CHAPTER IV

SKEW RINGS

In this chapter we shall classify complete ordered skew ring up to isomorphism.

<u>Definition 4.1</u> A system $(R,+,\cdot,\leqslant)$ is called an <u>ordered skew ring</u> iff $(R,+,\cdot)$ is a skew ring and $\leqslant$ is an order on R satisfying the following properties:

(i) For any $x, y \in R$, $x \leqslant y$ implies that $x+z \leqslant y+z$ and $z+x \leqslant z+y$ for all $z \in R$,

(ii) For any $x, y \in R$, $x \leqslant y$ implies that $x\cdot z \leqslant y\cdot z$ and $z\cdot x \leqslant z\cdot y$ for all $z \geqslant 0$ in R.

<u>Lemma 4.2</u> Let $f: \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

(1) $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$,

(2) $x \leqslant y$ implies that $f(x) \leqslant f(y)$ for all $x, y \in \mathbb{R}$.

Then there exists an $a \geqslant 0$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

<u>Proof</u>: Let $g(x) = f(x)-xf(1)$ for all $x \in \mathbb{R}$. So we have that $g(1) = f(1)-1\cdot f(1) = f(1)-f(1) = 0$. Let $x, y \in \mathbb{R}$ be arbitrary. Thus $g(x+y) = f(x+y)-(x+y)f(1) = f(x)+f(y)-xf(1)-yf(1) = (f(x)-xf(1))+(f(y)-yf(1)) = g(x)+g(y)$. Therefore, substituting 1 for y we have that $g(x+1) = g(x)+g(1) = g(x)$ for every $x \in \mathbb{R}$. Hence g is a periodic function of period $\leqslant 1$. Now, for every $x \in (-1,1)$ $f(x) \leqslant f(1)$

so f is bounded on $(-1,1)$ and $f(1)$ is an upper bound. We get that for every $x \in (-1,1)$, $g(x) = f(x)-xf(1) \leqslant f(1)+f(1) = 2f(1)$ which implies that g is bounded on $(-1,1)$ and $2f(1)$ is an upper bound. Since $(-1,1)$ contains an interval of length 1, g is a bounded function on $\mathbb{R}$ by periodicity. Clearly, $g(0) = 0$, therefore $0 = g(0) = g(x-x) = g(x)+g(-x)$ for all $x \in \mathbb{R}$. Thus $g(-x) = -g(x)$ for all $x \in \mathbb{R}$.

Let $B = 2f(1)$. Thus $g(x) \leqslant B$ for all $x \in \mathbb{R}$. So we have that $-B \leqslant g(-x)$ for all $x \in \mathbb{R}$. Therefore $-B \leqslant g(x)$ for all $x \in \mathbb{R}$. It follows that $-B \leqslant g(x) \leqslant B$ for all $x \in \mathbb{R}$.

Case 1. Suppose that $B \leqslant 0$. Thus $g(x) = 0$ for all $x \in \mathbb{R}$. Therefore $f(x) = x\, f(1)$ for all $x \in \mathbb{R}$. Let $a = f(1)$. Since $0 = f(0) \leqslant f(1)$, we get that $a \geqslant 0$ and we have the lemma.

Case 2. Suppose that $B > 0$.

Subcase 2.1. Suppose that $g(x) = 0$ for all $x \in \mathbb{R}$. Therefore we have that $f(x) = xf(1)$ for all $x \in \mathbb{R}$ and letting $a = f(1)$, we get that $a \geqslant 0$ as before so we are done.

Subcase 2.2. Suppose that there exists an element x_o in $\mathbb{R}$ such that $g(x_o) \neq 0$. We can assume that $g(x_o) > 0$. Choose $C \in \mathbb{R}$ such that $g(x_o) > C > 0$. Since g is an additive homomorphism, for every $n \in \mathbb{Z}^+$, $g(nx) = ng(x)$. Since for every $x \in \mathbb{R}$ $-B \leqslant g(nx) \leqslant B$, we get that for every $x \in \mathbb{R}$ $-B \leqslant ng(x) \leqslant B$ for all $n \in \mathbb{Z}^+$. But $0 < C$ and $0 = \inf\{\frac{B}{n} \mid n \in \mathbb{Z}^+\}$. Hence there exists an $N \in \mathbb{Z}^+$ such that $\frac{B}{N} < C$. Thus $g(x_o) = \frac{1}{N}\, g(Nx_o) \leqslant \frac{B}{N} < C$, a contradiction. So this case cannot occur. #

<u>Proposition 4.3</u> Let m, n ε $\mathbb{Z}^+$ be such that $m \neq n$. Then $(m\mathbb{Z},+,\cdot)$ is not isomorphic to $(n\mathbb{Z},+,\cdot)$

<u>Proof:</u> Let m, n ε $\mathbb{Z}^+$ be such that $m \neq n$. $(m\mathbb{Z},+)$ is an infinite cyclic group whose only generators are m and -m and $(n\mathbb{Z},+)$ is an infinite cyclic group whose only generators are n and -n.

