CHAPTER III

SKEW RINGS OF RIGHT [LEFT] DIFFERENCES OF SEMIRINGS

In this chapter, we shall generalize the concept of the ring of differences of a commutative semiring to the skew ring of right [left] differences of a semiring which gives P.Sinutoke's construction when the semiring is commutative.

<u>Definition 3.1</u>. Let S be a semiring. A skew ring R is said to be a <u>skew ring of right [left] differences of S</u> iff there exists a monomorphism $i : S \to R$ such that for all $x \in R$ there exist $a, b \in S$ such that $x = i(a) - i(b)$ $[x = -i(b) + i(a)]$. A monomorphism i satisfying the above property is said to be a <u>right [left] difference embedding</u> of S into R.

<u>Example 3.2</u>. Let $S = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}\}$ and $A,B \in S$. Define $A \oplus B = AB$ and $A \odot B = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(S,\oplus,\odot)$ is a semiring. Let $R = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q}\}$ and $X,Y \in R$. Define $X \oplus Y = XY$ and $X \odot Y = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(R,\oplus,\odot)$ is a skew ring of right [left] differences of S.

<u>Remark 3.3</u>. In this chapter, we shall prove some theorems for skew rings of right differences of a semiring S. The theorems are true for skew rings of left differences of S and the proofs are similar so we shall not give the proofs for skew rings of left differences.

Remark 3.4. Let S be a semiring with an additive zero a. If a skew ring R of right [left] differences of S exists then S = {a}.

Proof. Let 0 be an additive identity of R and $i : S \to R$ a right difference embedding. Then $i(a) = i(a) + 0 = i(a) + i(a) - i(a) = i(a + a) - i(a) = i(a) - i(a) = 0$. Let $x \in S$ be arbitary. Then $i(x) = i(a) + i(x) = i(a + x) = i(a)$, so x = a and hence S = {a}.

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From now on, in this chapter the word "semiring" means a semiring without an additive zero.

Theorem 3.5. Let S be a semiring. Then a skew ring of right [left] differences of S exists iff

(i) S is additively cancellative

and (ii) (S,+) satisfies the right [left] Ore condition.

Proof. Assume that (i) and (ii) hold. Consider $S \times S$. Let $(a,b),(c,d) \in S \times S$. Define a relation $\sim$ on $S \times S$ by $(a,b) \sim (c,d)$ iff there exist $x,y \in S$ such that $a + x = c + y$ and $b + x = d + y$. A proof similar to the one given in Theorem 2.4 shows that $\sim$ is an equivalence relation on $S \times S$. Let $R = \frac{S \times S}{\sim}$.

Let $\alpha,\beta \in R$. Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. There exist $x,y \in S$ such that $b + x = c + y$. Define $\alpha + \beta = [(a + x,d + y)]$. A proof similar to the one given in Theorem 2.4 for $(K,\cdot)$ shows that (R,+) is a group which having $[(z,z)]$, where $z \in S$, as an additive identity which we denote by 0 and $[(b,a)] = -[(a,b)]$ i.e. $[(b,a)]$ is the additive inverse of $[(a,b)]$.

Define $\alpha \cdot \beta = [(ac + bd,ad + bc)]$. We must show that $\cdot$ is well-defined. 1) Fix (a,b). Suppose that $(c,d) \sim (c',d')$. Then there exist $x,y \in S$ such that $c + x = c' + y$ and $d + x = d' + y$.

Since S is A.C.,

$$\begin{aligned}(ac + bd) + (ax + bx) &= (ac + ax) + (bd + bx)\\ &= (ac' + ay) + (bd' + by)\\ &= (ac' + bd') + (ay + by).\end{aligned}$$

Similarly, $(ad + bc) + (ax + bx) = (ad' + bc') + (ay + by)$. Thus $(ac + bd, ad + bc) \sim (ac' + bd', ad' + bc')$. 2) Fix (c,d). Suppose that $(a,b) \sim (a',b')$. A proof similar to the one just given shows that $(ac + bd, ad + bc) \sim (a'c + b'd, a'd + b'c)$. Hence $\cdot$ is well-defined.

