CHAPTER III

PARTIALLY ORDERED DISTRIBUTIVE NEAR-RINGS

In this chapter, some fundamental theorems of partially ordered distributive near-rings are given.

<u>Definition 3.1</u>. A partial order $\leqslant$ on a distributive near-ring R is said to be <u>compatible</u> if it satisfies the following properties:

(i) For any $x,y,z \in R$, $x \leqslant y$ implies $x + z \leqslant y + z$ and $z + x \leqslant z + y$.

(ii) For any $x,y,z \in R$, $x \leqslant y$ and $0 \leqslant z$ imply $xz \leqslant yz$ and $zx \leqslant zy$.

<u>Definition 3.2</u>. A system $(R,+,\cdot,\leqslant)$ is called a <u>partially ordered distributive near-ring</u> if $(R,+,\cdot)$ is a distributive near-ring and $\leqslant$ is a compatible partial order on R.

<u>Example 3.3</u>. (1) Every distributive near-ring is a partially ordered distributive near-ring with respect to the trivial partial order.

(2) Every subnear-ring of a partially ordered distributive near-ring is a partially ordered distributive near-ring.

(3) $(\mathbb{Z},+,\cdot,\leqslant)$, $(\mathbb{Q},+,\cdot,\leqslant)$ and $(\mathbb{R},+,\cdot,\leqslant)$ are partially ordered distributive near-rings.

(4) Let $(G,+,\leqslant)$ be a partially ordered group. Define the operation $\cdot$ on G by $x\cdot y = 0$ for all $x,y \in G$. Then $(G,+,\cdot,\leqslant)$ is a

partially ordered distributive near-ring.

(5) Let $n \in \mathbb{Z}^+$ and $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices having entries in $\mathbb{R}$. Define the relation $\leqslant$ on $M_n(\mathbb{R})$ by $(a_{ij}) \leqslant (b_{ij})$ if and only if $a_{ij} \leqslant b_{ij}$ for all $i,j \in \{1,2,\ldots,n\}$. Then $(M_n(\mathbb{R}),+,\cdot,\leqslant)$ is a partially ordered distributive near-ring.

(6) Let $\mathbb{R}[X]$ be the set of all polynomials with coefficients in $\mathbb{R}$. Define the relation $\leqslant$ on $\mathbb{R}[X]$ by $\sum_{n=0}^{\infty} a_n X^n \leqslant \sum_{n=0}^{\infty} b_n X^n$ if and only if $a_n \leqslant b_n$ for all $n \in \mathbb{Z}_0^+$. Then $(\mathbb{R}[X],+,\cdot,\leqslant)$ is a partially ordered distributive near-ring.

Let R be a partially ordered distributive near-ring and A a subset of R. The positive cone of A, denoted by P_A, is $\{x \in A \mid x \geqslant 0\}$. The following statements hold:

(1) $(P_R,+,\cdot)$ is a distributive seminear-ring containing 0.

(2) $P_R \cap (-P_R) = \{0\}$.

(3) $-x + P_R + x = P_R$ for all $x \in R$.

(4) $P_H = P_R \cap H$ where H is a subset of R.

Proposition 3.4. Let R be a partially ordered distributive near-ring. Then the subnear-ring H is convex in R if and only if P_H is a convex subset of P_R.

Proof: This proof is similar to the proof of Proposition 2.6. #

Proposition 3.5. Let R be a partially ordered distributive near-ring. Then the following statements hold:

(1) R is directed if and only if P_R generates $(R,+)$.

(2) R is a lattice if and only if it is directed and P_R is a lattice.

(3) R is complete if and only if every subset of P_R has an infimum.

(4) R is totally ordered if and only if $R = P_R \cup (-P_R)$.

Proof: This proof is similar to the proof of Proposition 2.8(4), (5) and (6). #

Definition 3.6. A subset A of a distributive near-ring R is called an O-set of R if it satisfies the following conditions:

(i) $A \cap (-A) = \{0\}$.

(ii) $A^2 \subseteq A$.

(iii) $A + A \subseteq A$.

(iv) $-x + A + x \subseteq A$ for all $x \in R$.

Note that for any distributive near-ring R, {0} is an O-set of R and for any partially ordered distributive near-ring R′, the positive cone of R′ is an O-set of R′.

