CHAPTER V

SOLUTIONS OF CLASS 2

In this chapter, we shall determine all the solutions of (*) on S of class 2. The main result of this chapter is Theorem 5.10.

Theorem 5.1. (f,g) is a class 2 negative-type solution of (*) if and only if (f,g) is the trivial solution.

Proof. It is clear that the trivial solution is a class 2 negative-type solution of (*)

Conversely, assume that (f,g) is a class 2 negative-type solution of (*), i.e. f,g satisfying the conditions:

(*) $$g(xy^{-1}) = g(x)g(y) + f(x)f(y)$$

for all x,y in S,

(5.1.1) $$f(x) = -f(x^{-1})$$

for all x in S,

(5.1.2) $$g(e) \neq 1$$

for any e in E(S).

We claim that $f(x) = f(x^{-1})$ for all x in S. To prove this, let x in S. If $f(xx^{-1}) = 0$ then, by (5.1.2) and (3.4.1), we have $g(xx^{-1}) = 0$. Therefore, by (3.4.4) and using (3.4.3), we have that $f(x) = 0 = f(x^{-1})$. In the case $f(xx^{-1}) \neq 0$, it follows from (3.3.3) that

$$f(x)f(xx^{-1}) = g(x)\,[1-g(xx^{-1})\,],$$
$$f(x^{-1})f(x^{-1}x) = g(x^{-1})\,[1-g(x^{-1}x)].$$

Thus, by (3.3.2) we have that

$$f(x)f(xx^{-1}) = f(x^{-1})f(x^{-1}x) = f(x^{-1})f(xx^{-1}).$$

It follows from $f(xx^{-1}) \neq 0$ that

$$f(x) = f(x^{-1}).$$

Hence we have our claim. From this and (5.1.1) we have that $f(x) = 0$ for all x in S. Therefore for each x in S, $f(xx^{-1}) = 0$ so, by (3.4.1) and (5.1.2), $g(xx^{-1}) = 0$. Thus, by (3.4.4), we have that $g(x) = 0 = f(x)$. Hence f and g are identically zero, i.e. (f,g) is the trivial solution. #

<u>Lemma 5.2</u>. Let (f,g) be any solution of (*) on S. Then the followings hold for all x in S:

(5.2.1) if $g(x) = g(xx^{-1})$ then $f(x) = f(xx^{-1})$,

(5.2.2) if $g(x) = -g(xx^{-1})$ then $f(x) = -f(xx^{-1})$,

(5.2.3) $[1-g(xx^{-1})]\,[g(x)^2 - g(xx^{-1})^2] = 0.$

<u>Proof</u>. Let $x \in S$. To show (5.2.1) we assume that $g(x) = g(xx^{-1})$. Thus, by (3.3.3) and (3.3.1) we have that

$$\begin{aligned} f(x)f(xx^{-1}) &= g(x) - g(x)g(xx^{-1}) \\ &= g(xx^{-1}) - g(xx^{-1})^2 \\ &= g(xx^{-1}xx^{-1}) - g(xx^{-1})^2 \\ &= f(xx^{-1})^2. \end{aligned}$$

Hence

$$f(xx^{-1})\,[f(x) - f(xx^{-1})] = 0.$$

Case 1. Assume that $f(xx^{-1}) \neq 0$. It follows that $f(x) = f(xx^{-1})$.

Case 2. Assume that $f(xx^{-1}) = 0$. By (3.3.1) we have that

$$\begin{aligned} f(x)^2 &= g(xx^{-1}) - g(x)^2 \\ &= g(xx^{-1}) - g(xx^{-1})^2 \\ &= g(xx^{-1}xx^{-1}) - g(xx^{-1})^2 \\ &= f(xx^{-1})^2 \\ &= 0. \end{aligned}$$

Hence $f(x) = 0 = f(xx^{-1})$.

To show (5.2.2), we assume that $x \in S$ is such that $g(x) = -g(xx^{-1})$. Thus by (3.3.3) and (3.3.1), we have that

$$\begin{aligned} f(x)f(xx^{-1}) &= g(x) - g(x)g(xx^{-1}) \\ &= -g(xx^{-1}) + g(xx^{-1})^2 \\ &= -g(xx^{-1}xx^{-1}) + g(xx^{-1})^2 \\ &= -f(xx^{-1})^2. \end{aligned}$$

Hence

$$f(xx^{-1})\,[f(x) + f(xx^{-1})] = 0.$$

If $f(xx^{-1}) \neq 0$, then $f(x) = -f(xx^{-1})$. On the other hand, if $f(xx^{-1}) = 0$, it can be verified in the same way as in case 2 that $f(x)^2 = 0$ because $g(xx^{-1})^2 = (-g(xx^{-1}))^2$. Thus $f(x) = 0 = -f(xx^{-1})$. Therefore (5.2.2) holds.

