CHAPTER I

PRELIMINARIES

Let S be a semigroup. An element e of S is called an <u>identity</u> of S if xe = ex = x for all $x \in S$. An element z of S is called a <u>zero</u> of S if xz = zx = z for all $x \in S$. Note that S can have at most one identity and at most one zero. The symbols 1 and 0 usually denote the identity of S (if it exists) and the zero of S (if it exists), respectively.

A subset H of a semigroup S is called a <u>subgroup</u> of S if H is a group under the operation of S.

If S is a semigroup with zero 0 such that $S \setminus \{0\}$ is a subgroup of S, then S is called a <u>group with zero</u>.

A triple $(S,+,\cdot)$ is called a <u>semiring</u> if

(i) $(S,+)$ and $(S,\cdot)$ are semigroups and

(ii) $x\cdot(y + z) = (x\cdot y) + (x\cdot z)$ and $(y + z)\cdot x = (y\cdot x) + (z\cdot x)$

for all $x, y, z \in S$, and the operations + and $\cdot$ are called the <u>addition</u> and <u>multiplication</u> of the semiring, respectively.

An element 1 of the semiring $S = (S,+,\cdot)$ is called an <u>identity</u> of the semiring S if 1 is the identity of the semigroup $(S,\cdot)$ and an element 0 of the semiring S is called a <u>zero</u> of the semiring S if 0 is the zero of $(S,.)$ and the identity of $(S,+)$.

A semiring $(S,+,\cdot)$ with zero 0 is called a <u>skew-semifield</u> if

(i) $(S,+)$ is a commutative semigroup and

(ii) $(S,\cdot)$ is a group with zero 0.

A commutative skew-semifield is called a <u>semifield</u>. Then every skew-field and every semifield is a skew-semifield.

<u>Example</u>. For each positive integer $n \geq 2$, let SK_n be the set of all matrices in the form

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & . & . & . & b \\ 0 & a_2 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & a_3 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$ and $a_1, a_2, \ldots, a_n > 0$, including the $n \times n$ zero matrix over $\mathbb{R}$. Then for every positive integer $n \geq 2$, SK_n is a skew-semifield which is neither a semifield nor a skew-field. To show this, let n be a positive integer such that $n \geq 2$. Clearly, under usual addition and multiplication of matrices, SK_n is a semiring with zero 0_n and identity I_n where 0_n is the $n \times n$ zero matrix over $\mathbb{R}$ and I_n is the $n \times n$ identity matrix over $\mathbb{R}$, and it is commutative under addition.

Let

$$A = \begin{bmatrix} a_1 & 0 & 0 & 0 & . & . & . & b \\ 0 & a_2 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & a_3 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & a_n \end{bmatrix}$$

be a nonzero element of SK_n. Then $\det A = a_1 a_2 \ldots a_n > 0$ since $a_1, a_2, \ldots, a_n >$

Then the matrix

$$\begin{bmatrix} \frac{1}{a_1} & 0 & 0 & 0 & . & . & . & \frac{-b}{a_1 a_n} \\ 0 & \frac{1}{a_2} & 0 & 0 & . & . & . & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \frac{1}{a_n} \end{bmatrix}$$

is an element of SK_n and it is the multiplicative inverse of A. Hence under matrix multiplication, SK_n is a group with zero 0_n. Therefore the semiring SK_n is a skew-semifield.

$$\text{Since } \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix}$$

are elements of SK_n and

$$\begin{bmatrix} 1 & 0 & 0 & 0\ldots0 \\ 0 & 1 & 0 & 0\ldots0 \\ 0 & 0 & 1 & 0\ldots0 \\ . & . & . & \ldots\ldots \\ 0 & 0 & 0 & 0\ldots2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\ldots1 \\ 0 & 1 & 0 & 0\ldots0 \\ 0 & 0 & 1 & 0\ldots0 \\ . & . & . & \ldots\ldots \\ 0 & 0 & 0 & 0\ldots\frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 & 0\ldots1 \\ 0 & 1 & 0 & 0\ldots0 \\ 0 & 0 & 1 & 0\ldots0 \\ . & . & . & \ldots\ldots \\ 0 & 0 & 0 & 0\ldots\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\ldots0 \\ 0 & 1 & 0 & 0\ldots0 \\ 0 & 0 & 1 & 0\ldots0 \\ . & . & . & \ldots\ldots \\ 0 & 0 & 0 & 0\ldots2 \end{bmatrix},$$

it follows that the semiring SK_n is not a semifield. Clearly, every nonzero element of SK_n has no inverse under addition in SK_n. Then the semiring SK_n is not a skew-field.

The above example shows that skew-semifields are a generalization of skew-fields and also a generalization of semifields.

