ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของกึ่งกรุปการแปลงนัยทั่วไปและริงของการแปลงเชิงเส้นนัยทั่วไป

นาย รณสรรพ์ ชินรัมย์

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MINIMAL QUASI-IDEALS OF GENERALIZED TRANSFORMATION SEMIGROUPS AND GENERALIZED RINGS OF LINEAR TRANSFORMATIONS

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รณสรรพ์ ชินรัมย์ :ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของกึ่งกรุปการแปลงนัยทั่วไปและริงของการแปลงเชิงเส้นนัยทั่วไป (MINIMAL QUASI-IDEALS OF GENERALIZED TRANSFORMATION SEMIGROUPS AND GENERALIZED RINGS OF LINEAR TRANSFORMATIONS)
อ.ที่ปรึกษา : ศ.ดร. ยุพาภรณ์ เข็มประสิทธิ์, **83 หน้า, ISBN 974-17-6105-8**

เราเรียกกึ่งกรุปย่อย Q ของกึ่งกรุป S ว่า *ควอซี-ไอดีล* ของ S ถ้า $SQ \cap QS \subseteq Q$ *ควอซี-ไอดีล* ของริง R คือ ริงย่อย Q ของ R ซึ่ง $RQ \cap QR \subseteq Q$ โดย RQ $[QR]$ เป็นเซตของผลบวกจำกัดทั้งหมดในรูปแบบ $\sum r_i q_i$ $[\sum q_i r_i]$, $r_i \in R$ และ $q_i \in Q$ โอ สไตน์เฟลด์ ได้แนะนำความคิดเกี่ยวกับควอซี-ไอดีล ในปี 1953 และ 1956 สำหรับริงและกึ่งกรุป ตามลำดับ *ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่ม* ของกึ่งกรุป S คือ ควอซี-ไอดีลที่ไม่ใช่ศูนย์ของ S และ ไม่บรรจุควอซี-ไอดีลที่ไม่ใช่ศูนย์อื่นๆ ของ S *ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่ม* ของริงมีบทนิยามในทำนองเดียวกัน ในปี 1956 โอ สไตน์เฟลด์ให้ลักษณะของควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของกึ่งกรุปที่ไม่มีศูนย์ดังนี้ ควอซี-ไอดีล Q ของกึ่งกรุป S ที่ไม่มีศูนย์เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่ม ก็ต่อเมื่อ Q เป็นกรุปย่อยของ S ในปี 1957 เราได้แสดงว่าควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของริง [กึ่งกรุปที่มีศูนย์] A ไม่เป็นริงย่อยการหาร [กรุปย่อยที่มีศูนย์] ก็เป็นริงศูนย์ [กึ่งกรุปศูนย์] ของ A และในกรณีแรก บทกลับเป็นจริง

เราให้นัยทั่วไปของกึ่งกรุปและริงที่สำคัญหลากหลายโดยใช้การคูณแบบแซนด์วิช กึ่งกรุปและริงเหล่านี้คือ กึ่งกรุปการแปลง กึ่งกรุปการแปลงเชิงเส้น กึ่งกรุปเมทริกซ์ ริงของการแปลงเชิงเส้นและเมทริกซ์ริง ในการวิจัยนี้เราให้ลักษณะควอซี-ไอดีลเล็กสุดเฉพาะของกึ่งกรุปและริงนัยทั่วไปที่ตั้งเป้าหมายไว้อย่างสมบูรณ์

ภาควิชา คณิตศาสตร์	ลายมือชื่อนิสิต ....
สาขาวิชา คณิตศาสตร์	ลายมือชื่ออาจารย์ที่ปรึกษา..
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RONNASON CHINRAM : MINIMAL QUASI-IDEALS OF GENERALIZED TRANSFORMATION SEMIGROUPS AND GENERALIZED RINGS OF LINEAR TRANSFORMATIONS. THESIS ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 83 pp. ISBN 974-17-6105-8

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. A *quasi-ideal* of a ring R is a subring Q of R such that $RQ \cap QR \subseteq Q$ where RQ [QR] is the set of all finite sums of the form $\sum r_i q_i$ [$\sum q_i r_i$], $r_i \in R$ and $q_i \in Q$. The notion of quasi-ideal was introduced by O. Stienfeld in 1953 and 1956 for rings and semigroups, respectively. By a *minimal quasi-ideal* of a semigroup S we mean a nonzero quasi-ideal of S which does not properly contain any nonzero quasi-ideal of S. A *minimal quasi-ideal* of a ring is defined similarly. In 1956, O. Stienfeld characterized minimal quasi-ideals of a semigroup without zero as follows : A quasi-ideal Q of a semigroup S without zero is minimal if and only if Q is a subgroup of S. Also in 1957, he showed that a quasi-ideal of a ring [semigroup with zero] A is either a division subring [subgroup with zero] or a zero subring [zero subsemigroup] of A, and for the first case, the converse holds.

Various important semigroups and rings are generalized by using sandwich multiplication. These semigroups and rings are as follows : transformation semigroups, linear transformation semigroups, matrix semigroups, rings of linear transformations and matrix rings. Minimal quasi-ideals of our target generalized semigroups and generalized rings are completely characterized in this research.

Department Mathematics	Student's signature Ronnason Chinram
Field of study Mathematics	Advisor's signature Yupaporn Kemprasit
Academic year 2004	Co-advisor's signature –

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

In this introductory chapter, we present a number of elementary concepts, notations and propositions on semigroups and rings which will be used for this research.

Let $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers (positive integers), the set of integers and the set of real numbers. For any set X, let $|X|$ denote the cardinality of X.

A semigroup S with zero 0 is called a *zero semigroup* if $xy = 0$ for all $x, y \in S$.

A *subgroup* of a semigroup S is a subsemigroup of S which is also a group. A *group with zero* is a semigroup S with zero 0 such that $S \setminus \{0\}$ is a subgroup of S.

By a *nonzero subsemigroup* of a semigroup S we mean a subsemigroup of S if S has no zero and a subsemigroup T with $T \neq \{0\}$ if S has a zero 0. A *nonzero element* of a semigroup is considered analogously.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$.

For nonempty subsets X and Y of a ring R, let XY denote the set of all finite sums of the form $\sum x_i y_i$ where $x_i \in X$ and $y_i \in Y$. We note here that for $x \in R$, $Rx = \{rx \mid r \in R\}$ and $xR = \{xr \mid r \in R\}$. Also, for a nonempty subset X of a ring $R = (R, +, \cdot)$, let $\mathbb{Z}X$ be the set of all finite sums of a form $\sum k_i x_i$ where $k_i \in \mathbb{Z}$ and $x_i \in X$. It then follows that $\mathbb{Z}X$ is a subgroup of $(R, +)$ for every nonempty subset X of R.

A subring Q of a ring R is called a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$.

The notion of quasi-ideal for rings and semigroups was introduced by O. Steinfeld in 1953 [16] and 1956 [17], respectively. Every left ideal and every right ideal of a semigroup or a ring is clearly a quasi-ideal. The following example shows that the converse is not generally true. Then quasi-ideals of both semigroups and rings are generalizations of one-sided ideals.

Example 1.1. Let n be a positive integer greater than 1 and $M_n(F)$ the set of all $n \times n$ matrices over a field F. For $k, l \in \{1, 2, \ldots, n\}$, let

$$Q_n^{(kl)}(F) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ if } i \neq k \text{ or } l \neq j\}.$$

where A_{ij} denotes the entry of the matrix A in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column. Then for $k, l \in \{1, 2, \ldots, n\}$,

$$M_n(F)Q_n^{(kl)}(F) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ if } j \neq l\}, \tag{1}$$

$$Q_n^{(kl)}(F)M_n(F) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ if } i \neq k\} \tag{2}$$

in both the semigroup $(M_n(F), \cdot)$ and the ring $(M_n(F), +, \cdot)$ where $+$ and $\cdot$ are the usual addition and multiplication of matrices. From (1) and (2), we clearly have

$$M_n(F)Q_n^{(kl)}(F) \cap Q_n^{kl}(F)M_n(F) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ if } i \neq k \text{ or } j \neq l\} = Q_n^{kl}(F)$$

in both $(M_n(F), \cdot)$ and $(M_n(F), +, \cdot)$. Also, we can see from (1) and (2) that $Q_n^{(kl)}(F)$ is neither a left ideal nor a right ideal of the semigorup $(M_n(F), \cdot)$ and the ring $(M_n(F), +, \cdot)$ if $n > 1$.

It is known that the intersection of any set of quasi-ideals of a semigroup S is either empty or a quasi-ideal of S ([19], page 10). Also, the intersection of any set of quasi-ideals of a ring [semigroup with zero] A is a quasi-ideal of A ([19], page 10).

In particular, if H and K are a left ideal and a right ideal of a semigroup [ring] A, then $H \cap K$ is a quasi-ideal of A, and hence $AX \cap XA$ is a quasi-ideal of A for every nonempty subset X of A.

For a nonempty subset X of a semigroup [ring] A, let $(X)_q$ denote the quasi-ideal of A generated by X, that is, $(X)_q$ is the intersection of all quasi-ideals of A containing X. The following two facts are well-known. The second one was given by H. J. Weinert [22].

Theorem 1.2 ([19], page 85). *For a nonempty subset X of a semigroup S,*

$$(X)_q = (SX \cap XS) \cup X.$$

In particular, $(X)_q = SX \cap XS$ if S has an identity.

Theorem 1.3 ([22]). *For a nonempty subset X of a ring R,*

$$(X)_q = \mathbb{Z}X + RX \cap XR.$$

By a *minimal quasi-ideal* of a semigroup S (with or without zero) we mean a nonzero quasi-ideal of S which does not properly contain a nonzero quasi-ideal of S. A *minimal quasi-ideal* of a ring is defined analogously. For convenience, a trivial semigroup [ring] will be considered as the (unique) minimal quasi-ideal of itself.

The following remark is obvious but it is one of our main tools for this research.

Remark 1.4. A nonzero quasi-ideal Q of a semigroup [ring] A is minimal if and only if $(x)_q = Q$ for every nonzero element x of Q.

Remark 1.4 shows that every minimal quasi-ideal of a semigroup or a ring is principal, that is, it is generated by one element.

In 1956, O. Steinfeld [17] showed that a minimal quasi-ideal of a semigroup S without zero must be a subgroup of S and a quasi-ideal of S which is also a subgroup of S must be minimal. This result can be seen in [19], the book written by O. Steinfeld in 1978.

Theorem 1.5 ([19], page 27). *A quasi-ideal Q of a semigroup S without zero is minimal if and only if Q is a subgroup of S.*

Necessary conditions for any minimal quasi-ideal of a semigroup with zero were proved originally by O. Steinfeld [17] in 1956 which also appear in [19] as follows:

Theorem 1.6 ([19], page 35). *A minimal quasi-ideal of a semigroup S with zero is either a zero subsemigroup or a subgroup with zero of S.*

Theorem 1.6 has a partial converse as follows:

Theorem 1.7 ([19], page 37). *If a quasi-ideal of a semigroup S with zero is a subgroup with zero of S, then Q is a minimal quasi-ideal of S.*

Necessary conditions for any minimal quasi-ideal of a ring were given by O. Steinfeld [17] in 1956. These can be seen in [19] as follows:

Theorem 1.8 ([19], page 35). *A minimal quasi-ideal Q of a ring R is either a zero subring or a division subring of R.*

A partial converse of Theorem 1.8 is obtained similarly as above.

Theorem 1.9 ([19], page 37). *If a quasi-ideal Q of a ring R is a division subring of R, then Q is a minimal quasi-ideal of R.*

Example 1.10. Let $Q_n^{(kl)}(F)$ be defined as in Example 1.1. Recall that $Q_n^{(kl)}(F)$ is a quasi-ideal of both the semigroup $(M_n(F), \cdot)$ and the ring $(M_n(F), +, \cdot)$. Let $A \in Q_n^{(kl)}(F)$ be a nonzero matrix, that is, $A_{kl} \neq 0$. Since F is a field, $FA_{kl} = F = A_{kl}F$. This implies that

$$M_n(F)A = \{B \in M_n(F) \mid B_{ij} = 0 \text{ if } j \neq l\},$$

$$AM_n(F) = \{B \in M_n(F) \mid B_{ij} = 0 \text{ if } i \neq k\},$$

and hence

$$M_n(F)A \cap AM_n(F) = Q_n^{(kl)}(F) \tag{1}$$

in both $(M_n(F), \cdot)$ and $(M_n(F), +, \cdot)$. In the ring $(M_n(F), +, \cdot), \mathbb{Z}A \subseteq Q_n^{(kl)}(F)$, so

$$\mathbb{Z}A + M_n(F)A \cap AM_n(F) = \mathbb{Z}A + Q_n^{(kl)}(F) = Q_n^{(kl)}(F). \tag{2}$$

Since A is an abitrary nonzero matrix in $Q_n^{(kl)}(F)$, it follows from (1), Theorem 1.2 and Remark 1.4 that $Q_n^{(kl)}(F)$ is a minimal quasi-ideal of the semigroup $(M_n(F), \cdot)$. Also, from (2), Theorem 1.3 and Remark 1.4, $Q_n^{(kl)}(F)$ is a minimal quasi-ideal of the ring $(M_n(F), +, \cdot)$. Observe that if $k = l$, then

$$Q_n^{(kk)}(F) \cong (F, \cdot) \text{ in } (M_n(F), \cdot)$$

and

$$Q_n^{(kk)}(F) \cong (F, +, \cdot) \text{ in } (M_n(F), +, \cdot)$$

which imply that $Q_n^{(kk)}(F)$ is a subgroup with zero of $(M_n(F),\cdot)$ and it is a division subring of $(M_n(F),+,\cdot)$. If $k \neq l$, then it is clear that $Q_n^{(kl)}$ is a zero subsemigroup of $(M_n(F),\cdot)$ and it is a zero subring of $(M_n(F),+,\cdot)$.

Theorem 1.2 and Remark 1.4 are main tools for our works in Chapter II and Chapter III. Theorem 1.3 and Remark 1.4 are useful to obtain interesting results in Chapter IV. In this research, the explicit forms of minimal quasi-ideals $(\alpha)_q$ are provided in terms of α. We know respectively from Theorem 1.5, Theorem 1.6 and Theorem 1.8 that a minimal quasi-ideal of a semigroup without zero is a subgroup, a minimal quasi-ideal of a semigroup S with zero is either a zero subsemigroup or a subgroup with zero of S and a minimal quasi-ideal of a ring R is either a zero subring or a division subring of R.

For a map α from a set into a set, let the domain and the range (image) of α be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively; and the *rank* of α is $|\operatorname{ran}\alpha|$ which is denoted by $\operatorname{rank}\alpha$.

For a set X, let P_X, T_X, I_X and G_X denote respectively the partial transformation semigroup on X, the full transformation semigroup on X, the one-to-one partial transformation on X (the symmetric inverse semigroup on X) and the symmetric group on X (the permutation group on X). By a *transformation* of X we mean a map from X into X while a *transformation semigroup* on X we mean a subsemigroup of P_X. Also, we let M_X and E_X denote the subsemigroups of T_X defined respectively by

$$M_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}, E_X = \{\alpha \in T_X \mid \operatorname{ran}\alpha = X\}.$$

For $\alpha \in T_X$, α is said to be *one-to-one* at $x \in X$ if $(x\alpha)\alpha^{-1} = \{x\}$ and we call α *almost one-to-one* if the set $\{x \in X \mid \alpha \text{ is not one-to-one at } x\}$ is finite. Let AM_X be the set of all almost one-to-one transformations of X. Then $M_X \subseteq AM_X \subseteq$

T_X. It is easy to verify that for $\alpha, \beta \in T_X$, $A(\alpha\beta) \subseteq A(\alpha) \cup (A(\beta))\alpha^{-1}$ where $A(\gamma) = \{x \in X \mid \gamma \text{ is not one-to-one at } x\}$ for every $\gamma \in T_X$. This implies that AM_X is a subsemigroup of T_X. It then follows that AM_X is a subsemigroup of T_X containing M_X. A transformation $\alpha \in T_X$ is said to be *almost onto* if $|X \smallsetminus \text{ran } \alpha| < \infty$. It is easily seen that $X \smallsetminus \text{ran } \alpha\beta \subseteq (X \smallsetminus \text{ran } \beta) \cup (X \smallsetminus \text{ran } \alpha)\beta$ for all $\alpha, \beta \in T_X$. Then the set AE_X of all almost onto transformations of X is a subsemigroup of T_X containing E_X. Note that if X is finite, then $M_X = E_X = G_X$ and $AM_X = AE_X = T_X$. In fact, $M_X[E_X] = G_X$ or $AM_X[AE_X] = T_X$ implies that X must be finite. To see this, suppose that X is infinite. Let X_1 and X_2 be subsets of X such that $X = X_1 \cup X_2, X_1 \cap X_2 = \varnothing$ and $|X_1| = |X_2| = |X|$. Then there is a bijection $\alpha : X \to X_1$. Hence $X \smallsetminus \text{ran } \alpha = X \smallsetminus X_1 = X_2$. Thus $\alpha \in M_X \smallsetminus G_X$ and $\alpha \in T_X \smallsetminus AE_X$. If $a \in X_2$ is fixed and define $\beta : X \to X$ by

$$x\beta = \begin{cases} x\alpha^{-1} & \text{if } x \in X_1, \\ a & \text{if } x \in X_2, \end{cases}$$

then $(x\beta)\beta^{-1} = a\beta^{-1} = X_2 \cup a\alpha^{-1}$ for every $x \in X_2$ which implies that $\beta \in E_X \smallsetminus G_X$ and $\beta \in T_X \smallsetminus AM_X$.

Next, let X be an infinite set,

$$BL_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one and } X \smallsetminus \text{ran } \alpha \text{ is infinite}\} \text{ and}$$

$$OBL_X = \{\alpha \in T_X \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.$$

Let X_1 and X_2 be subsets of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$ and $|X_1| = |X| = |X_2|$. Then there is a bijection $\alpha : X \to X_1$. Thus $X \smallsetminus \text{ran } \alpha = X_2$ which is infinite, so $\alpha \in BL_X$. We shall show that $OBL_X \neq \varnothing$. Since $|X \times X| = |X|$, there is a bijection $\varphi : X \to X \times X$. Then

$$X = \bigcup_{x \in X} (\{x\} \times X)\varphi^{-1} \text{ which is a disjoint union,} \tag{1}$$

$$(\{x\} \times X)\varphi^{-1} \text{ is infinite for every } x \in X. \tag{2}$$

Define $\beta : X \to X$ by

$$((\{x\} \times X)\varphi^{-1})\beta = \{x\} \text{ for every } x \in X. \tag{3}$$

From (1), β is well-defined and onto. Moreover, if $a \in X$, then by (1), $a \in (\{b\} \times X)\varphi^{-1}$ for some $b \in X$, and thus from (3), we have

$$(a\beta)\beta^{-1} = (((\{b\} \times X)\varphi^{-1})\beta)\beta^{-1} = b\beta^{-1} = (\{b\} \times X)\varphi^{-1}$$

which is infinite by (2). Consequently, $\beta \in OBL_X$.

For $\alpha, \beta \in T_X$, ran $\alpha\beta \subseteq$ ran β and

$$(x\alpha\beta)(\alpha\beta)^{-1} = (x\alpha\beta)\beta^{-1}\alpha^{-1} \supseteq (x\alpha)\alpha^{-1} \text{ for every } x \in X.$$

Therefore we have that BL_X and OBL_X are subsemigroups of T_X. If X is countably infinite, BL_X is called the *Baer-Levi semigroup* on X ([3], page 14). The semigroup OBL_X may be considered as the "opposite semigroup" of BL_X.

In 1975, R. P. Sullivan [20] generalized transformation semigroups which are called *generalized transformation semigroups* in this paper as follows: If X and Y are any two sets, let $P(X,Y)$ denote the set of all mappings with domain in X and range in Y, that is,

$$P(X,Y) = \{\alpha : A \to Y \mid A \subseteq X\}.$$

Note that $0 \in P(X,Y)$ where 0 is the empty transformation. A *generalized transformation semigroup* of X into Y is a semigroup $(S(X,Y), \theta)$ where $S(X,Y)$ is a nonempty subset of $P(X,Y)$ and $\theta \in P(Y,X)$ with $\alpha\theta\beta \in S(X,Y)$ for all $\alpha, \beta \in S(X,Y)$ and the operation on $S(X,Y)$ is $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S(X,Y)$. Note that in [20], this system was called a *generalized partial transformation semigroup.* The notation $T(X,Y)$ will denote the set $\{\alpha \in P(X,Y) \mid \text{dom } \alpha = X\}$. The definition of *almost one-to-one maps* and *almost onto maps* in $T(X,Y)$ are given analogously as above. Then $P(X,X) = P_X$ and $T(X,X) = T_X$. The notations $I(X,Y), G(X,Y), M(X,Y), E(X,Y), AM(X,Y)$ and $AE(X,Y)$ are defined analogously, and we then have $I(X,X) = I_X$, $G(X,X) = G_X$, $M(X,X) = M_X$, $E(X,X) = E_X$, $AM(X,X) = AM_X$ and $AE(X,X) = AE_X$. Moreover, for $\theta \in G_X, (G_X, \theta)$ is a group having θ^{-1} as its identity. We note that $AM(X,Y) = T(X,Y)$ if $|X| < \infty$ and $AE(X,Y) = T(X,Y)$ if $|Y| < \infty$.

Clearly, if $M(X,Y) \neq \varnothing \neq M(Y,X)$, then the semigroup $(M(X,Y), \theta)$ where $\theta \in M(Y,X)$ is defined, and so is the semigroup $(E(X,Y), \theta)$ where $\theta \in E(Y,X)$.

For a map $\alpha : P \to Q$, α is said to *one-to-one* at $x \in P$ if $(x\alpha)\alpha^{-1} = \{x\}$, and also let $A(\alpha)$ be the set $\{x \in P \mid \alpha \text{ is not one-to-one at } x\}$. We can verify directly that if $\alpha : P \to Q$ and $\beta : Q \to Z$ are maps, then

$$A(\alpha\beta) \subseteq A(\alpha) \cup (A(\beta))\alpha^{-1} \tag{1}$$

and

$$Z \setminus \text{ ran } \alpha\beta \subseteq (Z \setminus \text{ ran } \beta) \cup (Q \setminus \text{ ran } \alpha)\beta. \tag{2}$$

From (1) and (2), we have respectively that if $AM(X,Y) \neq \varnothing \neq AM(Y,X)$, then the semigroup $(AM(X,Y), \theta)$ where $\theta \in AM(Y,X)$ is defined and so is the semigroup $(AE(X,Y), \theta)$ where $\theta \in AE(Y,X)$.

For a nonempty subset A of X and $y \in Y$, let A_y denote the element of $P(X,Y)$ with domain A and range $\{y\}$. A subset $S(X,Y)$ of $P(X,Y)$ is said to *cover* X and Y if for every pair $(x,y) \in X \times Y$, there exists an element $A_y \in S(X,Y)$ for some nonempty subset A of X containing x. Our definitions of A_y and the word "cover" are motivated by the paper written by R. P. Sullivan [21].

Observe that if $S(X,Y)$ is $P(X,Y)$ or $I(X,Y)$, then we always obtain the corresponding semigroup $(S(X,Y),\theta)$ where $\theta \in S(Y,X)$ and $S(X,Y)$ covers X and Y. Also, it is clear that if both X and Y are either empty or nonempty, then we obtain the semigroup $(T(X,Y),\theta)$ where $\theta \in T(Y,X)$ and $T(X,Y)$ covers X and Y.

