ไฮเพอร์มอดูลย่อยเฉพาะและเฉพาะอย่างอ่อนและไฮเพอร์มอดูลย่อยฟัซซีเฉพาะ

นายอภิรัฐ ศิระวรกุล

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PRIME AND WEAKLY PRIME SUBHYPERMODULES AND PRIME FUZZY SUBHYPERMODULES

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In this research, prime and weakly prime subhypermodules of a hypermodule over a hyperring are studied. We characterize such subhypermodules in several ways under various conditions on the hyperring and hypermodule. Moreover, we provide some conditions that imply prime and weakly prime subhypermodules are identical. Finally, fuzzy subhypermodules of hypermodules are defined and some characterizations of prime fuzzy subhypermodules are investigated.

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CONTENTS

CHAPTER I
HYPERRINGS AND HYPERMODULES

The theory of hyperstructures (also called multialgebras) started with the communication of F. Marty in 1934 at the 8^{th} Congress of Scandinavian Mathematicians. Marty introduced the notion of hypergroups and since then many researchers have worked on and developed this topic. The concept of hyperrings was introduced by M. Krasner. Later, J. Mittas and D. Stratigopoulos, two students of Krasner, earned their theses by studying the structure of hyperrings. These details can be found in [15].

P. Corsini gathered the fundamental concepts in his book "*Prolegomena of hypergroup theory*" and its applications in "*Application of hyperstructure theory*". The structure of hypermodules over hyperrings is defined analogously to that of modules over rings. It has been known that there are many different types of hyperrings, for examples, a Krasner hyperring, a feeble hyperring, a multiplicative hyperring, a D-hyperring and a V-S-hyperring. As a result, it is not surprising that a hypermodule over a hyperring can be defined in various ways. However, in this research, we choose hyperrings and hypermodules such that all operations are hyperoperations. These can be viewed as generalizations of Krasner hyperrings and hypermodules over Krasner hyperrings, respectively. In fact, hypermodules over hyperrings generalize modules over rings.

This chapter contains three sections. The first section introduces the basic notations and examples of hyperstructures. The second section gives definitions of hyperrings and hypermodules and provides with proofs of some elementary properties. Moreover, hyperideals and subhypermodules are introduced and some properties which will be used in this dissertation are investigated. The last section discusses the differences between modules over rings and hypermodules over hyperrings.

1.1 Preliminaries

In this section, we give some definitions of hyperstructures gathered by P. Corsini, [9]. Many examples of hyperstructures are also given.

For a set H, let $\wp(H)$ denote the power set of H, $\wp^*(H) = \wp(H) \smallsetminus \{\varnothing\}$ and $|H|$ the cardinality of H.

Definition 1.1.1. [9] A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $\wp^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets X and Y of H, let

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y),$$

and let $X \circ y = X \circ \{y\}$ and $y \circ X = \{y\} \circ X$ for all $y \in H$.

A hypergroupoid $(H, \circ)$ is said to be *commutative* if

$$x \circ y = y \circ x \quad \text{for all } x, y \in H.$$

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that

$$(x \circ y) \circ z = x \circ (y \circ z) \quad \text{for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that

$$x \circ H = H \circ x = H \quad \text{for all } x \in H.$$

Definition 1.1.2. [9] Let $(H, \circ)$ be a hypergroupoid.

An element e of H is called an *identity* of H if

$$x \in (x \circ e) \cap (e \circ x) \quad \text{for all } x \in H.$$

An element e of H is called a *scalar identity* of H if

$$x \circ e = e \circ x = \{x\} \quad \text{for all } x \in H.$$

In general, an identity of a hypergroupoid may not be unique. However, a scalar identity is unique since if x and y are scalar identities of a hypergroupoid $(H, \circ)$, then $\{x\} = x \circ y = \{y\}$, so that $x = y$.

Definition 1.1.3. [9] Let $(H, \circ)$ be a semihypergroup. An element x of H is said to be an *inverse* of an element y of H if there exists an identity e of H such that

$$e \in (x \circ y) \cap (y \circ x),$$

that is, $(x \circ y) \cap (y \circ x)$ contains at least one identity of H.

Example 1.1.4. [17] Let H be a nonempty set. Define

$$x \circ y = H \quad \text{for all } x, y \in H.$$

Then $(H, \circ)$ is a commutative hypergroup with the following properties.

(i) Every element of H is an identity of H. Consequently, H has a scalar identity if and only if $|H| = 1$.

(ii) All pairs of elements of H are inverses of each other.

This hypergroup $(H, \circ)$ is usually called the *total hypergroup.*

Definition 1.1.5. [9] A hypergroup $(H, \circ)$ is called a *canonical hypergroup* if

(i) $(H, \circ)$ is commutative,

(ii) $(H, \circ)$ has a scalar identity,

(iii) every element x of H has a unique inverse, denoted by x^{-1}, in H, and

(iv) $x \in y \circ z$ implies $z \in y^{-1} \circ x$ for all $x, y, z \in H$.

Note that if $(H, \circ)$ is a canonical hypergroup, then $x \in y \circ z$ also implies $z \in x \circ y^{-1}$ for all $x, y, z \in H$.

Definition 1.1.6. [9] Let $(H, \circ)$ be a canonical hypergroup. For a nonempty subset X of H, let

$$X^{-1} = \{x^{-1} \mid x \in X\}.$$

Proposition 1.1.7. [18] *Let $(H, \circ)$ be a canonical hypergroup. Then $(x^{-1})^{-1} = x$ and $(x \circ y)^{-1} = x^{-1} \circ y^{-1}$ for all $x, y \in H$.*

Proposition 1.1.8. *Let* $(H, \circ)$ *be a canonical hypergroup with the scalar identity* 0*. Then for all nonempty subsets* A*,* B *and* C *of* H*, we have*

(i) $A \circ B = B \circ A$,

(ii) $A \circ \{0\} = A$,

(iii) $(A \circ B) \circ C = A \circ (B \circ C)$, *and*

(iv) $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$.

Proof. (i), (ii), (iii) are straightforward.

(iv) First, let $x \in (A \circ B)^{-1}$. Then $x^{-1} \in A \circ B$. There exist $a \in A$ and $b \in B$ such that $x^{-1} \in a \circ b$, so that $b \in a^{-1} \circ x^{-1} = x^{-1} \circ a^{-1}$. Then $a^{-1} \in b \circ x$. Thus $x \in b^{-1} \circ a^{-1} = a^{-1} \circ b^{-1}$. Hence $x \in A^{-1} \circ B^{-1}$.

Conversely, let $x \in A^{-1} \circ B^{-1}$. Then $x \in a \circ b$ for some $a \in A^{-1}$ and $b \in B^{-1}$. Then $b \in a^{-1} \circ x$, so that $a^{-1} \in b \circ x^{-1}$. Thus $x^{-1} \in b^{-1} \circ a^{-1} = a^{-1} \circ b^{-1}$. Hence $x^{-1} \in A \circ B$, i.e., $x \in (A \circ B)^{-1}$. □

We give some examples of canonical hypergroups.

Example 1.1.9. [18] Let H be a nonempty set of cardinality at least 2. Choose an element in H and denote it by 0. Define a hyperoperation $\circ$ on H by, for any $a, b \in H$,

$$a \circ b = \begin{cases} \{a\}, & \text{if } b = 0, \\ \{b\}, & \text{if } a = 0, \\ H, & \text{if } a = b \neq 0, \\ \{a, b\}, & \text{if } a \neq b,\ a \neq 0 \text{ and } b \neq 0. \end{cases}$$

Then $(H, \circ)$ is a canonical hypergroup with 0 as the scalar identity and a as the inverse of $a \in H$.

The next examples are examples of canonical hypergroups constructed from real intervals. Let $\mathbb{R}$ be the set of real numbers.

Example 1.1.10. [18] Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $R = [0, a]$ or $[0, a)$. Define a hyperoperation $\oplus$ on R by, for any $x, y \in R$,

$$x \oplus y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y, \\ [0, x], & \text{if } x = y. \end{cases}$$

Then $(R, \oplus)$ is a canonical hypergroup.

Example 1.1.11. [18] Let $a \in \mathbb{R}$ be such that $a \geq 1$ and $R = [a, \infty) \cup \{0\}$ or $(a, \infty) \cup \{0\}$. Define a hyperoperation $\oplus$ on R by

$$\begin{aligned} x \oplus 0 = 0 \oplus x &= \{x\} && \text{for all } x \in R, \\ x \oplus x &= [x, \infty) \cup \{0\} && \text{for all } x \in R \smallsetminus \{0\} \text{ and} \\ x \oplus y &= \{\min\{x, y\}\} && \text{for all } x, y \in R \smallsetminus \{0\} \text{ with } x \neq y. \end{aligned}$$

Then $(R, \oplus)$ is a canonical hypergroup.

Example 1.1.12. [18] Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $R = [-a, a]$ or $(-a, a)$. Define a hyperoperation $\oplus$ on R by

$$\begin{aligned} x \oplus x &= \{x\} && \text{for all } x \in R, \\ x \oplus (-x) &= [-|x|, |x|\,] && \text{for all } x \in R \text{ and} \\ x \oplus y = y \oplus x &= \{x\} && \text{for all } x \in R \text{ with } |y| < |x|. \end{aligned}$$

Then $(R, \oplus)$ is a canonical hypergroup.

Definition 1.1.13. [9] Let $(H, \circ)$ be a canonical hypergroup. A nonempty subset H' of H is called a *canonical subhypergroup* of $(H, \circ)$ if

(i) $x \circ y \subseteq H'$ for all $x, y \in H'$,

(ii) $e \in H'$ where e is the scalar identity of H and

(iii) $x^{-1} \in H'$ for every $x \in H'$.

Remark 1.1.14. [18] *Let H' be a canonical subhypergroup of a canonical hypergroup $(H, \circ)$. It is easy to see that $(H', \circ)$ is a canonical hypergroup such that the scalar identity of H is the scalar identity of H' and the inverse of x in H' is the same as the inverse of x in H for each $x \in H'$.*

The following proposition gives a practical method for verifying whether a nonempty subset of a given canonical hypergroup is a canonical subhypergroup.

Proposition 1.1.15. [18] *Let $(H, \circ)$ be a canonical hypergroup and H' a nonempty subset of H. Then H' is a canonical subhypergroup of $(H, \circ)$ if and only if $x \circ y^{-1} \subseteq H'$ for all $x, y \in H'$.*

In group theory, if $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$ for all subgroups A, B and C of the same group. We extend this result to canonical hypergroups.

Proposition 1.1.16. *Let $(H, \circ)$ be a canonical hypergroup and A, B and C canonical subhypergroups of H. If $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$.*

Proof. Assume that $A \subseteq B \cup C$. Suppose that $A \nsubseteq B$ and $A \nsubseteq C$. There exist $c \in A \setminus B$ and $b \in A \setminus C$. By assumption, $b \in B$ and $c \in C$. Since $b, c \in A$, $b \circ c \subseteq A$. Let $x \in b \circ c$. Then $x \in A \subseteq B \cup C$, so $x \in B$ or $x \in C$. If $x \in B$, then $c \in x^{-1} \circ b \subseteq B$, a contradiction. If $x \in C$, then $b \in x^{-1} \circ c \subseteq C$, a contradiction. Hence $A \subseteq B$ or $A \subseteq C$. □

For a canonical hypergroup $(H, \circ)$, we define na, where n is an integer and $a \in H$, by

$$na = \begin{cases} \underbrace{\{a\} \circ \{a\} \circ \cdots \circ \{a\}}_{n \text{ copies}}, & \text{if } n > 0, \\ \underbrace{\{a^{-1}\} \circ \{a^{-1}\} \circ \cdots \circ \{a^{-1}\}}_{-n \text{ copies}}, & \text{if } n < 0, \\ \{0\}, & \text{if } n = 0, \end{cases}$$

where 0 is the scalar identity of H.

1.2 Hyperrings and Hypermodules

First, we introduce hyperrings and hypermodules in which both operations are hyperoperations.

Definition 1.2.1. A *hyperring* is a structure $(R, +, \odot)$ that satisfies the following peroperties:

(i) $(R, +)$ is a canonical hypergroup with scalar identity 0.

(ii) $(R, \odot)$ is a semihypergroup.

(iii) For all $a, b, c \in R$,
$a \odot (b + c) \subseteq a \odot b + a \odot c$ and $(b + c) \odot a \subseteq b \odot a + c \odot a$.

(iv) For all $a, b \in R$, $a \odot (-b) = (-a) \odot b = -(a \odot b)$.

