CHAPTER VI

GENERAL SOLUTIONS OF $g(xy^{-1}) = g(x)g(y) + f(x)f(y)$ ON COMMUTATIVE INVERSE SEMIGROUPS

All the solutions of the functional equation $g(xy^{-1}) = g(x)g(y)+f(x)f(y)$ on any commutative inverse semigroup are determined in this chapter.

Theorem 6.1. The negative-type solutions of (*) on S are those and only those (f,g) of the form:

$$(6.1.1) \qquad f(x) = \begin{cases} 0 , & x \in A \\ \dfrac{h(x) - h(x^{-1})}{2i} , & x \notin A \end{cases} , \quad g(x) = \begin{cases} 0 , & x \in A \\ \dfrac{h(x)+h(x^{-1})}{2} , & x \notin A \end{cases}$$

where A is a completely prime ideal of S or A is the empty set and h is a homomorphism from $S \setminus A$ into M(F).

Proof. By a straight forward verification, it can be shown that if $f,g : S \to F$ are of the form (6.1.1) then (f,g) is a negative-type solution of (*) on S.

Conversely, assume that (f,g) is a negative-type solution of (*) on S. From Theorem 3.6, we have

$$f = f_1 \cup f_2 \qquad \text{and} \qquad g = g_1 \cup g_2$$

where (f_i,g_i) is a class i negative-type solution of (*) on $S_i(f,g)$, for i = 1,2. Therefore, by Remark 4.10, we have

$$f_1(x) = \frac{h(x)-h(x^{-1})}{2i} \quad , \quad g_1(x) = \frac{h(x)+h(x^{-1})}{2}$$

for all x in $S_1(f,g)$ where h is a homomorphism from $S_1(f,g)$ into $M(F)$. By Theorem 5.1 we have that (f_2,g_2) is the trivial solution on $S_2(f,g)$, i.e. $f_2(x) = 0 = g_2(x)$ for all x in $S_2(f,g)$. Thus, if we let A = $S_2(f,g)$, then $S \setminus A = S_1(f,g)$ and by (3.5.2), we have that A is a completely prime ideal of S or A is the empty set. Hence f and g are of the form (6.1.1). #

<u>Lemma 6.2.</u> Let (f,g) be a positive-type solution of (*) on S such that

$$f(x) = \begin{cases} 0 & , x \in A \\ bh(x) & , x \in S_2(f,g) \setminus A \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x \in A \\ ah(x) & , x \in S_2(f,g) \setminus A \end{cases}$$

where A is a completely prime ideal of $S_2(f,g)$ or A is the empty set and h is a homomorphism from $S_2(f,g) \setminus A$ into {1,-1} and $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$. Then the following hold:

(6.2.1) A is a completely prime ideal of S or A is the empty set,

(6.2.2) $S_2(f,g) \setminus A$ is a completely prime ideal of $S \setminus A$ or $S_2(f,g) \setminus A$ is the empty set.

<u>Proof.</u> To show (6.2.1), assume that $A \neq \emptyset$. Then it follows from Lemma 3.5 and Theorem 2.3 that A is an ideal of S. Next we shall show that for $x,y \in S$, if $xy \in A$, then $x \in A$ or $y \in A$. Let $x,y \in S$ are such that $xy \in A$. Since $A \subseteq S_2(f,g)$, $xy \in S_2(f,g)$. By Lemma 3.5 we have that $S_2(f,g)$ is a completely prime ideal of S. Therefore $x \in S_2(f,g)$ or $y \in S_2(f,g)$. Let us assume that $x \in S_2(f,g)$. If $y \in S_2(f,g)$, then $x \in A$ or $y \in A$ since A is a completely prime ideal of $S_2(f,g)$. On the other hand, if

$y \notin S_2(f,g)$, then $y \in S_1(f,g)$, so $g(yy^{-1}) = 1$. Thus, it follows from (3.4.1) that $f(yy^{-1}) = 0$. Since A is an ideal, $xy(xy)^{-1} \in A$. Therefore

$$\begin{aligned} 0 = g(xy(xy)^{-1}) &= g(xx^{-1}(yy^{-1})^{-1}) \\ &= g(xx^{-1})g(yy^{-1}) + f(xx^{-1})f(yy^{-1}) \\ &= g(xx^{-1})1 + f(xx^{-1})0 \\ &= g(xx^{-1}). \end{aligned}$$

Thus, by (3.4.4), we have that $g(x) = 0$. Since $x \in S_2(f,g)$ and $g(x) = 0$, so $x \in A$. Therefore A is a completely prime ideal of S.

