CHAPTER IV

FORMULATION OF FIELD EQUATIONS

In this chapter we will show the roles that the continuity equation plays in constructing fundamental field equations and, as a result, the universal constant velocity k appeared in the last chapter, will be determined. The same approach is then used to derive the classical theory of electromagnetism which we have already studied in Chapter I.

Relations of Source to Field

It is indeed in doubt that *source* or *field* is primary, but, however, any view gives the same final result that integral of source current-density over any three-dimensional region equals integral of field over boundary of this region (Misner, Thorne, and Wheeler,1973). The laws governing the relation of sources to fields are called the *field equations.* In Chapter I, we have already shown that, from gauge field theory, for every true conservation law there is a complete theory of a gauge field for which the given conserved quantity is the source (Mills,1989). From this astonishing fact, we are convinced that our conserved quantity denoted by Q, of which conservation law is associated with an inertial transformation, Eq.(3.23), can provide some real field which exists in nature. Thus, for local consideration, the source current-density J^{ν} shall be a source of some second-rank tensor field $\mathbf{F} \equiv F^{\mu\nu}$. The investigation to obtain the fields can be illuminated as follows.

Firstly, it is reasonable to assume that the field equations be linear and contain no derivatives of the fields higher than the second order. The purpose of this assumption is not only to limit the choice of possible field equations, but also to affirm that the field due to two sources will be the sum of the fields due to each source (superposition of fields), and knowledge of the fields at spacetime infinity will be sufficient to determine the field everywhere (Dirichlet boundary conditions). Then, there is only one field equation we can write down that satisfies these assumptions, and relates the field $F^{\mu\nu}$ to the source current-density J^{ν}, namely,

$$\partial_{\mu}F^{\mu\nu} = \alpha J^{\nu}, \tag{4.1}$$

where α is to be some invariant scaling constant, determined later by the units which will ultimately emerge, and the conventional summation convention is used. The notations $\partial_{\mu} \equiv (\partial/\partial(kt), \nabla)$ is the derivative four-vector and $J^{\nu} \equiv (k\rho, \mathbf{j})$ is the current-density four-vector as already explained in the previous chapter. If we take the divergence of Eq.(4.1) and use the requirement of local conservation of source, $\partial_{\mu}J^{\mu} = 0$, we finally obtain

$$\partial_{\nu}\partial_{\mu}F^{\mu\nu} = \alpha\,\partial_{\nu}J^{\nu} = 0. \tag{4.2}$$

If we generalize Eq.(4.2), and use the symmetric of $\partial_{\nu}\partial_{\mu}$, we will find that it can be written as a linear equation for ten unknowns F^{00}, F^{11}, F^{22}, F^{33}, $F^{01}+F^{10}$, $F^{02}+F^{20}$, $F^{03}+F^{30}$, $F^{12}+F^{21}$, $F^{13}+F^{31}$, $F^{23}+F^{32}$, or, in complete form,

$$\left[\partial_0\partial_0F^{00}+\partial_1\partial_1F^{11}+\partial_2\partial_2F^{22}+\partial_3\partial_3F^{33}\right] + \partial_0\partial_1\left(F^{01}+F^{10}\right) + \partial_0\partial_2\left(F^{02}+F^{20}\right)$$
$$+ \partial_0\partial_3\left(F^{03}+F^{30}\right) + \partial_1\partial_2\left(F^{12}+F^{21}\right) + \partial_1\partial_3\left(F^{13}+F^{31}\right) + \partial_2\partial_3\left(F^{23}+F^{32}\right) = 0. \tag{4.3}$$

The simplest solution of this equation is achieved if $F^{\mu\nu}$ is being an antisymmetric tensor. Thus, Eq.(4.2) will be satisfied exactly in all inertial frames if $F^{\mu\nu}$ is an antisymmetric tensor, or,

$$F^{\mu\nu} = -F^{\nu\mu}. \tag{4.4}$$

As we have learned in Chapter II, an antisymmetric second-rank tensor in four-space has only six independent components which can be shown in matrix forms as follows:

$$F^{\mu\nu} \equiv \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & -F^{10} & -F^{20} & -F^{30} \\ F^{10} & 0 & -F^{21} & -F^{31} \\ F^{20} & F^{21} & 0 & -F^{32} \\ F^{30} & F^{31} & F^{32} & 0 \end{bmatrix} . \quad (4.5)$$

We find that these six components may be discriminated into two groups, one group which is containing F^{10}, F^{20}, F^{30}, involves with the zeroth-component, or scalar part, of the current-density four-vector J^{ν}, while the other group containing F^{21}, F^{31}, F^{32}, involves with vector part of J^{ν}.