Suppose that $(m\mathbb{Z},+,\cdot)$ is isomorphic to $(n\mathbb{Z},+,\cdot)$. Let $f: m\mathbb{Z} \rightarrow n\mathbb{Z}$ be an isomorphism. Thus $f(m) = \pm n$.

<u>Case 1</u>: $f(m) = n$. Therefore $f(m\ell) = n\ell$ for all ℓ ε $\mathbb{Z}$. Let x, y ε $m\mathbb{Z}$. So there exist unique p and q in $\mathbb{Z}$ such that $x = mp$ and $y = mq$. Thus $f(xy) = f(mp\ mq) = f(m(mpq)) = n(mpq)$ and $f(x)f(y) = f(mp)f(mq) = (np)(nq)$. Thus $f(xy) \neq f(x)f(y)$, a contradiction.

<u>Case 2</u>: $f(m) = -n$. The proof is similar to Case 1.#

<u>Theorem 4.4.</u> Let $(R,+,*,\leqslant)$ be a complete ordered skew ring. Then $(R,+,*,\leqslant)$ is isomorphic to exactly one of the following rings:

(1) $(\mathbb{R},+,\cdot,\leqslant)$.

(2) $(\mathbb{R},+,\circ,\leqslant)$ where $x \circ y = 0$ for all x, y ε $\mathbb{R}$.

(3) $(n\mathbb{Z},+,\cdot,\leqslant)$ for some n ε $\mathbb{Z}_o^+$.

(4) $(\mathbb{Z},+,\circ,\leqslant)$ where $x \circ y = 0$ for all x, y ε $\mathbb{Z}$.

<u>Proof:</u> Since $(R,+,*,\leqslant)$ is complete, by Theorem 1.26 and Theorem 1.31 either $(R,+,\leqslant) \cong (\mathbb{R},+,\leqslant)$ or $(R,+,\leqslant) \cong (\mathbb{Z},+,\leqslant)$ or $(R,+,\leqslant) \cong (\{0\},+,\leqslant)$.

<u>Case 1.</u> Suppose that $(R,+,\leqslant) \cong (\mathbb{R},+,\leqslant)$. For simplicity, we shall assume that $R = \mathbb{R}$.

Fix $a \in \mathbb{R}_o^+$. Define $f: \mathbb{R} \to \mathbb{R}$ by $f_a(x) = a * x$ for all $x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$ be arbitrary. Thus $f_a(x+y) = a*(x+y) = a*x + a*y = f_a(x) + f_a(y)$. If $x \leqslant y$ then $a*x \leqslant a*y$, it follows that $f(x) \leqslant f(y)$. Therefore f_a satisfies the hypothesis of Lemma 4.2, so there exists an $r_a \in \mathbb{R}_o^+$ such that $f_a(x) = r_a x$ for all $x \in \mathbb{R}$.

Let $a \in \mathbb{R}^-$. Then $-a \in \mathbb{R}^+$. So by the above, there exists an $r_{-a} \in \mathbb{R}_o^+$ such that $f_{-a}(x) = r_{-a}x$ for all $x \in \mathbb{R}$. Now for all $x \in \mathbb{R}$, $f_a(x) = a*x = -(-a)*x = -(r_{-a}x) = -r_{-a}x$. Let $r_a = -r_{-a}$. Then for all $x \in \mathbb{R}$, $f_a(x) = a*x = r_a x$. Hence for every $a \in \mathbb{R}$, there exists an $r_a \in \mathbb{R}$ such that $f_a(x) = r_a x$ for all $x \in \mathbb{R}$.

Let $F: \mathbb{R} \to \mathbb{R}$ be defined by $F(a) = r_a$ for all $a \in \mathbb{R}$. Let $a, b \in \mathbb{R}$. We shall show that F is an additive homomorphism. Let $x \in \mathbb{R}$ be arbitrary. Then $r_{a+b}x = (a+b)*x = a*x + b*x = r_a x + r_b x = (r_a + r_b)x$. Putting $x = 1$, we get that $r_{a+b} = r_a + r_b$. Hence $F(a+b) = r_{a+b} = r_a + r_b = F(a) + F(b)$.

Let $a_1, a_2 \in \mathbb{R}$ be such that $a_1 \leqslant a_2$. We shall show that $F(a_1) \leqslant F(a_2)$. Now $a_2 - a_1 \geqslant 0$ and $1 > 0$. Therefore $0 \leqslant (a_2 - a_1)*1 = a_2*1 - a_1*1$, it follows that $a_1*1 \leqslant a_2*1$ so $f_{a_1}(1) \leqslant f_{a_2}(1)$. Therefore $(r_{a_1})1 \leqslant (r_{a_2})1$. Thus $r_{a_1} \leqslant r_{a_2}$. We get that $F(a_1) \leqslant F(a_2)$.