From (6.2.1), Lemma 3.5 and Theorem 2.4 we have that $S_2(f,g) \setminus A$ is a completely prime ideal of $S \setminus A$ or $S_2(f,g) \setminus A$ is empty. Thus (6.2.2) holds. #

In what follow we shall make use of the concept of adjoining a semigroup S' by a semigroup S as zeroes. This concept is introduced at the end of Theorem 2.6.

Lemma 6.3. Let (f,g) be any solution of (*) on S. Assume that $S_i(f,g) \neq \emptyset$ for i = 1,2. Let η be any congruence on $S_1(f,g)$ and

$$\mu = \{(x,y) \ / \ x,y \in S_2(f,g) \text{ and } g(xx^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(yy^{-1})\}.$$

Then $\eta \cup \mu$ is a congruence on S and $(S_1(f,g)/\eta) \cup (S_2(f,g)/\mu) = S/\eta\cup\mu$ is the semigroup $S_1(f,g)/\eta$ with $S_2(f,g)/\mu$ adjoined as zeroes.

Proof. Let η be a congruence on $S_1(f,g)$ and

$$\mu = \{(x,y) \ / \ x,y \in S_2(f,g) \text{ and } g(xx^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(yy^{-1})\}.$$

It follows from the proof of Theorem 5.6 that μ is a congruence on $S_2(f,g)$. First, we claim that for all $c \in S_1(f,g)$ and $x \in S_2(f,g)$ we have

(6.3.1) $(xc,x) \in \mu$

To prove this, let $c \in S_1(f,g)$ and $x \in S_2(f,g)$. Since $S_2(f,g)$ is a completely prime ideal of S (Lemma 3.5), $xc \in S_2(f,g)$. If $g(xc(xc^{-1}))$ $\neq 0$, then $g((xx^{-1})(xc(xc)^{-1})) = g(xc(xc)^{-1}) \neq 0$. Hence it follows from Lemma 5.3 that $g(xx^{-1}) = g(xc(xc)^{-1})$ and $f(xx^{-1}) = f(xc(xc)^{-1})$. Thus $(xc,x) \in \mu$. On the other hand, if $g(xc(xc)^{-1}) = 0$, then

$$\begin{aligned} 0 = g(xc(xc)^{-1}) &= g(xx^{-1}(cc^{-1})^{-1}) \\ &= g(xx^{-1})g(cc^{-1}) + f(xx^{-1})f(cc^{-1}) \\ &= g(xx^{-1})1 + f(xx^{-1})0 \\ &= g(xx^{-1}). \end{aligned}$$

In the third equality we make use of the assumption that $c \in S_1(f,g)$ and (3.4.1). Thus, it follows from (3.4.1) that $f(xx^{-1}) = 0$. Therefore $(xc,x) \in \mu$. Hence we have (6.3.1).

Since η and μ are equivalence relation on $S_1(f,g)$ and $S_2(f,g)$, respectively, and $S_1(f,g) \cap S_2(f,g) = \emptyset$, we have that $\eta \cup \mu$ is an equivalence relation on $S_1(f,g) \cup S_2(f,g) = S$. Let (x,y), $(a,b) \in \eta\cup\mu$. If (x,y), $(a,b) \in \eta$ or (x,y), $(a,b) \in \mu$ then $(xa,yb) \in \eta\cup\mu$ since η and μ are congruences on $S_1(f,g)$ and $S_2(f,g)$, respectively. Now assume that $(a,b) \in \eta$ and $(x,y) \in \mu$. Then $a,b \in S_1(f,g)$ and $x,y \in S_2(f,g)$. It follows from (6.3.1) that (xa,x), $(yb,y) \in \mu$, so $(xa,yb) \in \mu \subseteq \eta\cup\mu$. Thus $\eta\cup\mu$ is a congruence on S. It is clear that