Since each group contains only three components, we can treat it as a vector in three-space. This means our second-rank tensor field $F^{\mu\nu}$ in four-space will appear to be two vector fields in three-space. The first vector field, which we will introduce, corresponding to the group (F^{10}, F^{20}, F^{30}) is called the polar field **P** and the second vector field corresponding to the group (F^{21}, F^{31}, F^{32}) is called the axial field **A**. (Please do not confuse with the vector potential **A** we have stated in Chapter I. In this chapter, the vector potential will be defined alternatively later by the notation **u**.) Thus, components of these two vector fields can be written as

$$\mathbf{P} = (P_1, P_2, P_3) = (F^{10}, F^{20}, F^{30}), \quad (4.6)$$
$$\mathbf{A} = (A_1, A_2, A_3) = (-F^{23}, -F^{31}, -F^{12}).$$

At this point the physical significance of $\mathbf{P} = (P_1, P_2, P_3)$ and $\mathbf{A} = (A_1, A_2, A_3)$ is not yet shown. Now, the tensor field $F^{\mu\nu}$ can be presented in terms of P_i and A_i $(i=1,2,3)$,

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & -P_1 & -P_2 & -P_3 \\ P_1 & 0 & -A_3 & A_2 \\ P_2 & A_3 & 0 & -A_1 \\ P_3 & -A_2 & A_1 & 0 \end{bmatrix}. \qquad (4.7)$$

To evaluate the significance of vector fields **P** and **A**, we write out the $\nu = 0$ component of Eq.(4.1),

$$\partial_\mu F^{\mu 0} = \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \alpha J^0,$$

or,

$$\nabla \cdot \mathbf{P}(\mathbf{x},t) = \alpha k \rho(\mathbf{x},t). \qquad (4.8a)$$

Evidently, **P** may be a field diverging from the Q-monopole. The vector components $(\nu=1,2,3)$ of Eq.(4.1) give the field equation,

$$\partial_\mu F^{\mu 1} + \partial_\mu F^{\mu 2} + \partial_\mu F^{\mu 3} = \alpha(J^1+J^2+J^3),$$

or,

$$\left[\partial_0 F^{01} + \partial_0 F^{02} + \partial_0 F^{03}\right] + \left[\partial_2 F^{21} + \partial_3 F^{31}\right] + \left[\partial_1 F^{12} + \partial_3 F^{32}\right] + \left[\partial_1 F^{13} + \partial_2 F^{23}\right] = \alpha \mathbf{j}.$$

Replace the components of **P** and **A** as defined in Eq.(4.6) in the above equation, we finally obtain the relation,

$$\nabla \times \mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) = \alpha \mathbf{j}(\mathbf{x},t). \qquad (4.8b)$$

The two inhomogeneous Eqs.(4.8a) and (4.8b) show the relations of the vector fields **P** and **A** with their sources ρ and **j**. We notice that the vector part of the source $J^{\nu} \equiv (k\rho, \mathbf{j})$, in four-space, produces both polar field and axial field, in three-space, of which relation is shown in Eq.(4.8b).