To show that $\cdot$ is associative, let $\alpha,\beta,\gamma \in R$. Choose $(a,b) \in \alpha$, $(c,d) \in \beta$ and $(e,f) \in \gamma$. Since $\alpha\beta = [(ac + bd, ad + bc)]$ and $\beta\gamma = [(ce + df, cf + de)]$,

$$\begin{aligned}(\alpha\beta)\gamma &= [((ac + bd)e + (ad + bc)f, (ac + bd)f + (ad + bc)e)]\\ &= [(ace + bde + adf + bcf, acf + bdf + ade + bce)]\\ &= [(ace + adf + bcf + bde, acf + ade + bce + bdf)]\\ &= [(a(ce + df) + b(cf + de), a(cf + de) + b(ce + df))]\\ &= \alpha(\beta\gamma).\end{aligned}$$

Hence $\cdot$ is associative.

To show that multiplication distributes over addition, let $\alpha,\beta,\gamma \in R$. Choose $(a,b) \in \alpha$, $(c,d) \in \beta$ and $(e,f) \in \gamma$. First, we shall show that $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. There exist $x,y \in S$ such that $d + x = e + y$, so $\beta + \gamma = [(c + x, f + y)]$, thus

$$\begin{aligned}\alpha(\beta + \gamma) &= [(a(c + x) + b(f + y), a(f + y) + b(c + x))]\\ &= [(ac + ax + bf + by, af + ay + bc + bx)].\end{aligned}$$

There exist $z,w \in S$ such that $ad + bc + z = ae + bf + w$, so

$$\alpha\beta + \alpha\gamma = [(ac + bd, ad + bc)] + [(ae + bf, af + be)]$$
$$= [(ac + bd + z, af + be + w)].$$

There exist $u,v \in S$ such that $ax + bf + by + u = bd + z + v$. Then

$$ad + ax + bf + by + ay + bc + bx + u$$
$$= ad + ay + bc + bx + ax + bf + by + u$$
$$= ad + ay + bc + bx + bd + z + v$$
$$= ad + ay + bc + be + by + z + v$$
$$= ay + be + by + ad + bc + z + v$$
$$= ay + be + by + ae + bf + w + v$$
$$= ae + ay + bf + by + be + w + v$$
$$= ad + ax + bf + by + be + w + v,$$

so $ay + bc + bx + u = be + w + v$. Hence

$(ac + ax + bf + by, af + ay + bc + bx) \sim (ac + bd + z, af + be + w)$.

Thus $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. Similarly, we can show that $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ so $(R,+,\cdot)$ is distributive and hence $(R,+,\cdot)$ is a skew ring.

Define $i : S \to R$ by $i(x) = [(x + x,x)]$ for all $x \in S$. To show that i is a homomorphism, let $a,b \in S$. Then

$$i(ab) = [(ab + ab,ab)]$$
$$= [(ab + ab + ab + ab + ab,ab + ab + ab + ab)]$$
$$= [((a + a)(b + b) + ab, (a + a)b + a(b + b))]$$
$$= [(a + a,a)][(b + b,b)]$$
$$= i(a)i(b).$$

There exist $x,y \in S$ such that $a + x = b + b + y$, so $i(a) + i(b) = [(a + a,a)] + [(b + b,b)] = [(a + a + x,b + y)]$. There exist $z,w \in S$ such that $a + b + a + b + z = a + a + x + w$, so $a + b + a + b + z = a + b + b + y + w$, thus $a + b + z = b + y + w$. Hence $(a + b + a + b,a + b) \sim (a + a + x,b + y)$, so $i(a + b) = i(a) + i(b)$. Thus i is a homomorphism. Let $a,b \in S$ be such that $i(a) = i(b)$. Then

$[(a + a, a)] = [(b + b, b)]$, so there exist $x, y \in S$ such that $a + a + x = b + b + y$ and $a + x = b + y$, thus $a = b$. Hence i is a monomorphism. Let $\alpha \in R$. Choose $(a,b) \in \alpha$. There exist $x, y \in S$ such that $a + x = b + y$, so $[(a + a, a)] + [(b, b + b)] = [(a + a + x, b + b + y)]$. Let $z \in S$ and $w = a + x + z$. Then $a + w = a + a + x + z$ and $b + w = b + b + y + z$, so $[(a,b)] = [(a + a + x, b + b + y)]$. Hence $\alpha = [(a,b)] = [(a + a, a)] + [(b, b + b)] = i(a) - i(b)$. Therefore $(R, +, \cdot)$ is a skew ring of right differences of S.