By (3.3.3) we have that

$$f(x)f(xx^{-1}) = g(x) - g(x)g(xx^{-1})$$
$$= g(x)\,[1-g(xx^{-1})].$$

Therefore

$$f(x)^2 f(xx^{-1})^2 = g(x)^2\,[1-g(xx^{-1})]^2.$$

Hence it follows from (3.3.1) that

$$\begin{aligned} 0 &= g(x)^2\,[1-g(xx^{-1})]^2 - f(x)^2 f(xx^{-1})^2 \\ &= g(x)^2\,[1-g(xx^{-1})]^2 - [g(xx^{-1}) - g(x)^2] \\ &\quad [g(xx^{-1}xx^{-1}) - g(xx^{-1})^2] \\ &= g(x)^2\,[1-g(xx^{-1})]^2 - [g(xx^{-1}) - g(x)^2] \\ &\quad [g(xx^{-1}) - g(xx^{-1})^2] \\ &= g(x)^2\,[1-g(xx^{-1})]^2 - [g(xx^{-1}) - g(x)^2] \\ &\quad g(xx^{-1})\,[1-g(xx^{-1})] \\ &= [1-g(xx^{-1})]\,[g(x)^2 - g(x)^2 g(xx^{-1}) - \\ &\quad g(xx^{-1})^2 + g(x)^2 g(xx^{-1})\,] \\ &= [1-g(xx^{-1})]\,[g(x)^2 - g(xx^{-1})^2]. \end{aligned}$$

This proves (5.2.3). #

<u>Lemma 5.3</u>. Let (f,g) be any solution of (*) of class 2. For e,e' in E(S), if $g(ee') \neq 0$, then

$$g(e) = g(ee') = g(e') \quad \text{and} \quad f(e) = f(ee') = f(e').$$

<u>Proof</u>. Assume that (f,g) is a solution of (*) of class 2, i.e. g satisfies

(5.3.1) $\quad g(e) \neq 1$

for any e in E(S). Let $e,e' \in E(S)$. Replacing x,y in (*) by e,ee', respectively, we find that

$$g(e(ee')^{-1}) \quad = \quad g(e)g(ee') + f(e)f(ee').$$

But

$$g(e(ee')^{-1}) \quad = \quad g(eee') \quad = \quad g(ee').$$

Therefore

$$(5.3.2) \qquad g(ee') \quad = \quad g(e)g(ee') + f(e)f(ee').$$

Thus

$$g(ee')\,[1-g(e)] \quad = \quad f(e)f(ee')$$
$$g(ee')^2\,[1-g(e)]^2 \quad = \quad f(e)^2 f(ee')^2.$$

Consequently, using (3.3.1), we obtain

$$g(ee')^2\,[1-g(e)]^2 \quad = \quad [g(ee^{-1}) - g(e)^2]\,[g(ee'(ee')^{-1}) - g(ee')^2].$$

Therefore

$$g(ee')^2 - 2g(ee')^2 g(e) + g(ee')^2 g(e)^2 \quad = \quad [g(e) - g(e)^2]\,[g(ee') - g(ee')^2]$$
$$= \quad g(e)g(ee') - g(e)g(ee')^2 - g(e)^2 g(ee') + g(e)^2 g(ee')^2.$$

Hence

$$0 \quad = \quad g(e)g(ee') + g(e)g(ee')^2 - g(e)^2 g(ee') - g(ee')^2$$
$$= \quad g(ee')\,[g(e) + g(e)g(ee') - g(e)^2 - g(ee')].$$

Suppose $g(ee') \neq 0$. Then

$$0 \quad = \quad g(e) + g(e)g(ee') - g(e)^2 - g(ee')$$
$$= \quad g(e) - g(e)^2 + g(e)g(ee') - g(ee')$$
$$= \quad g(e)\,[1-g(e)] - g(ee')\,[1-g(e)]$$
$$= \quad [1-g(e)]\,[g(e) - g(ee')].$$

Thus, by (5.3.1) we have that

$$g(e) = g(ee').$$

Subsitute g(e) by g(ee') in (5.3.2) we have that

$$g(ee') = g(ee')^2 + f(e)f(ee').$$

But, from (*) we have that

$$\begin{aligned} g(ee') &= g(ee'(ee')^{-1}) \\ &= g(ee')^2 + f(ee')^2. \end{aligned}$$

Thus

$$f(e)f(ee') = f(ee')^2.$$

If $f(ee') = 0$ then, by (3.4.2), we have that $g(ee') = 0$ or $g(ee') = 1$ which is a contradiction. Thus $f(ee') \neq 0$, so

$$f(e) = f(ee').$$

Hence $\quad g(ee') = g(e) \quad$ and $\quad f(ee') = f(e).$

Now, replacing x,y in (*) by e', ee', respectively, we can verify in the same way as above that

$$g(ee') = g(e') \quad \text{and} \quad f(ee') = f(e').$$

Thus $\quad g(e) = g(ee') = g(e')$ and $\quad f(e) = f(ee') = f(e').$ #

<u>Definition 5.4</u>. Let A be any subset of S and (f,g) be a solution of (*) on A. We say that (f,g) is <u>one-to-one</u> if

$$(f(x), g(x)) \neq (f(y), g(y))$$

for all x,y in A such that $x \neq y$.