Dual Tensor Fields

From Helmholtz theorem in three-space, (see for detail in appendix C), we find that both relations in Eq.(4.8) are not sufficient to determine the complete information of **P** and **A**. If we consider for the second-rank antisymmetric tensor field in four-space, we find, as shown in the previous chapter, that its dual will contain the same information but manifest in the different characteristics. Therefore, we write for the dual of the tensor field $F^{\mu\nu}$ as

$$^{*}F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{4.10}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Cavita symbol in four-space of which values are defined in Eq.(2.35). The covariant tensor $F_{\alpha\beta}$ is defined as

$$F_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu} = \begin{bmatrix} 0 & P_1 & P_2 & P_3 \\ -P_1 & 0 & -A_3 & A_2 \\ -P_2 & A_3 & 0 & -A_1 \\ -P_3 & -A_2 & A_1 & 0 \end{bmatrix}, \tag{4.11}$$

where the covariant metric tensor $g_{\mu\nu}$ for our inertial transformation is diagonal which can usually be written as

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1).$$

Then the dual tensor $^{*}F^{\mu\nu}$ in Eq.(4.10), can be written in matrix form as

$$
{}^{*}F^{\mu\nu} = \begin{bmatrix} 0 & -A_1 & -A_2 & -A_3 \\ A_1 & 0 & P_3 & -P_2 \\ A_2 & -P_3 & 0 & P_1 \\ A_3 & P_2 & -P_1 & 0 \end{bmatrix} . \qquad (4.12)
$$

We notice that the elements of the dual tensor ${}^{*}F^{\mu\nu}$ are simply obtained from $F^{\mu\nu}$ by putting $\mathbf{P} \rightarrow \mathbf{A}$ and $\mathbf{A} \rightarrow -\mathbf{P}$ in Eq.(4.7).

The source that relates to this dual field is called the dual source denoted by ${}^{*}J^{\nu} \equiv (k{}^{*}\rho, {}^{*}\mathbf{j})$. To give the dual tensor field, dual source must be locally conserved, this fact can be asserted by the covariance of the continuity equation in four-space,

$$
\partial_{\nu}{}^{*}J^{\nu} = 0, \qquad (4.13)
$$

where ${}^{*}J^{\nu}$ is the dual source current-density four-vector. Then the dual tensor field ${}^{*}F^{\mu\nu}$, corresponding to the source ${}^{*}J^{\nu}$, can be written as

$$
\partial_{\mu}{}^{*}F^{\mu\nu} = \alpha{}^{*}J^{\nu}. \qquad (4.14)
$$

Therefore, the conservation of dual source, shown in Eq.(4.13), follows as a consequence of the antisymmetry of the field tensor $F^{\mu\nu}$. This because if we take the four-dimensional divergence of Eq.(4.10), we find

$$
\partial_{\nu}{}^{*}J^{\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\,\partial_{\nu}\partial_{\mu}F_{\alpha\beta}, \qquad (4.15)
$$

the right-hand side vanishing identically because $\partial_{\nu}\partial_{\mu}$ is symmetric whereas $\varepsilon^{\mu\nu\alpha\beta}$ is antisymmetric under $\mu \leftrightarrow \nu$.

From Eqs.(4.14) and (4.10), we can obtain the other two field equations in three-space of **P** and **A**,

$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = \alpha k^{*}\rho(\mathbf{x},t), \qquad (4.16a)$$

$$-\nabla\times\mathbf{P}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} = \alpha^{*}\mathbf{j}(\mathbf{x},t). \qquad (4.16b)$$

Therefore, we can write the complete set of field equations for every point in four-space if both source J^{ν} and its dual $^{*}J^{\nu}$ exist as follows:

$$\partial_{\mu}F^{\mu\nu} = \alpha J^{\nu}, \qquad (4.17a)$$

$$\partial_{\mu}{}^{*}F^{\mu\nu} = \alpha^{*}J^{\nu}. \qquad (4.17b)$$

These two field equations will apparently manifest in three-space according to the following four equations,

$$\nabla\cdot\mathbf{P}(\mathbf{x},t) = \alpha k\rho(\mathbf{x},t), \qquad (4.18a)$$

$$\nabla\times\mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{P}(\mathbf{x},t)}{\partial t} = \alpha\,\mathbf{j}(\mathbf{x},t), \qquad (4.18b)$$

$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = \alpha k^{*}\rho(\mathbf{x},t), \qquad (4.18c)$$

$$-\nabla\times\mathbf{P}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} = \alpha^{*}\mathbf{j}(\mathbf{x},t). \qquad (4.18d)$$

These four equations are considered as the ***fundamental field equations*** in three-space while the two equations in Eq.(4.17) be the fundamental field equations in four-space related to the source J^{ν} and its dual $^{*}J^{\nu}$.

