end vertices.

Let $\mathcal{P}_m = x_1x_2\ldots x_m, \mathcal{P}_n = y_1y_2\ldots y_n$ and $\mathcal{P}_r = z_1z_2\ldots z_r$ where $x_1 = y_1 = z_1 = x$ and $x_m = y_n = z_r = y$.

We may assume that G is connected (otherwise G^2 is clearly not panconnected). As such H has at most two components. Further, if H has two components H_1 and H_2, we may assume without loss of generality that u is in H_1 and v is in H_2.

Then we assert that $J = G^2 - \{x,y\}$, has no Hamilton path and this contradicts the assumption that $P(x,y)$ is a Hamilton path in G^2 with x and y as end vertices.

To see this, let $S = \{x_2, \ldots, x_{m-1}, y_2, \ldots, y_{n-1}, z_2, \ldots, z_{r-1}\}$. Then $|S| = m+n+r-6$ and $J - S$ has at least $m+n+r-4$ components, namely $H_1 \cup A_x$, $H_2 \cup A_y$, $J[A_{x_i}]$, $J[A_{y_j}]$, $J[A_{z_k}]$ where $i = 2, \ldots, m-1$, $j = 2, \ldots, n-1$ and $k = 2, \ldots, r-1$.

Hence we assume that H is a connected graph. Further, we may assume that, for some vertices w_1 and w_2 in H such that w_1 and w_2 are neighbors of u and v respectively, there is a Hamilton path $P_1(w_1, w_2)$ in $H^2 - \{u, v\}$ with w_1 and w_2 as end vertices. If there is no such path in $H^2 - \{u, v\}$, then the following argument shows that $P(x, y)$ does not exist.

Let

$L_1 = x_{m-1}P_{x_{m-1}}x_{m-2}P_{x_{m-2}} \cdots x_2P_{x_2}$, $L_2 = y_{n-1}P_{y_{n-1}}y_{n-2}P_{y_{n-2}} \cdots y_2P_{y_2}$,

$L_3 = z_{r-1}P_{z_{r-1}}z_{r-2}P_{z_{r-2}} \cdots z_2P_{z_2}$, $L_4 = x_2P_{x_2}x_3P_{x_3} \cdots x_{m-1}P_{x_{m-1}}$,

$L_5 = y_2P_{y_2}y_3P_{y_3} \cdots y_{n-1}P_{y_{n-1}}$ and $L_6 = z_2P_{z_2}z_3P_{z_3} \cdots z_{r-1}P_{z_{r-1}}$.

If $uv \notin E(H)$, then we may assume without loss of generality that $P(x, y)$ must begin with a subpath of the form

$M_1 = xP_xL_4$, or

$M_2 = xP_xw_1L_4$, or

$M_3 = xP_xP_1(w_1, w_2)$, or

$M_4 = xP_{x_2}x_2P_{x_3} \cdots x_{m-2}P_{x_{m-1}}x_{m-1}$, or

$M_5 = xL_4$, or

$M_6 = xw_1L_4$, or

$M_7 = xP_1(w_1, w_2)$.

If $uv \in E(H)$, then $P(x, y)$ may also begin with a subpath of the form

$M_8 = xP_y$, or $M_9 = xP_1(w_2, w_1)$ or $M_{10} = xL_i$ for some $i \in \{1, 2, 3\}$

in addition to those given by $M_1, M_2, \ldots, M_7$.

Since M_1, M_2, M_5, M_6 and M_{10} cannot be extended to cover the rest of the vertices in G^2 (because vertices in a pendent set from one path are adjacent neither to vertices from another path nor to vertices of the graph $H - \{u, v\}$), it follows that $P(x, y)$ must begin with M_3, M_4, M_7, M_8 or M_9.

If $P(x, y)$ begins with M_3, then it must take the form M_3L_i for some $i \in$

$\{1,2,3\}$.