To generalize the semigroups BL_X and OBL_X, let X and Y be infinite sets,

$$BL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } Y \smallsetminus \operatorname{ran}\alpha \text{ is infinite}\},$$

$$OBL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.$$

If $\theta \in M(Y,X)$ and $\alpha,\beta \in BL(X,Y)$, then $\alpha\theta\beta$ is one-to-one and $Y \smallsetminus \operatorname{ran}\alpha\theta\beta \supseteq Y \smallsetminus \operatorname{ran}\beta$. Also, if $\theta \in E(Y,X)$ and $\alpha,\beta \in OBL(X,Y)$, then $\alpha\theta\beta$ is onto and for every $x \in X, (x\alpha\theta\beta)(\alpha\theta\beta)^{-1} = ((x\alpha\theta\beta)\beta^{-1}\theta^{-1})\alpha^{-1} \supseteq (x\alpha)\alpha^{-1}$. Then we can define the semigroup $(BL(X,Y),\theta)$ where $\theta \in M(Y,X)$ if $BL(X,Y) \neq \varnothing$ and $M(Y,X) \neq \varnothing$ and the semigroup $(OBL(X,Y),\theta)$ where $\theta \in E(Y,X)$ if $OBL(X,Y) \neq \varnothing$ and $E(Y,X) \neq \varnothing$.

The aim of Chapter II is to characterize minimal quasi-ideals of all the semigroups $(S(X,Y),\theta)$ introduced above when they are defined.

Next, we shall define linear transformation semigroups corresponding to transformation semigroups defined previously. If α is a linear transformation between two vector spaces over a field, we shall use the notation $\operatorname{Im}\alpha$, not $\operatorname{ran}\alpha$, to denote the image (range) of α. Let V be a vector space over a field F. Let $L_F(V)$ be the semigroup under composition of all linear transformations $\alpha : V \to V$,

$$G_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is an isomorphism }\},$$

$$M_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is one-to-one }\},$$

$$E_F(V) = \{\alpha \in L_F(V) \mid \operatorname{Im}\alpha = V\},$$

$$AM_F(V) = \{\alpha \in L_F(V) \mid \dim_F \operatorname{Ker}\alpha \text{ is finite }\} \text{ and}$$

$$AE_F(V) = \{\alpha \in L_F(V) \mid \dim_F (V/\operatorname{Im}\alpha) \text{ is finite }\}.$$

Then $G_F(V) \subseteq M_F(V) \subseteq AM_F(V)$ and $G_F(V) \subseteq E_F(V) \subseteq AE_F(V)$. We have that $M_F(V), E_F(V), AM_F(V)$ and $AE_F(V)$ are subsemigroups of $L_F(V)$. Note that $AM_F(V)$ and $AE_F(V)$ are subsemigroups of $L_F(V)$ because for $\alpha, \beta \in L_F(V)$,

$$\dim_F \operatorname{Ker}\alpha\beta \leqslant \dim_F \operatorname{Ker}\alpha + \dim_F \operatorname{Ker}\beta, \tag{1}$$

$$\dim_F (V/\operatorname{Im}\alpha\beta) \leqslant \dim_F (V/\operatorname{Im}\alpha) + \dim_F (V/\operatorname{Im}\beta). \tag{2}$$

The proofs of these two inequlities can be seen in [14]. We know that $G_F(V)$ is a group under composition. The semigroups $AM_F(V)$ and $AE_F(V)$ can be referred to respectively as the semigroup of all "almost one-to-one linear transformations" of V and the semigroup of all "almost onto linear transformations" of V. Observe that if $\dim_F V$ is finite, then $M_F(V) = G_F(V) = E_F(V)$ and $AM_F(V) = L_F(V) = AE_F(V)$. Moreover, if $\dim_F V$ is infinite, then $G_F(V) \neq M_F(V)$ $[E_F(V)]$ and $AM_F(V)$ $[AE_F(V)] \neq L_F(V)$. Assume that $\dim_F V$ is infinite and B is a basis of V. Let B_1 and B_2 be subsets of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let φ be a bijection from B onto B_1 and define $\alpha, \beta \in L_F(V)$ by

$$v\alpha = v\varphi \text{ for every } v \in B \text{ and } v\beta = \begin{cases} v\varphi^{-1} & \text{if } v \in B_1, \\ 0 & \text{if } v \in B_2. \end{cases}$$

Then $\operatorname{Ker}\alpha = \{0\}$, $\operatorname{Im}\alpha = \langle B_1 \rangle \neq V$, $\dim_F(V/\operatorname{Im}\alpha) = |\{v + \langle B_1 \rangle \mid v \in B_2\}| = |B_2| = |B|$, $\operatorname{Ker}\beta = \langle B_2 \rangle$, $\dim_F \operatorname{Ker}\beta = |B_2| = |B|$ and $\operatorname{Im}\beta = \langle B \rangle = V$. Therefore $\alpha \in M_F(V) \smallsetminus G_F(V)$, $\alpha \in L_F(V) \smallsetminus AE_F(V)$, $\beta \in E_F(V) \smallsetminus G_F(V)$ and $\beta \in L_F(V) \smallsetminus AM_F(V)$.

Let V be an infinite dimensional vector spaces over a field F,

$$BL_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is one-to-one and } \dim_F(V/\operatorname{Im}\alpha) \text{ is infinite}\} \text{ and}$$

$$OBL_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is onto and } \dim_F\operatorname{Ker}\alpha \text{ is infinite}\}.$$

Let B be a basis of V. Then B is infinite. Let B_1 and B_2 be subsets of B such that $B = B_1 \cup B_2$, $B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Thus there is a bijection $\varphi : B \to B_1$. Define $\alpha, \beta \in L_F(V)$ as above. Then we have that $\alpha \in BL_F(V)$ and $\beta \in OBL_F(V)$. This shows that both $BL_F(V)$ and $OBL_F(V)$ are nonempty. Also for any $\gamma, \lambda \in L_F(V)$, $\operatorname{Im}\gamma\lambda \subseteq \operatorname{Im}\lambda$ and $\operatorname{Ker}\gamma\lambda \supseteq \operatorname{Ker}\gamma$. These imply that $BL_F(V)$ and $OBL_F(V)$ are subsemigroups of $L_F(V)$.

To generalize the above linear transformation semigroups analogously, let V and W be vector spaces over a field F,

$$L_F(V,W) = \text{ the set of all linear transformations } \alpha : V \to W,$$

$$M_F(V,W) = \{\alpha \in L_F(V,W) \mid \alpha \text{ is one-to-one }\},$$

$$E_F(V,W) = \{\alpha \in L_F(V,W) \mid \operatorname{Im}\alpha = W\},$$

$$AM_F(V,W) = \{\alpha \in L_F(V,W) \mid \dim_F\operatorname{Ker}\alpha \text{ is finite }\} \text{ and}$$

$$AE_F(V,W) = \{\alpha \in L_F(V,W) \mid \dim_F(W/\operatorname{Im}\alpha) \text{ is finite }\}.$$

If $S_F(V,W)$ is $M_F(V,W), E_F(V,W), AM_F(V,W)$ or $AE_F(V,W)$, $S_F(V,W) \neq \varnothing$ and $\theta \in S_F(W,V) \neq \varnothing$, let $(S_F(V,W),\theta)$ be the semigroup $S_F(V,W)$ under the operation $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in S_F(V,W)$. If $\alpha,\beta \in AM_F(V,W)$ and $\theta \in AM_F(W,V)$, then $\alpha\theta\beta \in AM_F(V,W)$ because we have the fact as in (1), page 10 that

$$\dim_F\operatorname{Ker}\alpha\theta\beta \leqslant \dim_F\operatorname{Ker}\alpha + \dim_F\operatorname{Ker}\theta + \dim_F\operatorname{Ker}\beta.$$

Also, $\alpha\theta\beta \in AE_F(V,W)$ for all $\alpha,\beta \in AE_F(V,W)$ and $\theta \in AE_F(W,V)$ since

$$\dim_F(W/\operatorname{Im}\alpha\theta\beta) \leqslant \dim_F(W/\operatorname{Im}\alpha) + \dim_F(V/\operatorname{Im}\theta) + \dim_F(W/\operatorname{Im}\beta)$$

which is similar to (2), page 10. Notice that $AM_F(V,W) = L_F(V,W)$ if $\dim_F V$ is finite and $AE_F(V,W) = L_F(V,W)$ if $\dim_F W$ is finite.

For infinite dimensional vector spaces V and W over a field F, let

$$BL_F(V,W) = \{\alpha \in L_F(V,W) \mid \alpha \text{ is one-to-one and } \dim_F(W/\operatorname{Im}\alpha) \text{ is infinite}\},$$

$$OBL_F(V,W) = \{\alpha \in L_F(V,W) \mid \alpha \text{ is onto and } \dim_F \operatorname{Ker}\alpha \text{ is infinite}\}.$$

We define the semigroup $(BL_F(V,W),\theta)$ where $\theta \in M_F(W,V)$ if $BL_F(V,W) \neq \varnothing$ and $M_F(W,V) \neq \varnothing$ and the semigroup $(OBL_F(V,W),\theta)$ where $\theta \in E_F(W,V)$ if $OBL_F(V,W) \neq \varnothing$ and $E_F(W,V) \neq \varnothing$ similarly as above. Note that $\operatorname{Im}\alpha\theta\beta \subseteq \operatorname{Im}\beta$ and $\operatorname{Ker}\alpha\theta\beta \supseteq \operatorname{Ker}\alpha$ for all $\alpha,\beta \in L_F(V,W)$ and $\theta \in L_F(W,V)$.

Next, let V and W be vector spaces over a field F, k a cardinal number,

$$L_F(V,W,k) = \{\alpha \in L_F(V,W) \mid \operatorname{rank}\alpha < k\} \text{ if } k > 0,$$

$$\overline{L}_F(V,W,k) = \{\alpha \in L_F(V,W) \mid \operatorname{rank}\alpha \leqslant k\}.$$

Then for $\theta \in L_F(W,V)$, let $(L_F(V,W),\theta)[(L_F(V,W,k),\theta),(\overline{L}_F(V,W,k),\theta)]$ be the semigroup $L_F(V,W)[L_F(V,W,k),\overline{L}_F(V,W,k)]$ under the operation $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in L_F(V,W)$ $[L_F(V,W,k),\overline{L}_F(V,W,k)]$. In fact, $(L_F(V,W,k),\theta)$ $[(\overline{L}_F(V,W,k),\theta)]$ is an ideal of $(L_F(V,W),\theta)$ since $\operatorname{rank}\alpha\theta\beta \leqslant \min\{\operatorname{rank}\alpha, \operatorname{rank}\theta, \operatorname{rank}\beta\}$.

In Chapter III, minimal quasi-ideals of all generalized linear transformation semigroups defined above are characterized.

For an infinite cardinal number k and $\theta \in L_F(W,V)$, $(L_F(V,W),+,\theta)[(L_F(V,W,k),+,\theta),(\overline{L}_F(V,W,k),+,\theta)]$ denote the ring $L_F(V,W)[L_F(V,W,k),\overline{L}_F(V,W,k)]$ *under the usual addition of linear transformations* $+$ *and the multiplication* $*$ *defined* by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in L_F(V,W)[L_F(V,W,k),\overline{L}_F(V,W,k)]$. Notice that $\operatorname{rank}(\alpha+\beta)$ can be greater than $\operatorname{rank}\alpha$ and $\operatorname{rank}\beta$ if $\operatorname{rank}\alpha < \infty$ and $\operatorname{rank}\beta < \infty$.

To distinguish between quasi-ideals in the semigroup $(L_F(V,W,k),\theta)$ and the ring $(L_F(V,W,k),+,\theta)$ obtaining from Theorem 1.2 and Theorem 1.3, repectively, the following examples are given for better understanding. In the semigroup $(L_F(V,W,k),\theta)$,

$$(\alpha)_q = (L_F(V,W,k)\theta\alpha \cap \alpha\theta L_F(V,W,k)) \cup \{\alpha\}$$

for every $\alpha \in L_F(V,W,k)$, and in the ring $(L_F(V,W,k),+,\theta)$,

$$(\alpha)_q = \mathbb{Z}\alpha + (L_F(V,W,k)\theta\alpha \cap \alpha\theta L_F(V,W,k))$$

for every $\alpha \in L_F(V,W,k)$.

Let $m,n \in \mathbb{N}$ and F a field. Let $M_{m,n}(F)$ denote the set of all $m\times n$ matrices over F. For $P \in M_{n,m}(F)$, let $(M_{m,n}(F),P)$ denote the semigroup $M_{m,n}(F)$ under the operation $*$ defined by $A*B = APB$ for all $A,B \in M_{m,n}(F)$ and $(M_{m,n}(F),+,P)$ denote the ring $M_{m,n}(F)$ under the usual addition of matrices $+$ and the multiplication $*$ defined by $A*B = APB$ for all $A,B \in M_{m,n}(F)$. Next, let V and W be vector spaces over F. Assume that $\dim_F V = m$ and $\dim_F W = n, B = \{v_1, v_2, \ldots, v_m\}$ an ordered basis of V and $B' = \{w_1, w_2, \ldots, w_n\}$ an ordered basis of W. For $\alpha \in L_F(V,W)$, let $[\alpha]_{B,B'}$ denote the $m \times n$ matrices (r_{ij}) where

$$\begin{aligned}
v_1\alpha &= r_{11}w_1 + r_{12}w_2 + \ldots + r_{1n}w_n \\
v_2\alpha &= r_{21}w_1 + r_{22}w_2 + \ldots + r_{2n}w_n \\
&\vdots \qquad\qquad \vdots \\
v_m\alpha &= r_{m1}w_1 + r_{m2}w_2 + \ldots + r_{mn}w_n.
\end{aligned}$$

Then

$$(L_F(V,W),\theta) \cong (M_{m,n}(F),[\theta]_{B',B}) \text{ and}$$

$$(L_F(V,W),+,\theta) \cong (M_{m,n}(F),+,[\theta]_{B',B})$$

by $\alpha \mapsto [\alpha]_{B,B'}$ ([4], page 329-330). Moreover, for every $\alpha \in L_F(V,W)$,

$$\operatorname{rank}\alpha = \operatorname{rank}[\alpha]_{B,B'}$$

([4], page 337 and 339).

Let $SU_n(F)$ be the set of all strictly upper triangular $n \times n$ matrices over a field F. Then $SU_n(F)$ is a semigroup under usual multiplication and also a ring under usual addition and multiplication. If P is an upper triangular $n \times n$ matrix over F, then $AP, PA \in SU_n(F)$ for all $A \in SU_n(F)$. For an upper triangular $n \times n$ matrix P over F, let $(SU_n(F), P)$ be the semigroup $SU_n(F)$ under the operation $*$ defined by $A * B = APB$ for all $A, B \in SU_n(F)$ and $(SU_n(F), +, P)$ the ring $SU_n(F)$ under the usual addition of matrices and the multiplication $*$ defined by $A * B = APB$ for all $A, B \in SU_n(F)$. We can see that $(SU_n(F), I_n) = (SU_n(F), \cdot)$ and $(SU_n(F), +, I_n) = (SU_n(F), +, \cdot)$, where I_n is the identity $n \times n$ matrix over F and $\cdot$ is the usual multiplication of matrices. Y. Kemprasit and P. Juntarakajorn [8] gave some interesting results on minimal quasi-ideals of the ring $SU_n(F)$ as follows:

Theorem 1.11 ([8]). *If char $F = 0$, then the ring $SU_n(F)$ has no minimal quasi-ideals.*

Theorem 1.12 ([8]). *Let char $F > 0$ and $A \in SU_n(F)$.*

(i) If rank $A = 1$, then $(A)_q$ is a minimal quasi-ideal of the semigroup $SU_n(F)$.

(ii) If $n \leqslant 3$ and $(A)_q$ is a minimal quasi-ideal of the semigroup $SU_n(F)$, then rank $A = 1$ and this need not be true if the condition $n \leqslant 3$ is not given.

The following known result in [8] will be referred.

Theorem 1.13 ([8]). *For* $A \in SU_n(F)$, *if rank* $A = 1$, *then* $SU_n(F)A \cap ASU_n(F) = \{0\}$.

We study generalized matrix semigroups mentioned above in the last section of Chapter III. Their minimal quasi-ideals are determined.

Generalized rings of linear transformations defined above are studied in the first section of Chapter IV. We characterize all of their minimal quasi-ideals. Also, minimal quasi-ideals of generalized matrix rings provided above are determined in the second section of this chapter. Theorem 1.11 and Thorem 1.12 will become our spacial cases of our main results of this section.

A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$. Also, a subring B of a ring R is called a *bi-ideal* of R if $BRB \subseteq B$. *Minimal bi-ideals* for semigroups and rings are defined similarly as minimal quasi-ideals. The notion of bi-ideals for semigroup was introduced by R. A. Good and D. R. Hughes [2] in 1952. The notion of bi-ideals for rings was introduced much later. It was actually introduced in 1972 by S. Lajos and F. Szász [11]. Semigroups whose bi-ideals and quasi-ideals coincide have long been studied. They are sometimes called *BQ-semigroups.* There were characterizations of many transformation semigroups and linear transformation semigroups whose bi-ideals and quasi-ideals coincide given by C. Namnak and Y. Kemprasit. One can see in [6], [7], [9], [10], [13], [14] and [15]. Observe that in any BQ-semigroup S, the minimal bi-ideals and the minimal quasi-ideals of S are identical.

In the last section, we show that in any ring $(L_F(V,W,k),+,\theta)$ $[(\overline{L}_F(V,W,k),+,\theta)]$ defined previously, the bi-ideals and quasi-ideals are identical. Then its minimal quasi-ideals and minimal bi-ideals coincide. Moreover, we determine minimal bi-ideals in any ring $(SU_n(F),+,\theta)$ and give an example of minimal bi-ideals of $SU_n(F) =$

$(SU_n(F), +, I_n)$ which are not quasi-ideals for all $n \geqslant 4$. This implies that if $n \geqslant 4$, then the bi-ideals and the quasi-ideals of the rings $SU_n(F)$ do not coincide.

The bi-ideal of R generated by X is denoted by $(X)_b$, that is, the intersection of all bi-ideals of R containing X. The following known results will be referred.

Theorem 1.14 ([11]). *For a nonempty set X of R,*

$$(X)_b = \mathbb{Z}X + \mathbb{Z}X^2 + XRX.$$

Theorem 1.15 ([19], page 12). *If Q_1 and Q_2 are quasi-ideals of a ring [semigroup] A, then Q_1Q_2 is a bi-ideal of A.*

Since every left ideal and every right ideal of a ring [semigroup] A is a quasi-ideal of A, the following result is a consequence of Theorem 1.15.

Corollary 1.16. *If H and K are one-sided ideals of a ring [semigroup] A, then HK is a bi-ideal of A.*

CHAPTER II

MINIMAL QUASI-IDEALS OF GENERALIZED TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to characterize minimal quasi-ideals of various generalized transformation semigroups and generalized matrix semigroups mentioned in Chapter I.

Let us recall the notations which are used throughout this chapter. Let X and Y be sets and

$P_X =$ the partial transformation semigroup on X,

$T_X =$ the full transformation semigroup on X,

$I_X =$ the one-to-one partial transformation semigroup on X (the symmetric inverse semigroup on X),

$G_X =$ the symmetric group on X (the permutation group on X),

$M_X =$ the semigroup of one-to-one transformations of X,

$E_X =$ the semigroup of onto transformations of X,

$AM_X =$ the semigroup of almost one-to-one transformations of X,

$AE_X =$ the semigroup of almost onto transformations of X,

$BL_X = \{\alpha \in T_X \mid \alpha$ is one-to-one and $X \setminus \operatorname{ran}\alpha$ is infinite$\}$ where X is infinite,

$$OBL_X = \{\alpha \in T_X \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}$$

where X is infinite,

$$P(X,Y) = \{\alpha : A \to Y \mid A \subseteq X\},$$

$$T(X,Y) = \{\alpha \in P(X,Y) \mid \operatorname{dom}\alpha = X\},$$

$$I(X,Y) = \{\alpha \in P(X,Y) \mid \alpha \text{ is one-to-one}\},$$

$$M(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one}\},$$

$$E(X,Y) = \{\alpha \in T(X,Y) \mid \operatorname{ran}\alpha = Y\},$$

$$AM(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is almost one-to-one}\},$$

$$AE(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is almost onto}\},$$

$$BL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } Y \smallsetminus \operatorname{ran}\alpha \text{ is infinite}\}$$

where X and Y are infinite,

$$OBL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}$$

where X and Y are infinite.

It is clearly seen that for the case that only X or only Y is empty, $(S(X,Y),\theta)$ where $\theta \in S(Y,X)$ is not defined (as a semigroup) if $S(X,Y) = T(X,Y), M(X,Y)$, $E(X,Y), AM(X,Y)$ or $AE(X,Y)$. If $X = Y = \varnothing$, then it is a trivial semigroup. Thus when these five semigroups are considered, we may assume that both X and Y are nonempty.

2.1 The Semigroups $(P(X,Y),\theta), (T(X,Y),\theta)$ and $(I(X,Y),\theta)$

In the section, for the notation $(S(X,Y),\theta)$, unless otherwise metioned, we mean that $\varnothing \neq S(X,Y) \subseteq P(X,Y), \theta \in P(Y,X)$ and $\alpha\theta\beta \in S(X,Y)$ for all $\alpha,\beta \in S(X,Y)$.

We first give three lemmas which are used in this section. These also give us some general properties of generalized transformation semigroups.

Lemma 2.1.1. *In a semigroup* $(S(X,Y),\theta)$, *if* $\alpha \in S(X,Y)$ *is such that* $ran\,\alpha \cap dom\,\theta = \varnothing$ *or* $dom\,\alpha \cap ran\,\theta = \varnothing$, *then* $(\alpha)_q = \{0,\alpha\}$.

Proof. Let $\alpha \in S(X,Y)$ and assume that $\operatorname{ran}\alpha \cap \operatorname{dom}\theta = \varnothing$ or $\operatorname{dom}\alpha \cap \operatorname{ran}\theta = \varnothing$. Then $\alpha\theta = 0$ in P_X or $\theta\alpha = 0$ in P_Y. But $(\alpha)_q = (S(X,Y)\theta\alpha \cap \alpha\theta S(X,Y)) \cup \{\alpha\}$ by Theorem 1.2, so we have $(\alpha)_q = \{0\} \cup \{\alpha\} = \{0,\alpha\}$ in $(S(X,Y),\theta)$. □

Lemma 2.1.2. *In a semigroup* $(S(X,Y),\theta)$, *if* $\alpha \in S(X,Y)$ *is such that* $rank\,\alpha = 1$, *then in* $(S(X,Y),\theta)$, $(\alpha)_q = \{0,\alpha\}$ *if* $0 \in S(X,Y)$ *and* $(\alpha)_q = \{\alpha\}$ *if* $0 \notin S(X,Y)$.

Proof. Assume that $\alpha \in S(X,Y)$ and rank $\alpha = 1$. Let β be a nonzero element of $(\alpha)_q$. From Theorem 1.2, $\beta = \alpha$ or $\beta = \gamma\theta\alpha = \alpha\theta\lambda$ for some $\gamma, \lambda \in S(X,Y)$. Assume that the second case holds. Then ran $\beta =$ ran $(\gamma\theta\alpha) \subseteq$ ran α. From the fact that $\beta \neq 0$, rank $\alpha = 1$ and ran $\beta \subseteq$ ran α, we have ran $\beta =$ ran α. Since $0 \neq \beta = \alpha\theta\lambda$, it follows that ran $\alpha \cap$ dom $\theta\lambda \neq \varnothing$. But rank $\alpha = 1$, so ran $\alpha \subseteq$ dom $\theta\lambda$. Consequently, ran $\alpha =$ ran $\alpha \cap$ dom $\theta\lambda$, and hence dom $\alpha = (\operatorname{ran}\alpha)\alpha^{-1} = (\operatorname{ran}\alpha \cap \operatorname{dom}\theta\lambda)\alpha^{-1} = \operatorname{dom}(\alpha\theta\lambda) = \operatorname{dom}\beta$. Now we have dom $\alpha =$ dom β, ran $\alpha =$ ran β and rank $\alpha = 1$. This implies that $\beta = \alpha$.

Therefore the lemma is proved. □

Lemma 2.1.3. *Let* $(S(X,Y),\theta)$ *be a semigroup such that* $S(X,Y)$ *covers* X *and* Y. *If* $\alpha \in S(X,Y)$ *is such that* $ran\,\alpha \cap dom\,\theta \neq \varnothing$, $dom\,\alpha \cap ran\,\theta \neq \varnothing$ *and* $(\alpha)_q$ *is a minimal quasi-ideal of* $(S(X,Y),\theta)$, *then* $rank\,\alpha = 1$.