Let S be a semigroup and let 0 be a symbol not representing any element of S. Extend the operation in S to 0 in $S\cup\{0\}$ by defining $00 = 0$ and $x0 = 0x = 0$ for every $x \in S$. Then under this operation, $S \cup \{0\}$ is a semigroup with zero 0. Let

$$S^{o} = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S\cup\{0\} & \text{if } S \text{ has no zero.} \end{cases}$$

A semigroup S is said to <u>admit a ring structure</u> if there exists an operation + on S^{o} such that $(S^{o},+,\cdot)$ is a ring where $\cdot$ is the operation of S^{o}.

By the notation defined above, we have that for any group G, $G^{o} = G\cup\{0\}$ if $|G| > 1$ and $G^{o} = G$ if $|G| = 1$. But for convenience, we shall use G^{o} to denote $G\cup\{0\}$ for every group G.

A group G is said to <u>admit a skew-semifield</u> [semifield, skew-field, field] <u>structure</u> if there exists an operation + on G^{o} such that $(G^{o},+,\cdot)$ is a skew-semifield [semifield, skew-field, field] where $\cdot$ is the operation of G^{o}.

An $n \times n$ matrix A over a field F is said to be an <u>orthogonal matrix</u> if $AA^{t} = I_n$ where I_n is the $n \times n$ identity matrix over F.

A square matrix A over a field F is said to be a <u>unimodular matrix</u> if $\det A = 1$.

A square matrix A over a field F is said to be a <u>permutation matrix</u> if every member of A is either 0 or 1 and each row and each column contains exactly one 1.

For any field F and for any positive integer n, let $G_n(F)$ be the set of all $n \times n$ nonsingular matrices over F, so $G_n(F)$ is a group under usual matrix multiplication.

By a <u>matrix group</u> over a field F, we mean a subgroup of $G_n(F)$ under usual matrix multiplication for some positive integer n.

The following notation of matrix groups will be used in the thesis : For any field F and for any positive integer n, let

$U_n(F)[L_n(F)]$ = the matrix group of all n × n upper [lower] triangular nonsingular matrices over F (see [6], page 410),

$P_n(F)$ = the matrix group of all n × n permutation matrices over F (see [7], page 37 or [6], page 203),

$O_n(F)$ = the matrix group of all n × n orthogonal matrices over F,

$V_n(F)$ = the matrix group of all n × n unimodular matrices over F

and

$W_n(F)$ = the matrix group of all n × n matrices over F whose determinants are equal to 1 or -1.

Let X be a set and S_X the symmetric group on X. An element of S_X is called a permutation of X, that is, a permutation of X is a 1-1 map of X onto X. By a permutation group on X, we mean a subgroup of S_X.

For $\alpha \in S_X$, let

$$s(\alpha) = \{x \in X \mid x\alpha \neq x\}$$

which is called the shift of α. For $\alpha \in S_X$, is called an almost identical permutation of X if $s(\alpha)$ is finite. Let K_X be the set of all almost identical permutations of X. Then K_X is a subgroup of S_X.

Let A_X be a set of all almost identical even permutations of X. Then A_X is a subgroup of S_X and it is called the alternating group on X.

For any group G, let G' denote the commutator subgroup of G, that is, the subgroup of G generated by the set $\{aba^{-1}b^{-1} \mid a, b \in G\}$.

A system $(F,+,\cdot,\leq)$ is called an <u>ordered</u> <u>field</u> if $(F,+,\cdot)$ is a field and $\leq$ is a partial order on F satisfying the following properties :

(i) For any $x, y \in F$, exactly one of the relations $x < y$, $x = y$ or $y < x$ holds where for $a, b \in F$, $a < b$ means $a \leq b$ and $a \neq b$.

(ii) For $x, y \in F$, $x \leq y$ if and only if $0 \leq y - x$.

(iii) For $x, y \in F$, $0 \leq x$ and $0 \leq y$ imply $0 \leq x + y$ and $0 \leq xy$.

Note that if we replace the field of real numbers, $\mathbb{R}$, by any ordered field in the example given before, it is still an example of skew-semifields which are neither semifields nor skew-fields.

The following known results will be used in the thesis :

<u>Theorem 1.1</u>. ([1]) Let G be a group. If G is a cyclic group of order $p^n - 1$ for some prime p and positive integer n, then G admits a field structure.

<u>Theorem 1.2</u>. ([7]) If F is a field and n is a positive integer such that $n \geq 3$, then $G_n'(F) = V_n(F)$.

<u>Theorem 1.3</u>. ([9]) For any field F and for any positive integer n, $V_n'(F) = V_n(F)$ except the following two cases : (i) $n = 2$ and $|F| = 2$ and (ii) $n = 2$ and $|F| = 3$.

<u>Theorem 1.4</u>. ([9]) If F is a field such that $|F| = 3$, then $G_2'(F) = V_2(F)$.

<u>Theorem 1.5</u>. ([10]) If X is a finite set, then $S_X' = A_X$.

<u>Theorem 1.6</u>. ([10]) If X is an infinite set, then $S_X' = S_X$.