Note that $-r$ is the inverse of r in $(R, +)$ for any $r \in R$. If equality holds for both subset relations in (iii), the hyperring is called *strongly distributive*. A hyperring is *commutative* if $a \odot b = b \odot a$ for all $a, b \in R$. For convenience, we abbreviate a hyperring $(R, +, \odot)$ by a hyperring R and $a \odot b$ by ab for all $a, b \in R$.

Definition 1.2.2. Let R be a hyperring. An *R-hypermodule* is a structure $(M, +, \circ)$ such that $(M, +)$ is a canonical hypergroup and $\circ$ is a multivalued scalar operation, i.e., a function $R \times M \to \wp^*(M)$ such that for all $a, b \in R$ and $x, y \in M$,

(i) $a \circ (x + y) \subseteq a \circ x + a \circ y$,

(ii) $(a + b) \circ x \subseteq a \circ x + b \circ x$,

(iii) $(ab) \circ x = a \circ (b \circ x)$, and

(iv) $a \circ (-x) = (-a) \circ x = -(a \circ x)$ where $-a$ and $-x$ are the inverses of a and x, respectively, and $-(a \circ x) = \{-y \mid y \in a \circ x\}$.

If equality holds in (i), the R-hypermodule is said to be *strongly distributive on the right*. Similarly, if equality holds in (ii), the R-hypermodule is said to be *strongly distributive on the left*. Moreover, if equality holds in both (i) and (ii), then the R-hypermodule is said to be *strongly distributive*. For convenience, we abbreviate an R-hypermodule $(M, +, \circ)$ by an R-hypermodule M and $a \circ m$ by am for all $a \in R$ and $m \in M$.

It is easy to see that every hyperring R is an R-hypermodule.

This definition generalizes modules over rings. Moreover, it is a generalization of hypermodules over Krasner hyperrings.

To avoid any confusion about the meanings of AB and AX for any nonempty subsets A and B of a hypering R and a nonempty subset X of an R-hypermodule M, we define the following notations.

For nonempty subsets A and B of a hyperring R and a nonempty subset X of an R-hypermodule M,

$$\begin{aligned} AB &= \bigcup \{a_i b_i \mid a_i \in A, b_i \in B\} \\ [AB] &= \bigcup \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in A, b_i \in B, n \in \mathbb{N} \right\} \\ AX &= \bigcup \{a_i x_i \mid a_i \in A, x_i \in X\} \\ [AX] &= \bigcup \left\{ \sum_{i=1}^{n} a_i x_i \mid a_i \in A, x_i \in X, n \in \mathbb{N} \right\}. \end{aligned}$$

In particular, let $aB = \{a\}B, Ab = A\{b\}$, $Ax = A\{x\}$ and $aX = \{a\}X$ for all $a, b \in R$ and $x \in M$.

Next, we give some basic properties by extending the properties in the definition of hypermodules from the element point of view to the subset point of view.

Proposition 1.2.3. *Let M be an R-hypermodule. Then for nonempty subsets A and B of R and X and Y of M,*

(i) $A(X + Y) \subseteq AX + AY$,

(ii) $(A + B)X \subseteq AX + BX$,

(iii) $(AB)X = A(BX)$, *and*

(iv) $A(-X) = (-A)X = -(AX)$.

Proof. These are straightforward. □

Proposition 1.2.4. *Let M be an R-hypermodule. Then for nonempty subsets A and B of R and X and Y of M,*

(i) *if* $X, Y \subseteq X + Y$, *then* $[A(X+Y)] = [AX] + [AY]$, *and*

(ii) *if* $A, B \subseteq A + B$, *then* $[(A+B)X] = [AX] + [BX]$.

Proof. (i) Assume that $X, Y \subseteq X + Y$. Let $m \in [A(X+Y)]$. There exist $n \in \mathbb{N}$, $a_i \in A$ and $l_i \in X + Y$ such that $m \in \sum_{i=1}^{n} a_i l_i$. For each $i \in \{1, 2, \ldots, n\}$, there exist $x_i \in X$ and $y_i \in Y$ such that $l_i \in x_i + y_i$. Then

$$m \in \sum_{i=1}^{n} a_i l_i \subseteq \sum_{i=1}^{n} a_i(x_i + y_i) \subseteq \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_i y_i \subseteq [AX] + [AY].$$

For the reverse inclusion, since $X, Y \subseteq X + Y$,

$$[AX] + [AY] \subseteq [A(X+Y)] + [A(X+Y)] \subseteq [A(X+Y)].$$

Therefore $[A(X+Y)] = [AX] + [AY]$.

(ii) The proof is similar to (i). □

Since a hyperring R can be considered as an R-hypermodule, we obtain the following corollaries.

Corollary 1.2.5. *Let* R *be a hyperring. Then for all nonempty subsets* A, B *and* C *of* R, *we have*

(i) $(-A)B = A(-B) = -(AB)$,

(ii) $A(B+C) \subseteq AB + AC$, *and*

(iii) $(A+B)C \subseteq AC + BC$.

Corollary 1.2.6. *Let* R *be a hyperring. Then for all nonempty subsets* A, B *and* C *of* R, *we have*

(i) *if* $B, C \subseteq B + C$, *then* $[A(B+C)] = [AB] + [AC]$, *and*

(ii) *if* $A, B \subseteq A + B$, *then* $[(A+B)C] = [AC] + [BC]$.

Let M be an R-hypermodule. From the above notations, Rx and $[Rx]$ may not be equal for any $x \in M$. We give a condition which guarantees that equality holds in the next proposition.

Proposition 1.2.7. *Let $(M, +, \circ)$ be a strongly distributive R-hypermodule. Then for every $a \in R$, $x \in M$, subhypergroup P of $(R, +)$ and subhypergroup N of $(M, +)$, we have $Px = [Px]$ and $aN = [aN]$.*

Proof. It is obvious that $Px \subseteq [Px]$. Let $m \in [Px]$. Then $m \in \sum_{i=1}^{n} r_i x$ where $r_i \in P$ for all i. Thus

$$m \in (r_1 x) + (r_2 x) + (r_3 x) + \cdots + (r_n x) = (r_1 + r_2)x + (r_3 x) + \cdots + (r_n x).$$

Then there exists $l_1 \in r_1 + r_2 \subseteq P$ such that $m \in (l_1 x) + (r_3 x) + \cdots + (r_n x) = (l_1 + r_3)x + \cdots + (r_n x)$. Continuing this process, we eventually obtain $m \in (l_{n-2} + r_n)x \subseteq Px$. Hence $[Px] \subseteq Px$.

Consequently, $Px = [Px]$. Similarly, $aN = [aN]$. □

Corollary 1.2.8. *Let $(R, +, \cdot)$ be a strongly distributive hyperring. Then for every $a \in R$ and subhypergroup P of $(R, +)$, $Pa = [Pa]$ and $aP = [aP]$.*

The remainder of this section is divided into 3 subsections: hyperideals, subhypermodules and examples of hyperrings and hypermodules.

1.2.1 Hyperideals

In this subsection, we give a definition and some properties of hyperideals.

Definition 1.2.9. Let R be a hyperring. A nonempty subset I of R is called a *subhyperring* of R if I is a hyperring under the same hyperoperations. A subhyperring is a *hyperideal* of R if $ra \subseteq I$ and $ar \subseteq I$ for all $r \in R$ and $a \in I$.

The immediate result is the following.

Proposition 1.2.10. *Let I be a subhyperring of R. Then I is a hyperideal of R if and only if $[RI] \subseteq I$ and $[IR] \subseteq I$.*

Proof. This is obvious. □

The next result follows from Proposition 1.1.15.

Proposition 1.2.11. *Let I be a nonempty subset of a hyperring R. Then I is a hyperideal of R if and only if for every $a, b \in I$ and $r \in R$, $a - b \subseteq I$, $ra \subseteq I$ and $ar \subseteq I$.*

Proof. This proof is clear. □

The next proposition shows two ways to create new hyperideals from two hyperideals.

Proposition 1.2.12. *Let R be a hyperring and I, J hyperideals of R . Then $I + J$ and $[IJ]$ are hyperideals of R.*

Proof. First, we show that $I + J$ is a hyperideal of R. Let $x, y \in I + J$. Then $x \in a_1 + b_1$ and $y \in a_2 + b_2$ for some $a_1, a_2 \in I$ and $b_1, b_2 \in J$. Then $x - y = x + (-y) \subseteq (a_1 + b_1) + (-(a_2 + b_2)) = a_1 + b_1 + (-a_2) + (-b_2) = (a_1 + (-a_2)) + (b_1 + (-b_2)) \subseteq I + J$. Let $r \in R$. Then $rx \subseteq r(a_1 + b_1) \subseteq ra_1 + rb_1 \subseteq I + J$. Similarly, $xr \subseteq I + J$. Hence $I + J$ is a hyperideal of R.

Next, we show that $[IJ]$ is a hyperideal of R. Let $x, y \in [IJ]$. Then $x \in \sum_{i=1}^{n} a_i b_i$ and $y \in \sum_{i=1}^{m} c_i d_i$ where $a_i, c_i \in I$ and $b_j, d_j \in J$ for all i and j. Then

$$x - y \subseteq \sum_{i=1}^{n} a_i b_i + \left(- \sum_{i=1}^{m} c_i d_i \right) \subseteq \sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{m} -(c_i d_i) \right)$$
$$= \sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{m} (-c_i) d_i \right) \subseteq [IJ].$$

Let $r \in R$. Then $rx \subseteq r\big(\sum_{i=1}^{n} a_i b_i\big) \subseteq \sum_{i=1}^{n} r(a_i b_i) = \sum_{i=1}^{n} (ra_i) b_i \subseteq \sum_{i=1}^{n} I b_i$. Let $l \in rx$. Then $l \in \sum_{i=1}^{n} p_i b_i$ for some $p_i \in I$. Hence $l \in [IJ]$ so that $rx \subseteq [IJ]$. Similarly, $xr \subseteq [IJ]$. Hence $[IJ]$ is a hyperideal of R. □

Definition 1.2.13. An element e of a hyperring $(R, +, \cdot)$ is called an *identity* of R if $r \in er \cap re$ for all $r \in R$.

In ring theory, it is well known that an ideal which contains an identity is the whole ring. The following proposition shows that this is also true in hyperrings.

Proposition 1.2.14. *Let $(R, +, \cdot)$ be a hyperring with an identity e and I a hyperideal of R. If $e \in I$, then $I = R$.*

Proof. Assume that $e \in I$. Let $r \in R$. Then $r \in er \subseteq I$. Hence $r \in I$. $\square$

Next, we introduce the hyperideal generated by a subset of a hyperring.

Proposition 1.2.15. *Let A be a nonempty subset of a hyperring R. Then $[RA]+[AR]+[RAR]+[\mathbb{Z}A]$ is a hyperideal of R where $[\mathbb{Z}A] = \{\sum_{i=1}^{m} n_i a_i \mid m \in \mathbb{N}, n_i \in \mathbb{Z}, a_i \in A\}$.*

Proof. For any $b, c \in [RA]+[AR]+[RAR]+[\mathbb{Z}A]$, it follows that

$$\begin{aligned} b - c &\subseteq b + (-c) \\ &\subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]+(-[RA])+(-[AR])+(-[RAR])+(-[\mathbb{Z}A]) \\ &\subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]+[(-R)A]+[A(-R)]+[(-R)AR]+[(-\mathbb{Z})A] \\ &\subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]+[RA]+[AR]+[RAR]+[\mathbb{Z}A] \\ &\subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]. \end{aligned}$$

In particular, $0 \in b - b \subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]$ for some $b \in [RA]+[AR]+[RAR]+[\mathbb{Z}A]$. Let $r \in R$. Then

$$\begin{aligned} rb \subseteq r([RA]+[AR]+[RAR]+[\mathbb{Z}A]) &\subseteq (r[RA]+r[AR]+r[RAR]+r[\mathbb{Z}A]) \\ &\subseteq ([(rR)A]+[(rA)R]+[(rR)AR]+[\mathbb{Z}(rA)]) \\ &\subseteq [RA]+[RAR] \\ &\subseteq [RA]+[RAR]+0 \\ &\subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]. \end{aligned}$$

Similarly, $br \subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]$. Thus $[RA]+[AR]+[RAR]+[\mathbb{Z}A]$ is a hyperideal of R. $\square$

Definition 1.2.16. Let A be a nonempty subset of a hyperring R. Define $\langle A \rangle$ to be the smallest hyperideal of R containing A. The hyperideal $\langle A \rangle$ is called *the hyperideal generated by A*.