Theorem 3.7. Every distributive near-ring has a maximal O-set.

Proof: This proof is similar to the proof of Theorem 2.10. #

Let R be a distributive near-ring and A an O-set of R. Define a relation $\leqslant$ on R as follows: For $x,y \in R$, $x \leqslant y$ if and only if $y - x \in A$. The proof that $\leqslant$ is a partial order on R and for any $x,y,z \in R$, $x \leqslant y$ implies that $x + z \leqslant y + z$ and $z + x \leqslant z + y$ is similar to the proof of Theorem 1.19. Let $x,y,z \in R$ be such that $x \leqslant y$ and $z \geqslant 0$. Then $y - x, z \in A$. Since $A^2 \subseteq A$, so $yz - xz = (y-x)z \in A$ which implies that $xz \leqslant yz$. Similarly, $zx \leqslant zy$. Hence $\leqslant$ is a compatible partial order on R. The proof that $\leqslant$ is the unique compatible partial order on R having A as its positive cone is similar to the proof given in the note, page 9. Hence we have the following theorem.

<u>Theorem 3.8</u>. A subset A of a distributive near-ring R is an O-set of R if and only if there exists a unique compatible partial order $\leqslant$ on R such that A is the positive cone induced by $\leqslant$.

<u>Corollary 3.9</u>. Let R be a distributive near-ring, $\mathcal{A}$ the set of all O-sets of R and $\mathcal{B}$ the set of all compatible partial orders on R. Then $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic.

<u>Corollary 3.10</u>. Every distributive near-ring has a maximal compatible partial order.

<u>Corollary 3.11</u>. Let $\mathcal{B}$ be the set of all compatible partial orders on $\mathbb{Z}$ and $\mathcal{C}$ the set of all subseminear-rings of $\mathbb{Z}_0^+$ containing 0. Then $\mathcal{B}$ and $\mathcal{C}$ are order isomorphic.

<u>Proof</u>: Let A be an O-set of $\mathbb{Z}$. Suppose that there exists

an $n \in \mathbb{Z}^-$ such that $n \in A$. Then $n^2 \in A$. But $-n^2 = n + n + \dots + n$ (n times), so $-n^2 \in A$. Hence $n^2 \in A \cap (-A)$, so $n = 0$, a contradiction. Therefore every O-set of $\mathbb{Z}$ is a subset of $\mathbb{Z}_0^+$. We see that every subseminear-ring of $\mathbb{Z}_0^+$ containing 0 is an O-set of $\mathbb{Z}$. Then we get that $\mathcal{C}$ is order isomorphic to the set of all O-sets of $\mathbb{Z}$ by the identity map. Hence, by Corollary 3.9, $\mathcal{B}$ is order isomorphic to $\mathcal{C}$. #

Note that J is a subseminear-ring of $\mathbb{Z}_0^+$ containing 0 if and only if J is a seminear-ring ideal of $\mathbb{Z}_0^+$ (that is, $(J,+)$ is a subsemigroup of $\mathbb{Z}_0^+$, $J\mathbb{Z}_0^+ \subseteq J$ and $\mathbb{Z}_0^+ J \subseteq J$). Hence by Corollary 3.11, $\mathcal{B}$ is order isomorphic to the set of all seminear-ring ideals in $\mathbb{Z}_0^+$.

Definition 3.12. Let R and R′ be partially ordered distributive near-rings. A map $f: R \to R'$ is called an order homomorphism of R into R′ if f is isotone and a homomorphism. An order homomorphism $f: R \to R'$ is called an order monomorphism if f is injective and $f(P_R) = P_{f(R)}$, an order epimorphism if f is onto and $f(P_R) = P_{R'}$, an order isomorphism if f is a bijection and f^{-1} is isotone. R and R′ are said to be order isomorphic if there exists an order isomorphism of R onto R′ and we denote this by $R \simeq R'$.

Proposition 3.13. Let R and R′ be partially ordered distributive near-rings. Then the following statements hold:

(1) If $f: R \to R'$ is a homomorphism then f is isotone if and only if $f(P_R) \subseteq P_{R'}$.

(2) If $f: R \to R'$ is an order homomorphism then ker f is a convex ideal in R.