Lemma 5.5. Let S be a Kronecker semigroup of order greater than 1 and (f,g) be any one-to-one solution of (*) of class 2. Then we have

(5.5.1) $g(0) = 0 = f(0)$,

(5.5.2) $g(x) \neq 0$

for any x in S such that $x \neq 0$

(5.5.3) $g(x) + g(y) = 1$ and $f(x) + f(y) = 0$

for all x,y in $S \setminus \{0\}$ such that $x \neq y$.

Proof. Since S is a Kronecker semigroup, then $xy = 0$ for all x,y such that $x \neq y$. Furthermore $E(S) = S$, so $x = x^{-1}$ and $xx^{-1} = xx = x$. To show (5.5.1), suppose that $g(0) \neq 0$. Let $x \in S \setminus \{0\}$. Then $x0 = 0$, so $g(x0) = g(0) \neq 0$. Therefore, it follows from Lemma 5.3 that $g(x) = g(x0) = g(0)$ and $f(x) = f(x0) = f(0)$. This is contrary to the assumption that (f,g) is one-to-one. Thus $g(0) = 0$. It follows from (3.4.2) that $f(0) = 0$. Therefore (5.5.1) holds.

To show (5.5.2), suppose that there exists $x \in S \setminus \{0\}$ such that $g(x) = 0$. Then, by (3.4.2), we have $f(x) = 0$. Therefore, from (5.5.1) we have that $g(0) = 0 = g(x)$ and $f(0) = 0 = f(x)$, which is a contradiction. Thus $g(x) \neq 0$ for any x in $S \setminus \{0\}$.

To show (5.5.3), assume that $x,y \in S \setminus \{0\}$ are such that $x \neq y$. Thus we have that

(5.5.4) $g(x)^2 + f(x)^2 = g(xx^{-1}) = g(x)$,

(5.5.5) $g(y)^2 + f(y)^2 = g(yy^{-1}) = g(y)$.

From (5.5.1) we have that

(5.5.6) $\quad 0 = g(0) = g(xy) = g(xy^{-1}) = g(x)g(y) + f(x)f(y).$

Therefore

$$g(x)g(y) = -f(x)f(y),$$

$$g(x)^2 g(y)^2 = f(x)^2 f(y)^2.$$

By using (5.5.4), (5.5.5) we have

$$\begin{aligned} g(x)^2 g(y)^2 &= [g(x) - g(x)^2]\ [g(y) - g(y)^2] \\ &= g(x)g(y) - g(x)g(y)^2 - g(x)^2 g(y) + \\ &\quad\ g(x)^2 g(y)^2. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= g(x)g(y) - g(x)g(y)^2 - g(x)^2 g(y) \\ &= g(x)g(y)\,[1-g(y) - g(x)]. \end{aligned}$$

Since $x,y \in S\setminus\{0\}$ and (5.5.2) holds, so $g(x) \neq 0$ and $g(y) \neq 0$. Therefore

$$0 = 1-g(y) - g(x).$$

It follows that

(5.5.7) $\quad g(x) = 1-g(y) \quad$ and $\quad g(y) = 1-g(x).$

From (5.5.4), (5.5.5), (5.5.7) we have

$$\begin{aligned} f(x)^2 &= g(x) - g(x)^2 = g(x)\,[1-g(x)] = g(x)g(y) \\ &= [1-g(y)]\,g(y) = g(y) - g(y)^2 = f(y)^2. \end{aligned}$$

Thus $(f(x) - f(y))(f(x) + f(y)) = 0$. Suppose that $f(x) - f(y) = 0$. Then $f(x) = f(y)$. Therefore by (5.5.6), (5.5.7) we have

$$
\begin{aligned}
0 &= g(x)g(y) + f(x)f(y) \\
&= g(x)\,[1-g(x)] + f(x)^2 \\
&= g(x) - g(x)^2 + f(x)^2.
\end{aligned}
$$

Thus $$g(x)^2 = g(x) + f(x)^2.$$

But from (5.5.4) we have that

$$g(x)^2 = g(x) - f(x)^2.$$

Thus $f(x) = 0$ since the characteristic of F is different from 2. It follows from (3.4.2) and the assumption that $g(e) \neq 1$ for any e in $E(S) = S$ that $g(x) = 0$. This is contrary to (5.5.2). Hence $f(x) + f(y) = 0$. Therefore (5.5.3) holds. #

<u>Theorem 5.6</u>. (f,g) is a class 2 positive-type solution of (*) on S if and only if there exists a $\mathcal{K}$-congruence μ on S and a one-to-one class 2 positive-type solution (f_o, g_o) on S/μ and a function h from S into $\{1,-1\}$ whose restriction to any congruence class $a\mu$ is a homomorphism such that

$$f(x) = f_o(x\mu)h(x) \qquad \text{and} \qquad g(x) = g_o(x\mu)h(x)$$

for all x in S.

