CHAPTER III
GENERALIZATION UNDER IMPLICATION AND θ-PROOF

In this chapter we will study the theory of generalization under implication. We note that implication is difficult to work with, and therefore we will introduce a relation equivalent to implication, called θ-proof.

Implication

Implication is the most natural and straightforward basis for generalization, since the concept of inductive conclusion is defined in terms of logical consequence.

Definition 1. Let C and D be clauses. Then C **implies** D, denoted $C \Rightarrow D$, if and only if every model for C is also a model for D (i.e., $\{C\} \models D$). We also say that C is a **generalization under implication** of D.

Example. Consider the following clauses:

$$C = (p(x) \leftarrow p(f(x))),$$
$$D = (p(x) \leftarrow p(f^2(x))),$$
$$E = (p(x) \leftarrow p(f^2(y))), \text{ and}$$
$$F = (p(x) \leftarrow p(f^3(x))).$$

Then we have that both C and E imply both D and F, but C does not θ-subsume D, and neither C implies E nor E implies C.

Proposition 2. Let $C_1, C_2, \ldots, C_k, D_1, D_2, \ldots, D_n$, and E be clauses. If $\{C_1, C_2, \ldots, C_k\} \models D_j$ for all $j \in \{1, 2, \ldots, n\}$ and $\{D_1, D_2, \ldots, D_n\} \models E$, then $\{C_1, C_2, \ldots, C_k\} \models E$.

Proof. Assume that $\{C_1, C_2,\ldots, C_k\} \models D_j$ for all $j \in \{1, 2,\ldots, n\}$ and $\{D_1, D_2,\ldots, D_n\} \models E$. We must show that every model for $\{C_1, C_2,\ldots, C_k\}$ is a model for E. Let I be an interpretation which is a model for $\{C_1, C_2,\ldots, C_k\}$. Since $\{C_1, C_2,\ldots, C_k\} \models D_j$ for all $j \in \{1, 2,\ldots, n\}$, I is a model for D_j for all $j \in \{1, 2,\ldots, n\}$. And since $\{D_1, D_2,\ldots, D_n\} \models E$, I is a model for E. Thus, $\{C_1, C_2,\ldots, C_k\} \models E$. □

Proposition 3. Implication is reflexive and transitive.

Proof. Let C, D and E be clauses.

1) We must show that $C \Rightarrow C$. Since $\{C\} \models C$, $C \Rightarrow C$. So, implication is reflexive

2) We must show that if $C \Rightarrow D$ and $D \Rightarrow E$, then $C \Rightarrow E$. Assume that $C \Rightarrow D$ and $D \Rightarrow E$. Then $\{C\} \models D$ and $\{D\} \models E$, and so $\{C\} \models E$ by the previous proposition. Hence, $C \Rightarrow E$. Thus, implication is transitive. □

As in the case of θ-subsumption, implication between clauses is not anti-symmetric. Also, two clauses may be equivalent under implication without being equivalent under θ-subsumption.

Definition 4. Let C and D be clauses. Then C and D are **equivalent under implication**, denoted $C \Leftrightarrow D$, if and only if $C \Rightarrow D$ and $D \Rightarrow C$.

Example. Consider the following clauses:

$$C = (p(x), p(y) \leftarrow p(f(x)), p(f^2(y))), \text{ and}$$

$$D = (p(z) \leftarrow p(f^2(z))).$$

Then we have $C \Leftrightarrow D$. We also have $D \prec C$, but $C \not\prec D$.

The above example also shows that if a clause C implies a clause D then C does not necessary θ-subsume D. It is well known that implication is a strictly weaker relation between clauses than θ-subsumption.

Proposition 5. Let C and D be clauses. If $C \prec D$, then $C \Rightarrow D$.

Proof. Assume $C \prec D$. Then there exists a substitution θ such that $C\theta \subseteq D$. Let I be a model for C. Then every ground instance of C is true with respect to I, and thus every ground instance of $C\theta$ is true with respect to I. Note that if $C\theta$ is true with respect to I, then $C\theta \cup \Lambda$ is true with respect to I for any set of literals Λ. Consequently every model for C is a model for D, and $C \Rightarrow D$. □

Since there is no least general generalization of Horn clauses under implication, we instead turn our interest to minimally general generalizations under implication in the next definition.