<u>Proof</u>: This proof is similar to the proof of Proposition 2.15 by using Proposition 1.36(1). #

<u>Theorem 3.14</u>. Let $(R, \leqslant)$ be a partially ordered distributive near-ring and J a convex ideal of R. Then there exists a compatible partial order on R/J such that the projection map π is an order epimorphism.

<u>Proof</u>: Define a relation $\overset{*}{\leqslant}$ on R/J as follows: For $\alpha, \beta \in R/J$, $\alpha \overset{*}{\leqslant} \beta$ if and only if there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$. The proof that $\overset{*}{\leqslant}$ is a partial order on R/J and for any $\alpha, \beta, \gamma \in R/J$, $\alpha \overset{*}{\leqslant} \beta$ implies $\alpha + \gamma \overset{*}{\leqslant} \beta + \gamma$ and $\gamma + \alpha \overset{*}{\leqslant} \gamma + \beta$ is similar to the proof of Theorem 2.16. Let $\alpha, \beta, \gamma \in R/J$ be such that $\alpha \overset{*}{\leqslant} \beta$ and $[0] \overset{*}{\leqslant} \gamma$. Then there exist $a \in \alpha$, $b \in \beta$, $c \in [0]$ and $d \in \gamma$ such that $a \leqslant b$ and $c \leqslant d$. Thus $0 \leqslant d - c$ which implies that $a(d-c) \leqslant b(d-c)$. Since $[d-c] = [d]-[c] = [d]-[0] = [d]$, so we get that $[a][d] = [a][d-c] = [a(d-c)] \overset{*}{\leqslant} [b(d-c)] = [b][d-c] = [b][d]$. Hence $\alpha\gamma \overset{*}{\leqslant} \beta\gamma$. Similarly, $\gamma\alpha \overset{*}{\leqslant} \gamma\beta$. Therefore $\overset{*}{\leqslant}$ is compatible. The proof that π is an order epimorphism is similar to the proof of Theorem 2.16. #

<u>Definition 3.15</u>. Let R be a distributive near-ring and J an ideal of R. A <u>compatible partial order on J</u> is a partial order $\leqslant$ on J such that

(i) for any $x,y,z \in J$, $x \leqslant y$ implies $x + z \leqslant y + z$ and $z + x \leqslant z + y$,

(ii) $(P_J^*)^2 \subseteq P_J^*$ where $P_J^* = \{x \in J \mid x > 0\}$ and

(iii) $- x + P_J^* + x \subseteq P_J^*$ for all $x \in R$.

Remark 3.16. (1) If R is a partially ordered distributive near-ring and J an ideal of R then the restriction of the partial order on R to J gives a compatible partial order on J.

(2) Let R be a distributive near-ring and J a subnear-ring of R which is also an ideal and let $\leqslant$ be a partial order on J. If $\leqslant$ is a compatible partial order on J as an ideal then $\leqslant$ is a partial order compatible with the subnear-ring structure of J.

Theorem 3.17. Let R be a distributive near-ring and J a prime ideal of R. Assume that R/J has a compatible partial order $\leqslant$ and J has a compatible partial order $\leqslant^*$ such that ba, ab $\in P_J^*$ for all $a \in P_J^*$, $[b] \in P_{R/J} \setminus \{J\}$. Then there exists a compatible partial order on R such that $\leqslant^*$ is the restriction of the partial order on R and the projection map π is an order epimorphism.

Proof: Let $A = P_J^* \cup \left(\bigcup_{\alpha \in P_{R/J} \setminus \{J\}} \alpha \right)$. The proof that $A \cap (-A) = \{0\}$, $A + A \subseteq A$ and $- x + A + x \subseteq A$ is similar to the proof of Theorem 2.19. To show that $A^2 \subseteq A$, let $a,b \in A$. If $a,b \in P_J^*$ then $ab \in P_J^*$, so we are done. Assume that $a \notin P_J^*$ or $b \notin P_J^*$.

Case 1: $a,b \in \bigcup_{\alpha \in P_{R/J} \setminus \{J\}} \alpha$. Then $a \in \alpha$ and $b \in \beta$ for some $\alpha, \beta \in P_{R/J} \setminus \{J\}$. Also, $[ab] = [a][b] = \alpha\beta \geqslant [0]$. Since $a,b \notin J$ and J is a prime ideal in R, so $ab \notin J$ which implies that $[ab] > [0]$. Hence $ab \in \bigcup_{\alpha \in P_{R/J} \setminus \{J\}} \alpha$.