Proof. Let all the assumptions of the lemma be given. We shall show that rank $\alpha = 1$ by the help of Lemma 2.1.2. Since ran $\alpha \cap$ dom $\theta \neq \varnothing$ and dom $\alpha \cap$ ran $\theta \neq \varnothing$, there are elements $x \in$ dom $\alpha \cap$ ran θ and $y \in$ ran $\alpha \cap$ dom θ. Then $y'\theta = x$ and $x'\alpha = y$ for some $y' \in$ dom θ and $x' \in$ dom α. This implies that $x' \in$ dom $\alpha\theta$ and

$y' \in \operatorname{dom} \theta\alpha$. Since $S(X,Y)$ covers X and Y, there exists a subset A of X such that $x'\alpha\theta \in A$ and $A_{y'} \in S(X,Y)$. Consequently, ran $(\alpha\theta A_{y'}) = \{y'\}$. Since $y' \in$ dom $\theta\alpha$, it follows that ran $(\alpha\theta A_{y'}\theta\alpha) = \{y'\theta\alpha\} = \{x\alpha\}$, so rank $(\alpha\theta A_{y'}\theta\alpha) = 1$. Thus $\alpha\theta A_{y'}\theta\alpha \neq 0$ and $\alpha\theta A_{y'}\theta\alpha \in S(X,Y)\theta\alpha \cap \alpha\theta S(X,Y) \subseteq (\alpha)_q$ in $(S(X,Y),\theta)$ by Theorem 1.2. But $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$, so $(\alpha\theta A_{y'}\theta\alpha)_q = (\alpha)_q$. Because rank $(\alpha\theta A_{y'}\theta\alpha) = 1$, by Lemma 2.1.2, we have $\alpha = \alpha\theta A_{y'}\theta\alpha$, and hence rank $\alpha = 1$. □

Now, we characterize minimal quasi-ideals of the semigroup $(S(X,Y),\ \theta)$ where $S(X,Y)$ is $P(X,Y), T(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. Recall that all $P(X,Y)$, $T(X,Y)$ and $I(X,Y)$ cover X and Y.

Theorem 2.1.4. *Let $S(X,Y)$ be $P(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. Then for $\alpha \in S(X,Y) \setminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$ if and only if one of the following statements holds.*

(i) ran $\alpha \cap$ dom $\theta = \varnothing$.

(ii) dom $\alpha \cap$ ran $\theta = \varnothing$.

(iii) rank $\alpha = 1$.

If this is the case, $(\alpha)_q = \{0,\alpha\}$. If $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$, and if $\alpha\theta\alpha \neq 0$, then $(\alpha)_q$ is a subgroup with zero of $(S(X,Y),\theta)$.

Proof. To show sufficiency, first assume that (i) or (ii) holds. Then by Lemma 2.1.1, $(\alpha)_q = \{0,\alpha\}$ in $(S(X,Y),\theta)$, so $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$. If rank $\alpha = 1$, by Lemma 2.1.2, $(\alpha)_q = \{0,\alpha\}$ in $(S(X,Y),\theta)$, so it is a minimal quasi-ideal of $(S(X,Y),\theta)$. Clearly, if $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$, and $(\alpha)_q$ is a subgroup with zero of $(S(X,Y),\theta)$ if $\alpha\theta\alpha \neq 0$.

To show necessity, assume that $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$ and suppose that (i) and (ii) are false. Then $\operatorname{ran}\alpha \cap \operatorname{dom}\theta \neq \varnothing$ and $\operatorname{dom}\alpha \cap \operatorname{ran}\theta \neq \varnothing$. But $S(X,Y)$ covers X and Y, so by Lemma 2.1.3, we deduce that $\operatorname{rank}\alpha = 1$. Hence (iii) holds.

Therefore the theorem is proved. $\square$

Remark 2.1.5. In Theorem 2.1.4, if (i) or (ii) holds, then $\alpha\theta\alpha = 0$, so $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$. One might expect that if (iii) holds, then $(\alpha)_q$ must be a subgroup with zero of $(S(X,Y),\theta)$. In fact, for this case, there are X, Y, θ and α such that $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$, and also there are X, Y, θ and α such that $(\alpha)_q$ is a subgroup with zero of $(S(X,Y),\theta)$.

If $\operatorname{rank}\alpha = 1$ and (i) or (ii) holds (for an example, $\theta = 0$), then $\alpha\theta\alpha = 0$, so $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$.

If $X = Y, |X| = |Y| = 1$ and $\alpha = \theta \neq 0$ in $I(X,Y)(= I(Y,X))$, then $(\alpha)_q = \{0,\alpha\}$ and $\alpha\theta\alpha = \alpha$ since $|I(X,Y)| = 2$, and hence $(\alpha)_q$ is a subgroup with zero in $(I(X,Y),\theta)$.

It is natural to ask that in Theorem 2.1.4 if (i) and (ii) do not hold and (iii) holds, is $(\alpha)_q$ always a subgroup with zero of $(S(X,Y),\theta)$? The answer is "no". It can be shown by the following example. Let $X = Y = \{a,b\}$. Define θ and α by $\theta = 1_X$, the identity map on X, $\operatorname{dom}\alpha = \{a\}$ and $\operatorname{ran}\alpha = \{b\}$. Then $\operatorname{rank}\alpha = 1$. Since $\operatorname{dom}\theta = \operatorname{ran}\theta = X$, (i) and (ii) do not hold. Because $\operatorname{ran}\alpha\theta = \{b\}$, $\operatorname{ran}\alpha\theta \cap \operatorname{dom}\alpha = \varnothing$, so $\alpha\theta\alpha = 0$. We therefore deduce from Theorem 2.1.4 that $(\alpha)_q$ is a zero subsemigroup of $(I(X,Y),\theta)$ and of $(P(X,Y),\theta)$.

Theorem 2.1.6. *Assume that $X \neq \varnothing$ and $Y \neq \varnothing$ and let $\theta \in T(Y,X)$. Then for $\alpha \in T(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$.*

If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. Since dom $\theta = Y \supseteq$ ran $\beta \neq \varnothing$ and dom $\beta = X \supseteq$ ran $\theta \neq \varnothing$ for every $\beta \in T(X,Y)$, we have from Lemma 2.1.3 that if $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$, then rank $\alpha = 1$.

For the converse, assume that rank $\alpha = 1$. Then $(\alpha)_q = \{\alpha\}$ by Lemma 2.1.2, so $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$. $\square$

2.2 The Semigroups $(M(X,Y),\theta)$, $(E(X,Y),\theta)$, $(AM(X,Y),\theta)$ and $(AE(X,Y),\theta)$

In this section, we characterize minimal quasi-ideals of the following semigroups: $(M(X,Y),\theta)$ with $\theta \in M(Y,X)$, $(E(X,Y),\theta)$ with $\theta \in E(Y,X)$, $(AM(X,Y),\theta)$ with $\theta \in AM(Y,X)$ and $(AE(X,Y),\theta)$ with $\theta \in AE(Y,X)$ when these sets are nonempty. For each characterization, some lemmas are required.

Lemma 2.2.1. *The following statements hold.*

(i) $M(X,Y) \neq \varnothing$ *and* $M(Y,X) \neq \varnothing$ *if and only if* $|X| = |Y|$.

(ii) $E(X,Y) \neq \varnothing$ *and* $E(Y,X) \neq \varnothing$ *if and only if* $|X| = |Y|$.

(iii) If φ *is a bijection of* X *onto* Y *and* $\theta \in M(Y,X)$, *then* $(M(X,Y),\theta) \cong (M_X, \varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

(iv) If φ *is a bijection of* X *onto* Y *and* $\theta \in E(Y,X)$, *then* $(E(X,Y),\theta) \cong (E_X, \varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

Proof. (i) Assume that $M(X,Y) \neq \varnothing$ and $M(Y,X) \neq \varnothing$. Let $\alpha \in M(X,Y)$ and $\beta \in M(Y,X)$. Since $\alpha : X \to Y$ and $\beta : Y \to X$ are one-to-one, we have

$$|X| = |X\alpha| \leqslant |Y| = |Y\beta| \leqslant |X|,$$

and so $|X| = |Y|$.

Conversely, if $|X| = |Y|$, then there is a bijection λ from X onto Y, and therefore $\lambda \in M(X,Y)$ and $\lambda^{-1} \in M(Y,X)$.

(ii) Assume that $\alpha \in E(X,Y)$ and $\beta \in E(Y,X)$. Since $\alpha : X \to Y$ and $\beta : Y \to X$ are onto, it follows that

$$|X| \geqslant |X\alpha| = |Y| \geqslant |Y\beta| = |X|,$$

and thus $|X| = |Y|$.

For the converse, if $|X| = |Y|$, then λ in the proof of (i) is an element of $E(X,Y)$ and λ^{-1} is an element of $E(Y,X)$.

(iii) Define $\overline{\varphi} : M(X,Y) \to M_X$ by

$$\alpha\overline{\varphi} = \alpha\varphi^{-1} \text{ for all } \alpha \in M(X,Y).$$

Since $\varphi^{-1} : Y \to X$ is a bijection, we deduce that $\overline{\varphi}$ is one-to-one. If $\alpha \in M_X$, then $\alpha\varphi \in M(X,Y)$ and $(\alpha\varphi)\overline{\varphi} = (\alpha\varphi)\varphi^{-1} = \alpha$. We have that $\overline{\varphi}$ is a homomorphism from $(M(X,Y),\theta)$ into $(M_X, \varphi\theta)$ since for $\alpha, \beta \in M(X,Y)$,

$$(\alpha\theta\beta)\overline{\varphi} = (\alpha\theta\beta)\varphi^{-1} = (\alpha\varphi^{-1})\varphi\theta(\beta\varphi^{-1}) = (\alpha\overline{\varphi})\varphi\theta(\beta\overline{\varphi}).$$

(iv) can be proved similarly to the proof given for (iii). □

Lemma 2.2.2. *Assume that $X \neq \varnothing$ and $Y \neq \varnothing$. The following statements hold.*

(i) $AM(X,Y) \neq \varnothing$ *and* $AM(Y,X) \neq \varnothing$ *if and only if either both X and Y are finite or both X and Y are infinite and $|X| = |Y|$.*

(ii) $AE(X,Y) \neq \varnothing$ *and* $AE(Y,X) \neq \varnothing$ *if and only if either both X and Y are finite or both X and Y are infinite and $|X| = |Y|$.*

(iii) If φ is a bijection of X onto Y and $\theta \in AM(Y,X)$, then $(AM(X,Y),\theta) \cong$

$(AM_X, \varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

(*iv*) *If* φ *is a bijection of* X *onto* Y *and* $\theta \in AE(Y,X)$, *then* $(AE(X,Y),\theta) \cong (AE_X, \varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

Proof. First, let $\alpha \in AM(X,Y)$ and $\beta \in AM(Y,X)$. Then

$$A(\alpha) = \{x \in X \mid |(x\alpha)\alpha^{-1}| > 1\}, A(\beta) = \{y \in Y \mid |(y\beta)\beta^{-1}| > 1\},$$

both $A(\alpha)$ and $A(\beta)$ are finite, $\alpha|_{X \smallsetminus A(\alpha)} : X \smallsetminus A(\alpha) \to Y$ is one-to-one and $\beta|_{Y \smallsetminus A(\beta)} : Y \smallsetminus A(\beta) \to X$ is one-to-one. Thus $|X \smallsetminus A(\alpha)| \leqslant |Y|$ and $|Y \smallsetminus A(\beta)| \leqslant |X|$. If X is finite, then Y is finite since $A(\beta)$ is finite and $|Y \smallsetminus A(\beta)| \leqslant |X|$. Similarly, if Y is finite, then X is finite. Hence X is finite if and only if Y is finite. Consequently, X is infinite if and only if Y is infinite. Also, if X and Y are infinite, then

$$\begin{aligned} |X| &= |X \smallsetminus A(\alpha)| && \text{since } |A(\alpha)| < \infty \\ &\leqslant |Y| \\ &= |Y \smallsetminus A(\beta)| && \text{since } |A(\beta)| < \infty \\ &\leqslant |X| \end{aligned}$$

which implies that $|X| = |Y|$.

Next, let $\gamma \in AE(X,Y)$ and $\lambda \in AE(Y,X)$. Then $|Y \smallsetminus \operatorname{ran}\gamma| < \infty$ and $|X \smallsetminus \operatorname{ran}\lambda| < \infty$. If X is finite, then $\operatorname{ran}\gamma$ is a finite subset of Y, so Y is finite since $|Y \smallsetminus \operatorname{ran}\gamma| < \infty$. Similarly, X is finite if Y is finite. Consequently, X is infinite if and only if Y is infinite. Moreover, if both X and Y are infinite, then

$$\begin{aligned} |X| &= |\operatorname{ran}\lambda \cup (X \smallsetminus \operatorname{ran}\lambda)| \\ &= |\operatorname{ran}\lambda| + |X \smallsetminus \operatorname{ran}\lambda| \\ &\leqslant |Y| + |X \smallsetminus \operatorname{ran}\lambda| \\ &= |Y| && \text{since } |X \smallsetminus \operatorname{ran}\lambda| < \infty \end{aligned}$$

$$= |\operatorname{ran}\gamma \cup (Y \setminus \operatorname{ran}\gamma)|$$

$$= |\operatorname{ran}\gamma| + |Y \setminus \operatorname{ran}\gamma|$$

$$\leqslant |X| + |Y \setminus \operatorname{ran}\gamma|$$

$$= |X| \qquad \text{since } |Y \setminus \operatorname{ran}\gamma| < \infty,$$

and thus $|X| = |Y|$.

If both X and Y are finite, then $AM(X,Y) = T(X,Y) = AE(X,Y)$ and $AM(Y,X) = T(Y,X) = AE(Y,X)$. If $|X| = |Y|$, then there is a bijection α from X onto Y, so $\alpha \in AM(X,Y) \cap AE(X,Y)$ and $\alpha^{-1} \in AM(Y,X) \cap AE(Y,X)$.

Hence (i) and (ii) of the lemma are completely proved.

(iii) and (iv). The proof can be given similarly to that given for (iii) of Lemma 2.2.1. □

Lemma 2.2.3. *The following statements hold.*

(i) For $\theta \in M_X$, (M_X,θ) has a minimal quasi-ideal if and only if $|X| < \infty$. If $|X| < \infty$, then $(M_X,\theta) = (G_X,\theta)$, so M_X is itself a unique minimal quasi-ideal of (M_X,θ).

(ii) For $\theta \in E_X$, (E_X,θ) has a minimal quasi-ideal if and only if $|X| < \infty$. If $|X| < \infty$, then $(E_X,\theta) = (G_X,\theta)$, so E_X is itself a unique minimal quasi-ideal of (E_X,θ).

Proof. If $|X| < \infty$, then $M_X = G_X = E_X$, so M_X $[E_X]$ is itself a unique minimal quasi-ideal of (M_X,θ) $[(E_X,\theta)]$ where $\theta \in M_X$ $[E_X]$ since (G_X,θ) is a group.

Next, assume that X is infinite. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$. Then there is a bijection $\beta : X \to X \setminus \{a\}$. Let $\gamma : X \to X$ be defined by

$$x\gamma = x\beta^{-1} \text{ for all } x \in X \setminus \{a\} \text{ and } a\gamma = a.$$

Then $a\gamma = a = b\beta^{-1} = b\gamma$ for some $b \in X \smallsetminus \{a\}$, so ran $\beta \neq X$ and γ is not one-to-one. Moreover, $\beta \in M_X$ and $\gamma \in E_X$.

To show that (M_X, θ) has no minimal quasi-ideal where $\theta \in M_X$, let $\alpha \in M_X$ be arbitrary. Then $\alpha\theta\beta\theta\alpha \in (\alpha)_q$ in (M_X, θ), so $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $(\alpha\theta\beta\theta\alpha)_q = (\alpha)_q$. By Theorem 1.2, $\alpha = \alpha\theta\beta\theta\alpha$ or $\alpha = \lambda\theta\alpha\theta\beta\theta\alpha$ for some $\lambda \in M_X$. Hence $\theta\alpha = \theta\alpha\theta\beta\theta\alpha$ or $\theta\alpha = \theta\lambda\theta\alpha\theta\beta\theta\alpha$. Since $\theta\alpha$ is one-to-one, $1_X = \theta\alpha\theta\beta$ or $1_X = \theta\lambda\theta\alpha\theta\beta$ which implies that ran $\beta = X$, a contradiction. Hence $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$ in (M_X, θ).

Finally, to show that (E_X, θ) has no minimal quasi-ideal where $\theta \in E_X$, let $\alpha \in E_X$ be arbitrary. Then $(\alpha\theta\gamma\theta\alpha)_q \subseteq (\alpha)_q$ in (E_X, θ). If $(\alpha\theta\gamma\theta\alpha)_q = (\alpha)_q$ in (E_X, θ), then by Theorem 1.2, $\alpha = \alpha\theta\gamma\theta\alpha$ or $\alpha = \alpha\theta\gamma\theta\alpha\theta\mu$ for some $\mu \in E_X$. Thus $\alpha\theta = \alpha\theta\gamma\theta\alpha\theta$ or $\alpha\theta = \alpha\theta\gamma\theta\alpha\theta\mu\theta$. Since ran $\alpha\theta = X$, $1_X = \gamma\theta\alpha\theta$ or $1_X = \gamma\theta\alpha\theta\mu\theta$. This implies that γ is one-to-one, a contradiction.

Hence the proof of the lemma is complete. □

Lemma 2.2.4. *Let X be infinite.*

(i) For any $\theta \in AM_X$, (AM_X, θ) has no minimal quasi-ideal.

(ii) For any $\theta \in AE_X$, (AE_X, θ) has no minimal quasi-ideal.

Proof. (i) Let $\alpha \in AM_X$ be arbitrary. Then $|A(\alpha\theta)| < \infty$. Let $a, b \in X \smallsetminus A(\alpha\theta)$ be distinct. Then $a\alpha\theta \neq b\alpha\theta$ which implies that $a\alpha \neq b\alpha$. Let $c \in X \smallsetminus \{a\alpha\theta, b\alpha\theta\}$. Since $|X \smallsetminus \{a\alpha\theta, b\alpha\theta\}| = |X \smallsetminus \{c\}|$, there is a bijection $\varphi : X \smallsetminus \{a\alpha\theta, b\alpha\theta\} \to X \smallsetminus \{c\}$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x\varphi & \text{if } x \in X \smallsetminus \{a\alpha\theta, b\alpha\theta\}, \\ c & \text{if } x = a\alpha\theta \text{ or } x = b\alpha\theta. \end{cases}$$

Then $A(\beta) = \{a\alpha\theta, b\alpha\theta\}$, so $\beta \in AM_X$. Since $a\alpha \neq b\alpha$, $a\alpha\theta\beta\theta\alpha = c\theta\alpha = b\alpha\theta\beta\theta\alpha$ and $a\alpha\theta\beta\theta\alpha\theta\gamma = c\theta\alpha\theta\gamma = b\alpha\theta\beta\theta\alpha\theta\gamma$ for all $\gamma \in AM_X$, it follows that $\alpha \neq \alpha\theta\beta\theta\alpha$ and $\alpha \neq \alpha\theta\beta\theta\alpha\theta\gamma$ for all $\gamma \in AM_X$. Thus $\alpha \notin (\alpha\theta\beta\theta\alpha)_q$ by Theorem 1.2. Hence $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$.

(ii) Let $\alpha \in AE_X$. Then $\alpha\theta \in AE_X$, so ran $\alpha\theta$ is infinite since $X \setminus$ ran $\alpha\theta$ is finite. Let $a, b \in X$ be such that $a\alpha\theta \neq b\alpha\theta$. Then $a\alpha \neq b\alpha$. Since X is infinite, $|X \setminus \{a\alpha\theta, b\alpha\theta\}| = |X|$, so there is a bijection $\varphi : X \setminus \{a\alpha\theta, b\alpha\theta\}$ onto X. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x\varphi & \text{if } x \in X \setminus \{a\alpha\theta, b\alpha\theta\}, \\ a\alpha\theta & \text{if } x = a\alpha\theta \text{ or } x = b\alpha\theta. \end{cases}$$

Then $\beta \in E_X \subseteq AE_X$ and $a\alpha\theta\beta = b\alpha\theta\beta$, so $a\alpha\theta\beta\theta\alpha = b\alpha\theta\beta\theta\alpha$ and $a\alpha\theta\beta\theta\alpha\theta\gamma = b\alpha\theta\beta\theta\alpha\theta\gamma$ for all $\gamma \in AE_X$, By Theorem 1.2, $\alpha \notin (\alpha\theta\beta\theta\alpha)_q$. But $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$, so $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$.

Hence the proof is complete. $\square$

We show that the semigroup $(S(X,Y), \theta)$ where $S(X,Y)$ is any of $M(X,Y)$, $E(X,Y)$, $AM(X,Y)$ or $AE(X,Y)$ and $\theta \in S(Y,X)$ has no minimal quasi-ideal except for the case that X and Y are finite.

Theorem 2.2.5. *The semigroup $(M(X,Y), \theta)$, where $\theta \in M(Y,X)$, has a minimal quasi-ideal if and only if $|X| = |Y| < \infty$.*

If $|X| = |Y| < \infty$, then $(M(X,Y), \theta)$ is a group, so $M(X,Y)$ is itself a unique minimal quasi-ideal of $(M(X,Y), \theta)$.

Proof. By Lemma 2.2.1(i), $|X| = |Y|$, and from Lemma 2.2.1(iii), $(M(X,Y), \theta) \cong (M_X, \varphi\theta)$ where $\varphi : X \to Y$ is a bijection. By Lemma 2.2.3(i), $(M_X, \varphi\theta)$ has a minimal quasi-ideal if and only if $|X| < \infty$. If $|X| < \infty$, then $(M_X, \varphi\theta) = (G_X, \varphi\theta)$

which is a group, so M_X is a unique quasi-ideal of $(M_X, \varphi\theta)$. Hence the theorem is proved. $\square$

Theorem 2.2.6. *The semigroup $(E(X,Y),\theta)$, where $\theta \in E(Y,X)$, has a minimal quasi-ideal if and only if $|X| = |Y| < \infty$.*

If $|X| = |Y| < \infty$, then $(E(X,Y),\theta)$ is a group, so $E(X,Y)$ is itself a unique minimal quasi-ideal of $(E(X,Y),\theta)$.

Proof. It follows from Lemma 2.2.1(ii) and (iv) that $|X| = |Y|$ and $(E(X,Y),\theta) \cong (E_X, \varphi\theta)$ where $\varphi : X \to Y$ is a bijection. By Lemma 2.2.3(ii), $(E_X, \varphi\theta)$ has a minimal quasi-ideal if and only if $|X| < \infty$. If $|X| < \infty$, then $(E_X, \varphi\theta) = (G_X, \varphi\theta)$ which is a group, so E_X is a unique quasi-ideal of $(E_X, \varphi\theta)$. Therefore the theorem is proved, as required. $\square$

Theorem 2.2.7. *For $\theta \in AM(Y,X)$, the semigroup $(AM(X,Y),\theta)$ has a minimal quasi-ideal if and only if X and Y are finite.*

If X and Y are finite and nonempty, then for $\alpha \in AM(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X,Y),\theta)$ if and only if rank $\alpha = 1$. If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. First assume that X and Y are finite and nonempty. Then $AM(X,Y) = T(X,Y)$ and $AM(Y,X) = T(Y,X)$. By Theorem 2.1.6, for every $\alpha \in AM(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X,Y),\theta)$ if and only if rank $\alpha = 1$, and for this case, $(\alpha)_q = \{\alpha\}$.