If we assume that these field equation should also be covariant under parity transformation, as described in Eq.(2.33a), we find that with source Q a true conserved scalar under all transformations, because it does not depend on any coordinate system, its density ρ is also a true scalar. Then we see that, from Eq.(4.18a),

$$\nabla \cdot \mathbf{P}(\mathbf{x},t) = \alpha k \rho(\mathbf{x},t),$$

the field vector **P** must be a *polar vector* since both sides must transform in the same manner under the parity transformation. As a result from Eq.(4.18d),

$$\nabla \times \mathbf{P}(\mathbf{x},t) + \frac{1}{k}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\alpha\,{}^{*}\mathbf{j}(\mathbf{x},t),$$

the first term transforms as a pseudovector, or axial vector, under spatial inversion. To preserve the invariance of form it is therefore necessary that the vector field **A** must be an *axial vector* and the source $^{*}\mathbf{j}$ must be also an axial vector. Then, the left-hand side of Eq.(4.18b),

$$\nabla \times \mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) = \alpha\,\mathbf{j}.$$

can be seen to transform as a polar vector. This implies that the current density **j** is a polar vector, as expected from its definition in terms of charge density times velocity. These facts explain why we call vector fields **P** and **A** earlier as polar field and axial field respectively.

Potential Four-vectors

It is not necessary to introduce potential, but they can be introduced for convenience as follows. If we consider Eq.(4.2),

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0,$$

we find that this equation still be satisfied if we introduce $F^{\mu\nu}$ in terms of another four-vector called *potential four-vector*, denoted by $U^\mu \equiv (\Phi, \mathbf{u})$, where

$$F^{\mu\nu} = \left[\partial^\mu U^\nu - \partial^\nu U^\mu\right]. \qquad (4.19)$$

This definition is not only satisfied the properties of field tensor $F^{\mu\nu}$ in Eqs.(4.2) and (4.4), but also of the dual field tensor $^*F^{\mu\nu}$ defined in Eqs.(4.10) and (4.14). This because if we differentiate Eq.(4.10) with respect X^μ, we will obtain

$$\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left[\partial_\mu\partial_\alpha U_\beta - \partial_\mu\partial_\beta U_\alpha\right] = 0.$$

The first term in the bracket vanishes because $\partial_\mu\partial_\alpha$ is symmetric under $\mu\leftrightarrow\alpha$, while $\varepsilon^{\mu\nu\alpha\beta}$ is antisymmetric, and the second term vanishes because $\partial_\mu\partial_\beta$ is symmetric under $\mu\leftrightarrow\beta$, while $\varepsilon^{\mu\nu\alpha\beta}$ is antisymmetric. By considering the definitions of **P** and **A** in Eq.(4.6), we find that our vector fields **P** and **A** can be written in terms of $U^\mu \equiv (\Phi, \mathbf{u})$ as

$$\mathbf{P} = -\frac{1}{k}\frac{\partial \mathbf{u}}{\partial t} - \nabla\Phi \qquad (4.20a)$$

$$\mathbf{A} = \nabla \times \mathbf{u}. \qquad (4.20b)$$

With the definition of the field $F^{\mu\nu}$ in terms of the potential U^{μ} in Eq.(4.15), the source-field relation Eq.(4.1) becomes

$$\partial_{\mu}\partial^{\mu}U^{\nu} - \partial^{\nu}\partial_{\mu}U^{\mu} = \alpha J^{\nu}, \qquad (4.21)$$

where ∂_{μ} and ∂^{ν} in the second term on the left-side are interchanged. For the special case, in which $\partial_{\mu}U^{\mu}$ vanishes, we finally get the potential field equation, in four-space,