Case 2: $a \in P_J^*$ and $b \in \bigcup_{\alpha \in P_{R/J} \setminus \{J\}} \alpha$ or $a \in \bigcup_{\alpha \in P_{R/J} \setminus \{J\}} \alpha$ and $b \in P_J^*$.

By assumption, we get that $ab \in P_J^*$.

Hence $A^2 \subseteq A$. Therefore A is an O-set of R. By Theorem 3.8, there exists a compatible partial order on R such that A is the positive cone of R. Using a proof similar to the proof of Theorem 2.19 we can show that $\leqslant^*$ is the restriction of the partial order on R and π is an order epimorphism. #

Theorem 3.18 (First Isomorphism Theorem). Let R and R' be partially ordered distributive near-rings and $f: R \to R'$ an order epimorphism. Then $R/_{\ker f} \simeq R'$. Furthermore, there exists an order isomorphism between the set of all subnear-rings of R containing ker f and the set of all subnear-rings of R' and there exists an order isomorphism between the set of all ideals of R containing ker f and the set of all ideals of R'.

Proof: The proof is similar to the proof of Theorem 2.21 by using Proposition 1.36 and 3.13. #

Remark 3.19. Let R be a partially ordered distributive near-ring, H a subnear-rings of R and J a convex ideal in R. Then $H \cap J$ is a convex ideal of H and H + J is a subnear-ring of R.

Proof: If is clear that $H \cap J$ is a convex ideal of H. To show that H + J is a subnear-ring of R, let $x, y \in H + J$. Then $x = h_1 + a_1$ and $y = h_2 + a_2$ for some $h_1, h_2 \in H$, $a_1, a_2 \in J$. Hence $x - y = h_1 + a_1 - a_2 - h_2 = (h_1 - h_2) + (h_2 + (a_1 - a_2) - h_2) \in H + J$ and $xy = (h_1 + a_1)(h_2 + a_2) = h_1 h_2 + (h_1 a_2 + a_1 h_2 + a_1 a_2) \in H + J$. #

Theorem 3.20 (Second Isomorphism Theorem). Let R be a partially ordered distributive near-ring, H a subnear-ring of R and J a convex ideal in R such that $P_{H+J} = P_H$. Then $H/_{H \cap J} \simeq H + J/_J$.

Proof: The proof is similar to the proof of Theorem 2.23. #

Remark 3.21. Let R be a partially ordered distributive near-ring, H and K convex ideals in R such that $H \subseteq K$. Then $K/_H$ is a convex ideal of $R/_H$.

Proof: This proof is similar to the proof of Remark 2.24. #

Theorem 3.22 (Third Isomorphism Theorem). Let R be a partially ordered distributive near-ring, H and K convex ideals in R such that $H \subseteq K$. Then $(R/_H)/(K/_H) \simeq R/_K$.

Proof: The proof is similar to the proof of Theorem 2.25. #

Theorem 3.23. Let R and R´ be partially ordered distributive near-rings and $f: R \to R'$ an order epimorphism. If J' is a convex ideal in R' then $R/_{f^{-1}(J')} \simeq R'/_{J'}$.

<u>Proof</u>: The proof is similar to the proof of Theorem 2.26.#

<u>Definition 3.24</u>. Let $\{(R_\alpha, \leqslant_\alpha)\}_{\alpha\varepsilon I}$ be a family of partially ordered distributive near-rings. The <u>direct product</u> of the family $\{(R_\alpha, \leqslant_\alpha)\}_{\alpha\varepsilon I}$, denoted by $\prod_{\alpha\varepsilon I} R_\alpha$, is the set of all elements $(x_\alpha)_{\alpha\varepsilon I}$ in the Cartesian product of the family $\{(R_\alpha, \leqslant_\alpha)\}_{\alpha\varepsilon I}$ together with operations + and $\cdot$ and the partial order $\leqslant$ on $\prod_{\alpha\varepsilon I} R_\alpha$ defined by

$$(x_\alpha)_{\alpha\varepsilon I} + (y_\alpha)_{\alpha\varepsilon I} = (x_\alpha + y_\alpha)_{\alpha\varepsilon I},$$

$$(x_\alpha)_{\alpha\varepsilon I} \cdot (y_\alpha)_{\alpha\varepsilon I} = (x_\alpha y_\alpha)_{\alpha\varepsilon I} \quad \text{and}$$

$$(x_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I} \text{ if and only if } x_\alpha \leqslant_\alpha y_\alpha \text{ for all } \alpha \varepsilon I.$$