<u>Proof</u>. Assume that (f,g) is a class 2 positive-type solution of (*) on S, i.e. f,g satisfy

(5.6.1) $\quad g(e) \neq 1$

for any e in E(S),

(5.6.2) $\quad f(x) = f(x^{-1})$

for all x in S. Let

$$\mu = \{(x,y) \varepsilon S \times S \;/\; g(xx^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(yy^{-1})\}.$$

It is clear that μ is an equivalence relation. To verify that it is a congruence, let (x,y), $(u,v) \varepsilon \mu$. Then $g(xx^{-1}) = g(yy^{-1})$, $g(uu^{-1}) = g(vv^{-1})$, $f(xx^{-1}) = f(yy^{-1})$, $f(uu^{-1}) = f(vv^{-1})$. Therefore

$$\begin{aligned} g(xu(xu)^{-1}) &= g(xx^{-1}(uu^{-1})^{-1}) \\ &= g(xx^{-1})g(uu^{-1}) + f(xx^{-1})f(uu^{-1}) \\ &= g(yy^{-1})g(vv^{-1}) + f(yy^{-1})f(vv^{-1}) \\ &= g(yy^{-1}(vv^{-1})^{-1}) \\ &= g(yv(yv)^{-1}). \end{aligned}$$

If $g(xu(xu)^{-1}) = 0$. Then $g(xu(xu)^{-1}) = 0 = g(yv(yv)^{-1})$. Therefore it follows from (3.4.1) that $f(xu(xu)^{-1}) = 0 = f(yv(yv)^{-1})$. In the case $g(xu(xu)^{-1}) \neq 0$, we have $g(yv(yv)^{-1}) = g(xu(xu)^{-1}) \neq 0$, so $g(yy^{-1}vv^{-1}) = g(xx^{-1}uu^{-1}) \neq 0$. Thus it follows from Lemma 5.3 that $f(yy^{-1}) = f(yy^{-1}vv^{-1}) = f(vv^{-1})$ and $f(xx^{-1}) = f(xx^{-1}uu^{-1}) = f(uu^{-1})$. Since $f(xx^{-1}) = f(yy^{-1})$, so $f(yy^{-1}vv^{-1}) = f(xx^{-1}uu^{-1})$. Thus $(xu,yv) \varepsilon \mu$. Hence μ is a congruence on S. Note that $xx(xx)^{-1} = xx^{-1}xx^{-1} = xx^{-1}$. Hence $g(xx(xx)^{-1}) = g(xx^{-1})$ and $f(xx(xx)^{-1}) = f(xx^{-1})$. Therefore

(5.6.3) $\quad (xx, x) \varepsilon \mu$

for all x in S.

Now, we shall show that S/μ is a Kronecker semigroup.

<u>Case 1</u>. $g(e) \neq 0$ for any e in E(S). Let $x,y \varepsilon S$. Then $xx^{-1}yy^{-1} \varepsilon E(S)$. Therefore $g(xx^{-1}yy^{-1}) \neq 0$. Thus it follows from Lemma 5.3 that

$$g(xx^{-1}) = g(xx^{-1}yy^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(xx^{-1}yy^{-1}) = f(yy^{-1}).$$

Hence $(x,y) \varepsilon \mu$ for all x,y in S, i.e. $|S/\mu| = 1$, so S/μ is a Kronecker semigroup.

<u>Case 2</u>. $g(e) = 0$ for some e in $E(S)$. Let e be fixed element of $E(S)$ such that $g(e) = 0$. Therefore, by (3.4.2) we have that $f(e) = 0$. Claim that if $(x, y) \notin \mu$, then $g(xy(xy)^{-1}) = 0$. To show this, let $x,y \varepsilon S$ are such that $g(xy(xy)^{-1}) \neq 0$. It follows from Lemma 5.3 that

$$g(xx^{-1}) = g(xx^{-1}yy^{-1}) = g(yy^{-1}) \text{ and } f(xx^{-1}) = f(xx^{-1}yy^{-1}) = f(yy^{-1}).$$

Thus $(x,y) \varepsilon \mu$. Hence we prove that

(5.6.4) $\qquad g(xy(xy)^{-1}) = 0$

for all $(x,y) \notin \mu$.

