# CHAPTER IV

## POLYNOMIAL EXTENSIONS OF SEMIRINGS

In this chapter, the word "semiring" means an additively commutative semiring which contains an element 0 such that $x \cdot 0 = 0 \cdot x = 0$ and $x + 0 = 0 + x = x$ for all $x \in S$. We shall study polynomials with coefficients in a semiring. Before the study, we shall give some notation.

<u>Notation</u> Let S be a semiring. Then $(a_n)_{n\in\mathbb{Z}_o^+}$ denotes the infinite sequence in S whose $n^{\underline{th}}$ term is $a_n$.

In [3] P. Cohn studied skew polynomial rings. We shall now study skew polynomial semirings.

<u>Definition 4.1.</u> Let S be a semiring and $\alpha : S \to S$ a monomorphism such that $\alpha(0) = 0$. Let $S[X,\alpha] = \{(a_n)_{n\in\mathbb{Z}_o^+} \mid a_n \in S$ for all $n \in \mathbb{Z}_o^+$ and $a_n \neq 0$ for only finitely many $n\}$. Denote $(a_n)_{n\in\mathbb{Z}_o^+} \in S[X,\alpha]$ by $\sum_{i=0}^{\infty} a_i X^i$. Let $f = \sum_{i=0}^{\infty} a_i X^i$, $g = \sum_{i=0}^{\infty} b_i X^i \in S[X,\alpha]$. Define

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i)X^i$$

and

$$f \cdot g = \sum_{\ell=0}^{\infty} \Big( \sum_{i+j=\ell} a_i \alpha^i(b_j)\Big)X^{\ell}.$$

The proof that $(S[X,\alpha],+,\cdot)$ is a semiring is similar to the proof given in Example 2.15. The semiring $S[X,\alpha]$ is called the <u>skew polynomial semiring</u>.

We shall now give an example to show that a monomorphism $\alpha : S \to S$ may not have the property that $\alpha(0) = 0$ so the assumption above that $\alpha(0) = 0$ is necessary.

Example 4.2. Let $\mathbb{Z}_o^+$ have the usual multiplication and define addition by $m + n = \max\{m,n\}$ for all $m,n \in \mathbb{Z}_o^+$. Then $(\mathbb{Z}_o^+,+,\cdot)$ is a semiring. Let $S = \{(a_n)_{n\in\mathbb{Z}^+} | a_n \in \mathbb{Z}_o^+\}$. Define the addition and multiplication on S by

$$(a_n)_{n\in\mathbb{Z}^+} + (b_n)_{n\in\mathbb{Z}^+} = (a_n + b_n)_{n\in\mathbb{Z}^+}$$

and $$(a_n)_{n\in\mathbb{Z}^+} \cdot (b_n)_{n\in\mathbb{Z}^+} = (a_n \cdot b_n)_{n\in\mathbb{Z}^+} .$$

Then $(S,+,\cdot)$ is a semiring. Define $\alpha : S \to S$ by $\alpha((a_n)_{n\in\mathbb{Z}^+}) = (1,a_1,a_2,\ldots)$. Then $\alpha$ is a monomorphism such that $\alpha((0,0,0,\ldots)) = (1,0,0,\ldots) \neq (0,0,0,\ldots)$.

Theorem 4.3. Let S be a semiring. If S is additively cancellative then $S[X,\alpha]$ is additively cancellative.

Proof. Assume that S is additively cancellative. Let $f = \sum_{i=0}^{\infty} a_iX^i$, $g = \sum_{i=0}^{\infty} b_iX^i$ and $h = \sum_{i=0}^{\infty} c_iX^i \in S[X,\alpha]$ be such that $f + g = f + h$. Then $\sum_{i=0}^{\infty} (a_i + b_i)X^i = \sum_{i=0}^{\infty} (a_i + c_i)X^i$. Let $i \in \mathbb{Z}_o^+$ be arbitary. Since $a_i + b_i = a_i + c_i$ and S is A.C., $b_i = c_i$. Hence $g = h$. Therefore $S[X,\alpha]$ is additively cancellative.

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Theorem 4.4. Let S be a semiring. If S is cancellative then $S[X,\alpha]$ is cancellative.