Proposition 1.2.17. *Let I_λ be a hyperideal of a hyperring R for all $\lambda \in \Lambda$. Then $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a hyperideal of R. Moreover, for any nonempty subsets A of R, $\langle A \rangle = \bigcap\{I \mid I$ is a hyperideal of R containing $A\}$ which is a hyperideal of R.*

Proof. We know that $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Let $a, b \in \bigcap_{\lambda\in\Lambda} I_\lambda$ and $r \in R$. Then $a, b \in I_\lambda$ for all $\lambda \in \Lambda$. Since I_λ is a hyperideal of R, $a - b \subseteq I_\lambda$ and $ra, ar \subseteq I_\lambda$ for all $\lambda \in \Lambda$. Hence $a - b, ra, ar \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a hyperideal of R.

Let $\mathcal{A} = \{I \mid I \text{ is a hyperideal of } R \text{ containing } A\}$. It follows that $\bigcap \mathcal{A}$ is a hyperideal of R containing A. Then $\langle A\rangle \subseteq \bigcap \mathcal{A}$. Since $\langle A\rangle$ is a hyperideal of R containing A, $\langle A\rangle \in \mathcal{A}$. Hence $\bigcap \mathcal{A} \subseteq \langle A\rangle$. Therefore $\langle A\rangle = \bigcap \mathcal{A}$. □

Next, we give an explicit form of $\langle A\rangle$ for all nonempty subsets A of R.

Proposition 1.2.18. *Let A be a nonempty subset of a hyperring R. Then*

$$\langle A\rangle = [RA] + [AR] + [RAR] + [\mathbb{Z}A].$$

Proof. Note that $a \in 0 + a$ for every $a \in A$. Since $[RA] + [AR] + [RAR] + [\mathbb{Z}A]$ is a hyperideal of R, clearly, $0 \in [RA] + [AR] + [RAR] + [\mathbb{Z}A]$. Hence

$$a \in 0 + a \subseteq \big([RA] + [AR] + [RAR] + [\mathbb{Z}A]\big) + [\mathbb{Z}A] \subseteq [RA] + [AR] + [RAR] + [\mathbb{Z}A].$$

Therefore $A \subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]$, so $\langle A\rangle \subseteq [RA]+[AR]+[RAR]+[\mathbb{Z}A]$.

Next, since $A \subseteq \langle A\rangle$, we have $[RA] \subseteq [R\langle A\rangle] \subseteq \langle A\rangle$, $[AR] \subseteq [\langle A\rangle R] \subseteq \langle A\rangle$, $[RAR] \subseteq [R\langle A\rangle R] \subseteq \langle A\rangle$ and $[\mathbb{Z}A] \subseteq \langle A\rangle$. Thus $[RA]+[AR]+[RAR]+[\mathbb{Z}A] \subseteq \langle A\rangle$. Hence $\langle A\rangle = [RA] + [AR] + [RAR] + [\mathbb{Z}A]$. □

Corollary 1.2.19. *Let A be a nonempty subset of a hyperring R.*

(i) *If $A \subseteq RA$, then $\langle A\rangle = [RA] + [RAR]$.*

(ii) *If $A \subseteq AR$, then $\langle A\rangle = [AR] + [RAR]$.*

(iii) *If $A \subseteq RAR$, then $\langle A\rangle = [RAR]$.*

(iv) *If R is commutative, then $\langle A\rangle = [RA] + [\mathbb{Z}A]$.*

(v) *If R is commutative and $A \subseteq RA$, then $\langle A\rangle = [RA]$.*

Corollary 1.2.20. *Let A be a nonempty subset of a hyperring R.*

(i) *If $a \in Ra$ for all $a \in A$, then $\langle A\rangle = [RA] + [RAR]$.*

(ii) *If* $a \in aR$ *for all* $a \in A$*, then* $\langle A \rangle = [AR] + [RAR]$.

(iii) *If* $a \in RaR$ *for all* $a \in A$*, then* $\langle A \rangle = [RAR]$.

(iv) *If* R *is commutative and for every* $a \in R$, $a \in Ra$*, then* $\langle A \rangle = [RA]$.

Let $(R, +, \cdot)$ be a hyperring and P a hyperideal of R. We can construct a hyperring by defining the relation ρ on R by

$$a \rho b \quad \text{if and only if} \quad a + P = b + P \qquad \text{for all } a, b \in R.$$

It is obvious that ρ is an equivalence relation. We denote the collection of all equivalence classes by R/P. Note that $R/P = \{[a]_\rho \mid a \in R\}$ where $[a]_\rho$ is the equivalence class containing a.

Lemma 1.2.21. *Let* ρ *be the equivalence relation defined as above. Then* $[a]_\rho = a + P$ *for all* $a \in R$*. Moreover,* $R/P = \{a + P \mid a \in R\}$.

Proof. Fix $a \in R$. Let $x \in [a]_\rho$. Then $x \in x + P = a + P$. Next, let $x \in a + P$. We show that $x + P = a + P$. Since $x \in a + P$, we obtain that $x + P \subseteq a + P + P \subseteq a + P$ and there exists $p \in P$ such that $x \in a + p$. Hence $a \in x + (-p) \subseteq x + P$. Thus $a + P \subseteq x + P + P \subseteq x + P$. Therefore $x + P = a + P$. We can conclude that $x \in [a]_\rho$. □

As a consequence of Lemma 1.2.21, we can conclude that $a \in b + P$ if and only if $a + P = b + P$ for all $a, b \in R$.

Proposition 1.2.22. *Let* $(R, +, \cdot)$ *be a hyperring and* P *a hyperideal of* R*. Define* $\oplus : R/P \times R/P \to \wp^*(R/P)$ *by*

$$(a + P) \oplus (b + P) = \{x + P \mid x \in a + b\} \qquad \text{for all } a, b \in R.$$

Then $(R/P, \oplus)$ *is a canonical hypergroup.*

Proof. First, we show that $\oplus$ is well-defined. Let $a_1 + P = a_2 + P$ and $b_1 + P = b_2 + P$ where $a_1, a_2, b_1, b_2 \in R$. Moreover, let $A = \{v + P \mid v \in a_1 + b_1\}$ and $B = \{w + P \mid w \in a_2 + b_2\}$. To show that $A = B$, first let $v \in a_1 + b_1$. Then

$v \in a_1 + b_1 \subseteq (a_2 + P) + (b_2 + P) = (a_2 + b_2) + P$. So there exists $w \in a_2 + b_2$ such that $v \in w + P$, i.e., $v + P = w + P$. Hence $A \subseteq B$. The proof of the reverse inclusion is similar. Consequently, $\oplus$ is well-defined.

Next, we show that $(R/P, \oplus)$ is a hypergroup. Let $a_1, a_2, a_3 \in R$. Then

$$\begin{aligned}\big((a_1 + P) \oplus (a_2 + P)\big) \oplus (a_3 + P) &= \{v + P \mid v \in a_1 + a_2\} \oplus (a_3 + P) \\ &= \bigcup_{v \in a_1 + a_2} (v + P) \oplus (a_3 + P) \\ &= \bigcup_{v \in a_1 + a_2} \{w + P \mid w \in v + a_3\} \\ &= \{w + P \mid w \in (a_1 + a_2) + a_3\} \\ &= \{w + P \mid w \in a_1 + (a_2 + a_3)\} \\ &= \bigcup_{v \in a_2 + a_3} \{w + P \mid w \in a_1 + v\} \\ &= \bigcup_{v \in a_2 + a_3} (a_1 + P) \oplus (v + P) \\ &= (a_1 + P) \oplus \{v + P \mid v \in a_2 + a_3\} \\ &= (a_1 + P) \oplus \big((a_2 + P) \oplus (a_3 + P)\big).\end{aligned}$$

Thus $\oplus$ is associative. In order to show that $(a_1 + P) \oplus (R/P) = R/P$, let $a \in R$. Since R is a hypergroup, $R = a_1 + R$ so that there exists $b \in R$ such that $a \in a_1 + b$. Then $a + P \in (a_1 + P) \oplus (b + P) \subseteq (a_1 + P) \oplus R/P$.

Now, we prove that $(R/P, \oplus)$ is canonical. It is clear that $(R/P, \oplus)$ is commutative because $(R, +)$ is commutative. We see that P is the scalar identity of $(R/P, \oplus)$. To show that $-a + P$ is an inverse of $a + P$ for each $a \in R$, let $a \in R$. Then $(a + P) \oplus (-a + P) = \{v + P \mid v \in a + (-a)\}$. Thus $P \in (a + P) \oplus (-a + P)$. Hence $-a + P$ is an inverse of $a + P$. For the uniqueness of an inverse of $a + P$, we let $b \in R$ be such that $P \in (a + P) \oplus (b + P)$. There exists $t \in a + b$ such that $t + P = P$. Then $t \in P$ and $b \in -a + t$, so $b \in (-a + t) + 0 \subseteq (-a + t) + P = -a + P$. Hence $b + P = -a + P$.

Finally, assume that $a_1 + P \in (a_2 + P) \oplus (a_3 + P)$ where $a_1, a_2, a_3 \in R$. There exists $t \in a_2 + a_3$ such that $a_1 + P = t + P$. Then $t \in a_1 + u$ for some $u \in P$. Since $t \in a_2 + a_3$, we obtain that $a_3 \in t - a_2 \subseteq a_1 + u - a_2 = (a_1 - a_2) + u$. Then there exists $s \in a_1 - a_2$ such that $a_3 \in s + u$ so that $a_3 \in s + P$, i.e., $a_3 + P = s + P$. Hence $a_3 + P \in (a_1 + P) \oplus (-a_2 + P) = (a_1 + P) \oplus \big(-(a_2 + P)\big)$. □

Proposition 1.2.23. *Let $(R, +, \cdot)$ be a hyperring and P a hyperideal of R. Define $\circ : R/P \times R/P \to \wp^*(R/P)$ by*

$$(a + P) \circ (b + P) = \{x + P \mid x \in ab\} \quad \text{for all } a, b \in R.$$

Then $(R/P, \oplus, \circ)$ is a hyperring. This hyperring is called a quotient hyperring.

Proof. First, we show that $\circ$ is well-defined. Let $a_1+P = a_2+P$ and $b_1+P = b_2+P$ where $a_1, a_2, b_1, b_2 \in R$. To show that $A := \{v + P \mid v \in a_1b_1\} = \{w + P \mid w \in a_2b_2\} := B$, let $v \in a_1b_1$. Then $v \in a_1b_1 \subseteq (a_2 + P)(b_2 + P) \subseteq (a_2b_2) + P$. So there exists $w \in a_2b_2$ such that $v \in w + P$, i.e., $v + P = w + P$. Hence $A \subseteq B$. The proof of the reverse inclusion is similar. Consequently, $\circ$ is well-defined.

The proof of the associativity of $\circ$ is essentially the same as the proof of the associativity of $\oplus$ in Proposition 1.2.22.

Moreover, we see that

$$\begin{aligned}
(a_1 + P) \circ \big((a_2 + P) \oplus (a_3 + P)\big) &= (a_1 + P) \circ \{v + P \mid v \in a_2 + a_3\} \\
&= \bigcup_{v \in a_2 + a_3} (a_1 + P) \circ (v + P) \\
&= \bigcup_{v \in a_2 + a_3} \{w + P \mid w \in a_1 v\} \\
&= \{w + P \mid w \in a_1(a_2 + a_3)\} \\
&\subseteq \{w + P \mid w \in a_1a_2 + a_1a_3\} \\
&= \bigcup_{s \in a_1a_2,\ l \in a_1a_3} \{w + P \mid w \in s + l\} \\
&= \bigcup_{s \in a_1a_2,\ l \in a_1a_3} (s + P) \oplus (l + P) \\
&= \{s + P \mid s \in a_1a_2\} \oplus \{l + P \mid l \in a_1a_3\} \\
&= \big((a_1 + P) \circ (a_2 + P)\big) \oplus \big((a_1 + P) \circ (a_3 + P)\big),
\end{aligned}$$

i.e., $(a_1+P)\circ\big((a_2+P)\oplus(a_3+P)\big) \subseteq \big((a_1+P)\circ(a_2+P)\big)\oplus\big((a_1+P)\circ(a_3+P)\big)$. Similarly, $\big((a_2+P)\oplus(a_3+P)\big)\circ(a_1+P) \subseteq \big((a_2+P)\circ(a_1+P)\big)\oplus\big((a_3+P)\circ(a_1+P)\big)$.