A proof similar to the one given in Theorem 2.4 shows that the converse is true.

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<u>Remark 3.6.</u> Let S be a semiring having R as a skew ring of right [left] differences. In the proof of Theorem 3.5 we can see that

1) if S is multiplicatively commutative then R is multiplicatively commutative.

2) if S is additively commutative then R is a ring and we shall call it the ring of differences of S.

3) if S is commutative then R is the ring of differences of S given by P.Sinutoke in [1].

<u>Corollary 3.7.</u> Let S be a semiring having R as a skew ring of right [left] differences, i a right [left] difference embedding of S into R, T a skew ring and $f : S \to T$ a homomorphism. Then there exists a unique homomorphism $g : R \to T$ such that $g \circ i = f$. Furthermore, if f is a monomorphism then g is a monomorphism.

<u>Proof.</u> Define $g : R \to T$ in the following way : Let $x \in R$. Then there exist $a, b \in S$ such that $x = i(a) - i(b)$. Define $g(x) = f(a) - f(b)$.

A proof similar to the one given in Corollary 2.7 shows that g is well-defined and $g(x + y) = g(x) + g(y)$ for all $x,y \varepsilon R$. Let $x,y \varepsilon R$. Then $x = i(a) - i(b)$ and $y = i(c) - i(d)$ for some $a,b,c,d \varepsilon S$. Thus

$$\begin{aligned} g(xy) &= g((i(a) - i(b))(i(c) - i(d))) \\ &= g(i(a)i(c) - i(a)i(d) - i(b)i(c) + i(b)i(d)) \\ &= g(i(a)i(c) + i(b)i(d) - i(a)i(d) - i(b)i(c)) \\ &= g(i(ac + bd) - i(bc + ad)) \\ &= f(ac + bd) - f(bc + ad) \\ &= f(a)f(c) + f(b)f(d) - f(a)f(d) - f(b)f(c) \\ &= f(a)f(c) - f(a)f(d) - f(b)f(c) + f(b)f(d) \\ &= (f(a)- f(b))(f(c) - f(d)) \\ &= g(x)g(y). \end{aligned}$$

Hence g is a homomorphism. A proof similar to the one given in Corollary 2.7 shows that $g \circ i = f$, g is unique and if f is a monomorphism then g is a monomorphism. Hence we have the corollary.

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<u>Corollary 3.8.</u> Let S be a semiring having R as a skew ring of right [left] differences. If T is a skew ring and T contains an isomorphic copy of S then T contains an isomorphic copy of R.

<u>Corollary 3.9.</u> If S is a semiring having R and R' as skew rings of right or left differences then $R \cong R'$.

<u>Proof.</u> The proof of this theorem is similar to the proof of Corollary 2.9.

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Theorem 3.10. Let $(S,\leqslant)$ be a partially ordered semiring having R as a skew ring of right [left] differences and $i : S \to R$ a right [left] difference embedding. Then there exists a unique partial order $\leqslant^*$ on R such that $(R,\leqslant^*)$ is a partially ordered skew ring and i is an increasing map iff

(i) $\leqslant$ is additively regular

and (ii) $y \leqslant x$ and $w \leqslant z$ imply that $(xw + yz) \leqslant (xz + yw)$ for all $x,y,z,w \in S$.

Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

Proof. Assume that (i) and (ii) hold. Let $E = \{\alpha \in R \mid \alpha = i(x) - i(y)$ for some $x,y \in S$ such that $y \leqslant x\}$. Define a relation $\leqslant^*$ on R by $\alpha \leqslant^* \beta$ iff $\beta - \alpha \in E$ for all $\alpha,\beta \in R$. A proof similar to the one given in Theorem 2.30 shows that $\leqslant^*$ is a partial order on R, $\alpha \leqslant^* \beta$ implies that $(\alpha + \gamma) \leqslant^* (\beta + \gamma)$ and $(\gamma + \alpha) \leqslant^* (\gamma + \beta)$ for all $\alpha,\beta,\gamma \in R$ and i is an increasing map. We shall now show that $\alpha \leqslant^* \beta$ and $0 \leqslant^* \gamma$ imply $\alpha\gamma \leqslant^* \beta\gamma$ and $\gamma\alpha \leqslant^* \gamma\beta$ for all $\alpha,\beta,\gamma \in R$. Claim that if $\alpha,\beta \in E$ then $\alpha\beta \in E$. To prove this, let $\alpha,\beta \in E$. Then $\alpha = i(x) - i(y)$ and $\beta = i(z) - i(w)$ for some $x,y,z,w \in S$ such that $y \leqslant x$ and $w \leqslant z$. Thus $(xw + yz) \leqslant (xz + yw)$, so $\alpha\beta = i(xz + yw) - i(xw + yz) \in E$, hence we have the claim. Let $\alpha,\beta,\gamma \in R$ be such that $\alpha \leqslant^* \beta$ and $0 \leqslant^* \gamma$. Then $\beta - \alpha, \gamma \in E$. Thus $\beta\gamma - \alpha\gamma = (\beta - \alpha)\gamma \in E$ and $\gamma\beta - \gamma\alpha = \gamma(\beta - \alpha) \in E$, so $\alpha\gamma \leqslant^* \beta\gamma$ and $\gamma\alpha \leqslant^* \gamma\beta$. Hence $(R,\leqslant^*)$ is a partially ordered skew ring.

Conversely, assume that there exists a unique partial order $\leqslant^*$ on R such that $(R,\leqslant^*)$ is a partially ordered skew ring and i is an increasing map. To show (ii), let $x,y,z,w \in S$ be such that $y \leqslant x$ and $w \leqslant z$. Then $i(y) \leqslant^* i(x)$ and $i(w) \leqslant^* i(z)$, so $0 \leqslant^* (i(x) - i(y))(i(z) - i(w)) =$

$i(xz + yw) - i(xw + yz)$, thus $i(xw + yz) \leqslant^* i(xz + yw)$, hence $(xw + yz) \leqslant (xz + yw)$. Thus (ii) holds. A proof similar to the one given in Theorem 2.30 shows that (i) holds and if $\leqslant$ is total then $\leqslant^*$ is total.

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A partial order on a semiring S satisfying condition (ii) in Theorem 3.10 will be called normal.

Corollary 3.11. Let $(S,\leqslant)$ be a partially ordered additively commutative semiring having R as a ring of differences and $i : S \to R$ a difference embedding. Then there exists a unique partial order $\leqslant^*$ on R such that $(R,\leqslant^*)$ is a partially ordered ring and i is an increasing map iff $\leqslant$ is additively regular and normal. Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

Theorem 3.12. Let S be a semiring having R as a skew ring of right [left] differences of S, $i : S \to R$ a right [left] difference embedding, A the lower semilattice of additively regular, normal partial orders $\leqslant$ on S such that $(S,\leqslant)$ is a partially ordered semiring and B the lower semilattice of partial orders $\leqslant^*$ on R such that $(R,\leqslant^*)$ is a partially ordered skew ring and i is an increasing map. Then there exists an order isomorphism between A and B.

Proof. Define a map $f : A \to B$ in the following way : Let $\leqslant\ \varepsilon\ A$. Then Theorem 3.10 determines a unique $\leqslant^*\ \varepsilon\ B$. Define $f(\leqslant) = \leqslant^*$. To show that f is a surjection, let $\leqslant'\ \varepsilon\ B$. Define a relation $\leqslant$ on S by $x \leqslant y$ iff $i(x) \leqslant' i(y)$ for all $x,y\ \varepsilon\ S$. Clearly $\leqslant$ is an additively regular partial order on S such that $(S,\leqslant)$ is a partially ordered semiring.

A proof similar to the one given in Theorem 3.9 shows that $\leqslant$ is normal. Hence $\leqslant \varepsilon A$. By Theorem 3.10, $\leqslant' = \leqslant^*$, so $f(\leqslant) = \leqslant'$. Therefore f is a surjection. A proof similar to the one given in Theorem 2.32 shows that the remainder of this theorem is true.