Note that $(\prod_{\alpha\varepsilon I} R_\alpha, +, \cdot, \leqslant)$ is a partially ordered distributive near-ring and $P_{\prod_{\alpha\varepsilon I} R_\alpha} = \prod_{\alpha\varepsilon I} P_{R_\alpha}$. So we see that given some examples of partially ordered distributive near-rings we can construct new examples of partially ordered distributive near-rings using the direct product.

<u>Proposition 3.25</u>. Let $\{(R_\alpha, \leqslant_\alpha)\}_{\alpha\varepsilon I}$ be a family of partially ordered distributive near-rings. Then the following statements hold:

(1) $\prod_{\alpha\varepsilon I} R_\alpha$ is directed if and only if R_α is directed for all $\alpha \varepsilon I$.

(2) $\prod_{\alpha \in I} R_\alpha$ is a lattice if and only if R_α is a lattice for all $\alpha \in I$.

(3) $\prod_{\alpha \in I} R_\alpha$ is complete if and only if R_α is complete for all $\alpha \in I$.

(4) $\prod_{\alpha \in I} R_\alpha$ is totally ordered if and only if either $I = \{\alpha\}$ and D_α is totally ordered or there exists an $\alpha_o \in I$ such that D_{α_o} is totally ordered and $|D_\alpha| = 1$ for all $\alpha \in I \setminus \{\alpha_o\}$.

Proof: The proof is similar to the proof of Proposition 2.28 by using Proposition 3.5.

Finally, we shall characterize those distributive seminear-ringswhich can be the positive cone of a partially ordered distributive near-ring.

Theorem 3.26. Let P be a distributive seminear-ring with additive identity 0. Then there exists a partially ordered distributive near-ring having P as its positive cone if and only if P satisfies the following properties:

(i) P is additively cancellative.

(ii) $P + a = a + P$ for all $a \in P$.

(iii) For any $a,b \in P$, $a + b = 0$ implies $a = b = 0$.

(iv) $ab + cd = cd + ab$ for all $a,b,c,d \in P$.

Moreover, if P satisfies properties (i) - (iv) then there exist a partially ordered distributive near-ring R and a monomorphism $i: P \to R$ such that

(1) $i(P)$ is the positive cone of R and

(2) if R' is a partially ordered distributive near-ring and $j: P \to R'$ is a monomorphism such that $j(P)$ is the positive cone of R' then there exists a unique order monomorphism $f: R \to R'$ such that $f \circ i = j$, that is, R is the smallest partially ordered distributive near ring having P as its positive cone up to isomorphism. Furthermore, R is directed.

<u>Proof</u>: Since the positive cone of a partially ordered distributive near-ring R has properties (i)-(iv), so if P is isomorphic to the positive cone of R then P also has properties (i)-(iv).

Conversely, assume that P satisfies properties (i)-(iv). By properties (i) and (ii) of P, we get that for any $a,x \in P$ there exists a unique $x_a \in P$ such that $x + a = a + x_a$. Clearly, $a_a = a_0 = a$ and $0_a = 0$ for all $a \in P$. Using a proof similar to the proof of Theorem 2.29 we can show that

(1) $(x+y)_a = x_a + y_a$ and

(2) $(x_a)_b = x_{a+b}$

for all $a,b,x,y \in P$.

Define a relation $\sim$ on $P \times P$ as follows: For $a,b,c,d \in P$, $(a,b) \sim (c,d)$ if and only if $a + d_b = c + b$. The proof that $\sim$ is an equivalence relation is similar to the proof of Theorem 1.21. Let $R = \frac{P \times P}{\sim}$. Define operations $+$ and $\cdot$ on R by

$$[(a,b)]+[(c,d)] = [(a+c_b,\ d+b)] \quad \text{and}$$

$$[(a,b)]\cdot[(c,d)] = [(ac+bd,\ ad+bc)]$$

for all a,b,c,d ε P. Using a proof similar to the proof of Theorem 2.29 we can show that + is well-defined and (R,+) is a group with [(0,0)] as the identity and [(b,a)] as the inverse of [(a,b)] for all a,b ε P.