Next, we show that $(a_1 + P) \circ \big(-(a_2 + P)\big) = -\big((a_1 + P) \circ (a_2 + P)\big) = \big(-(a_1 + P)\big) \circ (a_2 + P)$. First,

$$\begin{aligned}(a_1 + P) \circ \big(-(a_2 + P)\big) = (a_1 + P) \circ (-a_2 + P) &= \{v + P \mid v \in a_1(-a_2)\}\\ &= \{v + P \mid v \in (-a_1)a_2\}\\ &= (-a_1 + P) \circ (a_2 + P)\\ &= \big(-(a_1 + P)\big) \circ (a_2 + P),\end{aligned}$$

i.e., $(a_1 + P) \circ \big(-(a_2 + P)\big) = \big(-(a_1 + P)\big) \circ (a_2 + P)$. Moreover,

$$\begin{aligned}(a_1 + P) \circ \big(-(a_2 + P)\big) = (a_1 + P) \circ (-a_2 + P) &= \{v + P \mid v \in a_1(-a_2)\}\\ &= \{v + P \mid v \in -(a_1a_2)\}\\ &= \{-v + P \mid v \in a_1a_2\}\\ &= -\{v + P \mid v \in a_1a_2\}\\ &= -\big((a_1 + P) \circ (a_2 + P)\big).\end{aligned}$$

i.e., $(a_1 + P) \circ \big(-(a_2 + P)\big) = -\big((a_1 + P) \circ (a_2 + P)\big)$.

Therefore $(R/P, \oplus, \circ)$ is a hyperring. □

1.2.2 Subhypermodules

In this subsection, we give a definition of subhypermodules and some properties that parallel the properties of hyperideals.

Definition 1.2.24. A nonempty subset N of an R-hypermodule M is called a *subhypermodule* of M if N is an R-hypermodule under the same hyperoperations on M.

By the same ideas as in the section on hyperideals, we obtain the following propositions. So the proof are omitted.

Proposition 1.2.25. *Let N be a nonempty subset of an R-hypermodule M. Then N is a subhypermodule of M if and only if for every $x, y \in N$ and $r \in R$, $x - y \in N$ and $rx \subseteq N$.*

Corollary 1.2.26. *Let N be a canonical subhypergroup of an R-hypermodule M. Then N is a subhypermodule of M if and only if $[RN] \subseteq N$.*

The following are some ways to construct new subhypermodules from given subhypermodules, hyperideals and elements in hyperrings and hypermodules.

Proposition 1.2.27. *Let M be an R-hypermodule, I a hyperideal of R, N and K subhypermodules of M, $a \in R$ and $m \in M$. Then $[aN]$, $[Im]$, $[IN]$, and $N + K$ are subhypermodules of M.*

Proof. The proofs are routine. □

Next, we introduce the subhypermodule generated by a subset of a hypermodule.

Proposition 1.2.28. *Let X be a nonempty subset of a hypermodule M. Then $[RX] + [\mathbb{Z}X]$ is a subhypermodule of M.*

Proof. Let $b, c \in [RX] + [\mathbb{Z}X]$. Then

$$\begin{aligned} b - c &\subseteq b + (-c) \\ &\subseteq [RX] + [\mathbb{Z}X] + (-[RX]) + (-[\mathbb{Z}X]) \\ &\subseteq [RX] + [\mathbb{Z}X] + [(-R)X] + [(-\mathbb{Z})X] \\ &\subseteq [RX] + [\mathbb{Z}X] + [RX] + [\mathbb{Z}X] \\ &\subseteq [RX] + [\mathbb{Z}X]. \end{aligned}$$

Hence $b - c \subseteq [RX] + [\mathbb{Z}X]$. Let $r \in R$. Then

$$\begin{aligned} rb \subseteq r([RX] + [\mathbb{Z}X]) &\subseteq (r[RX] + r[\mathbb{Z}X]) \\ &\subseteq ([(rR)X] + [\mathbb{Z}(rX)]) \\ &\subseteq [RX] \\ &\subseteq [RX] + [\mathbb{Z}X]. \end{aligned}$$

Thus $[RX] + [\mathbb{Z}X]$ is a subhypermodule of M. □

Definition 1.2.29. Let X be a nonempty subset of a hypermodule M. Define $\langle X\rangle$ to be the smallest subhypermodule of M containing X. The subhypermodule $\langle X\rangle$ is called *the subhypermodule generated by* X.

Proposition 1.2.30. *Let N_λ be a subhypermodule of a hypermodule M for all $\lambda \in \Lambda$. Then $\bigcap_{\lambda\in\Lambda} N_\lambda$ is a subhypermodule of M. Moreover, for all nonempty subsets X of M, $\langle X\rangle = \bigcap\{N \mid N \text{ is a subhypermodule of } M \text{ containing } X\}$ which is a subhypermodule of M.*

Proof. We know that $0 \in \bigcap_{\lambda\in\Lambda} N_\lambda$. Let $a, b \in \bigcap_{\lambda\in\Lambda} N_\lambda$ and $r \in R$. Since N_λ is a subhypermodule of M, $a - b \subseteq N_\lambda$ and $ra \subseteq N_\lambda$ for all $\lambda \in \Lambda$. Hence $a - b, ra \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a subhypermodule of M.

Let $\mathcal{N} = \{N \mid N \text{ is a subhypermodule of } M \text{ containing } X\}$. Then $\bigcap\mathcal{N}$ is a subhypermodule of M containing X. Thus $\langle X\rangle \subseteq \bigcap\mathcal{N}$. Since $\langle X\rangle$ is a subhypermodule of M containing X, $\langle X\rangle \in \mathcal{N}$. Hence $\bigcap\mathcal{N} \subseteq \langle X\rangle$. Therefore $\langle X\rangle = \bigcap\mathcal{N}$. □

Next, we give an explicit form for $\langle X\rangle$ for all subsets X of M.

Proposition 1.2.31. *Let X be a nonempty subset of an R-hypermodule M. Then $\langle X\rangle = [RX] + [\mathbb{Z}X]$.*

Proof. Note that for every $x \in X$, $x \in 0 + x$. Since $[RX] + [\mathbb{Z}X]$ is a subhypermodule of M, $0 \in [RX] + [\mathbb{Z}X]$. Hence

$$x \in 0 + x \subseteq \big([RX] + [\mathbb{Z}X]\big) + [\mathbb{Z}X] \subseteq [RX] + [\mathbb{Z}X].$$

Therefore $X \subseteq [RX] + [\mathbb{Z}X]$, so $\langle X\rangle \subseteq [RX] + [\mathbb{Z}X]$. Next, since $X \subseteq \langle X\rangle$, we have $[RX] \subseteq [R\langle X\rangle] \subseteq \langle X\rangle$ and $[\mathbb{Z}X] \subseteq \langle X\rangle$. Thus $[RX] + [\mathbb{Z}X] \subseteq \langle X\rangle$. Hence $\langle X\rangle = [RX] + [\mathbb{Z}X]$. □

Corollary 1.2.32. *Let X be a nonempty subset of an R-hypermodule M such that $X \subseteq RX$. Then $\langle X\rangle = [RX]$.*

Corollary 1.2.33. *Let X be a nonempty subset of an R-hypermodule M such that $x \in Rx$ for all $x \in M$. Then $\langle X\rangle = [RX]$.*

Proposition 1.2.34. *Let N be a subhypermodule of an R-hypermodule M. Then $N = \sum_{n \in N} \langle n \rangle$.*

Proof. The proof is straightforward. □

Similar to the way that we define a quotient hyperring, we can define a quotient hypermodule. For an R-hypermodule M and a subhypermodule N of M, we can define $M/N = \{x + N \mid x \in M\}$ where $x \in y + N$ if and only if $x + N = y + N$ for all $x, y \in M$ in the same way that R/P is defined where P is a hyperideal of R. We describe the details about quotient hypermodules in the next proposition.

Proposition 1.2.35. *Let $(M, +, \cdot)$ be an R-hypermodule and N a subhypermodule of M. Define $\oplus : M/N \times M/N \to \wp^*(M/N)$ and $\circ : R \times M/N \to \wp^*(M/N)$ by*

$$(x + N) \oplus (y + N) = \{t + N \mid t \in x + y\}$$
$$r \circ (x + N) = \{t + N \mid t \in rx\}$$

for all $r \in R$ and $x, y \in M$. Then $(M/N, \oplus, \circ)$ is an R-hypermodule. This R-hypermodule is called a quotient hypermodule.

Proof. The proof is similar to the proof of Proposition 1.2.22 and Proposition 1.2.23 combined. □

Proposition 1.2.36. *Let M be an R-hypermodule and N a subhypermodule of M. Then every subhypermodule of M/N is in the form K/N, where K is a subhypermodule of M containing in N.*

Proof. Let W be a subhypermodule of M/N and $K = \{w \in M \mid w + N \in W\}$. It is clear that $N \subseteq K$ and $W = K/N$. We show that K is a subhypermodule of M. Let $k_1, k_2 \in K$ and $r \in R$. To show that $k_1 - k_2 \subseteq K$, let $x \in k_1 - k_2 = k_1 + (-k_2)$. Then $x + N \in (k_1 + N) \oplus (-k_2 + N) \subseteq W$. Hence $x \in K$ so $k_1 - k_2 \subseteq K$. To show that $rk_1 \subseteq K$, let $x \in rk_1$. Then $x + N \in r \circ (k_1 + N) \subseteq W$. Hence $x \in K$ so $rk_1 \subseteq K$. Thus K is a subhypermodule of M. □

1.2.3 Examples of Hyperrings and Hypermodules

For convenience, we gather our examples of hyperrings and hypermodules together in this subsection. Before that, we recall the definition of a Krasner hyperring, [9].

Definition 1.2.37. [9] A system $(R, \oplus, \circ)$ is called a *Krasner hyperring* if

(i) $(R, \oplus)$ is a canonical hypergroup,

(ii) $(R, \circ)$ is a semigroup with zero 0 where 0 is the scalar identity of $(R, \oplus)$ and

(iii) $x \circ (y \oplus z) = x \circ y \oplus x \circ z$ and $(y \oplus z) \circ x = y \circ x \oplus z \circ x$ for all $x, y, z \in R$.

Example 1.2.38. Let $(A, +, \cdot)$ be a Krasner hyperring and let H be a hyperideal of A. Define $\circ : A \times A \to \wp^*(A)$ by $a \circ b = ab + H$ for all $a, b \in A$. Then $(A, +, \circ)$ is a strongly distributive hyperring.

To show that $(A, +, \circ)$ is a strongly distributive hyperring, first note that since $(A, +, \cdot)$ is a Krasner hyperring, $(A, +)$ is a canonical hypergroup. To show that $(A, +, \circ)$ is a hyperring, let $a, b, c \in A$. Then

$$\begin{aligned}(a \circ b) \circ c = \bigcup_{v \in a \circ b} v \circ c = \bigcup_{v \in ab+H} vc + H &= (ab + H)c + H \\ &= (ab)c + H \\ &= a(bc) + H \\ &= a(bc + H) + H \\ &= \bigcup_{w \in bc+H} aw + H \\ &= a \circ (b \circ c).\end{aligned}$$

Thus $(a \circ b) \circ c = a \circ (b \circ c)$. Next,

$$\begin{aligned}(a + b) \circ c = \bigcup_{v \in a+b} v \circ c = \bigcup_{v \in a+b} vc + H &= (a + b)c + H \\ &= \big(ac + H\big) + \big(bc + H\big) \\ &= (a \circ c) + (b \circ c).\end{aligned}$$

In the same way, $a \circ (b + c) = (a \circ b) + (a \circ c)$. Finally,

$$(-a) \circ b = (-a)b + H = a(-b) + H = a \circ (-b) = a(-b) + H = -(ab) + H$$
$$= -\big(ab + H\big) = -(a \circ b).$$

Hence $(A, +, \circ)$ is a strongly distributive hyperring.

Example 1.2.39. Let $(G, +)$ be a canonical hypergroup. Define $\circ : G \times G \to \wp^*(G)$ by $a \circ b = \langle a, b\rangle$, the subhypergroup of G generated by the set $\{a, b\}$, for all $a, b \in G$. Then $(G, +, \circ)$ is a hyperring.

To show that $(G, +, \circ)$ is a hyperring, first note that $\langle X\rangle = [\mathbb{Z}X]$ for all nonempty subsets X of G. Next, we show that $\langle a, b, c\rangle = \bigcup_{v\in\langle a,b\rangle}\langle v, c\rangle$ for all $a, b, c \in G$. Let $a, b, c \in G$. First, let $x \in \langle a, b, c\rangle$. Then $x \in n_1a + n_2b + n_3c$ for some $n_1, n_2, n_3 \in \mathbb{Z}$. Then $x \in y + n_3c$ for some $y \in n_1a + n_2b \subseteq \langle a, b\rangle$. Thus $x \in y + n_3c \subseteq \langle y, c\rangle$ and $y \in \langle a, b\rangle$. Hence $x \in \bigcup_{v\in\langle a,b\rangle}\langle v, c\rangle$.