Definition 6. A Horn clause C is a **minimally general generalization under implication (MinGGI)** of two Horn clauses D and E if and only if :

a) $C \Rightarrow D$ and $C \Rightarrow E$, and

b) for each Horn clause F such that $F \Rightarrow D$, $F \Rightarrow E$ and $C \Rightarrow F$, we also have $F \Rightarrow C$.

Example. Consider the following clauses:

$C = (\, p(a) \leftarrow p(f(a))),$

$D = (\, p(b) \leftarrow p(f^2(b))),$

$E = (\, p(x) \leftarrow p(f(y)))$, and

$F = (\, p(z) \leftarrow p(f(z))).$

The clause E is an LGGθ of $\{C, D\}$, and F is an MinGGI of $\{C, D\}$. The MinGGI is strictly more specific than the LGGθ, since $E \Rightarrow F$, but $F \not\Rightarrow E$.

Example. Consider the following clauses:

$$C = (p(x) \leftarrow p(f(x))),$$
$$D = (p(x) \leftarrow p(f^2(x))),$$
$$E = (p(x) \leftarrow p(f^2(y))), \text{ and}$$
$$F = (p(x) \leftarrow p(f^3(x))).$$

Then both clause C and clause E are MinGGI's of D and F.

θ**-proof**

Definition 7. Let $\boldsymbol{H}$ be a set of clauses and B a clause. We say that $\boldsymbol{H}$ **θ-proves** B if and only if there is a sequence $A_0, A_1, \ldots, A_n$ of clauses such that each A_i is either an element of $\boldsymbol{H}$, or follows from A_j and A_k by resolution for some $j, k < i$, and in addition A_n θ-subsumes B. We will write $\boldsymbol{H} \vdash_\theta B$ if and only if $\boldsymbol{H}$ θ-proves B. If $\boldsymbol{H} = \{A\}$, we often write $A \vdash_\theta B$, and say A θ-proves B

The following lemma is used to make it easier to prove the next theorem.

Lemma 8. Let $\boldsymbol{H}$ be a set of clauses, and let $c_1, c_2, \ldots, c_m$ be constant symbols which do not appear in any of the clauses of $\boldsymbol{H}$. Suppose we have a sequence $B_1, B_2, \ldots, B_n$ of clauses such that for each i either $B_i \in \boldsymbol{H}$ or there exist $j, l < i$ such that B_i follows from B_j and B_l by resolution.

Let $y_1, y_2, \ldots, y_m$ be variables which do not occur in any of the clauses $B_1, B_2, \ldots, B_n$. For each i ,construct B_i' by replacing all occurrences of c_j with y_j, for $j = 1, 2, \ldots, m$. Then for each i either $B_i' \in \boldsymbol{H}$ or there exist $j, l < i$ such that B_i' follows from B_j' and B_l' by resolution.

Proof. Let $i \in \{1, 2, \ldots, n\}$.

<u>Case I</u>. $B_i \in \boldsymbol{H}$. Then $c_1, c_2, \ldots, c_m$ do not occur in B_i so $B_i' = B_i$ and thus $B_i' \in \boldsymbol{H}$.

Case II. B_i follows from some B_j and B_l by resolution. Let $B_j^* \subseteq B_j$, and $B_l^* \subseteq B_l$, γ and μ be such that γ is an mgu of B_j^* and μ is an mgu of B_l^*. Let $A \in B_j\gamma$, $B \in B_l\mu$ and θ be such that θ is an mgu of $\{A, \overline{B}\}$ and $B_i = ((B_j\gamma)\backslash\{A\}) \cup (B_l\mu\backslash\{B\}))\theta$. Without loss of generality $y_1, y_2, \ldots, y_m$ do not occur in the domains of γ, μ and θ.