$$\partial_{\mu}\partial^{\mu}U^{\nu} = \alpha J^{\nu}. \qquad (4.22)$$

This equation can be written in three- space as

$$\Box\Phi = \alpha k\rho, \qquad (4.23a)$$

$$\Box\mathbf{u} = \alpha\mathbf{j}, \qquad (4.23b)$$

where $\Box$ is the d'Alembertian operator defined as $\Box = \partial^2/\partial(kt)^2 - \nabla^2$. Eq.(4.23) shows the wave equations for the vector potential $\mathbf{u}$ and the scalar potential Φ in three-space. Finally, let us note that the potential four-vector U^{μ} is *not* unique because we can modify it as,

$$U^{\mu} \rightarrow U^{\mu'} = U^{\mu} - \partial^{\mu}\chi, \qquad (4.24)$$

in order to obtain the same field tensor $F^{\mu\nu}$, where χ is an arbitrary spacetime-dependent scalar function. This fact can be proved as follows:

$$\begin{aligned} F^{\mu'\nu'} &= \partial^{\mu}[U^{\nu} - \partial^{\nu}\chi] - \partial^{\nu}[U^{\mu} - \partial^{\mu}\chi] \\ &= [\partial^{\mu}U^{\nu} - \partial^{\nu}U^{\mu}] = F^{\mu\nu}. \end{aligned} \qquad (4.25)$$

The invariance of $F^{\mu\nu}$ under the transformation of U^{μ} according to Eq.(4.24) is generally known as *gauge invariance*. The transformation law Eq.(4.24) is called the *gauge transformation*. The tensor field $F^{\mu\nu}$ is not only invariant under inertial transformation but also under gauge transformation.

Wave Equations

Among the most important consequences of the fundamental field equations are the existence of *wave equations*, which show that vector fields can propagate through space in the form of waves. For convenience, we will consider the region that both source J^{ν} and its dual $^{*}J^{\nu}$ are vanished, in which the field equations become

$$\nabla\cdot\mathbf{P}(\mathbf{x},t) = 0, \qquad (4.26a)$$

$$\nabla\times\mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) = 0, \qquad (4.26b)$$

$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = 0, \qquad (4.26c)$$

$$\nabla\times\mathbf{P}(\mathbf{x},t) + \frac{1}{k}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = 0. \qquad (4.26d)$$

If we take the curl of Eq.(4.26b) and make use of Eqs.(4.26c) and (4.26d), we find

$$\nabla^2\mathbf{A}(\mathbf{x},t) - \frac{1}{k^2}\frac{\partial^2\mathbf{A}(\mathbf{x},t)}{\partial t^2} = 0, \qquad (4.27a)$$

where the vector identity, $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$, is employed. Taking now the curl of Eq.(4.26d) and using Eqs.(4.26a) and (4.26b), we have

$$\nabla^2\mathbf{P}(\mathbf{x},t) - \frac{1}{k^2}\frac{\partial^2\mathbf{P}(\mathbf{x},t)}{\partial t^2} = 0. \qquad (4.27b)$$

The equations (4.27a) and (4.27b) have the basic form of wave equations in empty space,

$$\nabla^2\Psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{x},t)}{\partial t^2} = 0, \qquad (4.28)$$

where the factor c is the velocity of propagation in the medium. Thus, Eqs.(4.28a) and (4.28b) imply directly the existence of wave equations for vector fields **A** and **P** in empty space with velocity of propagation k. Therefore, the universal constant parameter k, which we have proposed in Chapter III to make the components of four-vector J^ν be equivalent, is exactly the velocity of propagation of vector fields, **P** and **A**, whose source are J^ν and $^*J^\nu$, in empty space. The propagating velocity k is universal and must be unique if we want our inertial transformation unique. As a result, we can conclude that, if there exists vector fields, whose sources are some conserved quantities, the propagating velocity of these fields must be the same, equal our universal constant velocity k.

Electromagnetic Fields

Before the invention of special relativity in 1905, no one knew that the electromagnetic field theory, which we have shown in Chapter I, could be deduced directly from the concept of tensor field in four-dimension of space and time. We know for a long time that *electric charge* is conserved then it can play a role as the source of some tensor field in four-space, as we have discussed earlier. The fields related to the electric charge are generally called the *electromagnetic fields*. These fields in three-space will appear as the constitution of two vector fields called the electric field **E** and the magnetic field **B**. The derivation of these two fields can be shown as follows.