If $P(x,y)$ begins with M_4, then it must take the form M_4L_i or $M_4P_yL_i$ for some $i \in \{2,3\}$, or the form $M_4P_1(w_2,w_1)L_j$ or $M_4P_1(w_2,w_1)P_xL_j$ for some $j \in \{5,6\}$.

If $P(x,y)$ begins with M_7, then it must take the form M_7L_i or $M_7P_yL_i$ for some $i \in \{1,2,3\}$.

If $P(x,y)$ begins with M_8, then it must take the form M_8L_i for some $i \in \{1,2,3\}$ or the form $M_8P_1(w_2,w_1)L_j$ or $M_8P_1(w_2,w_1)P_xL_j$ for some $j \in \{4,5,6\}$.

If $P(x,y)$ begins with M_9, then it must take the form $M_9P_xL_i$ or M_9L_i for some $i \in \{4,5,6\}$.

Since all these paths end with some pendent set, none of them can be extended to $P(x,y)$ (for the same reason as has been explained for the case with M_1, M_2, M_5, M_6 or M_{10}). This contradiction proves the proposition. $\square$

We now obtain necessary and sufficient conditions for the graph $\Theta(m_1, m_2, \ldots, m_r)$ to have panconnected square.

Theorem 4.8. *Let G denote the graph $\Theta(m_1, m_2, \ldots, m_r)$ where $r \geq 3$ and $m_1, m_2, \ldots, m_r \geq 3$. Then G^2 is panconnected if and only if G has at most 2 paths without vertices of degree 2.*

Proof. Suppose G has at least 3 paths without vertices of degree 2, say $\mathcal{P}_{m_1}, \mathcal{P}_{m_2}$ and $\mathcal{P}_{m_3}$. Then these three paths together with their pendent sets form the graph $\Theta(m_1, m_2, m_3)$ without vertices of degree 2. Let H denote the graph obtained from G by deleting all vertices of $(\mathcal{P}_{m_1} \cup \mathcal{P}_{m_2} \cup \mathcal{P}_{m_3}) - \{x,y\}$ together with their pendent sets. By Proposition 4.7, G^2 is not panconnected.

We shall establish the sufficiency by induction on r. For $r = 3$, G is the graph $\Theta(m_1, m_2, m_3)$. Since G has at most 2 paths without vertices of degree 2, by Theorem 3.8 (ii), $\Theta(m_1, m_2, m_3)^2$ is panconnected.

Hence we assume that $r \geq 4$. Suppose the theorem is true for any k-stripe cactus graph which has at most 2 paths without vertices of degree 2 for $k < r$. Let G be the graph $\Theta(m_1, m_2, ..., m_r)$ which has at most 2 paths without vertices of degree 2 and let u and v be two vertices in G. We shall show that there is a Hamilton path $P(u,v)$ in G^2 with u and v as end vertices. We may assume that neither u nor v is of degree 1.

Case (1): u and v are on different paths of G.

Without loss of generality, we may assume that u is in $\mathcal{P}_{m_r} = x_1x_2\ldots x_{m_r}$ and v is in $\mathcal{P}_{m_1}$.

Suppose $u = x_k$ for some $2 \le k \le m_r - 1$. Let H denote the graph obtained from G by deleting all the vertices of $\mathcal{P}_{m_r} - \{x, y\}$ together with their pendent sets. Then H is the graph $\Theta(m_1, m_2, ..., m_{r-1})$ which has at most 2 paths without vertices of degree 2. By the induction hypothesis, H^2 is panconnected. So there is a Hamilton path $P_1(x,v)$ in H^2 with x and v as end vertices.

If $k = m_r - 1$, let $P_2(u,x)$ denote the path $uP_{x_{m_r-1}}x_{m_r-2}P_{x_{m_r-2}}x_{m_r-3}\cdots x_2P_{x_2}x$.