To prove the converse, assume that X or Y is not finite. By Lemma 2.2.2(i), X and Y are infinite and $|X| = |Y|$. Let $\varphi : X \to Y$ be a bijection. By Lemma 2.2.2(iii), $(AM(X,Y),\theta) \cong (AM_X, \varphi\theta)$. We have from Lemma 2.2.4(i) that $(AM_X, \varphi\theta)$ has no minimal quasi-ideal. Hence $(AM(X,Y),\theta)$ has no minimal quasi-ideal. $\square$

Theorem 2.2.8. *For $\theta \in AE(Y,X)$, the semigroup $(AE(X,Y),\theta)$ has a minimal quasi-ideal if and only if X and Y are finite.*

If X and Y are finite and nonempty, then for $\alpha \in AE(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AE(X,Y),\theta)$ if and only if rank $\alpha = 1$. If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. If X and Y are finite and nonempty, then $AE(X,Y) = T(X,Y)$ and $AE(Y,X) = T(Y,X)$, so by Theorem 2.1.6, for $\alpha \in AE(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AE(X,Y),\theta)$ if and only if rank $\alpha = 1$ and for this case, $(\alpha)_q = \{\alpha\}$.

Conversely, assume that X or Y is not finite. By Lemma 2.2.2(ii), X and Y are infinite and $|X| = |Y|$. Let $\varphi : X \to Y$ be a bijection. Then $(AE(X,Y),\theta) \cong (AM_X, \varphi\theta)$ by Lemma 2.2.2(iv). From Lemma 2.2.4(ii), $(AE_X, \varphi\theta)$ has no minimal quasi-ideal. We therefore deduce that $(AE(X,Y),\theta)$ has no minimal quasi-ideal. $\square$

2.3 The Semigroups $(BL(X,Y),\theta)$ and $(OBL(X,Y),\theta)$

Let X and Y be infinite sets throughout this section and recall the following sets.

$$BL_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one and } X \setminus \operatorname{ran}\alpha \text{ is infinite}\},$$

$$OBL_X = \{\alpha \in T_X \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\},$$

$$BL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } Y \setminus \operatorname{ran}\alpha \text{ is infinite}\},$$

$$OBL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.$$

We have shown that if $BL(X,Y) \neq \varnothing$ and $M(Y,X) \neq \varnothing$, then $(BL(X,Y),\theta)$ where $\theta \in M(Y,X)$ is indeed a semigroup and so is $(OBL(X,Y),\theta)$ where $\theta \in E(Y,X)$. We shall obtain these semigroups if and only if $|X| = |Y|$.

Lemma 2.3.1. *The following statements hold.*

(i) $BL(X,Y) \neq \varnothing$ *and* $M(Y,X) \neq \varnothing$ *if and only if* $|X| = |Y|$.

(ii) $OBL(X,Y) \neq \varnothing$ *and* $E(Y,X) \neq \varnothing$ *if and only if* $|X| = |Y|$.

Proof. Since $BL(X,Y) \subseteq M(X,Y)$ and $OBL(X,Y) \subseteq E(X,Y)$, it follows from Lemma 2.2.1 that either $BL(X,Y) \neq \varnothing$ and $M(Y,X) \neq \varnothing$ or $OBL(X,Y) \neq \varnothing$ and $E(Y,X) \neq \varnothing$ implies that $|X| = |Y|$.

Assume that $|X| = |Y|$. By Lemma 2.2.1, $M(Y,X) \neq \varnothing$ and $E(Y,X) \neq \varnothing$. Since Y is infinite, there are subsets Y_1 and Y_2 of Y such that $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \varnothing$ and $|Y_1| = |Y_2| = |Y|$. Then $|X| = |Y_1| = |Y_2|$. Let $\alpha : X \to Y_1$ be a bijection. Then α is one-to-one and $Y \setminus \operatorname{ran}\alpha = Y \setminus Y_1 = Y_2$ which is infinite. Hence $\alpha \in BL(X,Y)$. Next, we shall show that $OBL(X,Y) \neq \varnothing$. Since X is infinite, $|X \times X| = |X| = |Y|$. Then there is a bijection $\varphi : X \to X \times X$. Consequently,

$$X = \bigcup_{x \in X} (\{x\} \times X)\varphi^{-1} \text{ which is a disjoint union,} \tag{1}$$

$$(\{x\} \times X)\varphi^{-1} \text{ is an infinite subset of } X \text{ for every } x \in X. \tag{2}$$

Let ψ be a bijection of X onto Y and define $\beta : X \to Y$ by

$$((\{x\} \times X)\varphi^{-1})\beta = x\psi \text{ for every } x \in X. \tag{3}$$

From (1), β is well-defined. Since $\operatorname{ran}\psi = Y$, $\operatorname{ran}\beta = Y$ by (3). Moreover, from (1), for each $x \in X, x \in (\{a\} \times X)\varphi^{-1}$ for some $a \in X$, so from (3),

$$(x\beta)\beta^{-1} = (((\{a\} \times X)\varphi^{-1})\beta)\beta^{-1} = (\{a\} \times X)\varphi^{-1}$$

which implies by (2) that $(x\beta)\beta^{-1}$ is infinite. Hence $\beta \in OBL(X,Y)$.

Therefore the lemma is proved. $\square$

Lemma 2.3.2. *If $\theta : Y \to X$ is a bijection, then $(BL(X,Y),\theta) \cong BL_X$ through the map $\alpha \mapsto \alpha\theta$.*

Proof. Let $\alpha \in BL(X,Y)$. Then α is one-to-one and $Y \setminus \operatorname{ran}\alpha$ is infinite. Since $\theta : Y \to X$ is a bijection, $\alpha\theta : X \to X$ is one-to-one and

$$X \setminus \operatorname{ran}\alpha\theta = Y\theta \setminus (\operatorname{ran}\alpha)\theta = (Y \setminus \operatorname{ran}\alpha)\theta.$$

Hence $|X \setminus \operatorname{ran}\alpha\theta| = |Y \setminus \operatorname{ran}\alpha|$ since θ is one-to-one. This shows that $\alpha \mapsto \alpha\theta$ is a map from $BL(X,Y)$ into BL_X. This map is one-to-one since θ is one-to-one. If $\alpha \in BL_X$, then we can show similarly as above that $\alpha\theta^{-1} \in BL(X,Y)$. Also, $(\alpha\theta^{-1})\theta = \alpha$ for every $\alpha \in BL_X$. We therefore deduce that the map $\alpha \mapsto \alpha\theta$ is a bijection of $BL(X,Y)$ onto BL_X. This map is a homomorphism from $(BL(X,Y),\theta)$ onto BL_X since for $\alpha, \beta \in BL(X,Y), (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta)$. □

Lemma 2.3.3. *If $\theta : Y \to X$ is a bijection, then $(OBL(X,Y),\theta) \cong OBL_X$ through the map $\alpha \mapsto \alpha\theta$.*

Proof. If $\alpha \in OBL(X,Y)$, then $\operatorname{ran}\alpha\theta = X$ since $\operatorname{ran}\alpha = Y$ and $\operatorname{ran}\theta = X$, and for $x \in X, (x\alpha\theta)(\alpha\theta)^{-1} = (x\alpha\theta)\theta^{-1}\alpha^{-1} = (x\alpha)\alpha^{-1}$ which is infinite. Hence $\alpha \mapsto \alpha\theta$ is a map from $OBL(X,Y)$ into OBL_X. This map is one-to-one since θ is one-to-one. If $\alpha \in OBL_X$, then we have similarly that $\alpha\theta^{-1} \in OBL(X,Y)$. Also, $(\alpha\theta^{-1})\theta = \alpha$ for every $\alpha \in OBL_X$. If $\alpha, \beta \in BL(X,Y), (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta)$. Hence $\alpha \mapsto \alpha\theta$ is an isomorphism of $(OBL(X,Y),\theta)$ onto OBL_X. □

Lemma 2.3.4. *The semigroup BL_X has no minimal quasi-ideal.*

Proof. Let $\alpha \in BL_X$. Then α is one-to-one and $X \setminus \operatorname{ran}\alpha$ is infinite. Since $\alpha^2 \in (\alpha)_q$, $(\alpha^2)_q \subseteq (\alpha)_q$. We claim that $(\alpha^2)_q \subsetneq (\alpha)_q$. Suppose on the contrary that $(\alpha^2)_q = (\alpha)_q$. By Theorem 1.2, $\alpha = \alpha^2$ or $\alpha = \beta\alpha^2$ for some $\beta \in BL_X$. But α is one-to-one, so we have $1_X = \alpha$ or $1_X = \beta\alpha$ which implies that $\operatorname{ran}\alpha = X$. This is contrary to the fact that $X \setminus \operatorname{ran}\alpha$ is infinite. Hence we have the claim. Since $\alpha \in BL_X$ is arbitrary, we deduce that BL_X has no minimal quasi-ideal. $\square$

Lemma 2.3.5. *The semigroup OBL_X has no minimal quasi-ideal.*

Proof. Let $\alpha \in OBL_X$ be arbitrary. Then $\operatorname{ran}\alpha = X$ and $|(x\alpha)\alpha^{-1}| > 1$ for every $x \in X$ and $(\alpha^2)_q \subseteq (\alpha)_q$. Suppose that $(\alpha^2)_q = (\alpha)_q$. By Theorem 1.2, $\alpha = \alpha^2$ or $\alpha = \alpha^2\beta$ for some $\beta \in OBL_X$. Since α is onto, $1_X = \alpha$ or $1_X = \alpha\beta$. Thus α is one-to-one. This is contrary to that $|(x\alpha)\alpha^{-1}| > 1$ for every $x \in X$. Hence $(\alpha^2)_q \subsetneq (\alpha)_q$. We therefore have that OBL_X has no minimal quasi-ideal. $\square$

The next two theorems show that the semigroup $(BL(X,Y),\theta)$ where $\theta \in M(Y,X)$ and the semigroup $(OBL(X,Y),\theta)$ where $\theta \in E(Y,X)$ have no minimal quasi-ideal.

Theorem 2.3.6. *For $\theta \in M(Y,X)$, the semigroup $(BL(X,Y),\theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in M(Y,X)$. Then $\theta : Y \to X$ is one-to-one.

Case 1: θ is onto. Then $\theta : Y \to X$ is a bijection. By Lemma 2.3.2, $(BL(X,Y),\theta) \cong BL_X$. But BL_X has no minimal quasi-ideal by Lemma 2.3.4, so $(BL(X,Y),\theta)$ has no minimal quasi-ideal.

Case 2: θ is not onto. Let $\alpha \in BL(X,Y)$ be arbitrary. Then $\alpha\theta\alpha \in (\alpha)_q$ in $(BL(X,Y),\theta)$, and so $(\alpha\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $(\alpha\theta\alpha)_q = (\alpha)_q$. By Theorem 1.2, $\alpha = \alpha\theta\alpha$ or $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in BL(X,Y)$. Since α is one-to-one, $1_X = \alpha\theta$ or $1_X = \beta\theta\alpha\theta$ which implies that θ is onto, a contradiction. This shows that $(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$. We then deduce that $(BL(X,Y),\theta)$ has no minimal quasi-ideal.

Hence the theorem is proved, as desired. $\square$

Theorem 2.3.7. *For $\theta \in E(Y, X)$, the semigroup $(OBL(X, Y), \theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in E(Y, X)$. Then $\theta : Y \to X$ is onto.

Case 1: θ is one-to-one. Then $\theta : Y \to X$ is a bijection. By Lemma 2.3.3, $(OBL(X, Y), \theta) \cong OBL_X$. But from Lemma 2.3.5, OBL_X has no minimal quasi-ideal, so $(OBL(X, Y), \theta)$ has no minimal quasi-ideal.

Case 2: θ is not one-to-one. Let $\alpha \in OBL(X, Y)$ be arbitrary. Then ran $\alpha = Y$ and $(x\alpha)\alpha^{-1}$ is infinite for every $x \in X$. Since $\alpha\theta\alpha \in (\alpha)_q$, $(\alpha\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $(\alpha\theta\alpha)_q = (\alpha)_q$. Then by Theorem 1.2, $\alpha = \alpha\theta\alpha$ or $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in OBL(X, Y)$. Since α is onto, $1_Y = \theta\alpha$ or $1_Y = \theta\alpha\theta\beta$ which implies that θ is one-to-one, a contradiction. Thus $(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$ and hence $(\alpha)_q$ is not a minimal quasi-ideal of $(OBL(X, Y), \theta)$. Therefore $(OBL(X, Y), \theta)$ has no minimal quasi-ideal.

Hence the theorem is proved, as required. $\square$

CHAPTER III

MINIMAL QUASI-IDEALS OF GENERALIZED LINEAR TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to characterize a linear transformation α so that $(\alpha)_q$ is a minimal quasi-ideal of our target generalized linear transformation semigroups.

Let us recall the notations which are used throughout this chapter. Let V and W be vector spaces over a field F, $m, n \in \mathbb{N}$,

$L_F(V) =$ the set of all linear transformations from V into V,

$G_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is an isomorphism }\}$,

$M_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is one-to-one }\}$,

$E_F(V) = \{\alpha \in L_F(V) \mid \operatorname{Im}\alpha = V\}$,

$AM_F(V) = \{\alpha \in L_F(V) \mid \dim_F \operatorname{Ker}\alpha \text{ is finite }\}$,

$AE_F(V) = \{\alpha \in L_F(V) \mid \dim_F (V/\operatorname{Im}\alpha) \text{ is finite }\}$,

$BL_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is one-to-one and } \dim_F (V/\operatorname{Im}\alpha) \text{ is infinite }\}$

where $\dim_F V$ is infinite ,

$OBL_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is onto and } \dim_F \operatorname{Ker}\alpha \text{ is infinite }\}$

where $\dim_F V$ is infinite,

$L_F(V, W) =$ the set of all linear transformations from V into W,

$L_F(V, W, k) = \{\alpha \in L_F(V, W) \mid \operatorname{rank}\alpha < k\}$

where k is a cardinal number greater than 0,

$(\mathrm{Im}\ \alpha)\theta\mu = \mathrm{Im}\ \lambda \subseteq \mathrm{Im}\ \alpha = Fu$. Consequently, $u\theta\mu = cu$ for some $c \in F$. By Lemma 3.1.1(ii), $\alpha\theta\mu = c\alpha$. Hence $\lambda = c\alpha \in F\alpha$. This proves that $(\alpha)_q = F\alpha$, as required. $\square$

Lemma 3.1.4. *Let $\alpha \in L_F(V, W, k)$ and $\theta \in L_F(W, V)$ be such that $Im\,\alpha \nsubseteq Ker\,\theta$ and $Im\,\theta \nsubseteq Ker\,\alpha$. If $rank\,\alpha = 1$, then $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(L_F(V, W, k), \theta)$.*

Proof. By Lemma 3.1.3, $(\alpha)_q = F\alpha$. Let $\beta \in (\alpha)_q \setminus \{0\}$. Then $\beta = a\alpha$ for some $a \in F \setminus \{0\}$. Consequently, $\mathrm{Im}\ \beta = \mathrm{Im}\ \alpha$ and $\mathrm{Ker}\ \beta = \mathrm{Ker}\ \alpha$. By applying Lemma 3.1.3 to β, we get $(\beta)_q = F\beta$. But $F\beta = Fa\alpha = F\alpha$, so $(\beta)_q = (\alpha)_q$. Hence $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$. $\square$

Lemma 3.1.5. *Let $\alpha \in L_F(V, W, k)$ and $\theta \in L_F(W, V)$ be such that $Im\,\alpha \nsubseteq Ker\,\theta$ and $Im\,\theta \nsubseteq Ker\,\alpha$. If $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(L_F(V, W, k), \theta)$, then $rank\,\alpha = 1$.*

Proof. Let $u \in \mathrm{Im}\ \alpha \setminus \mathrm{Ker}\ \theta$ and $u' \in \mathrm{Im}\ \theta \setminus \mathrm{Ker}\ \alpha$. Then $0 \neq u\theta \in V, u'\alpha \neq 0$ and $z\theta = u'$ for some $z \in W \setminus \{0\}$. Let B be a basis of V containing $u\theta$ and let $\beta \in L_F(V, W)$ be defined by

$$v\beta = \begin{cases} z & \text{if } v = u\theta, \\ 0 & \text{if } v \in B \setminus \{u\theta\}. \end{cases}$$

Then we have $\beta \in L_F(V, W, k)$ and

$$0 \neq u'\alpha = z\theta\alpha = u\theta\beta\theta\alpha \in (\mathrm{Im}\,\alpha)\theta\beta\theta\alpha = \mathrm{Im}\,(\alpha\theta\beta\theta\alpha).$$

Therefore $\alpha\theta\beta\theta\alpha \neq 0$, so $\mathrm{rank}\ (\alpha\theta\beta\theta\alpha) = 1$ since $\mathrm{rank}\ (\alpha\theta\beta\theta\alpha) \leqslant \mathrm{rank}\ \beta = 1$. Also, $\alpha\theta\beta\theta\alpha \in L_F(V, W, k)\theta\alpha \cap \alpha\theta L_F(V, W, k) \subseteq (\alpha)_q$ by Theorem 1.2. Since $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$, we have $(\alpha\theta\beta\theta\alpha)_q = (\alpha)_q$. By Lemma 3.1.2, rank $\alpha =$ rank $(\alpha\theta\beta\theta\alpha) = 1$. $\square$

Lemma 3.1.6. *If $\alpha \in L_F(V,W,k) \setminus \{0\}$ and $\theta \in L_F(W,V)$ are such that $Im\,\alpha \subseteq Ker\,\theta$ or $Im\,\theta \subseteq Ker\,\alpha$, then $(\alpha)_q = \{0,\alpha\}$ in the semigroup $(L_F(V,W,k),\theta)$. Hence $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$.*

Proof. We have that $\alpha\theta = 0$ if $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$ and $\theta\alpha = 0$ if $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$. Therefore, by Theorem 1.2, we have $(\alpha)_q = \{0,\alpha\}$ and hence $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$. $\square$

From Lemma 3.1.6, we directly obtain the following result.

Corollary 3.1.7. *Every nonzero principal quasi-ideal of the semigroup $(L_F(V,W,k),0)$ is minimal.*

We note here that if S is a zero semigroup with zero 0, then for every $x \in S \setminus \{0\}$, $(x)_q = \{0,x\}$ by Theorem 1.2, so it is minimal. We can see that $(L_F(V,W,k),0)$ is a zero semigroup. Hence Corollary 3.1.7 can be considered from this fact instead of a consequence of Lemma 3.1.6.

One of our main results of this section is the following .

Theorem 3.1.8. *For $\alpha \in L_F(V,W,k) \setminus \{0\}$ and $\theta \in L_F(W,V)$, $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(L_F(V,W,k),\theta)$ if and only if one of the following three conditions holds:*

(i) rank $\alpha = 1$, $Im\,\alpha \nsubseteq Ker\,\theta$ and $Im\,\theta \nsubseteq Ker\,\alpha$.

(ii) $Im\,\alpha \subseteq Ker\,\theta$.

(iii) $Im\,\theta \subseteq Ker\,\alpha$.

If (i) holds, then $(\alpha)_q = F\alpha$ and if (ii) or (iii) holds, then $(\alpha)_q = \{0, \alpha\}$.

Proof. Assume that $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$. To prove that one of (i), (ii) or (iii) holds, suppose that both (ii) and (iii) are fault. Then $\operatorname{Im}\alpha \not\subseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \not\subseteq \operatorname{Ker}\alpha$. By Lemma 3.1.5, we have that $\operatorname{rank}\alpha = 1$. Hence (i) holds.

The converse is obtained directly from Lemma 3.1.4 and Lemma 3.1.6.

The remaining conclusion of the theorem follows directly from Lemma 3.1 3 and Lemma 3.1.6. □

The proof can be given similarly to that of Theorem 3.1.8 for the semigroup $(\overline{L}_F(V, W, k), \theta)$ where $\theta \in L_F(W, V)$ to obtain the following.

Theorem 3.1.9. *For $\alpha \in \overline{L}_F(V, W, k) \setminus \{0\}$ and $\theta \in L_F(W, V)$, $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(\overline{L}_F(V, W, k), \theta)$ if and only if one of the following three conditions holds:*

(i) $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \not\subseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \not\subseteq \operatorname{Ker}\alpha$.

(ii) $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$.

(iii) $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$.

If (i) is true, then $(\alpha)_q = F\alpha$, and $(\alpha)_q = \{0, \alpha\}$ if (ii) or (iii) holds.

If k is a cardinal number greater than $\dim_F W$, then $L_F(V, W, k) = L_F(V, W)$. Hence from Theorem 3.1.8 we have

Corollary 3.1.10. *For $\alpha \in L_F(V, W) \setminus \{0\}$ and $\theta \in L_F(W, V)$, $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(L_F(V, W), \theta)$ if and only if one of the following conditions holds:*

(i) $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \not\subseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \not\subseteq \operatorname{Ker}\alpha$.

(ii) $Im\,\alpha \subseteq Ker\,\theta$.

(iii) $Im\,\theta \subseteq Ker\,\alpha$.

If (i) holds, then $(\alpha)_q = F\alpha$ *and if (ii) or (iii) holds, then* $(\alpha)_q = \{0, \alpha\}$.

The following corollary gives interesting characterizations of the standard linear transformation semigroup $L_F(V)$ (under composition).

Corollary 3.1.11. *For* $\alpha \in L_F(V) \smallsetminus \{0\}$, *the following statements are equivalent.*

(i) $(\alpha)_q$ *is a minimal quasi-ideal of* $L_F(V)$.

(ii) $rank\,\alpha = 1$.

(iii) $(\alpha)_q = F\alpha$.

Proof. From Corollary 3.1.10 and Lemma 3.1.3, we have (i)$\Leftrightarrow$ (ii) and (ii)$\Rightarrow$ (iii) follows from Lemma 3.1.3. To prove (iii)$\Rightarrow$ (i), assume that $(\alpha)_q = F\alpha$. Let $\beta \in (\alpha)_q \smallsetminus \{0\}$. Then $\beta = a\alpha$ for some $a \in F \smallsetminus \{0\}$. Consequently, we have

$$\begin{aligned}
(\beta)_q &= L_F(V)\beta \cap \beta L_F(V) && \text{from Theorem 1.2} \\
&= L_F(V)a\alpha \cap a\alpha L_F(V) \\
&= (aL_F(V))\alpha \cap \alpha(aL_F(V)) \\
&= L_F(V)\alpha \cap \alpha L_F(V) && \text{since } a \neq 0 \\
&= (\alpha)_q.
\end{aligned}$$

This proves that $(\beta)_q = (\alpha)_q$ for every nonzero $\beta \in (\alpha)_q$. Hence $(\alpha)_q$ is a minimal quasi-ideal of $L_F(V)$. □

Finally, we show by Theorem 3.1.8 the existence of a minimal quasi-ideal of every semigroup $(L_F(V, W, k), \theta)$.

Corollary 3.1.12. *A minimal quasi-ideal always exists in every semigroup* $(L_F(V, W, k), \theta)$ *where* $\theta \in L_F(W, V)$.

Proof. If $L_F(V, W, k) = \{0\}$, then $L_F(V, W, k)$ is itself the minimal quasi-ideal of $(L_F(V, W, k), \theta)$ (see Chapter I, page 3). Assume that $(L_F(V, W, k) \neq \{0\}$. Then $V \neq \{0\}$ $W \neq \{0\}$ and $k > 1$. If $\theta = 0$, then by Corollary 3.1.7, for every nonzero $\alpha \in L_F(V, W, k)$, $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$.

Next, assume that $\theta \neq 0$. Then $\operatorname{Ker} \theta \neq W$ and $\operatorname{Im} \theta \neq \{0\}$. Let $u \in \operatorname{Im} \theta \smallsetminus \{0\}$ and $w \in W \smallsetminus \operatorname{Ker} \theta$. Let B be a basis of V containing u and define $\alpha \in L_F(V, W, k)$ by

$$v\alpha = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v \in B \smallsetminus \{u\}. \end{cases}$$

Then $\operatorname{rank} \alpha = 1, w \in \operatorname{Im} \alpha \smallsetminus \operatorname{Ker} \theta$ and $u \in \operatorname{Im} \theta \smallsetminus \operatorname{Ker} \alpha$. It therefore follows from Theorem 3.1.8 that $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$. □

Remark 3.1.13. The aim of this remark is to clarify the nature of the product on a minimal quasi-ideal of the semigroup $(L_F(V, W, k), \theta)$. Let $\alpha \in L_F(V, W, k) \smallsetminus \{0\}$ be such that $(\alpha)_q$ is a minimal quasi-ideal of the semigroup $(L_F(V, W, k), \theta)$. From Theorem 1.6, $(\alpha)_q$ must be either a zero subsemigroup or a subgroup with zero of $(L_F(V, W, k), \theta)$. If $\operatorname{Im} \alpha \subseteq \operatorname{Ker} \theta$ or $\operatorname{Im} \theta \subseteq \operatorname{Ker} \alpha$, then $\alpha\theta\alpha = 0$, by Lemma 3.1.6, $(\alpha)_q = \{0, \alpha\}$ which is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$. For this case, $(\alpha)_q$ is a zero subsemigroup of $(L_F(V, W, k), \theta)$.