Proof. Assume that S is cancellative. By Theorem 4.3, $S[X,\alpha]$ is additively cancellative. To show that $S[X,\alpha]$ is multiplicatively cancellative, let $f = \sum_{i=0}^{\infty} a_i X^i \in S[X,\alpha]\setminus\{0\}$ and $g = \sum_{i=0}^{\infty} b_i X^i$, $h = \sum_{i=0}^{\infty} c_i X^i \in S[X,\alpha]$ be such that $fg = fh$. Then

$$\sum_{\ell=0}^{\infty} \Big( \sum_{i+j=\ell} a_i \alpha^i(b_j)\Big) X^\ell = \sum_{\ell=0}^{\infty} \Big( \sum_{i+j=\ell} a_i \alpha^i(c_j)\Big) X^\ell .$$

Since $f \neq 0$, there exists a smallest integer k such that $a_k \neq 0$. For $\ell = k$, we get that $a_k\alpha^k(b_o) = a_k\alpha^k(c_o)$. Since S is M.C. and $a_k \neq 0$, $\alpha^k(b_o) = \alpha^k(c_o)$, so $b_o = c_o$. Suppose that for all $j < n, b_j = c_j$. We must show that $b_n = c_n$. For $\ell = n + k$, we get that

$$a_k\alpha^k(b_n) + a_{k+1}\alpha^{k+1}(b_{n-1}) + \ldots + a_{n+k}\alpha^{n+k}(b_o)$$

$$= a_k\alpha^k(c_n) + a_{k+1}\alpha^{k+1}(c_{n-1}) + \ldots + a_{n+k}\alpha^{n+k}(c_o).$$ Since

$a_{k+1}\alpha^{k+1}(b_{n-1}) + \ldots + a_{n+k}\alpha^{n+k}(b_o) = a_{k+1}\alpha^{k+1}(c_{n-1}) + \ldots + a_{n+k}\alpha^{n+k}(c_o)$ and S is A.C., $a_k\alpha^k(b_n) = a_k\alpha^k(c_n)$. Since S is M.C. and $a_k \neq 0$, $\alpha^k(b_n) = \alpha^k(c_n)$, so $b_n = c_n$. Hence $g = h$. Suppose that $gf = hf$. Then

$$\sum_{\ell=0}^{\infty} \Big( \sum_{j+i=\ell} b_j \alpha^j(a_i)\Big) X^\ell = \sum_{\ell=0}^{\infty} \Big( \sum_{j+i=\ell} c_j \alpha^j(a_i)\Big) X^\ell .$$

For $\ell = k$, we get that $b_o a_k = c_o a_k$. Since S is M.C. and $a_k \neq 0$, $b_o = c_o$. Suppose that for all $j < n, b_j = c_j$. We must show that $b_n = c_n$. For $\ell = n + k$, we get that

$$b_n\alpha^n(a_k) + b_{n-1}\alpha^{n-1}(a_{k+1}) + \ldots + b_o a_{n+k}$$

$$= c_n\alpha^n(a_k) + c_{n-1}\alpha^{n-1}(a_{k+1}) + \ldots + c_o a_{n+k}.$$

Since $b_{n-1}\alpha^{n-1}(a_{k+1}) + \ldots + b_o a_{n+k} = c_{n-1}\alpha^{n-1}(a_{k+1}) + \ldots + c_o a_{n+k}$ and S is A.C., $b_n\alpha^n(a_k) = c_n\alpha^n(a_k)$. Since S is M.C. and $a_k \neq 0$, $b_n = c_n$. Hence $g = h$. Therefore $S[X,\alpha]$ is cancellative. #

Theorem 4.5. Let S be a semiring. If(S,+) satisfies the right [left] Ore condition then (S[X,α],+) satisfies the right [left] Ore condition.

Proof. Assume that (S,+) satisfies the right Ore condition. Let $\sum_{i=0}^{\infty} a_iX^i$, $\sum_{i=0}^{\infty} c_iX^i \in S[X,\alpha]$. Let $n \in \mathbb{Z}_o^+$ be such that $a_i = c_i = 0$ for all $i > n$. Let $i \in \mathbb{Z}_o^+$ be such that $i \leqslant n$. There exist $b_i, d_i \in S$ such that $a_i + b_i = c_i + d_i$. For $i \in \mathbb{Z}_o^+$ such that $i > n$, let $b_i = d_i = 0$. Thus $\sum_{i=0}^{\infty} b_iX^i$, $\sum_{i=0}^{\infty} d_iX^i \in S[X,\alpha]$ and

$$\sum_{i=0}^{\infty} a_iX^i + \sum_{i=0}^{\infty} b_iX^i = \sum_{i=0}^{\infty} c_iX^i + \sum_{i=0}^{\infty} d_iX^i.$$

Hence S[X,α] satisfies the right Ore condition.

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Next, we shall study polynomials with noncommutative variables such that the elements of the semiring commute with the variables.