Let $B_j^{*\prime}$, $B_l^{*\prime}$, A' and B' be obtained from B_j^*, B_l^*, A and B respectively, by replacing all occurrences of c_j with y_j, for $j = 1, 2, \ldots, m$. If $\gamma = \{x_1 / t_1, x_2 / t_2, \ldots, x_r / t_r\}$, for each s let t_s' be obtained by replacing all occurrences of c_j by y_j in t_s for $j = 1, 2, \ldots, m$. Let $\gamma' = \{x_1 / t_1', x_2 / t_2', \ldots, x_r / t_r'\}$ Define μ', θ' similarly. Then γ' is an mgu of $B_j^{*\prime}$, μ' is an mgu of $B_l^{*\prime}$, $A' \in B_j'\gamma'$, $B' \in B_l'\mu'$, θ' is an mgu of $\{A', \overline{B'}\}$ and $B_i' = ((B_j'\gamma' \setminus \{A'\}) \cup (B_l'\mu' \setminus \{B'\}))\theta'$. Thus B_i' follows from B_j' and B_l' by resolution. □

Theorem 9. (Resolution Theorem) Let T be a set of clauses. Then the empty clause is a member of $\mathcal{R}^n(T)$ for some n if and only if T is unsatisfiable.

Proof. A proof can be found in [9]. □

The following theorem shows that implication and θ-proof are equivalent.

Theorem 10. Let $\mathcal{H}$ be a finite set of clauses and C a clause which is not valid. Then $\mathcal{H} \models C$ if and only if $\mathcal{H} \vdash_\theta C$.

Proof.

Case ($\Leftarrow$). We must show that $\mathcal{H} \vdash_\theta C$ implies $\mathcal{H} \models C$. Assume $\mathcal{H} \vdash_\theta C$.

Let I be an interpretation which is a model for $\mathcal{H}$. Since $\mathcal{H} \vdash_\theta C$, there is a sequence $A_0, A_1, \ldots, A_n$ of clauses such that each A_i satisfies one of the following:

(i) $A_i \in \mathcal{H}$, or

(ii) A_i follows from A_j and A_k by resolution for some j, $k < i$,
and we also have that A_n θ-subsumes C.

Then we will prove by induction on $m \in \{0, 1, 2,..., n\}$ that A_m is true in I.

Basis Case. We have $m = 0$. Then $A_0 \in \boldsymbol{H}$, so A_0 is true in I.

Induction Case. Assume that $A_0, A_1,..., A_{m-1}$ are true in I.

Case I. $A_m \in \boldsymbol{H}$, then A_m is true in I.

Case II. A_m follows from A_j and A_k by resolution for some j, $k < m$. But A_j and A_k are true in I, so A_m is true in I.

In particular, by induction A_n is true in I. But $A_n \prec C$, so $\{A_n\} \models C$. Thus C is true in I. Since I was arbitrary, this shows every model for $\boldsymbol{H}$ is also a model for C. Thus, $\boldsymbol{H} \models C$.

Case ($\Rightarrow$). We must show that $\boldsymbol{H} \models C$ implies $\boldsymbol{H} \vdash_\theta C$. Assume that $\boldsymbol{H} \models C$.

Let $C = \{\lambda_1, \lambda_2,..., \lambda_m\}$. Let $x_1, x_2,..., x_k$ be the variables occurring in C. For each $i \in \{1, 2,..., k\}$, choose a constant symbol c_i not occurring in any of the clauses in $\boldsymbol{H}$, let $\sigma = \{x_1/c_1, x_2/c_2,...,x_k/c_k\}$ and let $C' = C\sigma$. Then C' is a ground clause. Also let $\lambda'_j = \lambda_j\sigma$, so that $C' = \{\lambda'_1, \lambda'_2,..., \lambda'_m\}$.

Note that $\boldsymbol{H} \models C'$. Indeed, if I is an interpretation which is a model for $\boldsymbol{H}$, then I is a model for C, which means that $C\theta$ is true with respect to I for every ground instance $C\theta$ of C. But C' is a ground instance of C, so C' is true with respect to I. Thus, I is a model for C'.

Then $\boldsymbol{H} \cup \{\{\overline{\lambda_1'}\}, \{\overline{\lambda_2'}\},\ldots, \{\overline{\lambda_m'}\}\}$ is unsatisfiable. By Theorem 9 there is a resolution proof of the empty clause from $\boldsymbol{H} \cup \{\{\overline{\lambda_1'}\}, \{\overline{\lambda_2'}\},\ldots, \{\overline{\lambda_m'}\}\}$. I claim that $\boldsymbol{H} \vdash_\theta C$.