We showed that F satisfies the hypothesis of Lemma 4.2, hence there exists an $s \in \mathbb{R}_o^+$ such that $F(a) = sa$ for all $a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be arbitrary.

Subcase 1.1 $s = 0$. Thus $F(a) = 0$ for all $a \in \mathbb{R}$. Let $u, v \in \mathbb{R}$. Then $u*v = r_u v = F(u)v = 0 \cdot v = 0$. Thus $(R,+,*,\leqslant) \cong (\mathbb{R},+,o,\leqslant)$ where $x \circ y = 0$ for all $x, y \in \mathbb{R}$.

<u>Subcase 1.2</u> $s > 0$. Clearly F is a bijection in this case since $F(a) = sa$ for all $a \in \mathbb{R}$. We shall show that $F(a*b) = F(a)\cdot F(b)$ for all $a, b \in \mathbb{R}$. To prove this, let $a, b \in \mathbb{R}$. Thus $a*b = f_a(b) = r_a b = (sa)b$ and $r_{a*b} = s(a*b) = s((sa)b) = (sa)(sb) = r_a \cdot r_b$. Hence $F(a*b) = r_{a*b} = r_a \cdot r_b = F(a)\cdot F(b)$. Therefore F is a ring homomorphism. Hence $(R,+,*,\leqslant) \cong (\mathbb{R},+,\cdot,\leqslant)$.

<u>Case 2</u>. Suppose that $(R,+,\leqslant) \cong (\mathbb{Z},+,\leqslant)$. Since $(\mathbb{Z},+)$ is an infinite cyclic group, R is an infinite cyclic group. Let $g_o \in R$ be a generator. Then $-g_o$ is a generator of R. Now either $g_o > 0$ or $-g_o > 0$. We can assume that $g_o > 0$. So $g_o^2 \geqslant 0$. Since $g_o^2 \in R$, there exists an $m \in \mathbb{Z}$ such that $g_o^2 = mg_o$, which implies that $m \geqslant 0$.

<u>Subcase 2.1</u> $m = 0$. Thus $g_o^2 = 0$. Let $g_1, g_2 \in R$ be arbitrary. Thus $g_1 = n_1 g_o$ for some $n_1 \in \mathbb{Z}$ and $g_2 = n_2 g_o$ for some $n_2 \in \mathbb{Z}$. We get that $g_1 * g_2 = (n_1 g_o)*(n_2 g_o) = (n_1 \cdot n_2)g_o^2 = 0$. Hence $(R,+,*,\leqslant) \cong (\mathbb{Z},+,o,\leqslant)$ where $x \circ y = 0$ for all $x, y \in \mathbb{Z}$.

<u>Subcase 2.2</u> $m > 0$. Define $h: R \rightarrow m\mathbb{Z}$ in the following way: Let $g_1 \in R$. So $g_1 = ng_o$ for some $n \in \mathbb{Z}$, define $h(g_1) = h(ng_o) = nm$.

To show that h is an injection. Suppose that $h(ng_o) = h(n'g_o)$. Thus $nm = n'm$, it follows that $n = n'$. Hence h is an injection.

To show that h is surjection. Let $p \in m\mathbb{Z}$, so $p = nm$ for some $n \in \mathbb{Z}$. We see that $ng_o \in R$. Thus $h(ng_o) = nm = p$. Hence h is a surjection.

To show that h is a homomorphism. Let $g_1, g_2 \in R$. Then there exist $n, n' \in \mathbb{Z}$ such that $g_1 = ng_o$, $g_2 = n'g_o$. Thus

$$h(g_1 + g_2) = h(ng_o + n'g_o)$$
$$= h((n+n')g_o)$$

$$= m(n+n')$$

$$= mn + mn'$$

$$= h(ng_o) + h(n'g_o)$$

$$= h(g_1) + h(g_2)$$

and

$$h(g_1 * g_2) = h((ng_o)(n'g_o)) = h((nn')g_o^2)$$

$$= h((nn')mg_o)$$

$$= (nn')m^2$$

$$= (nm)(n'm)$$

$$= h(ng_o)h(n'g_o)$$

$$= h(g_1)\, h(g_2).$$

Hence we proved that h is an isomorphism. Therefore $(R,+,*,\leqslant) \cong (m\mathbb{Z},+,\cdot,\leqslant)$ for some $m \in \mathbb{Z}^+$.

Case 3. $(R,+,\leqslant) \cong (\{0\},+,\leqslant)$. Clearly $(R,+,*,\leqslant) \cong (\{0\},+,\cdot,\leqslant)$. #

Corollary 4.5 A complete ordered skew field is isomorphic to $(\mathbb{R},+,\cdot,\leqslant)$.