If $\operatorname{rank} \alpha = 1$, $\operatorname{Im} \alpha \nsubseteq \operatorname{Ker} \theta$ and $\operatorname{Im} \theta \nsubseteq \operatorname{Ker} \alpha$, by Lemma 3.1.3, $(\alpha)_q = F\alpha$ and it is a minimal quasi-ideal of $(L_F(V, W, k), \theta)$ by Lemma 3.1.4. For this case, $(\alpha)_q$ can be either a zero subsemigroup or a subgroup with zero of $(L_F(V, W, k), \theta)$. If $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is clearly a zero subsemigroup of $(L_F(V, W, k), \theta)$. Next, assume that $\alpha\theta\alpha \neq 0$. Then $(\alpha)_q \setminus \{0\}$ is a subgroup of $(L_F(V,W,k),\theta)$ which is equivalent to that $(F \setminus \{0\})\alpha$ is a subgroup of $(L_F(V,W,k),\theta)$. Since $\alpha\theta\alpha \neq 0$, $\theta\alpha \neq 0$. Since rank $\alpha = 1$, there is an element $u \in W$ such that Im $\alpha = Fu$. Then $u\theta\alpha = cu$ for some $c \in F$. Thus we deduce from Lemma 3.1.1(ii) that $\alpha\theta\alpha = c\alpha$. Since $\alpha\theta\alpha \neq 0$, $c \neq 0$. Hence for all $a, b \in F \setminus \{0\}$,

$$(a\alpha)\theta(b\alpha) = ab\alpha\theta\alpha = abc\alpha \in (F \setminus \{0\})\alpha,$$

$$(a\alpha)\theta(c^{-1}\alpha) = ac^{-1}\alpha\theta\alpha = ac^{-1}c\alpha = a\alpha,$$

$$(a\alpha)\theta(a^{-1}(c^{-1})^2)\alpha = aa^{-1}(c^{-1})^2\alpha\theta\alpha = (c^{-1})^2c\alpha = c^{-1}\alpha$$

This shows that $(F \setminus \{0\})\alpha$ is indeed a subgroup of $(L_F(V,W,k),\theta)$ where $c^{-1}\alpha$ is its identity and for $a \in F \setminus \{0\}$, $a^{-1}(c^{-1})^2\alpha$ is the inverse of $a\alpha$. The following examples show that each of the cases $\alpha\theta\alpha = 0$ and $\alpha\theta\alpha \neq 0$ can occur. Let V be a vector space over F of dimension 4. Let $\{u_1, u_2, u_3, u_4\}$ be a basis of V. Define $\theta, \alpha_1, \alpha_2 \in L_F(V)$ by

$$u_1\theta = u_1, u_2\theta = u_2, u_3\theta = u_3, u_4\theta = 0,$$

$$u_1\alpha_1 = u_3\alpha_1 = u_1, u_2\alpha_1 = u_4\alpha_1 = 0,$$

$$u_1\alpha_2 = u_2\alpha_2 = u_3, u_3\alpha_2 = u_4\alpha_2 = 0.$$

Then Ker $\theta = \langle u_4 \rangle$, Im $\theta = \langle u_1, u_2, u_3 \rangle$, Ker $\alpha_1 = \langle u_1 - u_3, u_2, u_4 \rangle$, Im $\alpha_1 = \langle u_1 \rangle$, Ker $\alpha_2 = \langle u_1 - u_2, u_3, u_4 \rangle$ and Im $\alpha_2 = \langle u_3 \rangle$. Then rank $\alpha_1 = 1 =$ rank α_2, Im $\alpha_1 \not\subseteq$ Ker θ, Im $\theta \not\subseteq$ Ker α_1, Im $\alpha_2 \not\subseteq$ Ker θ and Im $\theta \not\subseteq$ Ker α_2. Moreover,

$$u_1\alpha_1\theta\alpha_1 = u_1 \text{ and } \{u_1, u_2, u_3, u_4\}\alpha_2\theta\alpha_2 = \{u_3, 0\}\theta\alpha_2 = \{u_3, 0\}\alpha_2 = \{0\},$$

so $\alpha_1\theta\alpha_1 \neq 0$ and $\alpha_2\theta\alpha_2 = 0$.

A nonzero ideal M of a semigroup S with zero 0 is called a 0-*minimal ideal* of S if M contains no ideals of S other than $\{0\}$ or itself. Since every ideal of S is a quasi-ideal, it follows that if a minimal quasi-ideal Q of S is an ideal, then Q is a 0-minimal ideal of S. The last thorem of this section characterizes when every minimal quasi-ideal of a nonzero semigroup $(L_F(V,W,k),\theta)$ is a 0-minimal ideal.

Theorem 3.1.14. *In a nonzero semigroup $(L_F(V,W.k),\theta)$ where $\theta \in L_F(W,V)$, every minimal quasi-ideal is a 0-minimal ideal if and only if $\theta = 0$ or $dim_F V = dim_F W = 1$.*

Proof. Let $0 \neq \alpha \in L_F(V,W,k)$ be such that $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$. First, assume that $\theta = 0$. We have by Lemma 3.1.6 that $(\alpha)_q = \{0,\alpha\}$. Since $\theta = 0$, $(L_F(V,W,k),\theta)$ is a zero semigroup. But $|(\alpha)_q| = 2$, so we have that $(\alpha)_q$ is a 0-minimal ideal of $(L_F(V,W,k),\theta)$. Next, assume that $\theta \neq 0$ and $\dim_F V = \dim_{FF} W = 1$. Then $\alpha : V \to W$ and $\theta : W \to V$ are isomorphisms. By Lemma 3.1.3, $(\alpha)_q = F\alpha$. Let $u \in V \setminus \{0\}$. Then $0 \neq u\alpha \in W$. Since $\dim_F V = \dim_F W = 1$, it follows that $V = Fu$ and $W = F(u\alpha)$. If $\beta \in L_F(V,W,k)$, then $u\beta\theta\alpha, u\alpha\theta\beta \in W = F(u\alpha)$, so $u\beta\theta\alpha = a(u\alpha) = u(a\alpha)$ and $u\alpha\theta\beta = b(u\alpha) = u(b\alpha)$ for some $a,b \in F$. This implies that $\beta\theta\alpha = a\alpha \in F\alpha$ and $\alpha\theta\beta = b\alpha \in F\alpha$ since $V = Fu$. Hence $(\alpha)_q$ is an ideal of $(L_F(V,W,k),\theta)$. But since every ideal of a semigroup is a quasi-ideal, we deduce that $(\alpha)_q$ is a 0-minimal ideal of $(L_F(V,W,k),\theta)$.

For the converse, first assume that $\theta \neq 0$ and $\dim_F V > 1$. Then $\operatorname{Ker}\theta \neq W$. Let $w \in W \setminus \operatorname{Ker}\theta$. Thus $0 \neq w\theta \in V$. Let B be a basis of V containing $w\theta$ and let $u \in B \setminus \{w\theta\}$. Define $\alpha \in L_F(V,W,k)$ by

$$v\alpha = \begin{cases} w & \text{if } v = w\theta \text{ or } v = u, \\ 0 & \text{if } v \in B \setminus \{w\theta, u\}. \end{cases}$$

Then $\operatorname{rank}\alpha = 1$, $w \in \operatorname{Im}\alpha \setminus \operatorname{Ker}\theta$ and $w\theta \in \operatorname{Im}\theta \setminus \operatorname{Ker}\alpha$, so by Lemma 3.1.3 and Lemma 3.1.4, we have respectively that $(\alpha)_q = F\alpha$ and $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$. Define $\beta \in L_F(V,W,k)$ by

$$v\beta = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v \in B \setminus \{u\}. \end{cases}$$

Then $u\beta\theta\alpha = w\theta\alpha = w \neq 0$, $(w\theta)\beta\theta\alpha = 0$ and $(w\theta)(c\alpha) = cw \neq 0$ for all $c \in F \setminus \{0\}$. Consequently, $\beta\theta\alpha \notin F\alpha = (\alpha)_q$. Therefore $(\alpha)_q$ is not an ideal of $(L_F(V,W,k),\theta)$.

Finally, assume that $\theta \neq 0$ and $\dim_F W > 1$. Let $w \in W \setminus \operatorname{Ker}\theta$. Since $\dim_F W > 1$, there exists an element $w' \in W$ such that w and w' are linearly independent over F. Let B be a basis of V containing $w\theta$. Define $\alpha \in L_F(V,W,k)$ by

$$v\alpha = \begin{cases} w & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \setminus \{w\theta\}. \end{cases}$$

Then $\operatorname{rank}\alpha = 1$, $w \in \operatorname{Im}\alpha \setminus \operatorname{Ker}\theta$ and $w\theta \in \operatorname{Im}\theta \setminus \operatorname{Ker}\alpha$. Thus $(\alpha)_q = F\alpha$ and $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),\theta)$ by Lemma 3.1.3 and Lemma 3.1.4, respectively. Define $\beta \in L_F(V,W,k)$ by

$$v\beta = \begin{cases} w' & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \setminus \{w\theta\}. \end{cases}$$

Then $(w\theta)\alpha\theta\beta = w\theta\beta = w' \neq cw = (w\theta)(c\alpha)$ for every $c \in F$ since w and w' are linearly independent. Hence $\alpha\theta\beta \notin F\alpha$ and thus $(\alpha)_q$ is not an ideal of $(L_F(V,W,k),\theta)$.

Hence the theorem is completely proved. $\square$

3.2 The Semigroups $(M_F(V,W),\theta)$, $(E_F(V,W),\theta)$, $(AM_F(V,W),\theta)$ and $(AE_F(V,W),\theta)$

In this section, we give characterizations determining minimal quasi-ideals of the semigroups $(M_F(V,W),\theta)$ where $\theta \in M_F(W,V)$, $(E_F(V,W),\theta)$ where $\theta \in E_F(W,V)$, $(AM_F(V,W),\theta)$ where $\theta \in AM_F(W,V)$ and $(AE_F(V,W),\theta)$ where $\theta \in AE_F(W,V)$.

The following three lemmas are needed.

Lemma 3.2.1. *The following statements hold.*

(i) $M_F(V,W) \neq \varnothing$ *and* $M_F(W,V) \neq \varnothing$ *if and only if* $\dim_F V = \dim_F W$.

(ii) $E_F(V,W) \neq \varnothing$ *and* $E_F(W,V) \neq \varnothing$ *if and only if* $\dim_F V = \dim_F W$.

(iii) If φ *is an isomorphism of* V *onto* W *and* $\theta \in M_F(W,V)$, *then* $(M_F(V,W),\theta) \cong (M_F(V),\varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

(iv) If φ *is an isomorphism of* V *onto* W *and* $\theta \in E_F(W,V)$, *then* $(E_F(V,W),\theta) \cong (E_F(V),\varphi\theta)$ *by* $\alpha \mapsto \alpha\varphi^{-1}$.

Proof. (i) Let $\alpha \in M_F(V,W)$ and $\beta \in M_F(W,V)$. Since α and β are both one-to-one linear transformations, we have $V \cong \operatorname{Im}\alpha$ and $W \cong \operatorname{Im}\beta$ as vector spaces over F. Also, $\operatorname{Im}\alpha$ is a subspace of W and $\operatorname{Im}\beta$ is a subspace of V. These imply that

$$\dim_F V = \dim_F \operatorname{Im}\alpha \leqslant \dim_F W = \dim_F \operatorname{Im}\beta \leqslant \dim_F V,$$

hence $\dim_F V = \dim_F W$.

Conversely, let $\dim_F V = \dim_F W$. Then $V \cong W$ as vector spaces over F. Let λ be an isomorphism of V onto W. Then $\lambda \in M_F(V,W)$ and $\lambda^{-1} \in M_F(W,V)$.

(ii) Let $\alpha \in E_F(V,W)$ and $\beta \in E_F(W,V)$. Then $V/\operatorname{Ker}\alpha \cong W$ and $W/\operatorname{Ker}\beta \cong V$. Hence

$$\dim_F V \geqslant \dim_F(V/\operatorname{Ker}\alpha) = \dim_F W \geqslant \dim_F(W/\operatorname{Ker}\beta) = \dim_F V,$$

and thus $\dim_F V = \dim_F W$.

Conversely, let $\dim_F V = \dim_F W$. Then $V \cong W$ as vector spaces over F. Let λ be an isomorphism of V onto W. Then $\lambda \in E_F(V,W)$ and $\lambda^{-1} \in E_F(W,V)$.

(iii) can be proved similarly to the proof of Lemma 2.2.1(iii).

(iv) can be proved similarly to the proof of Lemma 2.2.1(iv). $\square$

Lemma 3.2.2. *The following statements are ture.*

(i) $AM_F(V,W) \neq \varnothing$ *and* $AM_F(W,V) \neq \varnothing$ *if and only if either both* $\dim_F V$ *and* $\dim_F W$ *are finite or both* $\dim_F V$ *and* $\dim_F W$ *are infinite and* $\dim_F V = \dim_F W$.

(ii) $AE_F(V,W) \neq \varnothing$ *and* $AE_F(W,V) \neq \varnothing$ *if and only if either both* $\dim_F V$ *and* $\dim_F W$ *are finite or both* $\dim_F V$ *and* $\dim_F W$ *are infinite and* $\dim_F V = \dim_F W$.

(iii) If φ *is an isomorphism of* V *onto* W *and* $\theta \in AM_F(W,V)$, *then* $(AM_F(V,W),\theta) \cong (AM_F(V),\varphi\theta)$ *through* $\alpha \mapsto \alpha\varphi^{-1}$.

(iv) If φ *is an isomorphism of* V *onto* W *and* $\theta \in AE_F(W,V)$, *then* $(AE_F(V,W),\theta) \cong (AE_F(V),\varphi\theta)$ *through* $\alpha \mapsto \alpha\varphi^{-1}$.

Proof. (i) Let $\alpha \in AM_F(V,W)$ and $\beta \in AM_F(W,V)$. Then $\dim_F \mathrm{Ker}\,\alpha < \infty$ and $\dim_F \mathrm{Ker}\,\beta < \infty$. First, assume that $\dim_F V < \infty$. Then $\dim_F \mathrm{Im}\,\beta < \infty$. But $\dim_F W = \dim_F \mathrm{Ker}\,\beta + \dim_F \mathrm{Im}\,\beta$, so $\dim_F W < \infty$. Similarly, $\dim_F W < \infty$ implies that $\dim_F V < \infty$. This proves that $\dim_F V < \infty$ if and only if $\dim_F W < \infty$. Hence $\dim_F V$ is infinite if and only if $\dim_F W$ is infinite. Moreover, if $\dim_F V$ and $\dim_F W$ are infinite, then

$$\begin{aligned}
\dim_F V &= \dim_F \mathrm{Ker}\,\alpha + \dim_F \mathrm{Im}\,\alpha \\
&= \dim_F \mathrm{Im}\,\alpha && \text{since } \dim_F \mathrm{Ker}\,\alpha < \infty \\
&\leqslant \dim_F W \\
&= \dim_F \mathrm{Ker}\,\beta + \dim_F \mathrm{Im}\,\beta
\end{aligned}$$

$$= \dim_F \operatorname{Im}\beta \qquad \text{since} \ \dim_F \operatorname{Ker}\beta < \infty$$

$$\leqslant \dim_F V$$

which implies that $\dim_F V = \dim_F W$.

Conversely, assume that either $\dim_F V < \infty$ and $\dim_F W < \infty$ or $\dim_F V = \dim_F W$ which is infinite. If $\dim_F V < \infty$ and $\dim_F W < \infty$, then $AM_F(V, W) = L_F(V, W) \neq \varnothing$ and $AM_F(W, V) = L_F(W, V) \neq \varnothing$. Let $\dim_F V = \dim_F W$. Then $V \cong W$ as vector spaces over F. Let λ be an isomorphism from V onto W. Thus $\lambda \in M_F(V, W) \subseteq AM_F(V, W)$ and $\lambda^{-1} \in M_F(W, V) \subseteq AM_F(W, V)$.

(ii) Let $\alpha \in AE_F(V, W)$ and $\beta \in AE_F(W, V)$. Then $\dim_F(W/\operatorname{Im}\alpha) < \infty$ and $\dim_F(V/\operatorname{Im}\beta) < \infty$. Since

$$\begin{aligned} \dim_F W &= \dim_F(W/\operatorname{Im}\alpha) + \dim_F \operatorname{Im}\alpha \\ &\leqslant \dim_F(W/\operatorname{Im}\alpha) + \dim_F \operatorname{Im}\alpha + \dim_F \operatorname{Ker}\alpha \\ &= \dim_F(W/\operatorname{Im}\alpha) + \dim_F V \qquad (1) \end{aligned}$$

and similarly

$$\dim_F V \leqslant \dim_F(V/\operatorname{Im}\beta) + \dim_F W, \qquad (2)$$

it follows that $\dim_F V < \infty$ if and only if $\dim_F W < \infty$. Hence $\dim_F V$ is infinite if and only if $\dim_F W$ is infinite. If $\dim_F V$ and $\dim_F W$ are infinite, then

$$\begin{aligned} \dim_F W &\leqslant \dim_F(W/\operatorname{Im}\alpha) + \dim_F V && \text{from (1)} \\ &= \dim_F V && \text{since } \dim_F(W/\operatorname{Im}\alpha) < \infty \\ &\leqslant \dim_F(V/\operatorname{Im}\beta) + \dim_F W && \text{from (2)} \\ &= \dim_F W && \text{since } \dim_F(V/\operatorname{Im}\beta) < \infty \end{aligned}$$

which implies that $\dim_F V = \dim_F W$.

(iii) and (iv) can be proved similarly to the proof of Lemma 2.2.1(iii). $\square$

Lemma 3.2.3. *The following statements hold.*

(*i*) *For $\theta \in M_F(V)$, the semigroup $(M_F(V), \theta)$ has a minimal quasi-ideal if and only if $dim_F V < \infty$.*

(*ii*) *For $\theta \in E_F(V)$, the semigroup $(E_F(V), \theta)$ has a minimal quasi-ideal if and only if $dim_F V < \infty$.*

Proof. If $\dim_F V < \infty$, then $M_F(V) = G_F(V) = E_F(V)$, so $M_F(V)$ $[E_F(V)]$ is itself a minimal quasi-ideal of $(M_F(V), \theta)$ $[(E_F(V), \theta)]$ where $\theta \in M_F(V)$ $[E_F(V)]$ since (G_X, θ) is a group.

Next, to show that if $\dim_F V$ is infinite, then neither $(M_F(V), \theta)$ with $\theta \in M_F(V)$ nor $(E_F(V), \theta)$ with $\theta \in E_F(V)$ has a minimal qausi-ideal. Assume that $\dim_F V$ is infinite. Let B be a basis of V and $u \in B$. Then $|B| = |B \smallsetminus \{u\}|$ since B is infinite. Then there is a bijection $\psi : B \to B \smallsetminus \{u\}$. Define $\beta, \gamma \in L_F(V)$ by

$$v\beta = v\psi \text{ for all } v \in B,$$

$$v\gamma = v\psi^{-1} \text{ for all } v \in B \smallsetminus \{u\},$$

$$u\gamma = 0.$$

It then follows that $\beta \in M_F(V), \gamma \in E_F(V)$, $\operatorname{Im}\beta = \langle B \smallsetminus \{u\}\rangle \neq V$ and $\operatorname{Ker}\gamma = \langle u \rangle$.

Let $\theta \in M_F(V)$ and $\alpha \in M_F(V)$ be arbitrary. Then $\alpha\theta\beta\theta\alpha \in (\alpha)_q$ by Theorem 1.2, so $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$ in $(M_F(V), \theta)$. Suppose that $(\alpha\theta\beta\theta\alpha)_q = (\alpha)_q$. By Theorem 1.2, $\alpha = \alpha\theta\beta\theta\alpha$ or $\alpha = \lambda\theta\alpha\theta\beta\theta\alpha$ for some $\lambda \in M_F(V)$. Then $\theta\alpha = \theta\alpha\theta\beta\theta\alpha$ or $\theta\alpha = \theta\lambda\theta\alpha\theta\beta\theta\alpha$. Since $\theta\alpha$ is one-to-one, $1_V = \theta\alpha\theta\beta$ or $1_V = \theta\lambda\theta\alpha\theta\beta$ which implies $\operatorname{Im}\beta = V$, a contradiction. This shows that $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$.

Next, let $\theta, \alpha \in E_F(V)$. Then $(\alpha\theta\gamma\theta\alpha)_q \subseteq (\alpha)_q$ in $(E_F(V), \theta)$ by Theorem 1.2. Suppose that $(\alpha\theta\gamma\theta\alpha)_q = (\alpha)_q$. From Theorem 1.2, $\alpha = \alpha\theta\gamma\theta\alpha$ or $\alpha = \alpha\theta\gamma\theta\alpha\theta\lambda$ for some $\lambda \in E_F(V)$. Then $\alpha\theta = \alpha\theta\gamma\theta\alpha\theta$ or $\alpha\theta = \alpha\theta\gamma\theta\alpha\theta\lambda\theta$. Since $\operatorname{Im}\alpha\theta = V$,

$1_V = \gamma\theta\alpha\theta$ or $1_V = \gamma\theta\alpha\theta\lambda\theta$ which implies that γ is one-to-one, a contradiction. Hence $(\alpha\theta\gamma\theta\alpha)_q \subsetneq (\alpha)_q$.

Therefore the proof is complete. $\square$

Lemma 3.2.4. *Assume that V is of infinite dimentional.*

(i) For $\theta \in AM_F(V)$, $(AM_F(V), \theta)$ has no minimal quasi-ideal.

(ii) For $\theta \in AE_F(V)$, $(AE_F(V), \theta)$ has no minimal quasi-ideal.

Proof. (i) Let $\alpha \in AM_F(V)$. Then $\dim \operatorname{Ker} \alpha\theta$ is finite. Let $u \in V \setminus \operatorname{Ker} \alpha\theta$. Then $u\alpha\theta \neq 0$, so $u\alpha \neq 0$. Let B be a basis of V containing $u\alpha\theta$. Define $\beta \in L_F(V)$ by

$$v\beta = \begin{cases} v & \text{if } v \in B \setminus \{u\alpha\theta\}, \\ 0 & \text{if } v = u\alpha\theta. \end{cases}$$

Then $\operatorname{Ker} \beta = \langle u\alpha\theta \rangle$, so $\beta \in AM_F(V)$. Thus $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$. Since $u\alpha \neq 0$, $u\alpha\theta\beta\theta\alpha = 0\theta\alpha = 0$ and $u\alpha\theta\beta\theta\alpha\theta\gamma = 0\theta\alpha\theta\gamma = 0$ for all $\gamma \in AM_F(V)$, we have $\alpha \notin (\alpha\theta\beta\theta\alpha)_q$ by Theorem 1.2. Consequently, $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$.