Let $x,y \varepsilon S$. If $x\mu = y\mu$, then $(x,y) \varepsilon \mu$, so $(xy,yy) \varepsilon \mu$. From (5.6.3), we have that $(yy,y) \varepsilon \mu$. Therefore $(xy,y) \varepsilon \mu$, so $xy\mu = y\mu$. In the case $x\mu \neq y\mu$, by (5.6.4), we have $g(xy(xy)^{-1}) = 0$, so, by (3.4.1), we have that $f(xy(xy)^{-1}) = 0$. Thus $(xy,e) \varepsilon \mu$. Therefore

$$(x\mu)(y\mu) = xy\mu = \begin{cases} y\mu & \text{if } x\mu = y\mu, \\ e\mu & \text{if } x\mu \neq y\mu. \end{cases}$$

Hence S/μ is a Kronecker semigroup having $e\mu$ ($e \varepsilon E(S)$) as the zero element.

Let us define $f_o, g_o : S/\mu \to F$ by

$$f_o(x\mu) = f(xx^{-1}) \quad \text{and} \quad g_o(x\mu) = g(xx^{-1})$$

for all x in S. Since $(x,y) \varepsilon \mu$ iff $g(xx^{-1}) = g(yy^{-1})$ and $f(xx^{-1}) = f(yy^{-1})$, we have that $f_{\circ}$ and $g_{\circ}$ are well-defined and $(f_{\circ}, g_{\circ})$ is one-to-one. From (5.6.1), $g_{\circ}(x\mu) = g(xx^{-1}) \neq 1$ for all $x \in S$. For $x \in S$, $x\mu = (x\mu)^{-1}$ thus $f_{\circ}(x\mu) = f_{\circ}((x\mu)^{-1})$. Hence $(f_{\circ}, g_{\circ})$ is a one-to-one class 2 positive-type solution of (*) on the Kronecker semigroup S/μ.

From (5.2.3) and (5.6.1) we have that $g(x)^2 = g(xx^{-1})^2$ for all x in S. Thus $g(x) = g(xx^{-1})$ or $g(x) = -g(xx^{-1})$. Since F is a field of characteristic different from 2, we have that for $x \in S$, $g(xx^{-1}) \neq -g(xx^{-1})$ if $g(xx^{-1}) \neq 0$, so we can conclude that if $g(xx^{-1}) \neq 0$ then either $g(x) = g(xx^{-1})$ or $g(x) = -g(xx^{-1})$. Thus

$$g(x) = g(xx^{-1})h(x) = g_{\circ}(x\mu)h(x)$$

for all x in S, where

$$h(x) = \begin{cases} 1 & \text{if } g(x) = g(xx^{-1}) \neq 0 \text{ or } g(xx^{-1}) = 0, \\ -1 & \text{if } g(x) = -g(xx^{-1}) \neq 0. \end{cases}$$

By using (5.2.1), (5.2.2), (3.4.1) and (5.6.1) and the fact that characteristic of F is different from 2 we can conclude that

$$h(x) = \begin{cases} 1 & \text{if } f(x) = f(xx^{-1}) \neq 0 \text{ or } f(xx^{-1}) = 0, \\ -1 & \text{if } f(x) = -f(xx^{-1}) \neq 0. \end{cases}$$

Thus

$$f(x) = f(xx^{-1})h(x) = f_{\circ}(x\mu)h(x)$$

for all x in S. Now, to show that for each $a \in S$, the restriction of h to $a\mu$ is a homomorphism. Let a be a fixed element of S. If $g(aa^{-1}) = 0$, then for each $x \in a\mu$, $g(xx^{-1}) = g(aa^{-1}) = 0$. Therefore $h(x) = 1$ for all $x \in a\mu$. Thus the restriction of h to $a\mu$ is a homomorphism. In the case $g(aa^{-1}) \neq 0$, assume that $x, y \in a\mu$. Since S/μ is a Kronecker semigroup, so $xy \in (a\mu)(a\mu) = a\mu$, and so $g_\circ(xy\mu) = g(xy(xy)^{-1}) = g(aa^{-1}) \neq 0$. Thus it follows from (3.3.2) and (5.6.2) that

$$\begin{aligned}
g_\circ(xy\mu)h(xy) &= g(xy) \\
&= g(x)g(y^{-1}) + f(x)f(y^{-1}) \\
&= g(x)g(y) + f(x)f(y) \\
&= g_\circ(x\mu)h(x)g_\circ(y\mu)h(y) + f_\circ(x\mu)h(x)f_\circ(y\mu)h(y) \\
&= [\, g_\circ(x\mu)g_\circ(y\mu) + f_\circ(x\mu)f_\circ(y\mu) \,]\; h(x)h(y) \\
&= [\, g_\circ(x\mu)g_\circ(y^{-1}\mu) + f_\circ(x\mu)f_\circ(y^{-1}\mu) \,]\; h(x)h(y) \\
&= g_\circ(xy\mu)h(x)h(y).
\end{aligned}$$

Since $g_\circ(xy\mu) \neq 0$, so $h(xy) = h(x)h(y)$. Thus a restriction of h to $a\mu$ is a homomorphism for all a in S.