$$x\eta = x(\eta\cup\mu) \qquad \text{and} \qquad y\mu = y(\eta\cup\mu)$$

for all x in $S_1(f,g)$ and y in $S_2(f,g)$. Thus $(S_1(f,g)/\eta) \cup (S_2(f,g)/\mu)$ $= S/\eta\cup\mu$. Next we shall show that $S/\eta\cup\mu$ is the semigroup $S_1(f,g)/\eta$ with $S_2(f,g)/\mu$ adjoined as zeroes. Let $x \in S_1(f,g)$ and $y \in S_2(f,g)$. Then $xy \in S_2(f,g)$ since $S_2(f,g)$ is an ideal of S. By (6.3.1) we have (xy,y) $\in \mu$. Therefore

$$
\begin{aligned}
(x\eta)(y\mu) &= x(\eta\cup\mu)y(\eta\cup\mu) \\
&= xy(\eta\cup\mu) \\
&= xy\mu \\
&= y\mu .
\end{aligned}
$$

Since S is commutative, $(y\mu)(x\eta) = y(\eta\cup\mu)x(\eta\cup\mu) = x(\eta\cup\mu)y(\eta\cup\mu) = (x\eta)(y\mu)$. Thus $S/\eta\cup\mu$ is the semigroup $S_1(f,g)/\eta$ with $S_2(f,g)/\mu$ adjoined as zeroes. #

Theorem 6.4. The positive-type solutions of (*) on S are those and only those (f,g) of the forms:

$$
(6.4.1) \quad f(x) = \begin{cases} 0 & , x \varepsilon A \\ bh(x) & , x \varepsilon B \\ 0 & , x \varepsilon \bar{1} \\ dh(x) & , x \varepsilon x_1\eta \end{cases} , \quad g(x) = \begin{cases} 0 & , x \varepsilon A \\ ah(x) & , x \varepsilon B \\ h(x) & , x \varepsilon \bar{1} \\ ch(x) & , x \varepsilon x_1\eta \end{cases}
$$

where A is a completely prime ideal of S or A is the empty set and B is a completely prime ideal of $S\setminus A$ or B is the empty set and η is a $\mathcal{G}_{1,2}$-congruence on $S\setminus(A\cup B)$ where $(S\setminus(A\cup B))/\eta = \{\bar{1}\}$ or $(S\setminus(A\cup B))/\eta = \{\bar{1}, x_1\eta\}$, $x_1\eta \neq \bar{1}$ or $\eta = \emptyset$ and a,b,c,d ε F are such that

(1) $a \neq 1,0$ (2) $c \neq \pm 1$

(3) $a = a^2 + b^2$ (4) $c^2 + d^2 = 1$

(5) $a = ac + bd$

with corresponding homomorphism $h : S\setminus A \to \{1,-1\}$ or a,b,c,d ε F satisfy (1), (2), (3), (4) and

(5)' $-a = ac + bd$

with corresponding function $h : S \setminus A \to \{1,-1\}$ such that

$$h(xy) = \begin{cases} -h(x)h(y) & \text{if } (x,y) \in x_1\eta \times B, \\ h(x)h(y) & \text{otherwise;} \end{cases}$$

or

$$(6.4.2)\quad f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh_2(x) & , x \in e \\ -bh_2(x) & , x \in e'\mu \\ 0 & , x \in \bar{1} \\ dh_1(x) & , x \in x_1\eta \end{cases}, \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah_2(x) & , x \in e\mu \\ (1-a)h_2(x) & , x \in e'\mu \\ h_1(x) & , x \in \bar{1} \\ ch_1(x) & , x \in x_1\eta \end{cases}$$

where (I) μ is a $\mathcal{R}_3$-congruence on a completely prime ideal C of S such that $C/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero or $\mu = \emptyset$, and