(ii) Let $\alpha \in AE_F(V)$. Then $\dim_F(V/\operatorname{Im} \alpha\theta) < \infty$. But $\dim_F V$ is infinite, we have $\dim_F \operatorname{Im} \alpha\theta$ is infinite. Let $u \in V$ be such that $u\alpha\theta \neq 0$. Then $u\alpha \neq 0$. Let B be a basis of V containing $u\alpha\theta$ and define β as in (i). Then $\dim_F(V/\operatorname{Im} \beta) = |\{u\alpha\theta + \operatorname{Im} \beta\}| = 1$, so $\beta \in AE_F(V)$. We obtain similarly to the proof in (i) that $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$ in $(AE_F(V), \theta)$. $\square$

Theorem 3.2.5. *The semigroup $(M_F(V, W), \theta)$, where $\theta \in M_F(W, V)$, has a minimal quasi-ideal if and only if $\dim_F V = \dim_F W < \infty$.*

If $\dim_F V = \dim_F W < \infty$, then $(M_F(V, W), \theta)$ is a group, so $M_F(V, W)$ is itself a unique minimal quasi-ideal of $(M_F(V, W), \theta)$.

Proof. By Lemma 3.2.1(i), $\dim_F V = \dim_F W$, and from Lemma 3.2.1(iii), $(M_F(V,W), \theta) \cong (M_F(V), \varphi\theta)$ where $\varphi : V \to W$ is an isomorphism. From Lemma 3.2.3(i), $(M_F(V), \varphi\theta)$ has a minimal qausi-ideal if and only if $\dim_F V < \infty$. Therefore we conclude that $(M_F(V,W), \theta)$ has a minimal quasi-ideal if and only if $\dim_F V = \dim_F W < \infty$.

If $\dim_F V = \dim_F W < \infty$, then $M_F(V) = G_F(V)$, so $(M_F(V), \varphi\theta)$ is a group, and thus the remaining conclusion follows. □

Theorem 3.2.6. *The semigroup $(E_F(V,W), \theta)$, where $\theta \in E_F(W,V)$, has a minimal quasi-ideal if and only if $\dim_F V = \dim_F W < \infty$.*

If $\dim_F V = \dim_F W < \infty$, then $(E_F(V,W), \theta)$ is a group, so $E_F(V,W)$ is itself a unique minimal quasi-ideal of $(E_F(V,W), \theta)$.

Proof. It follows from Lemma 3.2.1(ii) and (iv) that $\dim_F V = \dim_F W$ and $(E_F(V,W), \theta) \cong (E_F(V), \varphi\theta)$ where $\varphi : V \to W$ is an isomorphism. We have from Lemma 3.2.3(ii), $(E_F(V), \varphi\theta)$ has a minimal qausi-ideal if and only if $\dim_F V < \infty$. It therefore follows that $(E_F(V,W), \theta)$ has a minimal quasi-ideal if and only if $\dim_F V = \dim_F W < \infty$.

If $\dim_F V = \dim_F W < \infty$, then $E_F(V) = G_F(V)$, so $(E_F(V), \varphi\theta)$ is a group, and hence the remaining conclusion is obtained. □

Theorem 3.2.7. *For $\theta \in AM_F(W,V)$, the semigroup $(AM_F(V,W), \theta)$ has a minimal quasi-ideal if and only if $\dim_F V$ and $\dim_F W$ are finite.*

If $\dim_F V$ and $\dim_F W$ are finite, then for $\alpha \in AM_F(V,W) \setminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM_F(V,W), \theta)$ if and only if one of the following three conditions holds:

(i) $\operatorname{rank} \alpha = 1$, $\operatorname{Im} \alpha \not\subseteq \operatorname{Ker} \theta$ and $\operatorname{Im} \theta \not\subseteq \operatorname{Ker} \alpha$.

(ii) $\mathit{Im}\,\alpha \subseteq \mathit{Ker}\,\theta$.

(iii) $\mathit{Im}\,\theta \subseteq \mathit{Ker}\,\alpha$.

Moreover, $(\alpha)_q = F\alpha$ *if (i) holds and* $(\alpha)_q = \{0, \alpha\}$ *if (ii) or (iii) holds.*

Proof. First assume that $\dim_F V$ and $\dim_F W$ are finite. Then $AM_F(V,W) = L_F(V,W)$ and $AM_F(W,V) = L_F(W,V)$. By Corollary 3.1.10, for every $\alpha \in AM_F(V,W) \smallsetminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM_F(V,W), \theta)$ if and only if one of the following three conditions holds:

(i) rank $\alpha = 1$, Im $\alpha \nsubseteq$ Ker θ and Im $\theta \nsubseteq$ Ker α.

(ii) Im $\alpha \subseteq$ Ker θ.

(iii) Im $\theta \subseteq$ Ker α.

Moreover, $(\alpha)_q = F\alpha$ if (i) holds and $(\alpha)_q = \{0, \alpha\}$ if (ii) or (iii) holds.

To prove the converse, assume that $\dim_F V$ or $\dim_F W$ is not finite. By Lemma 3.2.2(i), both $\dim_F V$ and $\dim_F W$ are infinite and $\dim_F V = \dim_F W$. Then $V \cong W$ as vector spaces over F, so there exists an isomorphism φ from V onto W. By Lemma 3.2.2(iii), $(AM_F(V,W), \theta) \cong (AM_F(V), \varphi\theta)$. From Lemma 3.2.4(i), $(AM_F(V), \varphi\theta)$ has no minimal quasi-ideal, and hence neither does $(AM_F(V,W), \theta)$.

□

Theorem 3.2.8. *For* $\theta \in AE_F(W,V)$, *the semigroup* $(AE_F(V,W), \theta)$ *has a minimal quasi-ideal if and only if* $\dim_F V$ *and* $\dim_F W$ *are finite.*

If $\dim_F V$ *and* $\dim_F W$ *are finite, then for* $\alpha \in AE_F(V,W) \smallsetminus \{0\}$, $(\alpha)_q$ *is a minimal quasi-ideal of* $(AE_F(V,W), \theta)$ *if and only if one of the following three conditions holds:*

(i) $\mathit{rank}\,\alpha = 1$, $\mathit{Im}\,\alpha \nsubseteq \mathit{Ker}\,\theta$ *and* $\mathit{Im}\,\theta \nsubseteq \mathit{Ker}\,\alpha$.

(ii) $\mathit{Im}\,\alpha \subseteq \mathit{Ker}\,\theta$.

(iii) $\mathit{Im}\,\theta \subseteq \mathit{Ker}\,\alpha$.

Moreover, $(\alpha)_q = F\alpha$ *if (i) holds and* $(\alpha)_q = \{0, \alpha\}$ *if (ii) or (iii) holds.*

Proof. If $\dim_F V$ and $\dim_F W$ are finite, then $AE_F(V,W) = L_F(V,W)$ and $AE_F(W,V) = L_F(W,V)$, so by Corollary 3.1.10, for $\alpha \in AE_F(V,W) \smallsetminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(AE_F(V,W),\theta)$ if and only if one of the following three conditions holds:

(i) rank $\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$.

(ii) $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$.

(iii) $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$.

Moreover, $(\alpha)_q = F\alpha$ if (i) holds and $(\alpha)_q = \{0,\alpha\}$ if (ii) or (iii) holds.

Conversely, assume that $\dim_F V$ or $\dim_F W$ is not finite. By Lemma 3.2.2(ii), both $\dim_F V$ and $\dim_F W$ are infinite and $\dim_F V = \dim_F W$. Then there is an isomorphism φ from V onto W. By Lemma 3.2.2(iv), $(AE_F(V,W),\theta) \cong (AE_F(V),\varphi\theta)$. But from Lemma 3.2.4(ii), $(AE_F(V),\varphi\theta)$ has no minimal quasi-ideal, it follows that $(AE_F(V,W),\theta)$ has no minimal quasi-ideal. □

3.3 The Semigroups $(BL_F(V,W),\theta)$ and $(OBL_F(V,W),\theta)$

Throughout this section, V and W are assumed to be infinite dimensional. We have indicated in Chapter I that if $BL_F(V,W) \neq \varnothing$ and $M_F(W,V) \neq \varnothing$, then $(BL_F(V,W),\theta)$ with $\theta \in M_F(W,V)$ is a semigroup and we obtain similarly for $(OBL_F(V,W),\theta)$ where $\theta \in E_F(W,V)$. The aim of this section is to show that both of these semigroups have no minimal quasi-ideal.

First, the following fact is provided.

Lemma 3.3.1. *The following statements hold.*

(i) $BL_F(V,W) \neq \varnothing$ *and* $M_F(W,V) \neq \varnothing$ *if and only if* $\dim_F V = \dim_F W$.

(ii) $OBL_F(V,W) \neq \varnothing$ *and* $E_F(W,V) \neq \varnothing$ *if and only if* $\dim_F V = \dim_F W$.

Proof. Since $BL_F(V,W) \subseteq M_F(V,W)$ and $OBL_F(V,W) \subseteq E_F(V,W)$, by Lemma

3.2.1(i) and Lemma 3.2.1(ii), we have that

$$BL_F(V,W) \neq \varnothing \text{ and } M_F(W,V) \neq \varnothing \Rightarrow \dim_F V = \dim_F W$$

and

$$OBL_F(V,W) \neq \varnothing \text{ and } E_F(W,V) \neq \varnothing \Rightarrow \dim_F V = \dim_F W.$$

Conversely, suppose that $\dim_F V = \dim_F W$. Then $V \cong W$ as vector spaces over F. Then there is an isomorphism α from W onto V. Hence $\alpha \in M_F(W,V)$ and $\alpha \in E_F(W,V)$. Let B and C be bases of V and W, respectively. Then $|B| = |C|$. Since B and C are infinite, there are $B_1, B_2 \subsetneq B$ and $C_1, C_2 \subsetneq C$ such that

$$B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing, |B_1| = |B_2| = |B|,$$

$$C = C_1 \cup C_2, C_1 \cap C_2 = \varnothing, |C_1| = |C_2| = |C|.$$

Thus $|B| = |C_1| = |C| = |B_1|$, so there are bijections $\varphi_1 : B \to C_1$ and $\varphi_2 : B_1 \to C$. Define $\beta, \gamma \in L_F(V,W)$ by

$$v\beta = v\varphi_1 \text{ for all } v \in B,$$

$$v\gamma = \begin{cases} v\varphi_2 & \text{if } v \in B_1, \\ 0 & \text{if } v \in B_2. \end{cases}$$

Then $\operatorname{Ker}\beta = \{0\}$, $\operatorname{Im}\beta = \langle C_1 \rangle$, $\operatorname{Im}\gamma = \langle C \rangle = W$ and $\operatorname{Ker}\gamma = \langle B_2 \rangle$. Thus

$$\begin{aligned} \dim_F(W/\operatorname{Im}\beta) &= \dim_F(W/\langle C_1 \rangle) \\ &= |\{v + \langle C_1 \rangle \mid v \in C \setminus C_1\}| \\ &= |C \setminus C_1| = |C_2| = |C| \end{aligned}$$

and

$$\dim_F \operatorname{Ker}\gamma = |B_2| = |B|.$$

Hence $\beta \in BL_F(V,W)$ and $\gamma \in OBL_F(V,W)$.

Therefore the lemma is proved. □

Lemma 3.3.2. *The following statements hold.*

(i) $BL_F(V)$ *has no minimal quasi-ideal.*

(ii) $OBL_F(V)$ *has no minimal quasi-ideal.*

Proof. (i) Let $\alpha \in BL_F(V)$ be arbitrary. Then $(\alpha^2)_q \subseteq (\alpha)_q$ in $BL_F(V)$. If $(\alpha^2)_q = (\alpha)_q$ in $BL_F(V)$, then from Theorem 1.2, $\alpha = \alpha^2$ or $\alpha = \beta\alpha^2$ for some $\beta \in BL_F(V)$. But α is one-to-one, so $1_V = \alpha$ or $1_V = \beta\alpha$. Hence $\operatorname{Im}\alpha = V$ and so $|V/\operatorname{Im}\alpha| = 1$, a contradiction. Hence $(\alpha^2)_q \subsetneq (\alpha)_q$.

(ii) Let $\alpha \in OBL_F(V)$. Then $(\alpha^2)_q \subseteq (\alpha)_q$ in $OBL_F(V)$. If $(\alpha^2)_q = (\alpha)_q$, then by Theorem 1.2, $\alpha = \alpha^2$ or $\alpha = \alpha^2\beta$ for some $\beta \in OBL_F(V)$. Since $\operatorname{Im}\alpha = V$, it follows that $1_V = \alpha$ or $1_V = \alpha\beta$ which implies that α is one-to-one, a contradiction. Thus $(\alpha^2)_q \subsetneq (\alpha)_q$.

Hence the lemma is proved, as required. □

Lemma 3.3.3. *The following statements hold.*

(i) If $\theta : W \to V$ *is an isomorphism, then* $(BL_F(V,W),\theta) \cong BL_F(V)$ *through the map* $\alpha \mapsto \alpha\theta$.

(ii) If $\theta : W \to V$ *is an isomorphism, then* $(OBL_F(V,W),\theta) \cong OBL_F(V)$ *through the map* $\alpha \mapsto \alpha\theta$.

Proof. (i) Since $\theta : W \to V$ is one-to-one, then $\alpha\theta : V \to V$ is one-to-one for all $\alpha \in BL_F(V,W)$. If $\alpha \in L_F(V,W)$, then

$$\begin{aligned}\dim_F(V/\operatorname{Im}\alpha\theta) &= \dim_F(W\theta/(\operatorname{Im}\alpha)\theta)\\ &= \dim_F(W/\operatorname{Im}\alpha) \qquad \text{since } \theta : W \to V \text{ is an isomorphism.}\end{aligned}$$

Consequently, $\alpha\theta \in BL_F(V)$ for all $\alpha \in BL_F(V,W)$. Since θ is one-to-one, $\alpha \mapsto \alpha\theta$ is a one-to-one map from $BL_F(V,W)$ into $BL_F(V)$. If $\alpha \in BL_F(V)$, then we have similarly that $\alpha\theta^{-1} \in BL_F(V,W)$. Also, $(\alpha\theta^{-1})\theta = \alpha$ for every $\alpha \in BL_F(V)$. Also, for $\alpha, \beta \in BL_F(V,W), (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta)$. Hence (i) is proved.

(ii) Since $\operatorname{Im}\alpha = W$ and $\operatorname{Im}\theta = V$, it follows that $\operatorname{Im}\alpha\theta = V$. For $\alpha \in L_F(V,W)$, $\operatorname{Ker}\alpha\theta = \operatorname{Ker}\alpha$ since $\operatorname{Ker}\theta = \{0\}$. Hence $\alpha\theta \in OBL_F(V)$ for all $\alpha \in OBL_F(V,W)$. We can show similarly as above that the map $\alpha \mapsto \alpha\theta$ is an isomorphism from $(OBL_F(V,W),\theta)$ onto $OBL_F(V)$. □

Theorem 3.3.4. *For $\theta \in M_F(W,V)$, the semigroup $(BL_F(V,W),\theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in M_F(W,V)$. Then $\theta : W \to V$ is one-to-one.

Case 1: θ is an isomorphism. By Lemma 3.3.3(i), $(BL_F(V,W),\theta) \cong BL_F(V)$. Hence from Lemma 3.3.2(i), so $(BL_F(V,W),\theta)$ has no minimal quasi-ideal.

Case 2: θ is not an isomorphism. Then $\operatorname{Im}\theta \neq V$ since θ is one-to-one. Let $\alpha \in BL_F(V,W)$ be arbitrary. Then $(\alpha\theta\alpha)_q \subseteq (\alpha)_q$ in $(BL_F(V,W),\theta)$. Suppose that $(\alpha\theta\alpha)_q = (\alpha)_q$. Then from Theorem 1.2, $\alpha = \alpha\theta\alpha$ or $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in BL_F(V,W)$. Since α is one-to-one, $1_V = \alpha\theta$ or $1_V = \beta\theta\alpha\theta$. Hence $\operatorname{Im}\theta = V$, a contradiction. Therefore $(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$. This shows that $(BL_F(V,W),\theta)$ has no minimal quasi-ideal.

Hence the theorem is proved. □

Theorem 3.3.5. *For $\theta \in E_F(W,V)$, the semigroup $(OBL_F(V,W),\theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in E_F(W,V)$. Then $\theta : W \to V$ with $\operatorname{Im}\theta = V$.

Case 1: θ is an isomorphism. By Lemma 3.3.3(ii), $(OBL_F(V,W),\theta) \cong OBL_F(V)$. Hence by Lemma 3.3.2(ii), $(OBL_F(V,W),\theta)$ has no minimal quasi-ideal.

Case 2: θ is not isomorphism. Then θ is not one-to-one. Let $\alpha \in OBL_F(V,W)$. Then $(\alpha\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $(\alpha\theta\alpha)_q = (\alpha)_q$. Then by Theorem 1.2, $\alpha = \alpha\theta\alpha$ or $\alpha = \alpha\theta\alpha\theta\beta$ for some $\beta \in OBL_F(V,W)$. Since $\operatorname{Im}\alpha = V$, $1_W = \theta\alpha$ or $1_W = \theta\alpha\theta\beta$ which implies that θ is one-to-one, a contradiction. Thus $(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$.

Therefore the theorem is proved. □

3.4 The Semigroups $(M_{m,n}(F),P)$ and $(SU_n(F),P)$

Let V and W be vector spaces over F, $\dim_F V = m$, $\dim_F W = n$, B a basis of V and B' a basis of W. We have mentioned in Chapter I that

$$(L_F(V,W),\theta) \cong (M_{m,n}(F),[\theta]_{B',B})$$

by $\alpha \mapsto [\alpha]_{B,B'}$ and for every $\alpha \in L_F(V,W)$,

$$\operatorname{rank}\alpha = \operatorname{rank}[\alpha]_{B,B'}.$$

Therefore from these facts and Corollary 3.1.10, we directly obtain the following result.

Theorem 3.4.1. *For $A \in M_{m,n}(F) \setminus \{0\}, (A)_q$ is a minimal quasi-ideal of the semigroup $(M_{m,n}(F),P)$ if and only if one of the following three conditions holds:*
(i) rank $A = 1, PA \neq 0$ and $AP \neq 0$.
(ii) $PA = 0$.
(iii) $AP = 0$.

Moreover, if (i) is ture, then $(A)_q = FA$, and if (ii) or (iii) holds, then $(A)_q = \{0,A\}$.

The following corollary follows from Theorem 3.4.1. It gives an interesting characterization of the standard matrix semigroup $M_n(F)$ of all $n \times n$ matrices under usual multiplication.

Corollary 3.4.2. *For $A \in M_n(F) \setminus \{0\}$, the following statements are equivalent.*

(i) $(A)_q$ is a minimal quasi-ideal of $M_n(F)$.

(ii) rank $A = 1$.

(iii) $(A)_q = FA$.

Next, we shall determine minimal quasi-ideals of the semigroup $(SU_n(F), P)$ where P is an upper triangular $n \times n$ matrix over F.

We prove the following fact which will be referred often later.

Lemma 3.4.3. *For $A \in SU_n(F) \setminus \{0\}, A \notin APSU_n(F)$.*

Proof. Let $A \in SU_n(F) \setminus \{0\}$ and let

$$l = \min\{j \in \{1, 2, \cdots, n\} \mid A_{ij} \neq 0 \text{ for some } i \in \{1, 2, \cdots, n\}\}.$$

Then $A_{kl} \neq 0$ for some $k \in \{1, 2, \ldots, n\}$. But for $B \in SU_n(F)$,

$$\begin{aligned}(APB)_{kl} &= \sum_{j=1}^{n} A_{kj}(PB)_{jl} \\ &= \sum_{j=l}^{n} A_{kj}(PB)_{jl} && \text{by the property of } l \\ &= 0 && \text{since } PB \in SU_n(F),\end{aligned}$$

so we have $A \neq APB$ for every $B \in SU_n(F)$. □

Theorem 3.4.4. *Let $A \in SU_n(F) \setminus \{0\}$. Then the following statements are equivalent.*

(i) $(A)_q$ is a minimal quasi-ideal of the semigroup $(SU_n(F), P)$.

(ii) $SU_n(F)PA \cap APSU_n(F) = \{0\}$.

(iii) $(A)_q = \{0, A\}$.

Proof. *(i)* $\Rightarrow$ *(ii)* By Theorem 1.2, $(A)_q = (SU_n(F)PA \cap APSU_n(F)) \cup \{A\}$. Then $SU_n(F)PA \cap APSU_n(F) \subsetneq (A)_q$. But $SU_n(F)PA \cap APSU_n(F)$ is a quasi-ideal of $(SU_n(F), P)$, so by (i) $SU_n(F)PA \cap APSU_n(F) = \{0\}$ or $SU_n(F)PA \cap APSU_n(F) = (A)_q$. But $A \notin APSU_n(F)$ by Lemma 3.4.3, so $SU_n(F)PA \cap APSU_n(F) \neq (A)_q$. Hence (ii) holds.

(ii) $\Rightarrow$ *(iii)*. From Theorem 1.2 and (ii), $(A)_q = \{0, A\}$.

(iii) $\Rightarrow$ *(i)*. Since $|(A)_q| = 2$, $(A)_q$ does not properly contain any nonzero quasi-ideal of $(SU_n(F), P)$. Consequently, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), P)$.

$\square$

Theorem 3.4.5. *For $A \in SU_n(F)$, if rank $A = 1$, then $(A)_q$ is a minimal quasi-ideal of the semigroup $(SU_n(F), P)$. If this is the case, $(A)_q = \{0, A\}$.*

Proof. Let $A \in SU_n(F)$ and assume that rank $A = 1$. By Theorem 1.13, $SU_n(F)A \cap ASU_n(F) = \{0\}$. But $SU_n(F)P \subseteq SU_n(F)$ and $PSU_n(F) \subseteq SU_n(F)$, so we have $SU_n(F)PA \cap APSU_n(F) = \{0\}$. By Theorem 3.4.4, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), P)$ and $(A)_q = \{0, A\}$. $\square$

Remark 3.4.6. It is natural to ask whether the converse of Thorem 3.4.5 is true. The following example shows that it is not generally true. If $P = 0$, then by Thorem 3.4.4, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), P)$ for every $A \in SU_n(F) \setminus \{0\}$. However even though $P \neq 0$, the converse of Thorem 3.4.5 need not true. To see this, let $P \in SU_n(F)$ be such this $P_{ij} = 0$ for all $(i,j) \neq (1,n)$. Clearly, $SU_n(F)PA \cap APSU_n(F) = \{0\}$. By Theorem 3.4.4, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), P)$ for every $A \in SU_n(F) \smallsetminus \{0\}$.

CHAPTER IV

MINIMAL QUASI-IDEALS OF GENERALIZED RINGS OF LINEAR TRANSFORMATIONS

The purpose of this chapter is to characterize minimal quasi-ideals in generalized rings of linear transformations and generalized matrix rings $(L_F(V,W,k),+,\theta)$ and $(\overline{L}_F(V,W,k),+,\theta)$ (see Section 3.1) $(M_{m,n}(F),+,P)$ and $(SU_n(F),+,P)$ (see Section 3.4). Moreover, some remarks on minimal bi-ideals of these rings comparing with their minimal quasi-ideals are povided

Let us recall the notations mentioned above first. Throughout, let V and W be vector spaces over a field F, k an infinite cardinal number, m and n positive integers,

$$\begin{aligned} L_F(V,W,k) &= \{\alpha \in L_F(V,W) \mid \operatorname{rank}\alpha < k\}, \\ \overline{L}_F(V,W,k) &= \{\alpha \in L_F(V,W) \mid \operatorname{rank}\alpha \leqslant k\}, \\ M_{m,n}(F) &= \text{ the set of all } m \times n \text{ matrices over } F, \\ SU_n(F) &= \text{ the set of all strictly upper } n \times n \text{ matrices over } F, \end{aligned}$$

$+$ is the usual addition of linear transformations or matrices and the multiplication is a sandwich multiplication mentioned previously.

4.1 The Rings $(L_F(V,W,k),+,\theta)$ and $(\overline{L}_F(V,W,k),+,\theta)$

In this section, all minimal quasi-ideals of the rings $(L_F(V,W,k),+,\theta)$ and $(\overline{L}_F(V,W,k),+,\theta)$ where $\theta \in L_F(W,V)$ are completely characterized. We also give necessary and sufficient conditions for them to have a minimal quasi-ideal.

To obtain the main result in this section, the following series of lemmas is needed.