Conversely, assume that μ is a $\mathcal{K}$-congruence on S and $(f_\circ, g_\circ)$ is a one-to-one class 2 positive-type solution of (*) on S/μ and h is a function from S into $\{1,-1\}$ whose restriction to any congruence class $a\mu$ is a homomorphism. Let

(5.6.5) $\qquad f(x) = f_\circ(x\mu)h(x) \quad$ and $\quad g(x) = g_\circ(x\mu)h(x)$

for all x in S. Observes that $x, x^{-1}, xx^{-1} \in x\mu$ since $S/\mu \in \mathcal{K}$. Thus, by assumption on h, we have that

(5.6.6) $$\begin{aligned} h(x) &= h(xx^{-1}x) = h(xx^{-1})h(x) = h(x)h(x^{-1})h(x) \\ &= h(x)^2 h(x^{-1}) = h(x^{-1}) \end{aligned}$$

for all x in S, and

(5.6.7) $\qquad h(e) = 1$

for all e in $E(S)$.

Now, to show that (f,g) is a solution of (*) on S, let $x,y \in S$. Therefore

$$g(xy^{-1}) = g_\circ(xy^{-1}\mu)h(xy^{-1}).$$

But

$$\begin{aligned} g(x)g(y) + f(x)f(y) &= g_\circ(x\mu)h(x)g_\circ(y\mu)h(y) + f_\circ(x\mu)h(x)f_\circ(y\mu)h(y) \\ &= [g_\circ(x\mu)g_\circ(y\mu) + f_\circ(x\mu)f_\circ(y\mu)]\, h(x)h(y) \\ &= g_\circ(xy^{-1}\mu)h(x)h(y) \\ &= g_\circ(xy^{-1}\mu)h(x)h(y^{-1}). \end{aligned}$$

The last equality follows from (5.6.6). If $(x,y) \in \mu$ then $(x,y^{-1}) \in \mu$ since S/μ is a Kronecker semigroup. Therefore $h(xy^{-1}) = h(x)h(y^{-1})$, so

$$g(xy^{-1}) = g(x)g(y) + f(x)f(y).$$

In the case $(x,y) \notin \mu$, we have that $xy^{-1}\mu = xy\mu$ is the zero element. Therefore, it follows from (5.5.1) that

$$g_\circ(xy\mu) = 0.$$

Thus we have that

$$g(xy^{-1}) = 0 = g(x)g(y) + f(x)f(y).$$

Thus (f,g) is a solution of (*) on S. From (5.6.5), (5.6.6), (5.6.7) and the assumption on $(f_\circ, g_\circ)$, we have that

$$f(x) = f_\circ(x\mu)h(x) = f_\circ(x^{-1}\mu)h(x) = f_\circ(x^{-1}\mu)h(x^{-1}) = f(x^{-1})$$

for all x in S, and

$$g(e) = g_\circ(e\mu)h(e) = g_\circ(e\mu) \neq 1$$

for all e in E(S). Thus (f,g) is a class 2 positive-type solution of (*) on S. #

Remark 5.7. By Theorem 5.6 we see that to determine all class 2 positive-type solutions of (*) on S, we need to determine all one-to-one class 2 positive-type solutions of (*) on a Kronecker semigroup S/μ Hence it is sufficient to look for all one-to-one class 2 positive-type solutions of (*) on a Kronecker semigroup S.

Theorem 5.8. Let S be a Kronecker semigroup. Then a one-to-one class 2 positive-type solution of (*) on S exists iff $|S| \leqslant 3$. In such these case any solution must be of the following forms:

Case 1: If $|S| = 1$, say $S = \{0\}$, then

(5.8.1) $$f(x) = b \quad , \quad g(x) = a$$

where $a,b \in F$ are such that $a \neq 1$ and $a = a^2 + b^2$.

Case 2: If $|S| = 2$, say $S = \{0, e\}$ with 0 as the zero, then

(5.8.2) $$f(x) = \begin{cases} 0 & , x = 0 \\ b & , x = e \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x = 0 \\ a & , x = e \end{cases}$$

where $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$.

Case 3: If $|S| = 3$, say $S = \{0,e,e'\}$ with 0 as the zero, then

(5.8.3) $$f(x) = \begin{cases} 0 & , x = 0 \\ b & , x = e \\ -b & , x = e' \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x = 0 \\ a & , x = e \\ 1-a & , x = e' \end{cases}$$

where $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$.

Proof. By straight forward verification, it can be shown that (f,g) in (5.8.1), (5.8.2) and (5.8.3) are one-to-one class 2 positive-type solutions of (*) on S.