Let $x \in \bigcup_{v\in\langle a,b\rangle}\langle v, c\rangle$. Then $x \in \langle v, c\rangle$ for some $v \in \langle a, b\rangle$. Thus $x \in n_1v + n_2c$ and $v \in n_3a + n_4b$ for some $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. Hence $x \in n_1v + n_2c \subseteq n_1(n_3a + n_4b) + n_2c = (n_1n_3)a + (n_1n_4)b + n_2c \subseteq \langle a, b, c\rangle$.

This shows that $\langle a, b, c\rangle = \bigcup_{v\in\langle a,b\rangle}\langle v, c\rangle$.

Now, we prove that $(G, +, \circ)$ is a hyperring. Let $a, b, c \in G$. Then

$$(a\circ b)\circ c = \bigcup_{v\in a\circ b} v\circ c = \bigcup_{v\in\langle a,b\rangle} \langle v, c\rangle = \langle a, b, c\rangle = \bigcup_{w\in\langle b,c\rangle} \langle a, w\rangle = \bigcup_{w\in b\circ c} a\circ w = a\circ(b\circ c).$$

Moreover, $(a+b)\circ c = \bigcup_{v\in a+b} v\circ c = \bigcup_{v\in a+b}\langle v, c\rangle \subseteq \langle a, c\rangle + \langle b, c\rangle = (a\circ c)+(b\circ c)$. We also obtain similarly that $a \circ (b + c) = (a \circ b) + (a \circ c)$. Finally,

$$(-a) \circ b = \langle -a, b\rangle = \langle a, b\rangle = \langle a, -b\rangle = a \circ (-b) = \langle a, -b\rangle$$
$$= \langle a, b\rangle = -\langle a, b\rangle = -(a \circ b).$$

Hence $(G, +, \circ)$ is a hyperring.

Example 1.2.40. Let R be a hyperring. Then R is an R-hypermodule.

Example 1.2.41. Let R be a Krasner hyperring, M an R-hypermodule and N a subhypermodule of M. Define $\circ : R \times M \to \wp^*(M)$ by $a \circ x = ax + N$ for all $a \in R$ and $x \in M$. Then $(M, +, \circ)$ is a strongly distributive hypermodule.

1.3 Differences between Modules and Hypermodules

Besides the diferrence of operations and hyperoperation, there are some properties in modules over rings that may not hold in hypermodules. In the previous subsection, we give a basic case study to show the differences between modules and hypermodules.

First, $\{0\}$ is always a submodule of any modules but not necessarily a subhypermodule of a hypermodule.

Example 1.3.1. Consider the hyperring R in Example 1.2.39 as an R-hypermodule when $R \neq \{0\}$. We obtain that R is the only subhypermodule of R. Hence $\{0\}$ is not a subhypermodule of R.

We give a necessary and sufficient condition for $\{0\}$ to be a subhypermodule.

Proposition 1.3.2. *Let M be an R-hypermodule. Then $\{0\}$ is a subhypermodule of M if and only if there exist a hyperideal I of R and a subhypermodule N of M such that $IN = \{0\}$.*

Proof. First, assume that $\{0\}$ is a subhypermodule of M. Choose $I = R$ and $N = \{0\}$. Then $IN = \{0\}$.

Conversely, assume that there exist a hyperideal I of R and a subhypermodule N of M such that $IN = \{0\}$. It follows that $[IN] = \{0\}$. Thus $\{0\}$ is a subhypermodule of M since $[IN]$ is a subhypermodule of M. □

Proposition 1.3.3. *Let R be a hyperring. Then $\{0\}$ is a hyperideal of R if and only if there exist hyperideals I and J such that $IJ = \{0\}$.*

Proof. This is similar to the proof of the previous proposition. □

For nonempty subsets X and Y of an R-hypermodule, we define

$$(X : Y) = \{r \in R \mid rY \subseteq X\}.$$

As the following example shows, even if N is a subhypermodule of M, the set $(N : M)$ may be empty. However, if N is a submodule of a module M, then $(N : M)$ is nonempty.

Example 1.3.4. Let $(R, +)$ be an abelian group such that $|R| \geq 2$. Define $\odot : R \times R \to \wp^*(R)$ by $a \odot b = \langle a, b \rangle$, the subgroup of R generated by the set $\{a, b\}$, for all $a, b \in R$. Then $(R, +, \odot)$ is a hyperring. Consider R as an R-hypermodule via the hypermodule action $\circ : R \times R \to \wp^*(R)$ defined by

$$r \circ m = \begin{cases} r \odot m & \text{if} \quad m \neq 0, \\ \{0\} & \text{if} \quad m = 0, \end{cases}$$

for all $r, m \in R$. Then $\{0\}$ is a subhypermodule of R and $(\{0\} : R) = \varnothing$.

The next proposition is about a property of $(N : M)$ when $(N : M)$ is not empty.

Proposition 1.3.5. *Let N be a subhypermodule of an R-hypermodule M. If $(N : M)$ is nonempty, then $(N : M)$ is a hyperideal of R.*

Proof. Let $a, b \in (N : M)$ and $r \in R$. Then $aM \subseteq N$ and $bM \subseteq N$. It follows that $(a - b)M \subseteq aM + bM \subseteq N + N \subseteq N$. Hence $a - b \in (N : M)$. Next, we show that $ra, ar \in (N : M)$. Consider

$$raM = r(aM) \subseteq rN \subseteq N \quad \text{and} \quad arM = a(rM) \subseteq aM \subseteq N.$$

Thus $ra, ar \in (N : M)$ as desired. Therefore $(N : M)$ is a hyperideal of R. $\square$

CHAPTER II
PRIME HYPERIDEALS AND PRIME SUBHYPERMODULES

In this chapter, we introduce prime and weakly prime hyperideals and subhypermodules. Many properties are investigated. However, we emphasize ones related to prime and weakly prime subhypermodules.

2.1 Prime Hyperideals and Prime Subhypermodules

The first section is seperated into two subsections, namely prime hyperideals and prime subhypermodules.

2.1.1 Prime Hyperideals

In this subsection, we give a definition of prime hyperideals and determine some characterizations of them.

Definition 2.1.1. Let R be a hyperring. A proper hyperideal P of R is called *prime* if for all hyperideals I and J of R,

$$[IJ] \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P.$$

Proposition 2.1.2. *Let R be a hyperring and P a proper hyperideal of R. Then P is a prime hyperideal if and only if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all hyperideals I and J of R.*

Proof. Assume that P is a prime hyperideal. Let I and J be hyperideals of R such that $IJ \subseteq P$. Since P is a hyperideal and $[IJ]$ is the hyperideal generated by IJ, we see that $[IJ] \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$.

The converse follows from the fact that $IJ \subseteq [IJ]$. □

Next, we give examples of prime hyperideals.

Example 2.1.3. 1. Every prime ideal of a ring can be considered as a prime hyperideal of a hyperring.

2. Let $R = [0,1]$. Then $\big(R, \oplus_{\text{max}}, \cdot\big)$ is a Krasner hyperring $\big($see [18]$\big)$, where $\oplus_{\text{max}}$ is defined as in Example 1.1.10 and $\cdot$ is the usual multiplication on real numbers. Furthermore, let $K = [0, 0.5]$. Then K is a hyperideal of R. It follows from Example 1.2.38 that $(R, \oplus_{\text{max}}, \circ)$ is a hyperring. Choose $P = [0,1)$. Thus P is a prime hyperideal of R.

3. Let $R = [1,\infty) \cup \{0\}$. Then $\big(R, \oplus_{\text{min}}, \cdot\big)$ is a Krasner hyperring $\big($see [18]$\big)$, where $\oplus_{\text{min}}$ is defined as in Example 1.1.11 and $\cdot$ is the usual multiplication on real numbers. Furthermore, let $K = [3,\infty) \cup \{0\}$. Then K is a hyperideal of R. It follows from Example 1.2.38 that $(R, \oplus_{min}, \circ)$ is a hyperring. Choose $P = (1,\infty) \cup \{0\}$. Thus P is a prime hyperideal of R.

4. Let $R = [-1,1]$. Then $\big(R, \oplus_{\text{abs}}, \cdot\big)$ is a Krasner hyperring $\big($see [18]$\big)$, where $\oplus_{\text{abs}}$ is defined as in Example 1.1.12 and $\cdot$ is the usual multiplication on real numbers. Furthermore, let $K = [-3,3] \cup \{0\}$. Then K is a hyperideal of R. It follows from Example 1.2.38 that $(R, \oplus_{\text{abs}}, \circ)$ is a hyperring. Choose $P = (-1,1)$. Thus P is a prime hyperideal of R.

Next, we characterize prime hyperideals under the condition that the hyperring is commutative.

Proposition 2.1.4. *Let R be a commutative hyperring and P a proper hyperideal of R. Then P is a prime hyperideal if and only if $ab \subseteq P$ implies $a \in P$* or *$b \in P$ for all $a, b \in R$,.*

Proof. First, assume that P is prime. Let $a, b \in R$ be such that $ab \subseteq P$. Choose $I = \langle a \rangle$ and $J = \langle b \rangle$. Then I and J are hyperideals of R. We show that $IJ \subseteq P$. Recall from Corollary 1.2.20 that

$$I = [Ra] + [\mathbb{Z}a] \text{ and } J = [Rb] + [\mathbb{Z}b].$$

Since $ab \subseteq P$, P is a hyperideal and R is commutative, we have $IJ \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$. Thus $a \in \langle a \rangle \subseteq P$ or $b \in \langle b \rangle \subseteq P$.

Conversely, assume that $ab \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$. Let I and J be hyperideals of R such that $IJ \subseteq P$. Suppose that $I \nsubseteq P$. There exists $a \in I \setminus P$. Let $b \in J$. Then $ab \subseteq IJ \subseteq P$. By assumption, we have $b \in P$. Therefore $J \subseteq P$. □

Finally, we characterize prime hyperideals under the condition that $a \in Ra$ for all $a \in R$ or $a \in aR$ for all $a \in R$.

Proposition 2.1.5. *Let R be a hyperring such that $a \in Ra$ for all $a \in R$ and P a proper hyperideal of R. Then P is a prime hyperideal if and only if $aRb \subseteq P$ implies $a \in P$* or *$b \in P$ for all $a, b \in R$.*

Proof. First, assume that P is prime. Let $a, b \in R$ be such that $aRb \subseteq P$. Then $I = \langle a \rangle$ and $J = \langle b \rangle$ are hyperideals of R. We show that $IJ \subseteq P$. Note that by Corollary 1.2.20

$$I = [Ra] + [RaR] \text{ and } J = [Rb] + [RbR].$$

Since $aRb \subseteq P$ and P is a hyperideal, $IJ \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$. Thus $a \in \langle a \rangle \subseteq P$ or $b \in \langle b \rangle \subseteq P$.

Conversely, assume that $aRb \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$. Let I and J be hyperideals of R such that $IJ \subseteq P$. Suppose that $I \nsubseteq P$. There exists $a \in I \setminus P$. Let $b \in J$. Then $Rb \subseteq J$. Hence $aRb \subseteq IJ \subseteq P$. By assumption, we have $b \in P$. Therefore $J \subseteq P$. □

Proposition 2.1.6. *Let R be a hyperring such that $a \in aR$ for all $a \in R$ and P a proper hyperideal of R. Then P is a prime hyperideal if and only if $aRb \subseteq P$ implies $a \in P$* or *$b \in P$ for $a, b \in R$.*

Proof. The proof is similar to that of the previous proposition. □

2.1.2 Prime Subhypermodules

We first give a defintion of prime subhypermodules.

Definition 2.1.7. Let R be a hyperring and M an R-hypermodule. A proper subhypermodule N of M is said to be *prime* if for every hyperideal I of R and every subhypermodule D of M,

$$[ID] \subseteq N \Rightarrow I \subseteq (N : M) \text{ or } D \subseteq N.$$

Next, we give some characterizations of prime subhypermodules via three differnt conditions. Our first characterization resembles Proposition 2.1.2.

Proposition 2.1.8. *Let R be a hyperring, M an R-hypermodule and N a proper subhypermodule of M. Then N is a prime subhypermodule if and only if $ID \subseteq N$ implies $I \subseteq (N : M)$ or $D \subseteq N$ for all hyperideals I of R and all subhypermodules D of M.*

Proof. This proof is similar to the proof of Proposition 2.1.2. □

Example 2.1.9. 1. Every prime submodule of a module can be considered as a prime subhypermodule of a hypermodule.