<u>Case I</u>. If there is a resolution proof of the empty clause from $\boldsymbol{H}$, then there is a sequence of clauses $A_1, A_2,\ldots, A_n$ with $A_n = \{\}$ and for each $i \in \{1, 2,\ldots, n\}$.

(i) $A_i \in \boldsymbol{H}$, or

(ii) A_i follows from A_j and A_l by resolution for some $j, l < i$.

But $\{\} \prec C$, so $\boldsymbol{H} \vdash_\theta$ C.

<u>Case II</u>. If there is no resolution proof of $\{\}$ from $\boldsymbol{H}$, then since $\boldsymbol{H} \cup \{\{\overline{\lambda_1'}\}, \{\overline{\lambda_2'}\},\ldots, \{\overline{\lambda_k'}\}\}$ is unsatisfiable, there is a sequence of clauses $A_1, A_2,\ldots, A_n$ with $A_n = \{\}$ such that for each $i \in \{1, 2,\ldots, n\}$,

(i) $A_i \in \boldsymbol{H}$,

(ii) $A_i = \{\overline{\lambda_j'}\}$ for some $j \in \{1, 2,\ldots, k\}$, or

(iii) A_i follows from A_j and A_l by resolution for some $j, l < i$.

We want to use $A_1, A_2,\ldots, A_n$ to construct a new sequence $B_1, B_2,\ldots, B_n$ such that for each $i \in \{1, 2,\ldots, n\}$,

(i) $B_i \in \boldsymbol{H}$, or

(ii) B_i follows from B_j and B_l by resolution for some $j, l < i$,

and $B_n \prec C$.

To construct the sequence $B_1, B_2, \ldots, B_n$, we will first construct two sequences $A'_1, A'_2, \ldots, A'_n$ and $A''_1, A''_2, \ldots, A''_n$, and a sequence of substitutions $\theta_1, \theta_2, \ldots, \theta_n$ such that for all $i \in \{1, 2, \ldots, n\}$.

(i) $A_i \subseteq A'_i\theta_i \subseteq A_i \cup A''_i$,

(ii) $A''_i \subseteq \{\lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$,

(iii) either $A'_i \in H$, A'_i follows from A'_j and A'_l by resolution for some j, $l < i$, or $A'_i = \{\overline{\lambda_p}\}$ for some $p \in \{1, 2, \ldots, k\}$.

We will construct A'_i, A''_i, and θ_i by induction on i.

<u>Basis Case</u>. $i = 1$. See Case I below, since $A_1 \in H$.

<u>Induction Case</u>. Assume $i > 1$ and that $A'_1, A'_2, \ldots, A'_{i-1}, A''_1, A''_2, \ldots, A''_{i-1}$, and $\theta_1, \theta_2, \ldots, \theta_{i-1}$ have been defined satisfying (i) - (iii) above. Now, let us define A'_i, A''_i, and θ_i, and check that they satisfy (i) - (iii). There are several cases, depending on A_i.

<u>Case I</u>. $A_i \in H$. Let $A'_i = A_i$, $A''_i = \{\}$, and $\theta_i = \varepsilon$. Clearly $A_i \subseteq A'_i\theta_i \subseteq A_i \cup A''_i$, $A''_i \subseteq \{\lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$, and $A'_i \in H$.

<u>Case II</u>. $A_i = \{\overline{\lambda'_j}\}$. Let $A_i' = \{\overline{\lambda_j}\}$, $A''_i = \{\}$, and $\theta_i = \{x_1/c_1, x_2/c_2, \ldots, x_k/c_k\}$. So, $A_i \subseteq A'_i\theta_i \subseteq A_i \cup A''_i$, $A''_i \subseteq \{\lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$, and $A'_i = \{\overline{\lambda_p}\}$ for some $p \in \{1, 2, \ldots, k\}$.

<u>Case III</u>. A_i follows from A_j, A_l by resolution:

<u>Case III.1</u>. $A_j = \{\overline{\lambda'_p}\}$ and $A_l = \{\overline{\lambda'_q}\}$. Since A_i follows from A_l and A_j by resolution, and $\overline{\lambda'_p}$ and $\overline{\lambda'_q}$ are ground literals, we do not do any substitution for

resolution, so $\overline{\lambda'_q} = \overline{(\overline{\lambda'_p})}$. That is, $\overline{\lambda'_q} = \lambda'_p$, so $\lambda_p = \overline{\lambda_q}$, and C is valid. Thus, this case cannot occur.