(II) η is a $\mathcal{Y}_{1,2}$-congruence on $S \setminus C$ such that $(S \setminus C)/\eta = \{\bar{1}\}$ or $(S \setminus C)/\eta = \{\bar{1}, x_1\eta\}$, $x_1\eta \neq \bar{1}$ or $\eta = \emptyset$, and

(III) $((S \setminus C)/\eta) \cup (C/\mu)$ is of the following: $((S \setminus C)/\eta) \cup (C/\mu) = (S \setminus C)/\eta$ if $\mu = \emptyset$, $((S \setminus C)/\eta) \cup (C/\mu) = C/\mu$ if $\eta = \emptyset$ and otherwise, $((S \setminus C)/\eta) \cup (C/\mu)$ is the semigroup $(S \setminus C)/\eta$ with C/μ adjoined as zeroes, and

(VI) a, b, c, d $\in$ F are such that

(1) $a \neq 1,0$	(2) $c \neq \pm 1$
(3) $a = a^2+b^2$	(4) $c^2+d^2 = 1$
(5) $a = ac+bd$	(6) $c = 2a-1$

with corresponding functions $h_1 : S \setminus C \to \{1,-1\}$ and $h_2 : C \setminus \bar{0} \to \{1,-1\}$ such that h_1 is a homomorphism and h_2 is a homomorphism on $e\mu$, $e'\mu$ such that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times e'\mu, \\ h_1(x)h_2(y) & \text{otherwise;} \end{cases}$$

or a, b, c, d $\in$ F satisfy (1), (2), (3), (4) and

(5)' $-a = ac + bd$ (6)' $c = 1 - 2a$

with corresponding functions $h_1 : S \setminus C \to \{1,-1\}$ and $h_2 : C \setminus \bar{O} \to \{1,-1\}$ such that h_1 is a homomorphism and h_2 is a homomorphism on eu, e'μ such that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times B \\ h_1(x)h_2(y) & \text{otherwise.} \end{cases}$$

In this Theorem, if η is a $\mathcal{J}_1$-congruence, then all condition of c and $x_1\eta$ are omitted.

Proof. By a straight forward verification, it can be shown that if $f,g : S \to F$ are of the forms (6.4.1) or (6.4.2), then (f,g) is a positive-type solution of (*).

To show the converse, assume that (f,g) is a positive-type solution of (*) , i.e. (*) holds for all x,y in S and $f(x) = f(x^{-1})$ for all x in S. Thus, by (3.3.2), we have that $g(x) = g(x^{-1})$ and $g(y) = g(y^{-1})$ for all x,y in S so that

(6.4.3) $g(xy) = g(x)g(y^{-1}) + f(x)f(y^{-1}) = g(x)g(y) + f(x)f(y)$

for all x,y in S. Let $C = S_2(f,g)$. It follows from Lemma 3.5 that C is a completely prime ideal of S or C is an empty set and $S \setminus C = S_1(f,g)$ is a filter of S or $S \setminus C$ is the empty set.

From $C = S_2(f,g)$ we have that (f,g) is a class 2 positive-type solution of (*) on C. If $C \neq \emptyset$, then by the proof of Theorem 5.9, we have that f,g are of the forms:

(6.4.4) $f(x) = \begin{cases} 0 & , x \in A \\ bh_2(x) & , x \in C \setminus A \end{cases}$, $g(x) = \begin{cases} 0 & , x \in A \\ ah_2(x) & , x \in C \setminus A \end{cases}$

where A is a completely prime ideal of C or A is the empty set and h_2 is a homomorphism from $C \setminus A$ into $\{1, -1\}$ and $a,b \in F$ are such that $a \neq 1,0$, $a = a^2 + b^2$; or

$$(6.4.5) \quad f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh_2(x) & , x \in e\mu \\ -bh_2(x) & , x \in e'\mu \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah_2(x) & , x \in e\mu \\ (1-a)h_2(x) & , x \in e'\mu \end{cases}$$

where $\mu = \{(x,y) \in C \times C \ / \ g(xx^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(yy^{-1})\}$ which is a $\mathcal{K}_3$-congruence on C, such that $C/\mu = \{\bar{0} , e\mu, e'\mu\}$ which $\bar{0}$ as the zero and h_2: $C \setminus \bar{0} \to \{1,-1\}$ is a homomorphism on $e\mu$, $e'\mu$ and $a,b \in F$ are such that $a \neq 1,0$, $a = a^2 + b^2$.