Because we know that electric charge is conserved globally, then it is also conserved locally, so the continuity equation in four-space exists,

$$\partial_\nu J_e^{\ \nu} = 0, \tag{4.29}$$

where $J_e^{\ \nu}$ is the electric current-density four-vector defined as $J_e^{\ \nu} \equiv (k\rho_e \mathbf{j}_e)$. Then the relation of tensor field $F^{\mu\nu}$ corresponding to source $J_e^{\ \nu}$ can be shown as

$$\partial_\mu F^{\mu\nu} = \alpha J_e^{\ \nu}. \tag{4.30}$$

By considering Eq.(4.29), we find that tensor field $F^{\mu\nu}$ should be a second-rank antisymmetric tensor of which components can be presented in matrix form as:

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & -F^{10} & -F^{20} & -F^{30} \\ F^{10} & 0 & -F^{21} & -F^{31} \\ F^{20} & F^{21} & 0 & -F^{32} \\ F^{30} & F^{31} & F^{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} , \tag{4.31}$$

where we have introduce the two field vectors in three-space **E** and **B**, defined as

$$\mathbf{E} \equiv (E_1, E_2, E_3) = (F^{10}, F^{20}, F^{30}), \tag{4.32a}$$
$$\mathbf{B} \equiv (B_1, B_2, B_3) = (F^{32}, -F^{31}, F^{21}). \tag{4.32b}$$

The field vector **E** is called *electric* field vector while the vector field **B** is called *magnetic* field vector, both are real fields appear in three-space.

The content of Eq.(4.30) can now be seen since $F^{\mu\nu}$ is an antisymmetric tensor. For $\nu = 0$, Eq.(4.30) gives

$$\nabla\cdot\mathbf{E}(\mathbf{x},t) = \alpha k\rho_e(\mathbf{x},t), \qquad (4.33a)$$

while the vector components (ν=1,2,3) of Eq.(4.30) give the law

$$\nabla\times\mathbf{B}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t} = \alpha\,\mathbf{j}_e(\mathbf{x},t), \qquad (4.33b)$$

if Eq.(4.32) is realized. As we have said in the last section that, for every antisymmetric second-rank field tensor $F^{\mu\nu}$ in four-space, there may be its dual field $^*F^{\mu\nu}$ exists in nature. This dual field is related to $F^{\mu\nu}$ through the equation

$$^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \qquad (4.34)$$

where the definitions of $\varepsilon^{\mu\nu\alpha\beta}$ and $F_{\alpha\beta}$ are explained in Eqs.(2.35) and (4.11), respectively.

Therefore, there exists source for this dual field called *magnetic charge*, defined as $J_m^{\ \nu} \equiv (k\rho_m\mathbf{j}_m)$, then the dual field tensor can be shown as

$$\partial_\mu{}^*F^{\mu\nu} = \alpha J_m^{\ \nu}. \qquad (4.35)$$

Then, we find from Eq.(4.34) that the magnetic charge is also conserved locally,

$$\alpha\partial_\nu J_m^{\ \nu} = \partial_\nu\partial_\mu{}^*F^{\mu\nu} = 0, \qquad (4.36)$$

as it should be. For ν=0, the components of Eq.(4.35) become

$$\nabla\cdot\mathbf{B}(\mathbf{x},t) = \alpha k\rho_m(\mathbf{x},t), \qquad (4.37a)$$

and for $\nu=1,2$, and 3, they become

$$-\nabla \times \mathbf{E}(\mathbf{x},t) - \frac{1}{k}\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} = \alpha \mathbf{j}_m(\mathbf{x},t), \qquad (4.37b)$$

where $\mathbf{j}_m = (J_m{}^1, J_m{}^2, J_m{}^3)$ is the magnetic current density. The constitution of Eqs.(4.33) and (4.37),

$$\nabla \cdot \mathbf{E}(\mathbf{x},t) = \alpha k \rho_e(\mathbf{x},t), \qquad (4.38a)$$

$$\nabla \times \mathbf{B}(\mathbf{x},t) - \frac{1}{k}\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} = \alpha \mathbf{j}_e(\mathbf{x},t), \qquad (4.38b)$$