<u>Case III.2</u>. $A_j = \{\overline{\lambda'_p}\}$ and $A_l \neq \{\overline{\lambda'_q}\}$. There exists a μ such that μ is an mgu of a subset of A_l, and there exists a literal $A \in A_l\mu$ such that there exists an mgu θ of $\overline{A}$ and $\overline{\lambda'_p}$ (so that $\overline{A}\,\theta = \overline{\lambda'_p}\theta$, i.e., $A\theta = \lambda'_p\theta$). Finally, $A_i = (A_l\mu \setminus \{A\})\theta$. Let $A'_i = A'_l$, $A''_i = A''_l \cup \{\lambda'_p\}$, and $\theta_i = \theta_l\mu\theta$.

We need to show that $A_l\mu\theta = A_i \cup \{\lambda'_p\}$.

<u>Case ($\subseteq$)</u>. Let $B \in A_l\mu\theta$. Then there exists $B^* \in A_l\mu$ such that $B = B^*\theta$. If $B^* = A$, then $B = B^*\theta = A\theta = \lambda'_p$, so, $B \in A_i \cup \{\lambda'_p\}$. If $B^* \neq A$, then $B^* \in A_l\mu \setminus \{A\}$, so $B = B^*\theta \in (A_l\mu \setminus \{A\})\theta = A_i$. Thus, $B \in A_i \cup \{\lambda'_p\}$. This shows $A_l\mu\theta \subseteq A_i \cup \{\lambda'_p\}$.

<u>Case ($\supseteq$)</u>. Since $A \in A_l\mu$, $\lambda'_p = A\theta \in A_l\mu\theta$. And since $A_l\mu \setminus \{A\} \subseteq A_l\mu$, $A_i = (A_l\mu \setminus \{A\})\theta \subseteq A_l\mu\theta$. Thus, $A_i \cup \{\lambda'_p\} \subseteq A_l\mu\theta$.

Hence, $A_l\mu\theta = A_i \cup \{\lambda'_p\}$. So, $A'_i\theta_i = A'_l\theta_l\mu\theta \subseteq (A_l \cup A''_l)\mu\theta = A_l\mu\theta \cup A''_l = A_i \cup \{\lambda'_p\} \cup A''_l = A_i \cup A''_i$, $A_i \subseteq A_i \cup \{\lambda'_p\} = A_l\mu\theta \subseteq A'_l\theta_l\mu\theta = A'_i\theta_i$, and $A''_i = A''_l \cup \{\lambda'_p\} \subseteq \{\lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$. Finally, $A'_i = A'_l$ and A'_l satisfies (iii), so A'_i satisfies (iii).

<u>Case III.3</u>. $A_j \neq \{\overline{\lambda'_p}\}$ and $A_l = \{\overline{\lambda'_q}\}$. This is similar to Case III.2.

<u>Case III.4</u>. $A_j \neq \{\overline{\lambda'_p}\}$ and $A_l \neq \{\overline{\lambda'_q}\}$. Then there exist γ, μ such that γ is an mgu of a subset of A_j, μ is an mgu of a subset of A_l, and $A_j\gamma$ and $A_l\mu$ have no variables in common. Also, there exists an $A \in A_j\gamma$, there exists a $B \in A_l\mu$, and there exists a θ such that θ is an mgu of $\{A, \overline{B}\}$ $(A\theta = \overline{B}\,\theta)$, and $A_i = ((A_j\gamma \setminus \{A\}) \cup (A_l\mu \setminus \{B\}))\theta$.

Since $A_j \subseteq A'_j\theta_j$, we have $A \in A_j\gamma \subseteq A'_j\theta_j\gamma$. Let A^*_j be the largest subset of A'_j such that $A^*_j\theta_j\gamma = \{A\}$. Similarly, let A^*_l be the largest subset of A'_l such that $A^*_l\theta_l\mu = \{B\}$. Thus, $\theta_j\gamma$ is a unifier of A^*_j, so let γ' be an mgu of A^*_j. Let γ'' be such that $\theta_j\gamma = \gamma'\gamma''$. Define μ' and μ'' similarly. Let $A' \in A'_j\gamma'$ be such that $A'\gamma'' = A$. Similarly, let $B' \in A'_l\mu'$ be such that $B'\mu'' = B$.