2. Every prime hyperideal P of a hyperring R is a subhypermodule of the R-hypermodule R.

3. Every proper subhypermodules N of M such that $(N : M) = R$ is always a prime subhypermodule. For example, let $R = [0, 1)$ and $M = [0, 1]$. Then $[R, \oplus_{\max}, \cdot]$ and $[M, \oplus_{\max}, \circ]$ are Krasner hyperring and hypermodules over Krasner hyperring R, respectively, (see [18]), where $\oplus_{\max}$ is defined as in Example 1.1.10 and $\cdot$ is the usual multiplication on real numbers and $\circ : R \times M \to M$ is defined by $r \circ m = r \cdot m$ for all $m \in M$. Let $N = [0, 1)$ which is a proper subhypermodule of M. Then $(N : M) = \{r \in R \mid rM \subseteq N\} = R$. Thus N is a prime subhypermodule of M.

For the second characterization, we consider the condition that the hyperring is commutative.

Proposition 2.1.10. *Let R be a commutative hyperring, M an R-hypermodule and N a proper subhypermodule of M. Then N is a prime subhypermodule if and only if $am \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$.*

Proof. Assume first that N is a prime subhypermodule of M. Let $a \in R$ and $m \in M$ be such that $am \subseteq N$. Then $I = \langle a \rangle$ and $D = \langle m \rangle$ are a hyperideal of R and a subhypermodule of M, respectively. We claim that $ID \subseteq N$. As a result of the commutativity of R,

$$I = [Ra] + [\mathbb{Z}a] \text{ and } D = [Rm] + [\mathbb{Z}m].$$

Since $am \subseteq N$, N is a subhypermodule and R is commutative, we have $ID \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$. Thus $a \in \langle a \rangle \subseteq (N : M)$ or $m \in \langle m \rangle \subseteq N$.

Conversely, assume that $am \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. Suppose that $D \nsubseteq N$. There exists $m \in D \setminus N$. Since $ID \subseteq N$, it follows that $am \subseteq N$, so by assumption, $a \in (N : M)$ for all $a \in I$. Thus $I \subseteq (N : M)$. □

For the third characterization, we are interested in the condition $a \in aR$ for all $a \in R$.

Proposition 2.1.11. *Let M be an R-hypermodule, N a proper subhypermodule of M and assume that $a \in aR$ for every $a \in R$. Then N is a prime subhypermodule if and only if $aRm \subseteq N$ implies $a \in (N : M)$* or *$m \in N$ for all $a \in R$ and $m \in M$.*

Proof. Assume that N is a prime subhypermodule. Let $a \in R$ and $m \in M$ be such that $aRm \subseteq N$. Consider the hyperideal $I = \langle a \rangle$ of R and the subhypermodule $D = \langle m \rangle$ of M. We show that $ID \subseteq P$. It follows from Corollary 1.2.20 and Proposition 1.2.31 that

$$I = [aR] + [RaR] \text{ and } D = [Rm] + [\mathbb{Z}m].$$

Since $aRm \subseteq N$, we have $ID \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$. Thus $a \in \langle a \rangle \subseteq (N : M)$ or $m \in \langle m \rangle \subseteq N$.

Assume for the converse that $aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. Suppose that $D \nsubseteq N$. There exists $m \in D \setminus N$.

To show that $I \subseteq (N : M)$, let $a \in I$. Then $aR \subseteq IR \subseteq I$. Thus $aRm \subseteq ID \subseteq N$. By assumption, $a \in (N : M)$. Hence $I \subseteq (N : M)$. □

We can also obtain the same characterization under the condition $m \in Rm$ for all $m \in M$.

Proposition 2.1.12. *Let M be an R-hypermodule, N a proper subhypermodule of M and assume that $m \in Rm$ for every $m \in M$. Then N is a prime subhypermodule if and only if $aRm \subseteq N$ implies $a \in (N : M)$* or *$m \in N$ for all $a \in R$ and $m \in M$.*

Proof. The proof is similar to that of the previous proposition. □

Next, we give some properties of prime subhypermodules.

Proposition 2.1.13. *Let N be a subhypermodule of an R-hypermodule M such that $\varnothing \neq (N : M) \neq R$. If N is a prime subhypermodule of M, then $(N : M)$ is a prime hyperideal of R.*

Proof. Proposition 1.3.5 guarantees that $(N : M)$ is a hyperideal of R. Let I and J be hyperideals of R such that $IJ \subseteq (N : M)$. Suppose that $J \nsubseteq (N : M)$. Then $JM \nsubseteq N$, so that $[JM] \nsubseteq N$ but $I[JM] \subseteq [I[JM]] = [(IJ)M] \subseteq N$. Since N is a prime subhypermodule, $I \subseteq (N : M)$. Thus $(N : M)$ is a prime hyperideal of R. □

Corollary 2.1.14. *Let M be an R-hypermodule such that $M = RM$ and N a subhypermodule of M such that $(N : M) \neq \varnothing$. If N is a prime subhypermodule of M, then $(N : M)$ is a prime hyperideal of R.*

Proof. Assume that N is a prime subhypermodule of M. Then $N \neq M$. Suppose that $(N : M) = R$. Then $M = RM \subseteq N$, a contradiction. Hence $(N : M) \neq R$. The conclusion follows from Proposition 2.1.13. □

Definition 2.1.15. A *simple hypermodule* is a non-zero hypermodule which has no subhypermodules besides the zero subhypermodule and itself.

Lemma 2.1.16. *Let N be a maximal subhypermodule of an R-hypermodule M. Then the quotient hypermodule M/N is simple.*

Proof. The proof is straightforward. $\square$

Example 2.1.17. 1. Consider the hyperring R in Example 1.2.39 as an R-hypermodule where $R \neq \{0\}$. We obtain that R is the only subhypermodule of R.

2. From Example 2.1.3, we can see that $[0, 1)$, $(1, \infty) \cup \{0\}$ and $(-1, 1)$ are maximal subhypermodules of $[0, 1]$-hypermodule $[0, 1]$, $[1, \infty) \cup \{0\}]$-hypermodule $[1, \infty) \cup \{0\}$ and $[-1, 1]$-hypermodule $[-1, 1]$. Hence $[0, 1]/[0, 1)$, $[1, \infty) \cup \{0\}]/(1, \infty) \cup \{0\}]$ and $[-1, 1]/(-1, 1)$ are simple by above lemma.

Proposition 2.1.18. *Let N be a maximal subhypermodule of an R-hypermodule M. Then N is a prime subhypermodule of M.*

Proof. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. Suppose that $D \nsubseteq N$. There exists $m \in D \smallsetminus N$. Then $m + N \neq N$ and $I(m + N) = N$. Since N is a maximal subhypermodule, M/N is simple.

Let $\overline{m}$ be the element $m + N$ of M/N. Then $\langle \overline{m} \rangle = M/N$ (note that $\langle \overline{m} \rangle$ is the subhypermodule of M/N generates by $\overline{m}$). We have $I(M/N) = I\langle \overline{m} \rangle = I\big([R\overline{m}] + [\mathbb{Z}\overline{m}]\big) \subseteq [IR\overline{m}] + [I(\mathbb{Z}\overline{m})] \subseteq [I\overline{m}] + [\mathbb{Z}(I\overline{m})] \subseteq N$. This shows that $I(M/N) \subseteq \{N\}$. We claim that $IM \subseteq N$. Suppose not. Then there exist $a \in I$ and $m \in M$ such that $am \nsubseteq N$. Thus there exists $t \in am$ such that $t \notin N$. Hence $t + N \neq N$. Since $a \in I$ and $m \in M$, we obtain that $a(m + N) \subseteq I(M/N) \subseteq \{N\}$, i.e., $\{l + N \mid l \in am\} \subseteq \{N\}$, a contradiction. Thus $IM \subseteq N$, i.e., $I \subseteq (N : M)$. Therefore N is a prime subhypermodule of M. $\square$

The following property is another characterization of prime subhypermodules.

Proposition 2.1.19. *Let N be a proper subhypermodule of an R-hypermodule M. Then N is a prime subhypermodule if and only if $(N : K) = (N : M)$ for every subhypermodule K of M such that $N \subset K \subseteq M$.*

Proof. First, assume that N is prime and let K be a subhypermodule of M such that $N \subset K \subseteq M$. It is obvious that $(N : M) \subseteq (N : K)$. Let $r \in (N : K)$. Then $rK \subseteq N$. We show that $\langle r \rangle K \subseteq N$. Note that $\langle r \rangle K = \big([Rr] + [rR] + [RrR] + [\mathbb{Z}r]\big)K \subseteq [RrK] + [rRK] + [RrRK] + [\mathbb{Z}rK] \subseteq N$. Since N is prime and $N \subset K$, we obtain that $\langle r \rangle \subseteq (N : M)$. Hence $r \in (N : M)$. This shows $(N : K) = (N : M)$.

Conversely, assume that $(N : K) = (N : M)$ for every subhypermodule K of M such that $N \subset K \subseteq M$ and let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. Suppose $D \nsubseteq N$. Set $K = D + N$. Then $N \subset K \subseteq M$ and $IK = I(D + N) \subseteq ID + IN \subseteq N$. Thus $I \subseteq (N : K)$. By assumption, $I \subseteq (N : M)$. Hence N is a prime subhypermodule of M. □

In the rest of this section, we introduce homomorphisms of hypermodules and give some properties of prime subhypermodules that are related to homomorphisms.

Definition 2.1.20. Let M and M' be R-hypermodules. A function $\phi : M \to M'$ is called a *(hypermodule) homomorphism* if

$$\phi(x + y) = \phi(x) + \phi(y) \quad \text{and} \quad \phi(rx) = r\phi(x)$$

for all $r \in R$ and $x, y \in M$.

We define the kernel and the image of ϕ , denoted by $\ker(\phi)$ and $\operatorname{im}(\phi)$, respectively, by

$$\ker(\phi) = \{m \in M \mid \phi(m) = 0\} \quad \text{and} \quad \operatorname{im}(\phi) = \{\phi(m) \mid m \in M\}.$$

Proposition 2.1.21. *Let M and M' be R-hypermodules and $\phi : M \to M'$ a homomorphism. If $\phi(0) = 0$, then $\phi(-x) = -\phi(x)$ and $\phi(nx) = n\phi(x)$ for all $x \in M$ and $n \in \mathbb{Z}$.*

Proof. Assume that $\phi(0) = 0$. Let $x \in M$. Since $0 \in x + (-x)$, it follows that $\phi(0) \in \phi\big(x + (-x)\big) = \phi(x) + \phi(-x)$, so that $0 = \phi(0) \in \phi(x) + \phi(-x)$. Hence

$\phi(-x) = -\phi(x)$. Moreover, to show that $\phi(nx) = n\phi(x)$ for all $n \in \mathbb{Z}$, let $n \in \mathbb{Z}$. If $n = 0$, then we are done. If $n > 0$, then $\phi(nx) = \phi(\underbrace{x + x + \cdots + x}_{n \text{ copies}}) = \underbrace{\phi(x) + \phi(x) + \cdots + \phi(x)}_{n \text{ copies}} = n\phi(x)$. Assume that $n < 0$. Then

$$\begin{aligned}\phi(nx) = \phi\big(\underbrace{(-x) + (-x) + \cdots + (-x)}_{-n \text{ copies}}\big) &= \underbrace{\phi(-x) + \phi(-x) + \cdots + \phi(-x)}_{-n \text{ copies}} \\ &= \underbrace{\big(-\phi(x)\big) + \big(-\phi(x)\big) + \cdots + \big(-\phi(x)\big)}_{-n \text{ copies}} = n\phi(x).\end{aligned}$$

Hence $\phi(nx) = n\phi(x)$. □

From above, we see that the condition $\phi(0) = 0$ gives useful results. Thus from now on, all homomorphisms satisfy the condition $\phi(0) = 0$.

Proposition 2.1.22. *Let M and M' be R-hypermodules and $\phi : M \to M'$ a homomorphism.*

(i) *If N is a subhypermodule of M, then $\phi(N)$ is a subhypermodule of M'.*

(ii) *If N' is a subhypermodule of M', then $\phi^{-1}(N')$ is a subhypermodule of M.*

Proof. The proof is easy. □

Lemma 2.1.23. *Let M and M' be R-hypermodules and $\phi : M \to M'$ a homomorphism. Let N be a subhypermodule of M such that $\ker\phi \subseteq N$. If $x \in M$ is such that $\phi(x) \in \phi(N)$, then $x \in N$.*

Proof. Assume that $x \in M$ is such that $\phi(x) \in \phi(N)$. Then $\phi(x) = \phi(n)$ for some $n \in N$. Then $0 \in \phi(x) - \phi(n) = \phi(x - n)$. There exists $p \in x - n$ such that $\phi(p) = 0$, i.e., $p \in \ker(\phi) \subseteq N$. Since $p \in x - n$, we have $x \in p + n \subseteq N$. □

Proposition 2.1.24. *Let M and M' be R-hypermodules and $\phi : M \to M'$ a surjective homomorphism. Let N be a prime subhypermodule of M such that $\ker\phi \subseteq N$. Then $\phi(N)$ is a prime subhypermodule of M'.*

Proof. First, we show that $\phi(N) \neq M'$. Suppose not. Since N is prime, there exists $m \in M \smallsetminus N$. Since $\phi(N) = M'$, we have $\phi(m) \in \phi(N)$. By the previous lemma, $m \in N$, a contradiction. Thus $\phi(N) \neq M'$.