To show the converse, assume that (f,g) is a one-to-one class 2 positive-type solution of (*) on S, i.e. f,g satisfying the conditions:

(5.8.4) $\qquad g(e) \neq 1$

for any e in E(S),

(5.8.5) $\qquad f(x) = f(x^{-1})$

for all x in S,

(5.8.6) $\qquad (f(x), g(x)) \neq (f(y), g(y))$

for any $x \neq y$ in S. To show that $|S| \leqslant 3$, suppose that $|S| > 3$. Let e, e', e'' ε S be distinct and different from zero element of S. Thus, by (5.5.3) we have that

$$g(e) + g(e') = 1 \quad \text{and} \quad f(e) + f(e') = 0$$

and

$$g(e) + g(e'') = 1 \quad \text{and} \quad f(e) + f(e'') = 0.$$

Thus $g(e') = g(e'')$ and $f(e') = f(e'')$, contrary to the assumption that (f,g) is one-to-one. Hence $|S| \leqslant 3$.

Next, we shall show that (f,g) must be of the form (5.8.1) or (5.8.2) or (5.8.3).

<u>Case 1</u>: Assume that $|S| = 1$. Then $S = \{0\}$. Let $a = g(0)$ and $b = f(0)$. It follows from (5.8.4) that $a \neq 1$. Since $0 = 0^{-1}$ and $0 = 00^{-1}$, $g(0) = g(00^{-1}) = g(0)g(0) + f(0)f(0)$, so $a = a^2 + b^2$. Thus f,g are of the form (5.8.1).

<u>Case 2</u>: Assume that $|S| = 2$, say $S = \{0,e\}$ with 0 as the zero. It follows from (5.5.1) and (5.5.2) that $f(0) = 0 = g(0)$ and $g(e) \neq 0$. Therefore, if we let $b = f(e)$ and $a = g(e)$, then it follows from (5.8.4) and $g(e) \neq 0$ that $a \neq 1,0$. Since $e = e^{-1}$ and $e = ee = ee^{-1}$, therefore $g(e) = g(ee^{-1}) = g(e)g(e) + f(e)f(e)$, so $a = a^2 + b^2$. Thus f,g are of the form (5.8.2).

Case 3: Assume that $|S| = 3$, say $S = \{0,e,e'\}$ with 0 as the zero. We can verify in the same way as case 2 that $f(0) = 0 = g(0)$ and $f(e) = b$, $g(e) = a$ where $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$. From (5.5.3) we have that

$$g(e) + g(e') = 1 \quad \text{and} \quad f(e) + f(e') = 0$$

Thus $g(e') = 1-a$ and $f(e') = -b$. Therefore f,g are of the form (5.8.3) #

Theorem 5.9. (f,g) is a class 2 positive-type solution of (*) on S iff f,g are of the forms:

$$(5.9.1) \qquad f(x) = \begin{cases} 0 & , x \in A \\ bh(x) & , x \notin A \end{cases} , \qquad g(x) = \begin{cases} 0 & , x \in A \\ ah(x) & , x \notin A \end{cases}$$

where A is a completely prime ideal of S or A is the empty set and h is a homomorphism from $S \setminus A$ into $\{1,-1\}$ and $a,b \in F$ are such that $a \neq 1,0$, $a = a^2 + b^2$; or

$$(5.9.2) \qquad f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh(x) & , x \in e\mu \\ -bh(x) & , x \in e'\mu \end{cases} \qquad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah(x) & , x \in e\mu \\ (1-a)h(x) & , x \in e'\mu \end{cases}$$

where μ is a $\mathcal{K}_3$-congruence on S such that $S/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero and $h : S \setminus \bar{0} \to \{1,-1\}$ is a homomorphism on $e\mu, e'\mu$ and $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$.

Proof. By straight forward verification, it can be shown that (f,g) in (5.9.1) and (f,g) in (5.9.2) are class 2 positive-type solutions of (*) on S.

To show the converse, assume that (f,g) is a class 2 positive-type solution of (*) on S. It follows from Theorem 5.6 that there exists a $\mathcal{K}$-congruence μ on S and a one-to-one class 2 positive-type solution (f_o, g_o) of (*) on S/μ and a function h from S into $\{1,-1\}$ whose restriction to $a\mu$ is a homomorphism for all $a \in S$ such that

(5.9.3) $\qquad f(x) = f_o(x\mu)h(x) \quad , \quad g(x) = g_o(x\mu)h(x)$

for all x in S. It follows from Theorem 5.8 that $|S/\mu| \leqslant 3$. We shall determine (f,g) according to the order of S/μ.

<u>Case 1</u>: Assume that S/μ is trivial. By Theorem 5.8, we have that

$$f_o(x\mu) = b \quad \text{and} \quad g_o(x\mu) = a$$

where $a,b \in F$ are such that $a \neq 1$ and $a = a^2 + b^2$. Thus from (5.9.3) we have that

$$f(x) = bh(x) \quad \text{and} \quad g(x) = ah(x)$$

for all x in S. Thus, we see that if $a = 0$, then $b = 0$ and so f,g are of the form (5.9.1) where $A = S$, and if $a \neq 0$, then f,g are of the form (5.9.1) where $A = \emptyset$.