If $k < m_r - 1$, let $L = x_{k+1}P_{x_k}x_{k-1}P_{x_{k-1}}\cdots x_2P_{x_2}x$. Also, let $P_2(u,x)$ denote the following path

$$x_kP_{x_{k+1}}x_{k+2}P_{x_{k+3}}\cdots P_{x_{m_r-3}}x_{m_r-2}P_{x_{m_r-1}}x_{m_r-1}P_{x_{m_r-2}}x_{m_r-3}\cdots L$$

or

$$x_kP_{x_{k+1}}x_{k+2}P_{x_{k+3}}\cdots x_{m_r-3}P_{x_{m_r-2}}x_{m_r-1}P_{x_{m_r-1}}x_{m_r-2}P_{x_{m_r-3}}x_{m_r-4}\cdots L$$

depending on whether k and m_r are of the same or of different parity.

Then $P(u,v) = P_2(u,x)P_1(x,v)$ is a suitable Hamilton path. Notice that if $\mathcal{P}_{m_r}$ contains a vertex of degree 2, then $P_2(u,x)P_1(x,v)$ is still a suitable Hamilton path in view of Remark 3.3.

Case (2): u and v are on the same path of G.

Without loss of generality, we may assume that u and v are on $\mathcal{P}_{m_r} = x_1x_2\ldots x_{m_r}$.

Let H denote the graph obtained from G by deleting all the vertices of $\mathcal{P}_{m_r} - \{x, y\}$ together with their pendent sets. Then H is the graph $\Theta(m_1, m_2, ..., m_{r-1})$ which has at most 2 paths without vertices of degree 2. By the induction hypothesis, H^2 is panconnected, and so there is a Hamilton path $P_1(x,y)$ in H^2 with x and y as end vertices.

Suppose $u = x_k$ and $v = x_l$ where $1 \leq k < l \leq m_r$.

Assume first that $2 \leq k < l \leq m_r - 1$.

Let $P_2(u, x)$ denote the path $x_k P_{x_k} x_{k-1} P_{x_{k-1}} \cdots x_2 P_{x_2} x$.

If $k < l - 1$, let $L = yP_{x_{m_r-1}} x_{m_r-1} P_{x_{m_r-2}} x_{m_r-2} \cdots x_{l+1} P_{x_l} x_{l-1} P_{x_{l-2}} x_{l-3}$. Also, let $P_3(y, v)$ denote the following path

$$L \cdots x_{k+3} P_{x_{k+2}} x_{k+1} P_{x_{k+1}} x_{k+2} P_{x_{k+3}} \cdots x_{l-2} P_{x_{l-1}} x_l$$

or

$$L \cdots P_{x_{k+3}} x_{k+2} P_{x_{k+1}} x_{k+1} P_{x_{k+2}} x_{k+3} \cdots x_{l-2} P_{x_{l-1}} x_l$$

depending on whether l and k are of the same or of different parity.

If $k = l-1$, then $P_3(y, v)$ reduces to the path $yP_{x_{m_r-1}} x_{m_r-1} P_{x_{m_r-2}} x_{m_r-2} \cdots P_{x_l} x_l$.

Then $P(u, v) = P_2(u, x) P_1(x, y) P_3(y, v)$ is a suitable Hamilton path.

Notice that this Hamilton path also covers the case $k = 1$ and $l \leq m_r - 1$ if we take $P_2(u, x) = u$.

By changing the labels of the vertices in $\mathcal{P}_{m_r}$ in reverse order, we see that the above Hamilton path also covers the case $k \geq 2$ and $l = m_r$.

It remains only to consider the case $k = 1$ and $l = m_r$.

Suppose $\mathcal{P}_{m_1}, \mathcal{P}_{m_2}, \ldots, \mathcal{P}_{m_{r-2}}$ are $r - 2$ paths of G each having a vertex of degree 2.