Now we shall show that $\cdot$ is well-defined. Let v,w,x,y ε P be such that (v,w) ε [(a,b)] and (x,y) ε [(c,d)]. Then (a,b) $\sim$ (v,w) and (c,d) $\sim$ (x,y), so $a + w_b = v + b$ ....(*)

and $c + y_d = x + d$ ....(**).

Multiplying (*) by c and d, multiplying (**) by v and w, adding the result equation and the terms wd + ad + bc + bd, we get that

$ac + w_bc + vd + bd + vc + vy_d + wx + wd + wd + ad + bc + bd$

$= vc + bc + ad + w_bd + vx + vd + wc + wy_d + wd + ad + bc + bd.$

By property (iv) of P, we have that

$ac + (bc+w_bc)+(vd+vy_d)+ bd + vc + wx + wd + wd + ad + bd$

$= vc + bc + ad + (bd+w_bd) + vx + vd + wc + (wd+wy_d) + ad + bc.$

Hence

$ac + wc + bc + vy + vd + bd + vc + wx + wd + wd + ad + bd$

$= vc + bc + ad + wd + bd + vx + vd + wc + wy + wd + ad + bc.$

By properties (i) and (iv) of P, we get that

$ac + bd + vy + wx + ad + bc = bc + ad + vx + wy + ad + bc,$

so that

$ac + bd + ad + bc + (vy+wx)_{ad+bc} = ad + bc + vx + wy + ad + bc.$

Therefore $ac + bd + (vy+wx)_{ad+bc} = (vx+wy)+(ad+bc)$, so

$(ac+bd,ad+bc)\sim(vx+wy,vy+wx)$. Hence $[(ac+bd,ad+bc)]=[(vx+wy,vy+wx)]$.

Therefore $\cdot$ is well-defined.

To show that $\cdot$ is associative, let $a,b,c,d,x,y \in P$. Then

$([(a,b)][(c,d)])[(x,y)]$

$$= [((ac+bd)x + (ad+bc)y, (ac+bd)y + (ad+bc)x)]$$

$$= [(acx+ady+bcy+bdx, acy+adx+bcx+bdy)]$$

$$= [(a(cx+dy) + b(cy+dx), a(cy+dx) + b(cx+dy))]$$

$$= [(a,b)][(cx+dy, cy+dx)]$$

$$= [(a,b)]([(c,d)][(x,y)]).$$

Hence $\cdot$ is associative.

To show that $\cdot$ is distributive over + in R, let $a,b,c,d,x,y \in P$. Then by properties (i) and (iv) of P, we get that

$$x(a+c_b) + y(d+b) + (xd+xb+ya+yc_b) + (xd+yc+xb+ya)_{xd+xb+ya+yc_b}$$

$$= xa + xc_b + yd + yb + xd + yc + xb + ya + xd + xb + ya + yc_b$$

$$= xa + (xb+xc_b) + yd + (yc+yb) + xd + ya + xd + xb + ya + yc_b$$

$$= xa + xc + xb + yd + yb + yc_b + xd + ya + x(d+b) + y(a+c_b)$$

$$= xd + yc_b + xa + yb + (xc+yd) + (xb+ya) + x(d+b) + y(a+c_b)$$

$$= xd + yc_b + xa + yb + xb + ya + (xc+yd)_{xb+ya} + x(d+b) + y(a+c_b),$$

so

$$x(a+c_b) + y(d+b) + (xd+yc+xb+ya)_{x(d+b)+y(a+c_b)}$$

$$= (xa+yb+(xc+yd)_{xb+ya}) + (x(d+b)+y(a+c_b)).$$

Hence

$$(x(a+c_b) + y(d+b), x(d+b) + y(a+c_b)) \sim (xa+yb+(xc+yd)_{xb+ya}, xd+yc+xb+ya)$$

..................(3).