Lemma 4.1.1. *For $\theta \in L_F(W,V)$ and $\alpha, \beta \in L_F(V,W,k)$, if $\beta \in (\alpha)_q$ in the ring $(L_F(V,W,k),+,\theta)$, then $Im\,\beta \subseteq Im\,\alpha$.*

Proof. From Theorem 1.3, $\beta = t\alpha + \gamma\theta\alpha$ for some $t \in \mathbb{Z}$ and $\gamma \in L_F(V,W,k)$. This implies that $\operatorname{Im}\beta = \operatorname{Im}(t1_V + \gamma\theta)\alpha \subseteq \operatorname{Im}\alpha$ where 1_V is the identity map on V. □

Lemma 4.1.2. *Let $\theta \in L_F(W,V)$ and $\alpha \in L_F(V,W,k)$ be such that $Im\,\alpha \nsubseteq Ker\,\theta$ and $Im\,\theta \nsubseteq Ker\,\alpha$. If $rank\,\alpha = 1$, then $(\alpha)_q = F\alpha$ in the ring $(L_F(V,W,k),+,\theta)$.*

Proof. Let $u \in \operatorname{Im}\alpha \setminus \operatorname{Ker}\theta$ and $u' \in \operatorname{Im}\theta \setminus \operatorname{Ker}\alpha$. Then $0 \neq u\theta \in V, 0 \neq u'\alpha \in W$ and $z\theta = u'$ for some $z \in W \setminus \{0\}$. Then $\operatorname{Im}\alpha = Fu$ since $\operatorname{rank}\alpha = 1$. Thus $u'\alpha = au$ for some $a \in F \setminus \{0\}$. Let B and B' be respectively a basis of V containing $u\theta$ and a basis of W containing u. Define $\beta \in L_F(V,W)$ and $\gamma \in L_F(W,W)$ by

$$v\beta = \begin{cases} u & \text{if } v = u\theta, \\ 0 & \text{if } v \in B \setminus \{u\theta\}, \end{cases} \qquad w\gamma = \begin{cases} a^{-1}z & \text{if } w = u, \\ 0 & \text{if } w \in B' \setminus \{u\}. \end{cases}$$

Then $\alpha\gamma \in L_F(V,W)$, rank $\beta = 1$ and rank $\alpha\gamma \leqslant 1$, so $\beta, \alpha\gamma \in L_F(V,W,k)$. From the definitions of β and γ, we have

$$u\gamma\theta\alpha = a^{-1}z\theta\alpha = a^{-1}u'\alpha = a^{-1}au = u = u\theta\beta.$$

Since $\operatorname{Im}\alpha = Fu$, by Lemma 3.1.1(i), $\alpha\gamma\theta\alpha = \alpha = \alpha\theta\beta$. Hence for every $b \in F, b\alpha = (b\alpha\gamma)\theta\alpha = \alpha\theta(b\beta) \in L_F(V,W,k)\theta\alpha \cap \alpha\theta L_F(V,W,k)$. It therefore follows from Theorem 1.3 that $F\alpha \subseteq (\alpha)_q$. To show the converse inclusion, let $\lambda \in (\alpha)_q$. By Lemma 4.1.1, $\operatorname{Im}\lambda \subseteq \operatorname{Im}\alpha$. We have from Theorem 1.3 that $\lambda = t\alpha + \alpha\theta\mu$ for some $t \in \mathbb{Z}$ and $\mu \in L_F(V,W,k)$. Hence $\operatorname{Im}(\alpha\theta\mu) = \operatorname{Im}(\lambda - t\alpha) \subseteq \operatorname{Im}\alpha$ since $\operatorname{Im}\lambda \subseteq$

$\operatorname{Im}\alpha$. But $u \in \operatorname{Im}\alpha = Fu$ and $\operatorname{Im}(\alpha\theta\mu) = (\operatorname{Im}\alpha)\theta\mu \subseteq \operatorname{Im}\alpha$, so $u\theta\mu = cu$ for some $c \in F$. Thus from Lemma 3.1.1(ii), $\alpha\theta\mu = c\alpha$. It now follows that

$$\lambda = t\alpha + c\alpha = (t1_F + c)\alpha \in F\alpha$$

where 1_F is the identity of F. This proves that $(\alpha)_q = F\alpha$, as required. $\square$

Lemma 4.1.3. *Let $\theta \in L_F(W,V)$ and $\alpha \in L_F(V,W,k)$ be such that Im $\alpha \nsubseteq$ Ker θ and Im $\theta \nsubseteq$ Ker α. If rank $\alpha = 1$, then $(\alpha)_q$ is a minimal quasi-ideal of the ring $(L_F(V,W,k),+,\theta)$.*

Proof. We have from Lemma 4.1.2 that $(\alpha)_q = F\alpha$. Let $\beta \in (\alpha)_q \smallsetminus \{0\}$. Then $\beta = a\alpha$ for some $a \in F \smallsetminus \{0\}$. It therefore follows that Ker β = Ker α and Im β = Im α. Thus rank $\beta = 1$. By applying Lemma 4.1.2 to β, we get $(\beta)_q = F\beta$. But since $F\beta = Fa\alpha = F\alpha$, we have $(\beta)_q = (\alpha)_q$. This shows that $(\beta)_q = (\alpha)_q$ for every nonzero element $\beta \in (\alpha)_q$. Hence $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$. $\square$

Lemma 4.1.4. *Let $\theta \in L_F(W,V)$ and $\alpha \in L_F(V,W,k)$ be such that Im $\alpha \nsubseteq$ Ker θ and Im $\theta \nsubseteq$ Ker α. If $(\alpha)_q$ is a minimal quasi-ideal of the ring $(L_F(V,W,k),+,\theta)$, then rank $\alpha = 1$.*

Proof. Let $u \in \operatorname{Im}\alpha \smallsetminus \operatorname{Ker}\theta$ and $u' \in \operatorname{Im}\theta \smallsetminus \operatorname{Ker}\alpha$. Then $0 \neq u\theta \in V, 0 \neq u'\alpha \in W$ and $z\theta = u'$ for some $z \in W \smallsetminus \{0\}$. Let B be a basis of V containing $u\theta$ and define $\beta \in L_F(V,W)$ by

$$v\beta = \begin{cases} z & \text{if } v = u\theta, \\ 0 & \text{if } v \in B \smallsetminus \{u\theta\}. \end{cases}$$

We then have $\beta \in L_F(V,W,k)$ and

$$0 \neq u'\alpha = z\theta\alpha = u\theta\beta\theta\alpha \in (\operatorname{Im}\alpha)\theta\beta\theta\alpha = \operatorname{Im}(\alpha\theta\beta\theta\alpha),$$

so $\alpha\theta\beta\theta\alpha \neq 0$ and rank $(\alpha\theta\beta\theta\alpha) \leqslant$ rank $\beta = 1$. Thus rank $(\alpha\theta\beta\theta\alpha) = 1$, and hence $\alpha\theta\beta\theta\alpha \in L_F(V,W,k)\theta\alpha \cap \alpha\theta L_F(V,W,k)$. Thus $\alpha\theta\beta\theta\alpha \in (\alpha)_q$ by Theorem 1.3. But since $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$ and $\alpha\theta\beta\theta\alpha \neq 0$, it follows that $(\alpha\theta\beta\theta\alpha)_q = (\alpha)_q$. From Lemma 4.1.1, rank $\alpha =$ rank $(\alpha\theta\beta\theta\alpha) = 1$. □

Lemma 4.1.5. *If $\theta \in L_F(W,V)$ and $\alpha \in L_F(V,W,k)$ are such that $Im\,\alpha \subseteq Ker\,\theta$ or $Im\,\theta \subseteq Ker\,\alpha$, then $(\alpha)_q = \mathbb{Z}\alpha$ in the ring $(L_F(V,W,k),+,\theta)$.*

Proof. We have that $\alpha\theta = 0$ if $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$ and $\theta\alpha = 0$ if $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$. It then follows from Theorem 1.3 that $(\alpha)_q = \mathbb{Z}\alpha$ in $(L_F(V,W,k),+,\theta)$. □

Lemma 4.1.6. *Let char $F = 0$, $\theta \in L_F(W,V)$ and $L_F(V,W,k) \neq \{0\}$. If $\alpha \in L_F(V,W,k)$ is such that $Im\,\alpha \subseteq Ker\,\theta$ or $Im\,\theta \subseteq Ker\,\alpha$, then the quasi-ideal $(\alpha)_q$ of the ring $(L_F(V,W,k),+,\theta)$ is not minimal.*

Proof. Clearly, $\operatorname{Im}(2\alpha) \subseteq \operatorname{Im}\alpha$. Since char $F = 0$, we have $\operatorname{Ker}(2\alpha) = \operatorname{Ker}\alpha$. It then follows from Lemma 4.1.5 that $(2\alpha)_q = \mathbb{Z}(2\alpha) = (2\mathbb{Z})\alpha \subseteq \mathbb{Z}\alpha = (\alpha)_q$. If $\alpha = 0$, then $(\alpha)_q = \{0\}$ which is not minimal. For the case that $\alpha \neq 0$, we have $u\alpha \neq 0$ for some $u \in V$. Since char $F = 0$, we deduce that $u\alpha \neq 2t(u\alpha) = u(2t\alpha)$ for all $t \in \mathbb{Z}$. Thus $\alpha \notin \mathbb{Z}(2\alpha)$, so $(2\alpha)_q \subsetneq (\alpha)_q$ in $(L_F(V,W,k),+,\theta)$. Hence $(\alpha)_q$ is not a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$. □

Lemma 4.1.7. *If char $F \neq 0$, $\theta \in L_F(W,V)$ and $\alpha \in L_F(V,W,k) \setminus \{0\}$ are such that $Im\,\alpha \subseteq Ker\,\theta$ or $Im\,\theta \subseteq Ker\,\alpha$, then the quasi-ideal $(\alpha)_q$ of the ring $(L_F(V,W,k),+,\theta)$ is minimal.*

Proof. Let char F be a prime p. Then

$$\mathbb{Z}\alpha = \{0, \alpha, 2\alpha, \ldots, (p-1)\alpha\}.$$

Since $\alpha \neq 0$, $\mathbb{Z}\alpha$ is a subgroup of $(L_F(V,W,k),+)$ of order p. Thus $\mathbb{Z}\alpha$ does not properly contain any nonzero subgroup of $(L_F(V,W,k),+)$. Consequently, $\mathbb{Z}\alpha$ does not properly contain any nonzero quasi-ideal of $(L_F(V,W,k),+,\theta)$. But $(\alpha)_q = \mathbb{Z}\alpha$ by Lemma 4.1.5, so $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$. □

The following two theorems completely characterize all minimal quasi-ideals of the ring $(L_F(V,W,k),+,\theta)$.

Theorem 4.1.8. Let char $F = 0$, $\theta \in L_F(W,V)$ and $L_F(V,W,k) \neq \{0\}$. Then for $\alpha \in L_F(V,W,k), (\alpha)_q$ *is a minimal quasi-ideal of the ring* $(L_F(V,W,k),+,\theta)$ *if and only if* $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ *and* $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$.

If this is the case, $(\alpha)_q = F\alpha$.

Proof. First, assume that $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$. By Lemma 4.1.6, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$. Hence by Lemma 4.1.4, $\operatorname{rank}\alpha = 1$, and also $(\alpha)_q = F\alpha$ by Lemma 4.1.2.

The converse holds by Lemma 4.1.3. □

Theorem 4.1.9. *Let char* $F = p > 0$, $\theta \in L_F(W,V)$ *and* $L_F(V,W,k) \neq \{0\}$. *Then for* $\alpha \in L_F(V,W,k), (\alpha)_q$ *is a minimal quasi-ideal of* $(L_F(V,W,k),+,\theta)$ *if and only if one of the following conditions holds:*

(*i*) $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ *and* $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$.

(*ii*) $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$.

(*iii*) $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$.

If (i) holds, then $(\alpha)_q = F\alpha$*, and* $(\alpha)_q = \{0, \alpha, \ldots, (p-1)\alpha\}$ *if (ii) or (iii) holds.*

Proof. Assume that $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V, W, k), +, \theta)$. To prove that (i), (ii) or (iii) holds, suppose that both (ii) and (iii) are fault. Then $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$. We therefore deduce from Lemma 4.1.4 that $\operatorname{rank}\alpha = 1$. Hence (i) holds. For this case, $(\alpha)_q = F\alpha$ by Lemma 4.1.2. If (ii) or (iii) holds, then $(\alpha)_q = \mathbb{Z}\alpha$ by Lemma 4.1.5.

The converse is obtained directly from Lemma 4.1.3 and Lemma 4.1.7. □

If k is an infinite cardinal number greater than $\dim_F W$, then $L_F(V, W, k) = L_F(V, W)$. Hence from Theorem 4.1.8, we have

Corollary 4.1.10. *Let* $\operatorname{char} F = 0$, $\theta \in L_F(W, V)$ *and* $L_F(V, W) \neq \{0\}$. *Then for* $\alpha \in L_F(V, W), (\alpha)_q$ *is a minimal quasi-ideal of* $(L_F(V, W), +, \theta)$ *if and only if* $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ *and* $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$.

If this is the case, $(\alpha)_q = F\alpha$.

Also by Theorem 4.1.9, we get

Corollary 4.1.11. *Let* $\operatorname{char} F = p > 0$, $\theta \in L_F(W, V)$ *and* $L_F(V, W) \neq \{0\}$. *Then for* $\alpha \in L_F(V, W), (\alpha)_q$ *is a minimal quasi-ideal of the ring* $(L_F(V, W), +, \theta)$ *if and only if one of the following conditions holds:*

(i) $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ *and* $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$.

(ii) $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$.

(iii) $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$.

If (i) holds, then $(\alpha)_q = F\alpha$*, and if (ii) or (iii) holds, then* $(\alpha)_q = \{0, \alpha, \ldots, (p-1)\alpha\}$.

The well-known ring $L_F(V)$ of all linear transformations $\alpha : V \to V$ under usual addition and composition has 1_V, the identity map on V, as its identity. Hence from Theorem 1.3, in the ring $L_F(V)$,

$$(\alpha)_q = L_F(V)\alpha \cap \alpha L_F(V)$$

for every $\alpha \in L_F(V)$. The following corollary gives interesting characterizations of all minimal quasi-ideals of the ring $L_F(V)$.

Corollary 4.1.12. *For $\alpha \in L_F(V) \setminus \{0\}$, the following statements are equivalent.*
(i) $(\alpha)_q$ is a minimal quasi-ideal of $L_F(V)$.
(ii) $\mathit{rank}\, \alpha = 1$.
(iii) $(\alpha)_q = F\alpha$.

Proof. The implication (i)$\Rightarrow$(ii) holds by Corollary 4.1.10 and Corollary 4.1.11. The implication (ii)$\Rightarrow$(iii) follows from Lemma 4.1.2. To prove (iii)$\Rightarrow$(i), assume that $(\alpha)_q = F\alpha$ and let $\beta \in (\alpha)_q \setminus \{0\}$. Then $\beta = a\alpha$ for some $a \in F \setminus \{0\}$. Hence

$$\begin{aligned}
(\beta)_q &= L_F(V)\beta \cap \beta L_F(V) && \text{from Theorem 1.3} \\
&= L_F(V)a\alpha \cap a\alpha L_F(V) \\
&= (aL_F(V))\alpha \cap \alpha(aL_F(V)) \\
&= L_F(V)\alpha \cap \alpha L_F(V) && \text{since } a \neq 0 \\
&= (\alpha)_q.
\end{aligned}$$

This shows that $(\beta)_q = (\alpha)_q$ for all $\beta \in (\alpha)_q \setminus \{0\}$. Therefore $(\alpha)_q$ is a minimal quasi-ideal of $L_F(V)$. $\square$

Finally, we show that every nonzero ring $(L_F(V, W, k), +, \theta)$ has a minimal quasi-ideal except only the case that char $F = 0$ and $\theta = 0$.

Theorem 4.1.13. *For $\theta \in L_F(W,V)$, a nonzero ring $(L_F(V,W,k),+,\theta)$ has a minimal quasi-ideal if and only if char $F \neq 0$ or $\theta \neq 0$.*

Proof. First, assume that char $F \neq 0$ or $\theta \neq 0$.

Case 1: $\theta \neq 0$. Then $\operatorname{Ker}\theta \neq W$ and $\operatorname{Im}\theta \neq \{0\}$. Let $u \in \operatorname{Im}\theta \smallsetminus \{0\}$ and $w \in W \smallsetminus \operatorname{Ker}\theta$. Let B be a basis of V containing u and define $\alpha \in L_F(V,W,k)$ by

$$v\alpha = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v \in B \smallsetminus \{u\}. \end{cases}$$

Then $\operatorname{rank}\alpha = 1, w \in \operatorname{Im}\alpha \smallsetminus \operatorname{Ker}\theta$ and $u \in \operatorname{Im}\theta \smallsetminus \operatorname{Ker}\alpha$. From Theorem 4.1.8 and Theorem 4.1.9, $(\alpha)_q$ is a minimal quasi-ideal of $(L_F(V,W,k),+,\theta)$.

Case 2: char $F \neq 0$ and $\theta = 0$. By Lemma 4.1.7, every nonzero principal quasi-ideal of $(L_F(V,W,k),+,\theta)$ is minimal.

Conversely, assume that char$F = 0$ and $\theta = 0$. It follows from Lemma 4.1.6 that for every $\alpha \in L_F(V,W,k)$, $(\alpha)_q$ is not a minimal quasi-ideal of $(L_F(V,W,k),\theta)$. Hence $(L_F(V,W,k),+,\theta)$ does not have a minimal quasi-ideal. □

Remark 4.1.14. Assume that $\alpha \in L_F(V,W,k)$ is such that $(\alpha)_q$ is a minimal quasi-ideal of the ring $(L_k(V,W,k),+,\theta)$. From Theorem 1.8, $(\alpha)_q$ must be either a zero subring or a division subring of $(L_F(V,W,k),+,\theta)$. By Theorem 4.1.8 and Theorem 4.1.9, we have (i) $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$, (ii) $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$ or (iii) $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$. If $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\theta$ or $\operatorname{Im}\theta \subseteq \operatorname{Ker}\alpha$, then $\alpha\theta\alpha = 0$. We therefore have from Lemma 4.1.5 that for this case, $(\alpha)_q$ is a zero subring of $(L_F(V,W,k),+,\theta)$.

Next, assume that $\operatorname{rank}\alpha = 1$, $\operatorname{Im}\alpha \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha$. Then by Lemma 4.1.2, $(\alpha)_q = F\alpha$. For this case, $(\alpha)_q$ can be either a zero subring or a division subring of $(L_F(V,W,k),+,\theta)$. If $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is clearly a zero subring of $(L_F(V,W,k),+,\theta)$. Next, assume that $\alpha\theta\alpha \neq 0$. Then $(\alpha)_q$ is a division subring of $(L_F(V,W,k),+,\theta)$ and hence $F\alpha \setminus \{0\} = (F \setminus \{0\})\alpha$ is a subgroup of $(L_F(V,W,k),\theta)$. Let $0 \neq u \in \operatorname{Im}\alpha$. Then $\operatorname{Im}\alpha = Fu$. Thus $u\theta\alpha = cu$ for some $c \in F$. We then deduce from Lemma 3.1.1(ii) that $\alpha\theta\alpha = c\alpha$. But $\alpha\theta\alpha \neq 0$, so $c \neq 0$. Hence for all $a, b \in F \setminus \{0\}$,

$$(a\alpha)\theta(b\alpha) = ab\alpha\theta\alpha = abc\alpha \in (F \setminus \{0\})\alpha,$$

$$(a\alpha)\theta(c^{-1}\alpha) = ac^{-1}\alpha\theta\alpha = ac^{-1}c\alpha = a\alpha,$$

$$(a\alpha)\theta(a^{-1}(c^{-1})^2)\alpha = aa^{-1}(c^{-1})^2\alpha\theta\alpha = (c^{-1})^2c\alpha = c^{-1}\alpha.$$

This shows that $(F \setminus \{0\})\alpha$ is indeed a subgroup of $(L_F(V,W,k),\theta)$ where $c^{-1}\alpha$ is its identity and for $a \in F \setminus \{0\}$, $a^{-1}(c^{-1})^2\alpha$ is the inverse of $a\alpha$. The following examples show that in this case, each of the subcases $\alpha\theta\alpha = 0$ and $\alpha\theta\alpha \neq 0$ can occur. Let V be a vector space over F of dimension 4. Let $\{u_1, u_2, u_3, u_4\}$ be a basis of V. Define $\theta, \alpha_1, \alpha_2 \in L_F(V)$ by

$$u_1\theta = u_1, u_2\theta = u_2, u_3\theta = u_3, u_4\theta = 0,$$

$$u_1\alpha_1 = u_3\alpha_1 = u_1, u_2\alpha_1 = u_4\alpha_1 = 0,$$

$$u_1\alpha_2 = u_2\alpha_2 = u_3, u_3\alpha_2 = u_4\alpha_2 = 0.$$

Then $\operatorname{Ker}\theta = \langle u_4 \rangle$, $\operatorname{Im}\theta = \langle u_1, u_2, u_3 \rangle$, $\operatorname{Ker}\alpha_1 = \langle u_1 - u_3, u_2, u_4 \rangle$, $\operatorname{Im}\alpha_1 = \langle u_1 \rangle$, $\operatorname{Ker}\alpha_2 = \langle u_1 - u_2, u_3, u_4 \rangle$ and $\operatorname{Im}\alpha_2 = \langle u_3 \rangle$. Then $\operatorname{rank}\alpha_1 = 1 = \operatorname{rank}\alpha_2$, $\operatorname{Im}\alpha_1 \nsubseteq \operatorname{Ker}\theta$, $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha_1$, $\operatorname{Im}\alpha_2 \nsubseteq \operatorname{Ker}\theta$ and $\operatorname{Im}\theta \nsubseteq \operatorname{Ker}\alpha_2$. Moreover,

$$u_1\alpha_1\theta\alpha_1 = u_1 \text{ and } \{u_1, u_2, u_3, u_4\}\alpha_2\theta\alpha_2 = \{u_3, 0\}\theta\alpha_2 = \{u_3, 0\}\alpha_2 = \{0\},$$

so $\alpha_1\theta\alpha_1 \neq 0$ and $\alpha_2\theta\alpha_2 = 0$.

We can see easily from the given proofs that Lamma 4.1.1-Lemma 4.1.7 are still true if we replace $(L_F(V,W,k),+,\theta)$ by $(\overline{L}_F(V,W,k),+,\theta)$. Hence the following three theorems are also obtained.

Theorem 4.1.15. *Let* $char\, F = 0$, $\theta \in L_F(W,V)$ *and* $\overline{L}_F(V,W,k) \neq \{0\}$. *Then for* $\alpha \in \overline{L}_F(V,W,k)$, $(\alpha)_q$ *is a minimal quasi-ideal of the ring* $(\overline{L}_F(V,W,k),+,\theta)$ *if and only if* $rank\, \alpha = 1$, $Im\, \alpha \nsubseteq Ker\, \theta$ *and* $Im\, \theta \nsubseteq Ker\, \alpha$.

If this is the case, $(\alpha)_q = F\alpha$.

Theorem 4.1.16. *Let* $char\, F = p > 0$, $\theta \in L_F(W,V)$ *and* $\overline{L}_F(V,W,k) \neq \{0\}$. *Then for* $\alpha \in \overline{L}_F(V,W,k)$, $(\alpha)_q$ *is a minimal quasi-ideal of the ring* $(\overline{L}_F(V,W,k),+,\theta)$ *if and only if one of the following conditions holds:*

(i) $rank\, \alpha = 1$, $Im\, \alpha \nsubseteq Ker\, \theta$ *and* $Im\, \theta \nsubseteq Ker\, \alpha$.

(ii) $Im\, \alpha \subseteq Ker\, \theta$.

(iii) $Im\, \theta \subseteq Ker\, \alpha$.

Moreover, $(\alpha)_q = F\alpha$ *if (i) holds, and* $(\alpha)_q = \{0, \alpha, \ldots, (p-1)\alpha\}$ *if (ii) or (iii) holds.*

Theorem 4.1.17. *For* $\theta \in L_F(W,V)$, *a nonzero ring* $(\overline{L}_F(V,W,k),+,\theta)$ *has a minimal quasi-ideal if and only if* $char\, F \neq 0$ *or* $\theta \neq 0$.