From $S \setminus C = S_1(f,g)$, we have that (f,g) is a class 1 positive-type solution of (*) on $S \setminus C$. Therefore if $S \setminus C \neq \emptyset$, then it follows from Remark 4.10 that f,g are of the form

$$(6.4.6) \quad f(x) = \begin{cases} 0 & , x \in \bar{1} \\ dh_1(x) & , x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} h_1(x) & , x \in \bar{1} \\ ch_1(x) & , x \in x_1\eta \end{cases}$$

where η is a $\mathcal{J}_{1,2}$-congruence on $S \setminus C$ such that $(S \setminus C)/\eta = \{\bar{1}\}$ or $(S \setminus C)/\eta = \{\bar{1}, x_1\eta\}$, $x_1\eta \neq \bar{1}$, and h_1 is a homomorphism from $S \setminus C$ into $\{1,-1\}$ and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

It is possible that a solution (f,g) is neither of class 1 nor of class 2. For such (f,g) we have that $C \neq \emptyset$ and $S \setminus C \neq \emptyset$. Let f_1,g_1 be the restrictions of (f,g) on $S \setminus C$, and f_2,g_2 be the restrictions of f,g on C. It is clear that (f_1,g_1) and (f_2,g_2) are solutions of (*) on $S \setminus C$ and on C, respectively. By Theorem 5.9, we see that (f_2,g_2) can be of the forms (6.4.4) or (6.6.5). So, there are two cases to be considered.

Case 1. (f_2,g_2) is of the form (6.4.4). Since (f_1,g_1) is a class 1 positive-type solution, by Theorem 4.10, we have (f_1,g_1) is of the form (6.4.6). It follows from Theorem 3.6 and $C = S_2(f,g)$ that

$$f(x) = \begin{cases} 0 & , x \in A \\ bh_2(x) & , x \in C \setminus A \\ 0 & , x \in \bar{1} \\ dh_1(x) & , x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in A \\ ah_2(x) & , x \in C \setminus A \\ h_1(x) & , x \in \bar{1} \\ ch_1(x) & , x \in x_1\eta \end{cases}$$

Let $B = C \setminus A$. By (6.2.1) and (6.2.2) and $C = S_2(f,g)$ we have that A is a completely prime ideal of S or A is the empty set and B is a completely prime ideal of $S \setminus A$ or B is the empty set. Since $A \cup B = C$, so $S_1(f,g) = S \setminus S_2(f,g) = S \setminus C = S \setminus (A \cup B)$. Let $x \in x_1\eta$ and $y \in B$. Then $xy \in B$ since B is an ideal of $S \setminus A$. Therefore, by (6.4.3) we have

(6.4.7) $ah_2(xy) = g(xy) = g(x)g(y) + f(x)f(y) = (ac + bd)h_1(x)h_2(y)$.

In the same way we can verify that

(6.4.8) $ah_2(xy) = ah_1(x)h_2(y)$

for all x in $\bar{1}$ and all y in B. From (6.4.7) we have that

$$ah_2(xy)\, h_1(x^{-1})h_2(y^{-1}) = ac + bd$$

for all $x \in x_1\eta$, $y \in B$. Since the ranges of h_1 and h_2 are subsets of $\{1,-1\}$, so

$$\pm a = ac + bd.$$

There are two case to be considered

(i) $a = ac + bd$,

or

(ii) $-a = ac + bd$.

First, we assume that (i) holds. From (6.4.7) and (6.4.8) and $a \neq 0$ we conclude that

$$h_2(xy) = h_1(x)h_2(y)$$

for all x in $\bar{1} \cup x_1\eta = S_1(f,g) = S\setminus C$ and $y \in B$. Thus $h_1 \cup h_2$ is a homomorphism. Now, assume that (ii) holds. From (6.4.7) and (6.4.8) and $a \neq 0$ we conclude that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times B, \\ h_1(x)h_2(y) & \text{otherwise.} \end{cases}$$

Thus, if we put $h = h_1 \cup h_2$, then f and g are of the form (6.4.1).