Note that we have γ', μ' such that γ' is an mgu of a subset of A'_j, and μ' is an mgu of a subset of A'_l. WLOG we can define γ', μ' such that $A'_j\gamma'$ and $A'_l\mu'$ have no variables in common, and thus, we may assume WLOG that $dom\ \gamma'' \cap dom\ \mu'' = \varnothing$.

Note that since θ is an mgu of $\{A, \overline{B}\}$, we must have $A\theta = \overline{B}\theta$, and thus $A'\gamma''\theta = A\theta = \overline{B}\theta = (\overline{B'\mu''})\theta = \overline{B'}\mu''\theta$. Thus, $(\gamma'' \cup \mu'')$ is a substitution, and $A'(\gamma'' \cup \mu'')\theta = A'\gamma''\theta = \overline{B'}\mu''\theta = \overline{B'}(\gamma'' \cup \mu'')\theta$. This shows $(\gamma'' \cup \mu'')\theta$ is a unifier of $\{A', \overline{B'}\}$. Let θ' be an mgu of $\{A', \overline{B'}\}$, and let θ'' be such that $(\gamma'' \cup \mu'')\theta = \theta'\theta''$. Define $A'_i = ((A'_j\gamma' \setminus \{A'\}) \cup (A'_l\mu' \setminus \{B'\}))\theta'$. Then A'_i follows from A'_j and A'_l by resolution. This shows (iii).

Let $A''_i = A''_j \cup A''_l$ and $\theta_i = \theta''$. So, $A''_i \subseteq \{\lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$ by the induction hypothesis. This shows (ii).

Now, let us prove property (i). Recall that $A_i = ((A_j\gamma \setminus \{A\}) \cup (A_l\mu \setminus \{B\}))\theta$, $A'_j\theta_j \subseteq A_j \cup A''_j$, and $A'_l\theta_l \subseteq A_l \cup A''_l$, by induction.

We need to show that $(A'_j\gamma' \setminus \{A'\})\gamma'' = A'_j\gamma'\gamma'' \setminus \{A'\gamma''\}$.

<u>Case (⊇)</u>. Clearly $(A'_j\gamma' \setminus \{A'\})\gamma'' \supseteq A'_j\gamma'\gamma'' \setminus \{A'\gamma''\}$.

Case ($\subseteq$). Let $D \in (A_j'\gamma' \setminus \{A'\})\gamma''$. Then $D = E\gamma''$, where $E \in A_j'\gamma'$ but $E \neq A'$. Suppose $E\gamma'' = A'\gamma''$. Then $E\gamma'' = A'\gamma'' = A$. Since $E \in A_j'\gamma'$, there exists an $F \in A_j'$ such that $E = F\gamma'$. I claim $F \in A_j^*$. Note that $F\theta_j\gamma = F\gamma'\gamma'' = E\gamma'' = A$. Thus, $(A_j^* \cup \{F\})\theta_j\gamma = A_j^*\theta_j\gamma \cup \{F\theta_j\gamma\} = \{A\} \cup \{A\} = \{A\}$. But A_j^* is the largest subset of A_j' such that $A_j^*\theta_j\gamma = \{A\}$, so $A_j^* \cup \{F\} = A_j^*$. This shows $F \in A_j^*$.

Since $A' \in A_j'\gamma'$, let $G \in A_j'$ be such that $A' = G\gamma'$. Since $A'\gamma'' = A$, we have $G\theta_j\gamma = G\gamma'\gamma'' = A'\gamma'' = A$, so the same argument as above shows $G \in A_j^*$ also. But then $A' = G\gamma' \in A_j^*\gamma'$ and $E = F\gamma' \in A_j^*\gamma'$, and γ' is an mgu of A_j^*, so $A_j^*\gamma'$ contains only one element. Thus $A' = E$, a contradiction. The contradiction came from the assumption $E\gamma'' = A'\gamma''$. Thus $E\gamma'' \neq A'\gamma''$. Hence, $D = E\gamma'' = F\gamma'\gamma'' \in A_j'\gamma'\gamma'' \setminus \{A'\gamma''\}$.