Let I be a hyperideal of R and D' a subhypermodule of M' such that $ID' \subseteq \phi(N)$. Proposition 2.1.22 yields that $\phi^{-1}(D')$ is a subhypermodule of M. We claim that $I\phi^{-1}(D') \subseteq N$. Let $a \in I$ and $x \in \phi^{-1}(D')$. Then $\phi(x) \in D'$, so that $\phi(ax) = a\phi(x) \subseteq \phi(N)$. To show that $ax \subseteq N$, let $l \in ax$. Then $\phi(l) \in \phi(N)$, so that $l \in N$ from the previous lemma. Therefore $ax \subseteq N$. This shows that $I\phi^{-1}(D') \subseteq N$ as claimed.

Since $I\phi^{-1}(D') \subseteq N$, $\phi^{-1}(D')$ is a subhypermodule of M and N is prime, we can conclude that $I \subseteq (N : M)$ or $\phi^{-1}(D') \subseteq N$. Then $IM \subseteq N$ or $\phi(\phi^{-1}(D')) \subseteq \phi(N)$. Since ϕ is surjective, $IM' = I\phi(M) = \phi(IM) \subseteq \phi(N)$ or $D' \subseteq \phi(\phi^{-1}(D')) \subseteq \phi(N)$. Hence $I \subseteq (\phi(N) : M')$ or $D' \subseteq \phi(N)$. Thus $\phi(N)$ is a prime subhypermodule of M'. □

Proposition 2.1.25. *Let M and M' be R-hypermodules and $\phi : M \to M'$ a homomorphism. Let N' be a prime subhypermodule of M' such that $\phi^{-1}(N') \neq M$. Then $\phi^{-1}(N')$ is a prime subhypermodule of M.*

Proof. Let I be a hyperideal of R and D a subhypermodule of M such that $ID \subseteq \phi^{-1}(N')$. Then $I\phi(D) = \phi(ID) \subseteq N'$. By Proposition 2.1.22, $\phi(D)$ is a subhypermodule of M'. Since N' is prime, $I \subseteq (N' : M')$ or $\phi(D) \subseteq N'$. Thus $\phi(IM) = I(\phi(M)) \subseteq IM' \subseteq N'$ or $D \subseteq \phi^{-1}(N')$. Hence $I \subseteq (\phi^{-1}(N') : M)$ or $D \subseteq \phi^{-1}(N')$. Therefore $\phi^{-1}(N')$ is a prime subhypermodule of M. □

Corollary 2.1.26. *Let N and K be subhypermodules of an R-hypermodule M such that $N \subseteq K$. Then K is a prime subhypermodule of M if and only if K/N is a prime subhypermodule of M/N.*

Proof. The proof follows from Proposition 2.1.24 and Proposition 2.1.25 by using the canonical projection $\phi : M \to M/N$. □

2.2 Weakly Primality

This section is divided into two subsections, discussing weakly prime hyperideals and weakly prime subhypermodules, respectively. Weakly prime hyperideals (subhypermodules) are a generalization of prime hyperideals (subhypermodules). We are interested in studying some properties regarding weakly prime hyperideals and weakly prime subhypermodules.

2.2.1 Weakly Prime Hyperideals

The definition of weakly prime hyperideals are extened from prime hyperideals in the same way as the extension of weakly prime ideals from prime ideals.

Definition 2.2.1. Let R be a hyperring. A proper hyperideal P is called *weakly prime* if for all hyperideals I and J,

$$\{0\} \neq [IJ] \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P.$$

We know from Chapter I that $\{0\}$ may not be a hyperideal. If $\{0\}$ is not a hyperideal, then, by Proposition 1.3.3, there are no hyperideals I and J such that $IJ = \{0\}$. Thus we obtain the following result.

Proposition 2.2.2. *Let R be a hyperring such that $\{0\}$ is not a hyperideal of R. Then prime hyperideals and weakly prime hyperideals of R are the same.*

In this subsection, it is reasonable for us to consider hyperrings such that $\{0\}$ is their hyperideal. For the rest of this subsection, we assume that $\{0\}$ is always a hyperideal. We characterize weakly prime hyperideals in the same ways as we did for prime hyperideals.

Proposition 2.2.3. *Let R be a hyperring and P a proper hyperideal of R. Then P is a weakly prime hyperideal of R if and only if $\{0\} \neq IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all hyperideals I and J of R.*

Proof. Assume that P is a weakly prime hyperideal of R. Let I and J be hyperideals of R such that $\{0\} \neq IJ \subseteq P$. Then $[IJ] \subseteq P$ and $\{0\} \neq IJ \subseteq [IJ]$. Thus $\{0\} \neq IJ \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$.

Conversely, assume that $\{0\} \neq IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all hyperideals I and J of R. Let I and J be hyperideals of R such that $\{0\} \neq [IJ] \subseteq P$. Since $[IJ] \neq \{0\}$, there exist $a_i \in I$ and $b_i \in J$ for all $i \in \{1, 2, \ldots, n\}$ such that $a_1b_1 + a_2b_2 + \cdots + a_nb_n \neq \{0\}$. Then $a_jb_j \neq \{0\}$ for some j. Hence $IJ \neq \{0\}$, so that $\{0\} \neq IJ \subseteq P$. By assumption, $I \subseteq P$ or $J \subseteq P$. This shows P is weakly prime. □

If a hyperring is strongly distributive and commutative, we obtain the following:

Proposition 2.2.4. *Let R be a strongly distributive commutative hyperring and P a proper hyperideal of R. Then P is a weakly prime hyperideal if and only if $\{0\} \neq ab \subseteq P$ implies $a \in P$* or *$b \in P$ for all $a, b \in R$.*

Proof. Assume that P is weakly prime. Let $a, b \in R$ be such that $\{0\} \neq ab \subseteq P$. Note that R is commutative. Then we consider the following hyperideals I and J of R:

$$I = \langle a \rangle = [Ra] + [\mathbb{Z}a] \text{ and } J = \langle b \rangle = [Rb] + [\mathbb{Z}b].$$

Then $a \in I$ and $b \in J$. Thus $ab \subseteq IJ$, so that $IJ \neq \{0\}$. Since R is commutative and $ab \subseteq P$, we have $IJ \subseteq P$. Thus $\{0\} \neq IJ \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$. Thus $a \in P$ or $b \in P$.

Assume for the converse that $\{0\} \neq ab \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$. Let I and J be hyperideals of R such that $\{0\} \neq IJ \subseteq P$. Suppose that $J \nsubseteq P$. There exists $x \in J \setminus P$. To show that $I \subseteq P$, let $a \in I$. If $ax \neq \{0\}$, then $\{0\} \neq ax \subseteq P$, so by assumption, we have $a \in P$. Now, assume that $ax = \{0\}$. Then there are two cases to be considered

Case 1 $aJ \neq \{0\}$. There exists $d \in J$ such that $ad \neq \{0\}$, so that $\{0\} \neq ad \subseteq P$. If $d \notin P$, then we are done. Assume that $d \in P$. Since R is strongly distributive, $a(x+d) = ax+ad = ad \neq \{0\}$. Hence there exists $l \in x+d$ such that $\{0\} \neq al \subseteq P$. We have $a \in P$ or $l \in P$. If $l \in P$, then $x \in l + (-d) \subseteq P$ since $l \in x + d$, a contradiction. Hence $a \in P$.

Case 2 $aJ = \{0\}$.

Case 2.1 $Ix \neq \{0\}$. There exists $r \in I$ such that $rx \neq \{0\}$. Then $\{0\} \neq rx \subseteq P$. We have $r \in P$. Note that $(r+a)x = rx + ax = rx \neq \{0\}$. Then there exists $p \in r + a$ such that $\{0\} \neq px \subseteq P$. We have $p \in P$. Since $p \in r + a$, $a \in (-r) + p \subseteq P$.

Case 2.2 $Ix = \{0\}$. Since $IJ \neq \{0\}$, there exist $b \in I$ and $d \in J$ such that $bd \neq \{0\}$, so that $\{0\} \neq bd \subseteq P$. If $d \notin P$, then we have $b \in P$. Note $(a+b)d = ad+bd = bd \neq \{0\}$. Hence there exists $p \in a+b$ such that $\{0\} \neq pd \subseteq P$. We have $p \in P$. Since $p \in a + b$, $a \in p + (-b) \subseteq P$.

Assume that $d \in P$. Then $b(x + d) = bx + bd = bd \neq \{0\}$. Hence there exists $l \in x + d$ such that $\{0\} \neq bl \subseteq P$. We have $b \in P$ or $l \in P$. If $l \in P$, then $x \in l + (-d) \subseteq P$ since $l \in x + d$, a contradiction. Therefore $l \notin P$ and $b \in P$. We have $(a + b)l = al + bl = bl \neq \{0\}$. Then there exists $p \in a + b$ such that $\{0\} \neq pl \subseteq P$. Hence $p \in P$. Since $p \in a + b$, $a \in p + (-b) \subseteq P$. $\square$

Finally the last two characterizations are considered.

Proposition 2.2.5. *Let R be a strongly distributive hyperring such that $a \in Ra$ for all $a \in R$ and P a proper hyperideal of R. Then P is a weakly prime hyperideal if and only if $\{0\} \neq aRb \subseteq P$ implies $a \in P$* or *$b \in P$ for all $a, b \in R$.*

Proof. Assume first that P is a weakly prime hyperideal. Let $a, b \in R$ be such that $\{0\} \neq aRb \subseteq P$. Here let

$$I = \langle a \rangle = [Ra] + [RaR] \text{ and } J = \langle b \rangle = [Rb] + [RbR].$$

Then I and J are hyperideals of R such that $a \in I$ and $b \in J$. Note $aRb \subseteq IJ$, so that $IJ \neq \{0\}$. Since $aRb \subseteq P$, we have $IJ \subseteq P$. Thus $\{0\} \neq IJ \subseteq P$. Then $I \subseteq P$ or $J \subseteq P$ because P is weakly prime. Hence $a \in P$ or $b \in P$.

Now assume that $\{0\} \neq aRb \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$. Let I and J be hyperideals of R such that $\{0\} \neq IJ \subseteq P$. Suppose that $J \nsubseteq P$. There exists $x \in J \setminus P$. We show that $I \subseteq P$. Let $a \in I$. Then $aRx \subseteq P$. If $aRx \neq \{0\}$, then we are done. Assume further that $aRx = \{0\}$. We obtain that $arx = \{0\}$ for all $r \in R$. Consider two cases as follow.

Case 1 $aRJ \neq \{0\}$. There exists $d \in J$ such that $aRd \neq \{0\}$, so that $\{0\} \neq aRd \subseteq P$. If $d \notin P$, then we are done. Assume that $d \in P$. Since $aRd \neq \{0\}$, there exists $r \in R$ such that $ard \neq \{0\}$. Then $ar(x+d) = arx + ard = ard \neq \{0\}$. There exists $l \in x + d$ such that $\{0\} \neq arl \subseteq P$. Therefore $\{0\} \neq aRl \subseteq P$. We have $a \in P$ or $l \in P$. If $l \in P$, then $x \in l + (-d) \subseteq P$ since $l \in x + d$, a contradiction. Hence $a \in P$.

Case 2 $aRJ = \{0\}$.

Case 2.1 $Ix \neq \{0\}$. There exists $r \in I$ such that $rx \neq \{0\}$. Then $\{0\} \neq rRx \subseteq IJ \subseteq P$. We have $r \in P$. Since $rRx \neq \{0\}$, there exists $s \in R$ such that $rsx \neq \{0\}$. Then $(r + a)sx = rsx + asx = rsx \neq \{0\}$. There exists $p \in r + a$ such that $\{0\} \neq psx$, so that $\{0\} \neq pRx \subseteq P$. We have $p \in P$. Since $p \in r + a$, $a \in (-r) + p \subseteq P$.