<u>Case 2</u>: Assume that S/μ is a Kronecker semigroup of order 2, say $S/\mu = \{\bar{0}, e\mu\}$ with $\bar{0}$ as the zero. By Theorem 5.8 we have that

$$f_o(x\mu) = \begin{cases} 0 & , x\mu = \bar{0} \\ b & , x\mu = e\mu \end{cases} \quad , \quad g_o(x\mu) = \begin{cases} 0 & , x\mu = \bar{0} \\ a & , x\mu = e\mu \end{cases}$$

where $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$. Therefore, by (5.9.3) we have

$$f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh(x) & , x \in e\mu \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah(x) & , x \in e\mu . \end{cases}$$

Let $A = \bar{0}$. Then $e\mu = S \setminus A$. Since $\bar{0}e\mu = e\mu\bar{0} = \bar{0}$ and $e\mu e\mu = e\mu$, A is a completely prime ideal of S. It follows from assumption on h that h is a homomorphism from $e\mu = S \setminus A$ into $\{1,-1\}$. Thus f,g are of the form (5.9.1).

<u>Case 3</u>: Assume that S/μ is a Kronecker semigroup of order 3, say $S/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero. By Theorem 5.8 we have that

$$f_\circ(x\mu) = \begin{cases} 0 & , x\mu = \bar{0} \\ b & , x\mu = e\mu \\ -b & , x\mu = e'\mu \end{cases} \quad , \quad g_\circ(xu) = \begin{cases} 0 & , x\mu = \bar{0} \\ a & , x\mu = e\mu \\ 1-a & , x\mu = e'\mu \end{cases}$$

where $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$. Thus, by (5.9.3) we have that

$$f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh(x) & , x \in e\mu \\ -bh(x) & , x \in e'\mu \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah(x) & , x \in e\mu \\ (1-a)h(x) & , x \in e'\mu \end{cases}$$

where $h : S \setminus \bar{0} \to \{1,-1\}$ is a homomorphism on $e\mu$ and $e'\mu$. Therefore f,g are of the form (5.9.2). #

<u>Theorem 5.10</u>. The class 2 solutions of (*) on S are those and only those (f,g) of the forms:

(5.10.1) $$f(x) = \begin{cases} 0 & , x \in A \\ bh(x) & , x \notin A \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x \in A \\ ah(x) & , x \notin A \end{cases}$$

where A is a completely prime ideal of S or A is the empty set and h is a homomorphism from $S\setminus A$ into $\{1,-1\}$ and $a, b \in F$ are such that $a \neq 1,0$, $a = a^2 + b^2$; or

$$(5.10.2) \quad f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh(x) & , x \in e\mu \\ -bh(x) & , x \in e'\mu \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah(x) & , x \in e\mu \\ (1-a)h(x) & , x \in e'\mu \end{cases}$$

where μ is a $\mathcal{K}_3$-congruence on S such that $S/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero and $h : S\setminus\bar{0} \to \{1,-1\}$ is a homomorphism on $e\mu, e'\mu$ and $a, b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$.

Proof. By straight forward verification, it can be shown that (f,g) in (5.10.1) and (f,g) in (5.10.2) are class 2 solutions of (*).

Conversely, assume that (f,g) is a class 2 solution of (*). By Theorem 3.9, (f,g) must be class 2 solution of negative-type or positive-type.

If (f,g) is a class 2 negative-type solution of (*), then by Theorem 5.1, (f,g) is the trivial solution. Thus f,g are of the form (5.10.1) where A = S.

If (f,g) is a class 2 positive-type solution of (*), then by Theorem 5.9, f,g are of the forms:

$$f(x) = \begin{cases} 0 & , x \in A \\ bh(x) & , x \notin A \end{cases} , \quad g(x) = \begin{cases} 0 & , x \in A \\ ah(x) & , x \notin A \end{cases}$$

where A is a completely prime ideal or A is the empty set and h is a homomorphism from $S\setminus A$ into $\{1,-1\}$ and $a,b \in F$ are such that $a \neq 1,0$ and $a = a^2 + b^2$; or

$$f(x) = \begin{cases} 0 & , x \in \bar{0} \\ bh(x) & , x \in e\mu \\ -bh(x) & , x \in e'\mu \end{cases} \quad , \quad g(x) = \begin{cases} 0 & , x \in \bar{0} \\ ah(x) & , x \in e\mu \\ (1-a)h(x) & , x \in e'\mu \end{cases}$$

where μ is a $\mathcal{K}_3$-congruence on S such that $S/\mu = \{\bar{0}, e\mu, e'\mu\}$ with $\bar{0}$ as the zero and $h : S \setminus \bar{0} \to \{1,-1\}$ is a homomorphism on $e\mu, e'\mu$ and $a, b \in F$ are such that $a \neq 1, 0$ and $a = a^2 + b^2$. #