Case 2. (f_2, g_2) are of the form (6.4.5). Then we have

$$f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh_2(x) & , x \in e\mu \\ -bh_2(x) & , x \in e'\mu \\ 0 & , x \in \bar{1} \\ dh_1(x) & , x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah_2(x) & , x \in e\mu \\ (1-a)h_2(x) & , x \in e'\mu \\ h_1(x) & , x \in \bar{1} \\ ch_1(x) & , x \in x_1\eta. \end{cases}$$

By Lemma 6.3 and $C = S_2(f,g)$, we have that $(S\setminus C)/\eta \cup (C/\mu) = S/\eta\cup\mu$ is a semigroup $(S\setminus C)/\eta$ with C/μ adjoined as zeroes. Thus, we can verify in the same way as above that

(6.4.9) $ah_2(xy) = (ac + bd)h_1(x)h_2(y)$

for all x in $x_1\eta$ and all y in $e\mu$, and

(6.4.10) $ah_2(xy) = ah_1(x)h_2(y)$

for all x in $\bar{1}$ and all y in $e\mu$, and

(6.4.11) $(1-a)h_2(xy) = ((1-a)c-bd)h_1(x)h_2(y) = (c-ac-bd)h_1(x)h_2(y)$

for all x in $x_1\eta$ and all y in $e'\mu$, and

(6.4.12) $\quad (1-a)h_2(xy) = (1-a)h_1(x)h_2(y)$

for all x in $\bar{1}$ and all y in $e'\mu$.

Since image h_1, image h_2 are subsets of {1,-1}, it follows from (6.6.9) that

$$\pm a = ac + bd.$$

There are two cases to be considered

(i) $\quad a = ac + bd,$

or

(ii) $\quad -a = ac + bd.$

First, we assume that (i) holds. Then, by (6.4.11), we have that $c-a = \pm(1-a)$. Therefore $c = 2a - 1$ or $c = 1$. Since $c \neq 1$, so $c = 2a - 1$. From (6.4.9), (6.4.10), (6.4.11), (6.4.12) and $a \neq 1,0$, we conclude that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times e'\mu, \\ h_1(x)h_2(y) & \text{otherwise.} \end{cases}$$

Next, we assume that (ii) holds. Then from (6.4.11), we have $c + a = \pm(1-a)$. Therefore $c = 1-2a$ or $c = -1$. Since $c \neq -1$, so $c = 1-2a$. From (6.4.9), (6.4.10), (6.4.11), (6.4.12) and $a \neq 1,0$ we conclude that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times e\mu, \\ h_1(x)h_2(y) & \text{otherwise.} \end{cases}$$

Thus f,g are of the form (6.4.2). #

We know, from Theorem 3.9, that a solution of (*) on S must be of positive-type or of negative-type. Hence any solution of (*)

on S must be of the form given in Theorem 6.1 or Theorem 6.4. Therefore Theorem 6.1 and Theorem 6.4 give us all the solutions of (*) on S. By combining Theorem 6.1 and Theorem 6.4 we have our main result as follows:

Theorem 6.5. The solutions of (*) on S are those and only those (f,g) of the forms:

$$(6.5.1)\qquad f(x) = \begin{cases} 0 & , x \in A \\ \dfrac{h(x)-h(x^{-1})}{2i} & , x \notin A \end{cases}, \qquad g(x) = \begin{cases} 0 & , x \in A \\ \dfrac{h(x)+h(x^{-1})}{2} & , x \notin A \end{cases}$$

where A is a completely prime ideal of S or A is the empty set and h is a homomorphism from $S\setminus A$ into M(F); or

$$(6.5.2)\qquad f(x) = \begin{cases} 0 & , x \in A \\ bh(x) & , x \in B \\ 0 & , x \in \bar{1} \\ dh(x) & , x \in x_1\eta \end{cases}, \qquad g(x) = \begin{cases} 0 & , x \in A \\ ah(x) & , x \in B \\ h(x) & , x \in \bar{1} \\ ch(x) & x \in x_1\eta \end{cases}$$