This shows $(A_j'\gamma' \setminus \{A'\})\gamma'' = A_j'\gamma'\gamma'' \setminus \{A'\gamma''\}$. Similarly, $(A_l'\mu' \setminus \{B'\})\mu'' = A_l'\mu'\mu'' \setminus \{B'\mu''\}$. Then

$$\begin{aligned}
A_i'\theta_i &= ((A_j'\gamma' \setminus \{A'\}) \cup (A_l'\mu' \setminus \{B'\}))\theta'\theta'' \\
&= ((A_j'\gamma' \setminus \{A'\}) \cup (A_l'\mu' \setminus \{B'\}))(\gamma'' \cup \mu'')\theta \\
&= ((A_j'\gamma' \setminus \{A'\})\gamma'' \cup (A_l'\mu' \setminus \{B'\})\mu'')\theta \\
&= ((A_j'\gamma'\gamma'' \setminus \{A'\gamma''\}) \cup (A_l'\mu'\mu'' \setminus \{B'\mu''\}))\theta \\
&= ((A_j'\theta_j\gamma \setminus \{A\}) \cup (A_l'\theta_l\mu \setminus \{B\}))\theta \\
&\subseteq (((A_j \cup A_j'')\gamma \setminus \{A\}) \cup ((A_l \cup A_l'')\mu \setminus \{B\}))\theta \\
&= (((A_j\gamma \cup A_j'')\setminus \{A\}) \cup ((A_l\mu \cup A_l'')\setminus \{B\}))\theta \\
&\subseteq ((A_j\gamma \setminus \{A\}) \cup (A_l\mu \setminus \{B\}) \cup A_j'' \cup A_l'')\theta \\
&= ((A_j\gamma \setminus \{A\}) \cup (A_l\mu \setminus \{B\}))\theta \cup A_j'' \cup A_l'' \\
&= A_i \cup A_i'',
\end{aligned}$$

and

$$\begin{aligned}
A_i &= ((A_j\gamma \setminus \{A\}) \cup (A_l\mu \setminus \{B\})\theta \\
&\subseteq ((A_j'\theta_j\gamma \setminus \{A\}) \cup (A_l'\theta_l\mu \setminus \{B\}))\theta \\
&= ((A_j'\gamma'\gamma'' \setminus \{A'\gamma''\}) \cup (A_l'\mu'\mu'' \setminus \{B'\mu''\}))\theta \\
&= ((A_j'\gamma' \setminus \{A'\})\gamma'' \cup (A_l'\mu' \setminus \{B'\})\mu'')\theta
\end{aligned}$$

$$\begin{aligned} A_i &= ((A_j'\gamma' \setminus \{A'\}) \cup (A_l'\mu' \setminus \{B'\}))(\gamma'' \cup \mu'')\theta \\ &= A_i'\theta_i. \end{aligned}$$

This shows property (i).

Thus, we have sequences $A_1', A_2', \dots, A_n'$, and $A_1'', A_2'', \dots, A_n''$, and a sequence of substitutions $\theta_1, \theta_2, \dots, \theta_n$ satisfying (i)-(iii).

Now fix $H_0 \in \boldsymbol{H}$. For each $i \in \{1, 2, \dots, n\}$ define B_i by

$$B_i = \begin{cases} A_i' & \text{if } A_i \in \boldsymbol{H} \text{ or } A_i \text{ follows from } A_j, A_l \text{ by resolution} \\ H_0 & \text{if } A_i = \{\overline{\lambda_j'}\}. \end{cases}$$

Then we have $B_1, B_2, \dots, B_n$. I claim that for each i, either $B_i \in \boldsymbol{H}$ or B_i follows from some earlier B_j and B_l by resolution. This will be shown by induction on i. If $i = 1$, then $A_1 \in \boldsymbol{H}$, so $B_1 = A_1' = A_1 \in \boldsymbol{H}$. Assume we have $B_1, B_2, \dots, B_{i-1}$ such that for each $j \in \{1, 2, \dots, i-1\}$, either $B_j \in \boldsymbol{H}$ or B_j follows from some earlier B_p and B_q by resolution. We want to show either $B_i \in \boldsymbol{H}$ or B_i follows from some earlier B_j and B_l by resolution.