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Corollary 3.13. Let S be a semiring having R as a ring of differences, $i : S \to R$ a difference embedding, A the lower semilattice of additively regular, normal partial orders $\leqslant$ on S such that $(S,\leqslant)$ is a partially ordered semiring and B the lower semilattice of partial orders $\leqslant^*$ on R such that $(R,\leqslant^*)$ is a partially ordered ring and i is an increasing map. Then there exists an order isomorphism between A and B.

Theorem 3.14. Let S be a semiring having R as a skew ring of right [left] differences, $i : S \to R$ a right [left] difference embedding and ρ a congruence on S. Then there exists a unique congruence ρ^* on R such that ($i(x) \rho^* i(y)$ iff $x \rho y$ for all $x,y \varepsilon S$) iff ρ is additively regular.

Proof. The proof of this theorem is similar to the proof of Theorem 3.10 by defining a relation ρ^* on R in the following way : Let $E = \{\alpha \varepsilon R | \alpha = i(x) - i(y)$ for some $x,y \varepsilon S$ such that $y \rho x\}$. Let $\alpha,\beta \varepsilon R$. Define $\alpha \rho^* \beta$ iff $\beta - \alpha \varepsilon E$. From the definition it is clear that ρ^* is symmetric. Thus it remains to show that $\alpha \rho^* \beta$ implies $\alpha\gamma \rho^* \beta\gamma$ and $\gamma\alpha \rho^* \gamma\beta$ for all $\alpha,\beta,\gamma \varepsilon R$. Claim that $\alpha\beta,\beta\alpha \varepsilon E$ for all $\alpha \varepsilon E, \beta \varepsilon R$. To prove this, let $\alpha \varepsilon E$ and $\beta \varepsilon R$. Then $\alpha = i(x) - i(y)$ and $\beta = i(z) - i(w)$ for some $x,y,z,w \varepsilon S$ such that $y \rho x$. Thus $yz \rho xz$, $yw \rho xw$, $zy \rho zx$ and $wy \rho wx$. So $(xz + yw) \rho (xw + yz)$ and $(zx + wy) \rho (zy + wx)$. Hence $\alpha\beta = i(xz + yw) - i(xw + yz)$ and $\beta\alpha = i(zy + wx) - i(zx + wy) \varepsilon E$. So we have the claim. Let $\alpha,\beta,\gamma \varepsilon R$

be such that $\alpha \rho^{*} \beta$. Then $\beta - \alpha \varepsilon E$. Thus $\beta\gamma - \alpha\gamma = (\beta - \alpha)\gamma \varepsilon E$ and $\gamma\beta - \gamma\alpha = \gamma(\beta - \alpha) \varepsilon E$. Therefore we have the theorem.

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Corollary 3.15. Let S be an additively commutative semiring having R as a ring of differences of S, $i : S \to R$ a difference embedding and ρ a congruence on S. Then there exists a unique congruence ρ^{*} on R such that ($i(x) \rho^{*} i(y)$ iff $x \rho y$ for all $x,y \varepsilon S$) iff ρ is additively regular.

Theorem 3.15. Let S be a semiring having R as a skew ring of right [left] differences, $i : S \to R$ a right [left] difference embedding, A the lattice of additively regular congruences on S and B the lattice of congruences on R. Then there exists an order isomorphism between A and B.

Proof. The proof of this theorem is similar to the proof of Theorem 2.38.

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Corollary 3.17. Let S be an additively commutative semiring having R as a ring of differences, $i : S \to R$ a difference embedding, A the lattice of additively regular congruences on S and B the lattice of congruences on R. Then there exists an order isomorphism between A and B.

Theorem 3.18. Let S be a semiring having R as a skew ring of right [left] differences. Then R is multiplicatively cancellative iff S is strongly multiplicatively cancellative.

Proof. Let $i : S \to R$ be a right difference embedding.