where A is a completely prime ideal of S or A is the empty set and B is a completely prime ideal of $S\setminus A$ or B is the empty set and η is a $\mathcal{J}_{1,2}$-congruence on $S\setminus(A\cup B)$ such that $(S\setminus(A\cup B))/\eta = \{\bar{1}\}$ or $(S\setminus(A\cup B))/\eta = \{\bar{1}, x_1\eta\}$, $x_1\eta \neq \bar{1}$ or $\eta = \emptyset$ and $a,b,c,d \in F$ are such that

(1) $a \neq 1,0$ (2) $c \neq \pm 1$

(3) $a = a^2 + b^2$ (4) $c^2 + d^2 = 1$

(5) $a = ac + bd$

with corresponding homomorphism $h : S\setminus A \to \{1,-1\}$ or $a,b,c,d \in F$ satisfy (1), (2), (3), (4) and

(5)' $-a = ac+bd$

with corresponding function $h : S \setminus A \to \{1,-1\}$ such that

$$h(xy) = \begin{cases} -h(x)h(y) & \text{if } (x,y) \in x_1\eta \times B, \\ h(x)h(y) & \text{otherwise;} \end{cases}$$

or

$$(6.5.3)\quad f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh_2(x) & , x \in e\mu \\ -bh_2(x) & , x \in e'\mu \\ 0 & , x \in \bar{1} \\ dh_1(x) & , x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah_2(x) & , x \in e\mu \\ (1-a)h_2(x) & , x \in e'\mu \\ h_1(x) & , x \in \bar{1} \\ ch_1(x) & , x \in x_1\eta \end{cases}$$

where (I) μ is a $\mathcal{K}_3$-congruence on completely prime ideal C of S such that $C/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero or $\mu = \emptyset$, and

(II) η is a $\mathcal{G}_{1,2}$-congruence on $S \setminus C$ such that $(S \setminus C)/\eta = \{\bar{1}\}$ or $(S \setminus C)/\eta = \{\bar{1}, x_1\eta\}$, $x_1\eta \neq \bar{1}$ or $\eta = \emptyset$, and

(III) $((S\ C)/\eta) \cup (C/\mu)$ is of the following: $((S \setminus C)/\eta) \cup (C/\mu) = (S \setminus C)/\eta$ if $\mu = \emptyset$, $((S \setminus C)/\eta) \cup (C/\mu) = C/\mu$ if $\eta = \emptyset$ and otherwise, $((S \setminus C)/\eta) \cup (C/\mu)$ is the semigroup $(S \setminus C)/\eta$ with C/μ adjoined as zeroes, and

(IV) a, b, c, d $\in$ F are such that

(1) $a \neq 1,0$ (2) $c \neq \pm 1$

(3) $a = a^2 + b^2$ (4) $c^2 + d^2 = 1$

(5) $a = ac+bd$ (6) $c = 2a-1$

with corresponding functions $h_1 : S \setminus C \to \{1,-1\}$ and $h_2 : C \setminus \bar{0} \to \{1,-1\}$ such that h_1 is a homomorphism and h_2 is a homomorphism on $e\mu$, $e'\mu$ such that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times e'\mu, \\ h_1(x)h_2(y) & \text{otherwise;} \end{cases}$$

or a, b, c, d $\in$ F satisfy (1), (2), (3), (4) and

(5)' $-a = ac+bd$ (6)' $c = 1-2a$

with corresponding functions h_1: $S \setminus C \to \{1,-1\}$ and h_2: $C \setminus \bar{0} \to \{1,-1\}$ such that h_1 is a homomorphism and h_2 is a homomorphism on $e\mu$, $e'\mu$ such that

$$h_2(xy) = \begin{cases} -h_1(x)h_2(y) & \text{if } (x,y) \in x_1\eta \times e\mu, \\ h_1(x)h_2(y) & \text{otherwise.} \end{cases}$$

In this theorem, if η is a $\mathcal{J}_1$-congruence, then all conditions of c and $x_1\eta$ are omitted.