<u>Case I</u>. $A_i \in \boldsymbol{H}$. Then $B_i = A_i' = A_i \in \boldsymbol{H}$.

<u>Case II</u>. A_i follows from A_j and A_l by resolution, Then $B_i = A_i'$.

<u>Case II.1</u>. $A_j = \{\overline{\lambda_p'}\}$. Then $B_i = A_i' = A_l'$ for some $l < i$. But A_l cannot be of the form $\{\overline{\lambda_q'}\}$, so $B_l = A_l'$, and thus $B_i = B_l$. By induction, $B_l \in \boldsymbol{H}$ or B_l follows from some earlier B_r and B_s by resolution.

<u>Case II.2</u>. $A_j \neq \{\overline{\lambda_p'}\}$ and $A_l = \{\overline{\lambda_q'}\}$. This is similar to Case II.1.

Case II.3. $A_j \neq \{\overline{\lambda'_p}\}$ and $A_l \neq \{\overline{\lambda'_q}\}$. We have already proved that A'_i follows from A'_j and A'_l by resolution. But $B_j = A'_j$, $B_l = A'_l$, and $B_i = A'_i$, so B_i follows from B_j and B_l by resolution.

Case III. $A_i = \{\overline{\lambda'_p}\}$. Then $B_i \in H$.

Now note that $A_n = \{\}$, so $B_n = A'_n$. Thus, $B_n\theta_n = A'_n\theta_n \subseteq A_n \cup A''_n = A''_n$. But, by property (ii), $A''_n \subseteq \{\lambda'_1, \lambda'_2,..., \lambda'_k\}$. That is $B_n\theta_n \subseteq \{\lambda'_1, \lambda'_2,..., \lambda'_k\}$. Now, choose variables $\{y_1, y_2,..., y_m\}$ such that $y_1, y_2,..., y_m$ do not appear in any of the clauses $B_1, B_2,..., B_n$. For each i, let B_i' be obtained from B_i by replacing all occurrences of c_j with y_j for $j = 1, 2,..., m$. By Lemma 8, either $B'_i \in H$ or B'_i follows from B'_j and B'_l by resolution for some j, $l < i$.

Let $\iota = \{y_1 / x_1, y_2 / x_2,..., y_m / x_m\}$, and $\theta_n = \{z_1 / t_1, z_2 / t_2,...,z_r / t_r\}$. For each $l \in \{1, 2,..., r\}$ let t_l' be t_l with c_j replaced by y_j for $j = 1, 2,..., m$, and let $\theta_n' = \{z_1 / t_1', z_2 / t_2',...,z_r / t_r'\}$. Since $B_n\theta_n \subseteq \{\lambda'_1, \lambda'_2,..., \lambda'_k\}$, $B_n'\theta_n' \subseteq C\iota^{-1}$, where $\iota^{-1} = \{x_1 / y_1, x_2 / y_2,..., x_m / y_m\}$. Then $B'_n\theta_n'\iota \subseteq C\iota^{-1}\iota = C$. So, $B'_n \prec C$. Thus, $H \vdash_\theta C$.

Therefore, $H \models C$ implies $H \vdash_\theta C$. Consequently, $H \models C$ if and only if $H \vdash_\theta C$.□

The previous theorem is similar to Theorem 4 in the paper by Nienhuys-Cheng and de Wolf [7], but the proof is different. The above proof has the advantage that in some ways it is more direct, and is done all in one case, whereas the proof in [7] uses three cases, two concerning ground clauses, and then the general case.

Corollary 11. Let $C_1, C_2,..., C_k, D_1, D_2,...,D_n$ and E be clauses. If $\{C_1, C_2,..., C_k\} \vdash_\theta D_j$ for all j $\in \{1, 2,..., n\}$ and $\{D_1, D_2,..., D_n\} \vdash_\theta E$, then $\{C_1, C_2,..., C_k\} \vdash_\theta E$.

Proof. This follows from Theorem 10 and Proposition 2. □

Corollary 12. θ-proof is reflexive and transitive.

Proof. This follows from Theorem 10 and Proposition 3. □