Also, a remark for the ring $(\overline{L}_F(V,W,k),+,\theta)$ can be given similarly to Remark 4.1.14 which is given for the ring $(L_F(V,W,k),+,\theta)$

4.2 The Rings $(M_{m,n}(F),+,P)$ and $(SU_n(F),+,P)$

The aim of this section is to determine minimal quasi-ideals of the rings $(M_{m,n}(F),+,P)$ where $P \in M_{n,m}(F)$ and $(SU_n(F),+,P)$ where P is an upper triangular $n \times n$ matrix over F. If V and W be vector spaces over F, $\dim_F V = m$ and $\dim_F W = n$,

$$(L_F(V,W),+,\theta) \cong (M_{m,n}(F),+,[\theta]_{B',B})$$

by $\alpha \mapsto [\alpha]_{B,B'}$ and for every $\alpha \in L_F(V,W)$,

$$\operatorname{rank}\alpha = \operatorname{rank}[\alpha]_{B,B'},$$

so from Corollary 4.1.10 and Corollary 4.1.11, we directly obtain the following results.

Theorem 4.2.1. *If char $F = 0$, then for $P \in M_{n,m}(F)$ and $A \in M_{m,n}(F), (A)_q$ is a minimal quasi-ideal of the ring $(M_{m,n}(F),+,P)$ if and only if rank $A = 1, PA \neq 0$ and $AP \neq 0$.*

If this is the case, $(A)_q = FA$.

Theorem 4.2.2. *If char $F = p > 0$, then for $P \in M_{n,m}(F)$ and $A \in M_{m,n}(F) \setminus \{0\}, (A)_q$ is a minimal quasi-ideal of the ring $(M_{m,n}(F),+,P)$ if and only if one of the following three conditions holds:*

(i) rank $A = 1, PA \neq 0$ and $AP \neq 0$.

(ii) $PA = 0$.

(iii) $AP = 0$.

Moreover, $(A)_q = FA$ if (i) holds, and $(A)_q = \{0, A, \dots, (p-1)A\}$ if (ii) or (iii) holds.

The following corollary gives interesting characterizations of minimal quasi-deals of the well-known ring $M_n(F)$ of all $n \times n$ matrices under usual addition and multiplication.

Corollary 4.2.3. *For $A \in M_n(F) \setminus \{0\}$, the following statements are equivalent.*

(i) $(A)_q$ *is a minimal quasi-ideal of the ring* $M_n(F)$.

(ii) $\operatorname{rank} A = 1$.

(iii) $(A)_q = FA$.

Next, minimal quasi-ideals of the ring $(SU_n(F), +, P)$ are determined. From Theorem 1.3, for a nonempty subset X of a ring R, $(X)_q = \mathbb{Z}X + RX \cap XR \supseteq RX \cap XR$. But $RX \cap XR$ is a quasi-ideal of R, so we have the following lemma which will be used.

Lemma 4.2.4. *For an element x of a ring R, if $(x)_q$ is a minimal quasi-ideal of R and $x \notin Rx \cap xR$, then $Rx \cap xR = \{0\}$ and $(x)_q = \mathbb{Z}x$.*

Theorem 4.2.5. *If $\operatorname{char} F = 0$ and $n > 1$, then for every an upper triangular $n \times n$ matrix P over F, the ring $(SU_n(F), +, P)$ has no minimal quasi-ideal.*

Proof. Since $n > 1, SU_n(F) \neq \{0\}$. Suppose that $A \in SU_n(F)$ and $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), +, P)$. By Lemma 3.4.3 and Lemma 4.2.4, $(A)_q = \mathbb{Z}A$. But $0 \neq 2A \in (A)_q$, so $(2A)_q = (A)_q$. Hence by Lemma 3.4.3 and Lemma 4.2.4, $(2A)_q = \mathbb{Z}(2A)$ because char $F \neq 2$. Since $A \neq 0$, $A_{kl} \neq 0$ for some $k, l \in \{1, 2, \cdots, n\}$, thus $A_{kl} = 2mA_{kl}$ for some $m \in \mathbb{Z} \setminus \{0\}$. Hence $2m - 1 = 0$, a contradiction. $\square$

Theorem 4.2.6. *Assume that $\operatorname{char} F = p > 0$ and $A \in SU_n(F) \setminus \{0\}$. Then the following statements are equivalent.*

(i) $(A)_q$ *is a minimal quasi-ideal of the ring* $(SU_n(F), +, P)$.

(ii) $SU_n(F)PA \cap APSU_n(F) = \{0\}$.

(iii) $(A)_q = \{0, A, 2A, \ldots, (p-1)A\}$.

Proof. The implication $(i) \Rightarrow (ii)$ is obtained from Lemma 3.4.3 and Lemma 4.2.4.

$(ii) \Rightarrow (iii)$. From Theorem 1.3 and (ii), $(A)_q = \mathbb{Z}A$. Since char $F = p$, (iii) holds.

$(iii) \Rightarrow (i)$. Since $|(A)_q| = p$, $(A)_q$ does not properly contain any nonzero subgroup of $(SU_n(F), +)$. Consequently, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), +, P)$. $\square$

Theorem 4.2.7. *Assume that char $F = p > 0$. For $A \in SU_n(F)$, if rank $A = 1$, then $(A)_q$ is a minimal quasi-ideal of the ring $(SU_n(F), +, P)$. If this is the case, $(A)_q = \{0, A, 2A, \cdots, (p-1)A\}$.*

Proof. Let $A \in SU_n(F)$ and assume that rank $A = 1$. By Theorem 1.13, $SU_n(F)A \cap ASU_n(F) = \{0\}$. But $SU_n(F)P \subseteq SU_n(F)$ and $PSU_n(F) \subseteq SU_n(F)$, so we have $SU_n(F)PA \cap APSU_n(F) = \{0\}$. By Theorem 4.2.6, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), +, P)$ and $(A)_q = \{0, A, 2A, \cdots, (p-1)A\}$. $\square$

Remark 4.2.8. (1) Let $A \in SU_n(F) \setminus \{0\}$ be such that $A_{ij} = 0$ for all $(i, j) \neq (1, n)$. Then $SU_n(F)PA \cap APSU_n(F) = \{0\}$. Hence if char $F > 0$, then by Theorem 4.2.6, $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), +, P)$. Hence if char $F > 0$ and $n > 1$, then a minimal quasi-ideal of $(SU_n(F), +, P)$ always exists for every an upper $n \times n$ matrix P over F.

(2) Let $P \in SU_n(F)$ be such that $P_{ij} = 0$ for all $(i, j) \neq (1, n)$. Then $SU_n(F)PA \cap APSU_n(F) = \{0\}$. By Theorem 4.2.6, if char $F > 0$, then $(A)_q$ is a minimal quasi-ideal of $(SU_n(F), +, P)$ for every $A \in SU_n(F) \setminus \{0\}$. This shows that the converse of Theorem 4.2.7 is not generally true.

4.3 Some Remarks on Minimal Bi-ideals of the Rings $(L_F(V,W,k),+,\theta)$, $(\overline{L}_F(V,W,k),+,\theta)$, $(M_{m,n}(F),+,P)$ and $(SU_n(F),+,P)$

The first purpose is to show that bi-ideals and quasi-ideals of the rings $(L_k(V,W,k),+,\theta)$, $(\overline{L}_k(V,W,k),+,\theta)$ and $(M_{m,n}(F),+,P)$ coincide. These imply that their minimal bi-ideals and minimal quasi-ideals are identical. The second purpose is to determine minimal bi-ideals of the ring $(SU_n(F),+,P)$. An example of a minimal bi-ideal of some ring $SU_n(F)$ $(=(SU_n(F),+,I_n))$ which is not a quasi-ideal is given.

The first theorem shows that the sets of bi-ideals and quasi-ideals of the ring $(L_F(V,W,k),+,\theta)$ coincide. In fact, the technique of the proof of Proposition 1.9 in [5] given by K. M. Kapp is helpful for this work. Howover, our proof is more complicated.

Theorem 4.3.1. *In the ring $(L_F(V,W,k),+,\theta)$ where $\theta\in L_F(W,V)$, every bi-ideal is a quasi-ideal. That is, the sets of bi-ideals and quasi-ideals of the ring $(L_F(V,W,k),+,\theta)$ coincide.*

Proof. Let B be a bi-ideal of $(L_F(V,W,k),+,\theta)$. Then

$$B\theta L_F(V,W,k)\theta B\subseteq B.$$

To show that $L_F(V,W,k)\theta B\cap B\theta L_F(V,W,k)\subseteq B$, let α be an element of $L_F(V,W,k)\theta B\cap B\theta L_F(V,W,k)$. Then

$$\alpha\in L_F(V,W,k)\theta B, \tag{1}$$

$$\begin{aligned}\alpha =&\beta_{11}\theta\gamma_1+\beta_{12}\theta\gamma_2+\ldots+\beta_{1n}\theta\gamma_n\\ &\text{for some } \beta_{11},\ldots,\beta_{1n}\in B \text{ and } \gamma_1,\ldots,\gamma_n\in L_F(V,W,k).\end{aligned} \tag{2}$$

Since each $\beta_{1i}\theta \in L_F(V)$ and $(L_F(V), +, \cdot)$ is a regular ring, there exists $\lambda_{1i} \in L_F(V)$ such that $\beta_{1i}\theta = \beta_{1i}\theta\lambda_{1i}\beta_{1i}\theta$. By (2), we have

$$\alpha = \beta_{11}\theta\lambda_{11}\beta_{11}\theta\gamma_1 + \beta_{12}\theta\lambda_{12}\beta_{12}\theta\gamma_2 + \ldots + \beta_{1n}\theta\lambda_{1n}\beta_{1n}\theta\gamma_n, \tag{3}$$

$$\begin{aligned}\beta_{11}\theta\lambda_{11}\beta_{11}\theta\gamma_1 &= \beta_{11}\theta\lambda_{11}(\alpha - \beta_{12}\theta\gamma_2 - \ldots - \beta_{1n}\theta\gamma_n) \\ &= \beta_{11}\theta\lambda_{11}\alpha - \beta_{11}\theta\lambda_{11}\beta_{12}\theta\gamma_2 - \ldots - \beta_{11}\theta\lambda_{11}\beta_{1n}\theta\gamma_n.\end{aligned} \tag{4}$$

It then follows from (3) and (4) that

$$\alpha = \beta_{11}\theta\lambda_{11}\alpha + (\beta_{12}\theta\lambda_{12}\beta_{12} - \beta_{11}\theta\lambda_{11}\beta_{12})\theta\gamma_2 + \ldots + (\beta_{1n}\theta\lambda_{1n}\beta_{1n} - \beta_{11}\theta\lambda_{11}\beta_{1n})\theta\gamma_n.$$

But from (1) and (2), $\beta_{11}\theta\lambda_{11}\alpha \in B\theta\lambda_{11}L_F(V, W, k)\theta B \subseteq B\theta L_F(V, W, k)\theta B$ and for $i \in \{2, 3, \ldots, n\}$, $\beta_{1i}\theta\lambda_{1i}\beta_{1i} - \beta_{11}\theta\lambda_{11}\beta_{1i} \in B\theta\lambda_{1i}B - B\theta\lambda_{11}B \subseteq B\theta L_F(V, W, k)$, so

$$\begin{aligned}\alpha =& \beta_1 + \beta_{22}\theta\gamma_2 + \ldots + \beta_{2n}\theta\gamma_n \\ & \text{for some } \beta_1 \in B\theta L_F(V, W, k)\theta B \text{ and } \beta_{22}, \ldots, \beta_{2n} \in B\theta L_F(V, W, k).\end{aligned} \tag{5}$$

Since for $i \in \{2, \ldots, n\}, \beta_{2i}\theta \in L_F(V)$, we have that for each $i \in \{2, \ldots, n\}$, $\beta_{2i}\theta = \beta_{2i}\theta\lambda_{2i}\beta_{2i}\theta$ for some $\lambda_{2i} \in L_F(V)$. Thus from (5),

$$\alpha = \beta_1 + \beta_{22}\theta\lambda_{22}\beta_{22}\theta\gamma_2 + \ldots + \beta_{2n}\theta\lambda_{2n}\beta_{2n}\theta\gamma_n, \tag{6}$$

$$\begin{aligned}\beta_{22}\theta\lambda_{22}\beta_{22}\theta\gamma_2 &= \beta_{22}\theta\lambda_{22}(\alpha - \beta_1 - \beta_{23}\theta\gamma_3 - \ldots - \beta_{2n}\theta\gamma_n) \\ &= \beta_{22}\theta\lambda_{22}\alpha - \beta_{22}\theta\lambda_{22}\beta_1 - \beta_{22}\theta\lambda_{22}\beta_{23}\theta\gamma_3 - \ldots - \beta_{22}\theta\lambda_{22}\beta_{2n}\theta\gamma_n.\end{aligned} \tag{7}$$

We then deduce from (6) and (7) that

$$\begin{aligned}\alpha =& \beta_1 + \beta_{22}\theta\lambda_{22}\alpha - \beta_{22}\theta\lambda_{22}\beta_1 + (\beta_{23}\theta\lambda_{23}\beta_{23} - \beta_{22}\theta\lambda_{22}\beta_{23})\theta\gamma_3 \\ & + \ldots + (\beta_{2n}\theta\lambda_{2n}\beta_{2n} - \beta_{22}\theta\lambda_{22}\beta_{2n})\theta\gamma_n.\end{aligned}$$

But we have from (1) and (5) that

$$\begin{aligned}\beta_1 \in B\theta L_F(V,W,k)\theta B, \beta_{22}\theta\lambda_{22}\alpha &\in B\theta L_F(V,W,k)\theta\lambda_{22}L_F(V,W,k)\theta B\\ &\subseteq B\theta L_F(V,W,k)\theta B,\\ \beta_{22}\theta\lambda_{22}\beta_1 &\in B\theta L_F(V,W,k)\theta\lambda_{22}B\theta L_F(V,W,k)\theta B\\ &\subseteq B\theta L_F(V,W,k)\theta B\end{aligned}$$

and for $i \in \{3, \ldots, n\}$,

$$\begin{aligned}\beta_{2i}\theta\lambda_{2i}\beta_{2i} - \beta_{22}\theta\lambda_{22}\beta_{2i} &\in B\theta L_F(V,W,k)\theta\lambda_{2i}B\theta L_F(V,W,k)\\ &\quad + B\theta L_F(V,W,k)\theta\lambda_{22}B\theta L_F(V,W,k)\\ &\subseteq B\theta L_F(V,W,k),\end{aligned}$$

so we have

$$\begin{aligned}\alpha =&\beta_2 + \beta_{33}\theta\gamma_3 + \ldots + \beta_{3n}\theta\gamma_n\\ &\text{for some } \beta_2 \in B\theta L_F(V,W,k)\theta B \text{ and } \beta_{33}, \ldots, \beta_{3n} \in B\theta L_F(V,W,k).\end{aligned}$$

Continuing in this fashion, we obtain the $n-1^{\underline{th}}$ step that

$$\begin{aligned}\alpha =&\beta_{n-1} + \beta_{nn}\theta\gamma_n\\ &\text{for some } \beta_{n-1} \in B\theta L_F(V,W,k)\theta B \text{ and } \beta_{nn} \in B\theta L_F(V,W,k).\end{aligned} \tag{8}$$

Let $\lambda_{nn} \in L_F(V)$ be such that $\beta_{nn}\theta = \beta_{nn}\theta\lambda_{nn}\beta_{nn}\theta$. Then from (8),

$$\alpha = \beta_{n-1} + \beta_{nn}\theta\lambda_{nn}\beta_{nn}\theta\gamma_n, \tag{9}$$

$$\begin{aligned}\beta_{nn}\theta\lambda_{nn}\beta_{nn}\theta\gamma_n &= \beta_{nn}\theta\lambda_{nn}(\alpha - \beta_{n-1})\\ &= \beta_{nn}\theta\lambda_{nn}\alpha - \beta_{nn}\theta\lambda_{nn}\beta_{n-1}.\end{aligned} \tag{10}$$

Thus we obtain from (9) and (10) that

$$\alpha = \beta_{n-1} + \beta_{nn}\theta\lambda_{nn}\alpha - \beta_{nn}\theta\lambda_{nn}\beta_{n-1}.$$

But by (1) and (8), we have that

$$\beta_{n-1} \in B\theta L_F(V,W,k)\theta B, \beta_{nn}\theta\lambda_{nn}\alpha \in B\theta L_F(V,W,k)\theta\lambda_{nn}L_F(V,W,k)\theta B$$
$$\subseteq B\theta L_F(V,W,k)\theta B \text{ and}$$
$$\beta_{nn}\theta\lambda_{nn}\beta_{n-1} \in B\theta L_F(V,W,k)\theta\lambda_{nn}B\theta L_F(V,W.k)\theta B \subseteq B\theta L_F(V,W,k)\theta B,$$

so we have that $\alpha \in B\theta L_F(V,W,k)\theta B$ which implies that $\alpha \in B$.

This proves that $L_F(V,W,k)\theta B \cap B\theta L_F(V,W,k) \subseteq B$, so B is a quasi-ideal of the ring $(L_F(V,W,k),+,\theta)$. Hence the theorem is completely proved. □

If $k > \dim_F W$, then $L_F(V,W) = L_F(V,W,k)$. Therefore the following corollary is directly obtained.

Corollary 4.3.2. *For $\theta \in L_F(W,V)$, every bi-ideal of the ring $(L_F(V,W),+,\theta)$ is a quasi-ideal. That is, the sets of bi-ideals and quasi-ideals of the ring $(L_F(V,W),+,\theta)$ coincide.*

Because of the relationship between the rings $(M_{m,n}(F),+,P)$ and $(L_F(V,W),+,\theta)$ mentioned in Section 4.2, we have

Corollary 4.3.3. *For $P \in M_{n,m}(F)$, every bi-ideal of the ring $(M_{m,n}(F),+,P)$ is a quasi-ideal. Hence the bi-ideals and the quasi-ideals of the ring $(M_{m,n}(F),+,P)$ are identical.*

Lemma 4.3.4. *Let X be a nonempty subset of a ring R. Then $\mathbb{Z}X^2 + XRX$ is a bi-ideal of R contained in $(X)_b$.*

Proof. It is clearly seen that $\mathbb{Z}X^2+XRX = (\mathbb{Z}X+XR)(\mathbb{Z}X+RX)$. Since $\mathbb{Z}X+XR$ and $\mathbb{Z}X + RX$ are a right ideal and a left ideal of R, respectively, from Corollary 1.16, $\mathbb{Z}X^2 + XRX$ is a bi-ideal of R. By Theorem 1.14,

$$(X)_b = \mathbb{Z}X + \mathbb{Z}X^2 + XRX,$$

so $\mathbb{Z}X^2 + XRX \subseteq (X)_b$. $\square$

The following lemma is directly obtained from Theorem 1.14 and Lemma 4.3.4.

Lemma 4.3.5. *For $x \in R$, if $(x)_b$ is a minimal bi-ideal of R and $x \notin \mathbb{Z}x^2 + xRx$, then $\mathbb{Z}x^2 + xRx = \{0\}$ and $(x)_b = \mathbb{Z}x$.*

Theorem 4.3.6. *If char $F = 0$ and $n > 1$, then the ring $(SU_n(F), +, P)$ has no minimal bi-ideal.*

Proof. Suppose that $A \in SU_n(F)$ is such that $(A)_b$ is a minimal bi-ideal of $(SU_n(F), +, P)$. Since $\mathbb{Z}APA + APSU_n(F)PA \subseteq APSU_n(F)$, by Lemma 3.4.3 and Lemma 4.3.5, $(A)_b = \mathbb{Z}A$. Hence $(2A)_b = (A)_b$ since $0 \neq 2A \in (A)_b$. Also, by Lemma 3.4.3 and Lemma 4.3.5, $(2A)_b = \mathbb{Z}(2A)$. Then we have a contraction, as before. $\square$

Theorem 4.3.7. *Assume that char $F = p > 0$ and $A \in SU_n(F) \smallsetminus \{0\}$. Then the following statements are equivalent.*

(i) $(A)_b$ *is a minimal bi-ideal of the ring* $(SU_n(F), +, P)$.

(ii) $APA = 0$ *and* $APSU_n(F)PA = \{0\}$.

(iii) $(A)_b = \{0, A, 2A, \cdots, (p-1)A\}$.

Proof. $(i) \Rightarrow (ii)$. Since $\mathbb{Z}APA + APSU_n(F)PA \subseteq APSU_n(F)$, by Lemma 3.4.3 and Lemma 4.3.5, $\mathbb{Z}APA + APSU_n(F)PA = \{0\}$. But APA and $APSU_n(F)PA$ are an element and a subset of $\mathbb{Z}APA + APSU_n(F)PA$, respectively, so (ii) holds.

$(ii) \Rightarrow (iii)$. From Theorem 1.14 and (ii), $(A)_b = \mathbb{Z}A$. Since char $F = p$, (iii) holds.

$(iii) \Rightarrow (i)$. Since $|(A)_b| = p$, $(A)_b$ does not properly contain any nonzero subgroup of $(SU_n(F), +)$. Hence $(A)_b$ is a minimal bi-ideal of $(SU_n(F), +, P)$. □

Theorem 4.3.8. *Assume that char $F = p > 0$. For $A \in SU_n(F)$, if rank $A = 1$, then $(A)_b$ is a minimal bi-ideal of the ring $(SU_n(F), +, P)$. If this is the case, $(A)_b = \{0, A, 2A, \cdots, (p-1)A\}$.*

Proof. By Theorem 1.13, $SU_n(F)A \cap ASU_n(F) = \{0\}$. But APA and $APSU_n(F)PA$ are an element and a subset of $SU_n(F)A \cap ASU_n(F)$, respectively, so $APA = 0$ and $APSU_n(F)PA = \{0\}$. By Theorem 4.3.7, $(A)_b$ is a minimal bi-ideal of $(SU_n(F), +, P)$ and $(A)_b = \{0, A, 2A, \cdots, (p-1)A\}$. □

Remark 4.3.9. (1) Let char $F > 0$ and let A be as in Remark 4.2.8(1). Then A satisfies (ii) of Theorem 4.3.7. By Theorem 4.3.7, $(A)_b$ is a minimal bi-ideal of $(SU_n(F), +, P)$. Hence if char $F > 0$ and $n > 1$, then $(SU_n(F), +, P)$ always has a minimal bi-ideal for every upper triangular $n \times n$ matrix P over F.

(2) Let P be as in Remark 4.2.8(2), Then (ii) of Theorem 4.3.7 is true for every $A \in SU_n(F)$. By Theorem 4.3.7, if char $F > 0$, then $(A)_b$ is a minimal bi-ideal of $(SU_n(F), +, P)$ for every $A \in SU_n(F) \smallsetminus \{0\}$. Because of this fact, the converse of Theorem 4.3.8 is not generally true.

(3) Since every quasi-ideal of a ring R is a bi-ideal, it follows that if a minimal bi-ideal B of R is also a quasi-ideal, then B is a minimal quasi-ideal of R. However,

there exists a minimal bi-ideal of some $SU_n(F)(= (SU_n(F), +, I_n))$ which is not a quasi-ideal. To see this, let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in SU_4(\mathbb{Z}_5).$$

Then $A^2 = 0$ and $ASU_4(\mathbb{Z}_5)A = \{0\}$. By Theorem 4.3.7, $(A)_b = \{0, A, 2A, 3A, 4A\}$ which is a minimal bi-ideal of $SU_4(\mathbb{Z}_5)$. Let

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $B = AC = DA \in ASU_4(\mathbb{Z}_5) \cap SU_4(\mathbb{Z}_5)A$ but $B \notin (A)_b$. Then $(A)_b$ is not a quasi-ideal of $SU_4(\mathbb{Z}_5)$.

From the above proof, we can see that if $n \geqslant 4$, then $SU_n(F)$ has a minimal bi-ideal which is not a quasi-ideal.

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