Definition 4.6. Let S be a semiring, $n \in \mathbb{Z}^+$, A the set of all finite sequences in $\{1,2,\ldots,n\}$. Let $\alpha = (m_1,\ldots,m_M) \in A$. Define $|\alpha| = M$ and M is called the degree of α. Let $\alpha = (m_1,\ldots,m_M)$ and $\beta = (\ell_1,\ldots,\ell_L) \in A$. Define $\alpha + \beta = (m_1,\ldots,m_M,\ell_1,\ldots,\ell_L)$, so $\alpha + \beta \in A$. Let 0 be a symbol not representing any element in A. Let $B = A \cup \{0\}$. Extend + from A to B by $0 + \alpha = \alpha + 0 = \alpha$ for all $\alpha \in B$. Define $|0| = 0$.

Let $S[X_1,\ldots,X_n] = \{f : B \to S \mid f(\alpha) \neq 0$ for only finitely many $\alpha \in B\}$. Let $\alpha \in B$. If $f(\alpha) = a_\alpha$ then we denote f by

$a_o + \sum_{i=1}^{n} a_iX^i + \sum_{i,j=1}^{n} a_{ij}X_iX_j + \ldots$ . To simplify notation we write this as $\sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha$ where if $\alpha = (m_1,\ldots,m_M)$ then $X^\alpha = X_{m_1}X_{m_2}\ldots X_{m_M}$ and $a_oX^o = a_o$ for all $a_o \in S$.

Let $F = \sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha$, $G = \sum_{|\beta|=0}^{\infty} b_\beta X^\beta$ and $H = \sum_{|\gamma|=0}^{\infty} c_\gamma X^\gamma$

$\varepsilon\, S[X_1,\ldots,X_n]$. Define

$$F + G = \sum_{|\alpha|=0}^{\infty} (a_\alpha + b_\alpha)X^\alpha$$

and

$$F \cdot G = \sum_{|\gamma|=0}^{\infty} \Big( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) X^\gamma.$$

To show that $(S[X_1,\ldots,X_n],+,\cdot)$ is a semiring, note that

$$\begin{aligned}
(F + G) + H &= \Big( \sum_{|\alpha|=0}^{\infty} (a_\alpha + b_\alpha)X^\alpha \Big) + \Big( \sum_{|\alpha|=0}^{\infty} c_\alpha X^\alpha \Big) \\
&= \sum_{|\alpha|=0}^{\infty} (a_\alpha + b_\alpha + c_\alpha)X^\alpha \\
&= \Big( \sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha \Big) + \Big( \sum_{|\alpha|=0}^{\infty} (b_\alpha + c_\alpha)X^\alpha \Big) \\
&= F + (G + H),
\end{aligned}$$

so + is associative. Also,

$$\begin{aligned}
(FG)H &= \Big( \sum_{|\delta|=0}^{\infty} \Big( \sum_{\alpha+\beta=\delta} a_\alpha b_\beta \Big) X^\delta \Big)\Big( \sum_{|\gamma|=0}^{\infty} c_\gamma X^\gamma \Big) \\
&= \sum_{|\sigma|=0}^{\infty} \Big( \sum_{\delta+\gamma=\sigma} \Big( \sum_{\alpha+\beta=\delta} a_\alpha b_\beta \Big) c_\gamma \Big) X^\sigma \\
&= \sum_{|\sigma|=0}^{\infty} \Big( \sum_{\alpha+\beta+\gamma=\sigma} a_\alpha b_\beta c_\gamma \Big) X^\sigma
\end{aligned}$$

and

$$\begin{aligned}
F(GH) &= \Big( \sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha \Big)\Big( \sum_{|\delta|=0}^{\infty} \Big( \sum_{\beta+\gamma=\delta} b_\beta c_\gamma \Big) X^\delta \Big) \\
&= \sum_{|\sigma|=0}^{\infty} \Big( \sum_{\alpha+\delta=\sigma} a_\alpha \Big( \sum_{\beta+\gamma=\delta} b_\beta c_\gamma \Big) \Big) X^\sigma \\
&= \sum_{|\sigma|=0}^{\infty} \Big( \sum_{\alpha+\beta+\gamma=\sigma} a_\alpha b_\beta c_\gamma \Big) X^\sigma,
\end{aligned}$$

therefore $(FG)H = F(GH)$, so $\cdot$ is associative. Since

$$\begin{aligned}
F(G + H) &= \Big( \sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha \Big)\Big( \sum_{|\beta|=0}^{\infty} (b_\beta + c_\beta)X^\beta \Big) \\
&= \sum_{|\delta|=0}^{\infty} \Big( \sum_{\alpha+\beta=\delta} a_\alpha (b_\beta + c_\beta) \Big) X^\delta
\end{aligned}$$

and $FG + FH = (\sum_{|\delta|=0}^{\infty} (\sum_{\alpha+\beta=\delta} a_\alpha b_\beta)X^\delta) + (\sum_{|\delta|=0}^{\infty} (\sum_{\alpha+\beta=\delta} a_\alpha c_\beta)X^\delta)$