$$\nabla \cdot \mathbf{B}(\mathbf{x},t) = \alpha k \rho_m(\mathbf{x},t), \qquad (4.38c)$$

$$-\nabla \times \mathbf{E}(\mathbf{x},t) - \frac{1}{k}\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} = \alpha \mathbf{j}_m(\mathbf{x},t), \qquad (4.38d)$$

give the complete set of *electromagnetic field equations*, if sources being the electric charge and magnetic charge. By comparison with the fundamental field equations in Eq.(4.26), we can conclude that the electric field and the magnetic field must be polar field and axial field respectively. Similarly, if electric charge is a conserved scalar, then the magnetic charge must be a conserved pseudo scalar. At this stage, the Maxwell equations, which can be written as:

$$\nabla \cdot \mathbf{E}(\mathbf{x},t) = 4\pi \rho_e(\mathbf{x},t), \qquad (4.39a)$$

$$\nabla \times \mathbf{B}(\mathbf{x},t) - \frac{1}{k}\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} = \frac{4\pi \mathbf{j}_e(\mathbf{x},t)}{k}, \qquad (4.39b)$$

$$\nabla \cdot \mathbf{B}(\mathbf{x},t) = 0, \qquad (4.39c)$$

$$\nabla \times \mathbf{E}(\mathbf{x},t) + \frac{1}{k}\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} = 0, \qquad (4.39d)$$

can be easily obtained as a special case of Eq.(4.38) when the magnetic charge is vanished and the scale factor α is set to be $4\pi/k$ (for Gaussian units).

To find out the usual definitions of **E** and **B** in names of Coulomb's law and Biot-Sawart law, we have to consider the special case of Maxwell equations, Eq.(4.39). For the electrostatic case, $\mathbf{j}_e \equiv (J_e^1, J_e^2, J_e^3) = 0$ and **B** equals zero, we find that

$$\nabla.\mathbf{E}(\mathbf{x},t) = 4\pi\rho_e(\mathbf{x},t), \tag{4.40a}$$

$$\nabla\mathrm{x}\mathbf{E}(\mathbf{x},t) = 0, \tag{4.40b}$$

notice that the right-hand side of both equations are not depend explicitly on time so we can write $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})$. These two equations can be readily solved, by using Helmholtz's theorem shown in Appendix C, to give **E** in integral form as

$$\mathbf{E}(\mathbf{x}) = -\nabla \int_v \frac{\nabla'\cdot\mathbf{E}(\mathbf{x}')\, d^3x'}{4\pi\,|\mathbf{x}-\mathbf{x}'|}, \tag{4.41a}$$

or,

$$\mathbf{E}(\mathbf{x}) = \int_v \rho_e(\mathbf{x}')\, \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x', \tag{4.41b}$$

where the vector notation $-\nabla \dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = \dfrac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$ is used. Eq.(4.41b) is generally called Coulomb's law for static field $\mathbf{E}(\mathbf{x})$.

To obtain Biot-Savart law, we consider the steady-state magnetic phenomena which are characterized by no charge in the net charge density anywhere in space. Consequently, in magnetostatics,

$$\nabla \cdot \mathbf{j}_e(\mathbf{x},t) = 0. \tag{4.42}$$

By taking this condition and the fact that **E** is vanished, the Maxwell equations are reduced to be

$$\nabla \cdot \mathbf{B}(\mathbf{x},t) = 0, \tag{4.43a}$$

$$\nabla \times \mathbf{B}(\mathbf{x},t) = (4\pi/k)\mathbf{j}_e(\mathbf{x},t). \tag{4.43b}$$

Again, by using Helmholtz's theorem, we can derive **B** in integral form as

$$\mathbf{B}(\mathbf{x}) = \frac{1}{k}\int_V \mathbf{j}_e(\mathbf{x}') \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'. \tag{4.44}$$

This equation is known as the Biot-Savart law for magnetostatic field **B(x)**. Coulomb's law and Biot-Savart law were originally derived from empirical facts, but, in fact, they are directly obtained from logical process as we have done above.