Suppose $i \in \{1, 2, \ldots, r-2\}$. Let $\mathcal{P}_{m_i} = w_{i,1} w_{i,2} \ldots w_{i,m_i}$ where $w_{i,j}$ is a vertex of degree 2. Let

$P_i(m_i) = w_{i,m_i-1} P_{w_{i,m_i-1}} w_{i,m_i-2} P_{w_{i,m_i-2}} \cdots P_{w_{i,j+1}} w_{i,j} P_{w_{i,j-1}} w_{i,j-1} \cdots P_{w_{i,2}} w_{i,2}$ if i is odd and let

$P_i(m_i) = w_{i,2} P_{w_{i,2}} w_{i,3} P_{w_{i,3}} \cdots P_{w_{i,j-1}} w_{i,j} P_{w_{i,j+1}} w_{i,j+1} \cdots P_{w_{i,m_i-1}} w_{i,m_i-1}$ if i is even.

Suppose $\mathcal{P}_{m_{r-1}} = y_1 y_2 \ldots y_{m_{r-1}}$ and $\mathcal{P}_{m_r} = z_1 z_2 \ldots z_{m_r}$.

Let $N_1 = P_{y_2} y_2 P_{y_3} y_3 \cdots P_{y_{m_{r-1}-1}} y_{m_{r-1}-1}$.

Suppose r is odd. Let $N_2 = z_2 P_{z_2} z_3 P_{z_3} \cdots z_{m_r-1} P_{z_{m_r-1}}$. Then

$$P(u, v) = xN_1 P_y P_1(m_1) P_2(m_2) \cdots P_{r-2}(m_{r-2}) P_x N_2 y$$

is a suitable Hamilton path.

Suppose r is even. Let

$N_2 = z_{m_r-1}P_{z_{m_r-2}}z_{m_r-3}\cdots z_4P_{z_3}z_2P_{z_2}z_3P_{z_4}z_5\cdots z_{m_r-4}P_{z_{m_r-3}}z_{m_r-2}P_{z_{m_r-1}}$

if m_r is odd and let

$N_2 = z_{m_r-1}P_{z_{m_r-2}}z_{m_r-3}\cdots z_5P_{z_4}z_3P_{z_2}z_2P_{z_3}z_4\cdots z_{m_r-4}P_{z_{m_r-3}}z_{m_r-2}P_{z_{m_r-1}}$

if m_r is even.

Then

$$P(u,v) = xN_1P_yP_1(m_1)P_2(m_2)\cdots P_{r-3}(m_{r-3})P_xP_{r-2}(m_{r-2})N_2y$$

is a suitable Hamilton path. □

In the r-stripe cactus graph $\Theta(m_1, m_2, \ldots, m_r)$, we say that $\mathcal{P}_{m_i}$ is a *long* path if and only if $m_i \geq 3$.

Theorem 4.9. *Let G denote the graph $\Theta(2, m_1, m_2, \ldots, m_r)$ where $r \geq 3$ and $m_1, m_2, \ldots, m_r \geq 3$. Then G^2 is panconnected if and only if G has at most 2 long paths without vertices of degree 2.*

Proof. Suppose G has at least 3 paths without vertices of degree 2, say $\mathcal{P}_{m_1}, \mathcal{P}_{m_2}$ and $\mathcal{P}_{m_3}$. Then these three paths form the graph $\Theta(m_1, m_2, m_3)$ having no vertices of degree 2. Let H denote the graph obtained from G by deleting all the vertices of $(\mathcal{P}_{m_1} \cup \mathcal{P}_{m_2} \cup \mathcal{P}_{m_3}) - \{x, y\}$ together with their pendent sets. By Proposition 4.7, G^2 is not panconnected.

On the other hand, if G has at most 2 paths without vertices of degree 2, then by deleting the edge xy from G, we obtain the graph $\Theta(m_1, m_2, \ldots, m_r)$. By Theorem 4.8, $\Theta(m_1, m_2, \ldots, m_r)^2$ is panconnected and this implies that G^2 is also panconnected. □

The classification of the other five families of graphs having panconnected square remains to be explored.

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