$= \sum_{|\delta|=0}^{\infty} (\sum_{\alpha+\beta=\delta} (a_\alpha b_\beta + a_\alpha c_\beta))X^\delta$,

$F(G + H) = FG + FH$. Similarly, $(F + G)H = FH + GH$. Hence $\cdot$ distributes over $+$. Therefore $(S[X_1,\ldots,X_n],+,\cdot)$ is a semiring.

Remark 4.7. If $n = 1$ then $S[X_1]$ is the polynomial semiring where the elements of the semiring commute with the variable.

Theorem 4.8. Let $S$ be a semiring. If $S$ is additively cancellative then $S[X_1,\ldots,X_n]$ is additively cancellative.

Proof. The proof of this theorem is similar to the proof of Theorem 4.3.

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Theorem 4.9. Let $S$ be a semiring. If $S$ is cancellative then $S[X_1,\ldots,X_n]$ is cancellative.

Proof. Assume that $S$ is cancellative. By Theorem 4.8, $S[X_1,\ldots,X_n]$ is additively cancellative. To show that $S[X_1,\ldots,X_n]$ is multiplicatively cancellative, let $F = \sum_{|\alpha|=0}^{\infty} a_\alpha X^\alpha \in S[X_1,\ldots,X_n]\setminus\{0\}$ and $G = \sum_{|\beta|=0}^{\infty} b_\beta X^\beta$, $H = \sum_{|\beta|=0}^{\infty} c_\beta X^\beta \in S[X_1,\ldots,X_n]$ be such that $FG = FH$.

Then $\sum_{|\gamma|=0}^{\infty} (\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta)X^\gamma = \sum_{|\gamma|=0}^{\infty} (\sum_{\alpha+\beta=\gamma} a_\alpha c_\beta)X^\gamma$. ......... (*)

Let $\beta \in B$ be arbitary. We must show that $b_\beta = c_\beta$. Since $F \neq 0$, there exist an $\alpha_o \in B$ such that $a_{\alpha_o} \neq 0$. Let $\alpha_o = (m_1,\ldots,m_M)$ for some $M \in \mathbb{Z}_o^+$ ($M = 0$ means that $\alpha_o = 0$). We shall prove that $b_\beta = c_\beta$ by induction on the degree of $\beta$. For $|\beta| = 0$, we get that $a_{\alpha_o} b_o = a_{\alpha_o} c_o$, so $b_o = c_o$. For $|\beta| = 1$, let $\beta = (\ell)$. Consider the term in (*) with index $\gamma = (m_1,\dots,m_M,\ell)$. We get that $a_{\alpha_o}b_\beta + a_\gamma b_o = a_{\alpha_o}c_\beta + a_\gamma c_o$, so $a_{\alpha_o}b_\beta = a_{\alpha_o}c_\beta$, hence $b_\beta = c_\beta$. Assume by induction that it is true for $|\beta| = L - 1 \geq 1$. Let $\beta = (\ell_1,\dots,\ell_L)$. Consider the term in (*) with index $\gamma = (m_1,\dots,m_M,\ell_1,\dots,\ell_L)$. We get that

$$a_{\alpha_o}b_\beta + a_{\alpha_o+(\ell_1)}b_{(\ell_2,\dots,\ell_L)} + a_{\alpha_o+(\ell_1,\ell_2)}b_{(\ell_3,\dots,\ell_L)} + \cdots + a_\gamma b_o =$$

$$a_{\alpha_o}c_\beta + a_{\alpha_o+(\ell_1)}c_{(\ell_2,\dots,\ell_L)} + a_{\alpha_o+(\ell_1,\ell_2)}c_{(\ell_3,\dots,\ell_L)} + \cdots + a_\gamma c_o,$$

so $a_{\alpha_o}b_\beta = a_{\alpha_o}c_\beta$, hence $b_\beta = c_\beta$. Therefore $G = H$. Similarly, if $GF = HF$ then $G = H$. Thus $S[X_1,\dots,X_n]$ is cancellative. #

Theorem 4.13. Let $S$ be a semiring. If $(S,+)$ satisfies the right [left] Ore condition then $(S[X_1,\dots,X_n],+)$ satisfies the right [left] Ore condition.

Proof. The proof of this theorem is similar to the proof of Theorem 4.5. #