Assume that S is strongly multiplicatively cancellative. Let $\alpha,\beta,\gamma \in R$ be such that $\alpha\beta = \alpha\gamma$ and $\alpha \neq 0$. Then $\alpha = i(x) - i(y)$, $\beta = i(z) - i(w)$ and $\gamma = i(u) - i(v)$ for some $x,y,z,w,u,v \in S$ such that $x \neq y$. There exist $a,b \in S$ such that $w + a = v + b$, so $\beta - \gamma = i(z + a) - i(u + b)$. Thus

$$\begin{aligned} & i(x(z + a) + y(u + b)) - i(x(u + b) + y(z + a)) \\ &= (i(x) - i(y))(i(z + a) - i(u + b)) \\ &= \alpha(\beta - \gamma) \\ &= \alpha\beta - \alpha\gamma \\ &= 0, \end{aligned}$$

hence $x(z + a) + y(u + b) = x(u + b) + y(z + a)$. Since S is strongly multiplicatively cancellative and $x \neq y$, $z + a = u + b$, so $\beta = \gamma$. Similarly, if $\beta\alpha = \gamma\alpha$ and $\alpha \neq 0$ then $\beta = \gamma$. Hence R is multiplicatively cancellative.

Conversely, assume that R is multiplicatively cancellative. It suffices to show that R is strongly multiplicatively cancellative. Let $\alpha,\beta,\gamma,\delta \in R$ be such that $\alpha\gamma + \beta\delta = \alpha\delta + \beta\gamma$. Then $\beta\delta - \beta\gamma = \alpha\delta - \alpha\gamma$, so $\beta(\delta - \gamma) = \alpha(\delta - \gamma)$. Suppose that $\delta \neq \gamma$, so $\beta = \alpha$. Thus R is strongly multiplicatively cancellative, hence S is also.

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Remark 3.19. Clearly R is multiplicatively cancellative iff R has no left zero divisors and no right zero divisors.

Corollary 3.20. Let S be an additively commutative semiring having R as a ring of differences. Then R is multiplicatively cancellative iff S is strongly multiplicatively cancellative.

Theorem 3.21. Let S be a semiring having R as a skew ring of right [left] differences. Then the skew field of right [left] quotients of R exists iff

(i) S is strongly multiplicatively cancellative

and (ii) for all $x,y,z,w \in S$ with $x \neq y$ and $z \neq w$ there exist $a,b,c,d \in S$ with $a \neq b$ and $c \neq d$ such that $xc + yd + zb + wa = za + wb + xd + yc$.

Proof. Let $i : S \to R$ be a right difference embedding.

Assume that (i) and (ii) hold. By Theorem 3.18, R has no left zero divisors and no right zero divisors. Let $\alpha,\beta \in R\setminus\{0\}$. Then $\alpha = i(x) - i(y)$ and $\beta = i(z) - i(w)$ for some $x,y,z,w \in S$ such that $x \neq y$ and $z \neq w$. By (ii), there exist $a,b,c,d \in S$ with $a \neq b$ and $c \neq d$ such that $xc + yd + zb + wa = za + wb + xd + yc$. Let $\gamma = i(a) - i(b)$ and $\delta = i(c) - i(d)$. Then $\gamma,\delta \in R\setminus\{0\}$ and $\alpha\delta = i(xc + yd) - i(xd + yc) = i(za + wb) - i(zb + wa) = \beta\gamma$. Hence $(R,\cdot)$ satisfies the right Ore condition. By Theorem 2.12, the skew field of right quotients of R exists.

Conversely, assume that the skew field K of right quotients of R exists. It follows from Theorem 3.18 that S is strongly multiplicatively cancellative. To show (ii), let $x,y,z,w \in S$ be such that $x \neq y$ and $z \neq w$. Then $i(x) - i(y)$ and $i(z) - i(w) \in R\setminus\{0\}$. By Theorem 2.12, $(R,\cdot)$ satisfies the right Ore condition, so there exist $a,b,c,d \in S$ with $a \neq b$ and $c \neq d$ such that $(i(x) - i(y))(i(c) - i(d)) = (i(z) - i(w))(i(a) - i(b))$. Thus $i(xc + yd) - i(xd + yc) = i(za + wb) - i(zb + wa)$. Hence $i(xc + yd + zb + wa) = i(za + wb + xd + yc)$, so $xc + yd + zb + wa = za + wb + xd + yc$.

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<u>Corollary 3.22</u>. Let S be a semiring which is additively cancellative, strongly multiplicatively cancellative and satisfies property ii) of Theorem 3.21. Then S is additively commutative.

<u>Proof</u>. We can embed S into a skew field by Theorem 3.21 and every skew field is additively commutative therefore S is additively commutative. #