Therefore

$$[(x,y)]([(a,b)]+[(c,d)]) = [(x(a+c_b) + y(d+b), x(d+b)+y(a+c_b))]$$

$$= [(xa+yb+(xc+yd)_{xb+ya},\ xd+yc+xb+ya)] \qquad \text{(by(3))}$$

$$= [(xa+yb,\ xb+ya)]+[(xc+yd,\ xd+yc)]$$

$$= [(x,y)][(a,b)]+[(x,y)][(c,d)].$$

From

$$ax + c_bx + dy + by + (ay+c_by+dx+bx)+(cy+dx+ay+bx)_{ay+c_by+dx+bx}$$

$$= ax + c_bx + dy + by + cy + dx + ay + bx + ay + c_by + dx + bx$$

and property (iv) of P, we have that

$$ax + c_bx + dy + (by+c_by) + ay + dx + bx + (cy + dx + ay + bx)_{ay+c_by+dx+bx}$$

$$= ax + (bx+c_bx) + dy + by + cy + dx + ay + ay + c_by + dx + bx.$$

Also,

$$ax + c_bx + dy + cy + by + ay + dx + bx + (cy+dx+ay+bx)_{ay+c_by+dx+bx}$$

$$= ax + cx + bx + dy + by + cy + dx + ay + ay + c_by + dx + bx$$

$$= cy + dx + ax + by + (cx+dy)+(ay+bx)+(a+c_b)y + (d+b)x$$

$$= cy + dx + ax + by + ay + bx + (cx+dy)_{ay+bx} + (a+c_b)y + (d+b)x.$$

By property (i) of P,

$$(ax + c_bx + dy + by)+(cy + dx + ay + bx)_{(a+c_b)y + (d+b)x}$$

$$= (ax + by + (cx+dy)_{ay+bx} + ((a+c_b)y + (d+b)x).$$

Thus

$$((a+c_b)x + (d+b)y,\ (a+c_b)y + (d+b)x) \sim (ax+by+(cx+dy)_{ay+bx},\ (cy+dx)+(ay+bx)) \qquad \text{...............(4).}$$

Hence

$$([(a,b)]+[(c,d)])[(x,y)] = [((a+c_b)x + (d+b)y,\ (a+c_b)y + (d+b)x)]$$

$$= [(ax+by+(cx+dy)_{ay+bx},\ (cy+dx)+(ay+bx))] \qquad \text{(by (4))}$$

$$= [(ax+by, ay+bx)]+[(cx+dy, cy+dx)]$$

$$= [(a,b)][(x,y)]+[(c,d)][(x,y)]$$

Therefore $(R,+,\cdot)$ is a distributive near-ring.

Define $i: P \to R$ by $i(a) = [(a,0)]$ for all $a \in P$. Using a proof similar to the proof of Theorem 2.29 we get that i is a monomorphism and $i(P)$ is an O-set of R. By Theorem 3.8, there exists a compatible partial order on R such that $i(P)$ is the positive cone. Since for any $a,b \in P$, $[(a,b)] = [(a,0)]+[(0,b)] = [(a,0)]-[(b,0)] = i(a)-i(b)$, $i(P) = P_R$ generates $(R,+)$. By Proposition 3.5(1), R is directed.

Assume that R' is a partially ordered distributive near-ring and $j: P \to R'$ is a monomorphism such that $j(P) = P_{R'}$. Define $f: R \to R'$ by $f([(a,b)]) = j(a) - j(b)$ for all $a,b \in P$. Using a proof similar to the proof of Remark 1.22 we get that f is well-defined, injective, $f(P_R) = P_{f(R)}$ and $f(\alpha+\beta) = f(\alpha) + f(\beta)$ for all $\alpha,\beta \in R$. Let $a,b,c,d \in P$. Then

$$\begin{aligned} f([(a,b)][(c,d)]) &= f([(ac+bd, ad+bc)]) \\ &= j(ac+bd) - j(ad+bc) \\ &= j(a)j(c) + j(b)j(d) - j(a)j(d) - j(b)j(c) \\ &= j(a)(j(c) - j(d)) - j(b)(j(c) - j(d)) \\ &= (j(a) - j(b))(j(c) - j(d)) \\ &= f([(a,b)])f([(c,d)]). \end{aligned}$$

Hence f is a homomorphism. Therefore f is an order monomorphism. Using a proof similar to the proof of Remark 1.22 we get that f is the unique order monomorphism such that $f \circ i = j$. #