Case 2.2 $Ix = \{0\}$. Since $IJ \neq \{0\}$, there exist $b \in I$ and $d \in J$ such that $bd \neq \{0\}$, so that $\{0\} \neq bRd \subseteq P$. If $d \notin P$, then we have $b \in P$. Since $bRd \neq \{0\}$, there exists $s \in R$ such that $bsd \neq \{0\}$. Then $(a + b)sd = asd + bsd = bsd \neq \{0\}$. Hence there exists $p \in a + b$ such that $psd \neq \{0\}$, so that $\{0\} \neq pRd \subseteq P$. We have $p \in P$. Since $p \in a + b$, $a \in p + (-b) \subseteq P$.

Assume that $d \in P$. Since $bRd \neq \{0\}$, there exists $s \in R$ such that $bsd \neq \{0\}$. Then $bs(x + d) = bsx + bsd = bsd \neq \{0\}$. Hence there exists $l \in x + d$ such that $bsl \neq \{0\}$, so that $\{0\} \neq bRl \subseteq P$. We have $b \in P$ or $l \in P$. If $l \in P$, then $x \in l + (-d) \subseteq P$ since $l \in x + d$, a contradiction. Therefore $l \notin P$ and $b \in P$. We have $(a + b)sl = asl + bsl = bsl \neq \{0\}$. Then there exists $p \in a + b$ such that $psl \neq \{0\}$, so that $\{0\} \neq pRl \subseteq P$. Hence $p \in P$. Since $p \in a + b$, $a \in p + (-b) \subseteq P$. □

Proposition 2.2.6. *Let R be a hyperring such that $a \in aR$ for all $a \in R$ and P a proper hyperideal of R. Then P is a weakly prime hyperideal if and only if $\{0\} \neq aRb \subseteq P$ implies $a \in P$* or *$b \in P$ for $a, b \in R$.*

Proof. The proof is nearly the same as the proof of the previous proposition. □

2.2.2 Weakly Prime Subhypermodules

This subsection is devoted to studying properties of weakly prime subhypermodules. First, we give a definition of weakly prime subhypermodules.

Definition 2.2.7. Let R be a hyperring and M an R-hypermodule. A proper subhypermodule N of M is called *weakly prime* if for all hyperideals I of R and all subhypermodules D of M,

$$\{0\} \neq [ID] \subseteq N \Rightarrow I \subseteq (N : M) \text{ or } D \subseteq N.$$

Recall from Chapter I that $\{0\}$ may not be a subhypermodule. By Proposition 1.3.2, if $\{0\}$ is not a subhypermodule, then there are no hyperideals I and subhypermodules N such that $IN = \{0\}$. Thus the following result is obtained.

Proposition 2.2.8. *Let M be an R-hypermodule such that $\{0\}$ is not a subhypermodule of M. Then prime subhypermodules and weakly prime subhypermodules of M are the same.*

In this section, we consider only hypermodules satisfying the property that $\{0\}$ is a subhypermodule. First, we characterize weakly prime subhypermodules in the same ways as prime subhypermodules.

Before starting on these characterizations, we consider a generalization of weakly prime subhypermodules, called L-prime subhypermodules. L-prime subhypermodules are defined in similar way to weakly prime subhypermodules but we change the subhypermodule $\{0\}$ to a subhypermodule L.

Definition 2.2.9. Let R be a hyperring, M an R-hypermodule and L a subhypermodule of M. A proper subhypermodule N of M is called **L-prime** if for all hyperideals I of R and all subhypermodules D of M,

$$L \neq [ID] \subseteq N \Rightarrow I \subseteq (N : M) \text{ or } D \subseteq N.$$

The following proposition gives a relationship between weakly prime subhypermodules and L-prime subhypermodules.

Proposition 2.2.10. *Let M be an R-hypermodule and L a subhypermodule of M. Then a subhypermodule N of M is L-prime if and only if $N/(N \cap L)$ is a weakly prime subhypermodule of $M/(N \cap L)$.*

Proof. First, assume that N is L-prime and let I and $D/(N \cap L)$ be a hyperideal of R and a subhypermodule of $M/(N \cap L)$, respectively, such that $\{N \cap L\} \neq [I(D/(N \cap L))] \subseteq N/(N \cap L)$. Then $L \neq [ID] \subseteq N$. Since N is L-prime, we obtain that $I \subseteq (N : M)$ or $D \subseteq N$. Hence $I \subseteq (N/(N \cap L) : M/(N \cap L))$ or $D/(N \cap L) \subseteq N/(N \cap L)$. Thus $N/(N \cap L)$ is a weakly prime subhypermodule of $M/(N \cap L)$.

Conversely, assume that $N/(N \cap L)$ is a weakly prime subhypermodule of $M/(N \cap L)$, and let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $L \neq [ID] \subseteq N$. Then $\{N \cap L\} \neq [I(D/(N \cap L))] \subseteq N/(N \cap L)$, so that $I \subseteq (N/(N \cap L) : M/(N \cap L))$ or $D/(N \cap L) \subseteq N/(N \cap L)$. Hence $IM \subseteq N$ or $D \subseteq N$. Thus $I \subseteq (N : M)$ or $D \subseteq N$, which shows that N is L-prime. □

This relation confirms us that it is sufficient to study only weakly prime subhypermodules. Next we present some characterizations of weakly prime subhypermodules. First we characterize under the same conditions as the above.

Proposition 2.2.11. *Let M be a hypermodule and N a proper subhypermodule of M. Then N is a weakly prime subhypermodule if and only if $\{0\} \neq ID \subseteq N$ implies $I \subseteq (N : M)$ or $D \subseteq N$ for all hyperideals I of R and all subhypermodules D of M.*

Proof. Similar to the proof of Proposition 2.2.3. □

Proposition 2.2.12. *Let R be a commutative hyperring, M a strongly distributive R-hypermodule and N a proper subhypermodule of M. Then N is a weakly prime subhypermodule if and only if $\{0\} \neq am \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$.*

Proof. This proof is much like the proof of Proposition 2.2.4. □

Proposition 2.2.13. *Let M be a a strongly distributive R-hypermodule, N a proper subhypermodule of M and assume that $a \in aR$ for every $a \in R$. Then N is a weakly prime subhypermodule if and only if $\{0\} \neq aRm \subseteq N$ implies $a \in (N : M)$* or *$m \in N$ for all $a \in R$ and $m \in M$.*

Proof. This proof is much like the proof of Proposition 2.2.6. □

Proposition 2.2.14. *Let M be a strongly distributive R-hypermodule, N a proper subhypermodule of M and assume that $x \in Rx$ for every $x \in M$. Then N is a weakly prime subhypermodule if and only if $\{0\} \neq aRm \subseteq N$ implies $a \in (N : M)$* or *$m \in N$ for all $a \in R$ and $m \in M$.*

Proof. This proof is also similar to the proof of Proposition 2.2.5. □

Proposition 2.2.15. *Let M be an R-hypermodule and N a proper subhypermodule of M. The following are equivalent.*

(i) *N is a weakly prime subhypermodule.*

(ii) *For any subhypermodule $D \nsubseteq N$, $(N : D) = (N : M) \cup (\{0\} : D)$.*

(iii) *For any subhypermodule $D \nsubseteq N$, $(N : D) = (N : M)$ or $(N : D) = (\{0\} : D)$.*

Proof. (i) ⇒ (ii) Assume that (i) holds. Let D be a subhypermodule of M such that $D \nsubseteq N$. It is obvious that $(N : M) \cup (\{0\} : D) \subseteq (N : D)$. Let $a \in (N : D)$. Then $aD \subseteq N$. If $aD = \{0\}$, then $a \in (\{0\} : D)$. On the other hand, let $aD \neq \{0\}$. Then $\{0\} \neq \langle a \rangle D \subseteq N$ so that $\langle a \rangle \subseteq (N : M)$ or $D \subseteq N$ by Proposition 2.2.11. Consequently, $a \in \langle a \rangle \subseteq (N : M)$ since $D \nsubseteq N$.

(ii) ⇒ (iii) It is obtained from Proposition 1.1.16.

(iii) ⇒ (i) Assume that (iii) is valid. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $\{0\} \neq ID \subseteq N$. Suppose that $D \nsubseteq N$. It follows from (iii) that $(N : D) = (N : M)$ or $(N : D) = (\{0\} : D)$. Note that $I \subseteq (N : D)$ because $ID \subseteq N$. Thus $I \subseteq (N : M)$ or $I \subseteq (\{0\} : D)$. If $I \subseteq (\{0\} : D)$, then $ID \subseteq \{0\}$ so that $ID = \{0\}$ leading to a contradiction. Thus $I \subseteq (N : M)$. □

Corollary 2.2.16. *Let M be an R-hypermodule and N a proper subhypermodule of M. If N is a weakly prime subhypermodule, then $(N : \langle m\rangle) = (N : M) \cup (\{0\} : \langle m\rangle)$ for every element m of M with $m \notin N$.*

Proof. Let m be an element of M with $m \notin N$ and let $D = \langle m\rangle$. Then D is a subhypermodule of M with $D \nsubseteq N$, so the conclusion follows from Proposition 2.2.15. □

Although the following characterization of weakly prime subhypermodules are quite similar to those in Propoition 2.2.15, the strongly distributivity of R-hypermodules is needed.

Corollary 2.2.17. *Let M be a strongly distributive R-hypermodule, N a proper subhypermodule of M and assume that $x \in Rx$ for every $x \in M$. The following are equivalent.*

(i) *N is a weakly prime subhypermodule.*

(ii) *For all elements $m \in M$ with $m \notin N$, $(N : Rm) = (N : M) \cup (\{0\} : Rm)$.*

(iii) *For all elements $m \in M$ with $m \notin N$, $(N : Rm) = (N : M)$ or $(N : Rm) = (\{0\} : Rm)$.*

Proof. The proofs of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from Proposition 2.2.15 and the facts for $m \in M \setminus N$ that $[Rm]$ is a subhypermodule of M with $[Rm] \nsubseteq N$, $(N : Rm) = \big(N : [Rm]\big)$ and $(\{0\} : Rm) = \big(\{0\} : [Rm]\big)$.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Let $a \in R$ and $m \in M$ be such that $\{0\} \neq aRm \subseteq N$. Suppose that $m \notin N$. Note that $a \in (N : Rm)$. By (iii), $a \in (N : M)$ or $a \in (\{0\} : Rm)$. It is not possible that $a \in (\{0\} : Rm)$ since $aRm \neq \{0\}$. Hence $a \in (N : M)$. Therefore, N is a weakly prime subhypermodule by Proposition 2.2.14. □

Proposition 2.2.18. *Let M be a strongly distributive R-hypermodule and N a proper subhypermodule of M and assume that $a \in aR$ for every $a \in R$. The following are equivalent.*

(i) *N is a weakly prime subhypermodule.*

(ii) *For all elements $m \in M$ with $m \notin N$, $(N : Rm) = (N : M) \cup (\{0\} : Rm)$.*

(iii) *For all elements $m \in M$ with $m \notin N$, $(N : Rm) = (N : M)$ or $(N : Rm) = (\{0\} : Rm)$.*

Proof. (i) $\Rightarrow$ (ii) Assume that N is a weakly prime subhypermodule. Let $m \in M \smallsetminus N$. It is obvious that $(N : M) \cup (\{0\} : Rm) \subseteq (N : Rm)$. Let $a \in (N : Rm)$. Thus $aRm \subseteq N$. If $aRm = \{0\}$, then $a \in (\{0\} : Rm)$. If $aRm \neq \{0\}$, then $a \in (N : M)$ by assumption together with Proposition 2.2.13.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) This is similar to the proof (iii) $\Rightarrow$ (i) of the above proposition by applying Proposition 2.2.13 instead. □

This subsection ends with an investigation of some properties of weakly prime subhypermodules.

Proposition 2.2.19. *Let M be an R-hypermodule and N and K subhypermodules of M with $K \subseteq N$.*

(i) *If N is a weakly prime subhypermodule of M, then N/K is a weakly prime subhypermodule of M/K.*

(ii) *If K and N/K are weakly prime subhypermodules of the appropriate hypermodules, then N is a weakly prime subhypermodule of M.*

Proof. (i) Assume that N is a weakly prime subhypermodule of M. Let I and D/K be a hyperideal of R and a subhypermodule of M/K, respectively, such that $\{K\} \neq I(D/K) \subseteq N/K$. If $ID = \{0\}$, then $I(D/K) = K$, a contradiction. Thus $\{0\} \neq ID \subseteq N$. Since N is weakly prime, $I \subseteq (N : M)$ or $D \subseteq N$. Hence $I \subseteq (N/K : M/K)$ or $D/K \subseteq N/K$. Therefore, N/K is a weakly prime subhypermodule of M/K.

(ii) Assume that K and N/K are weakly prime subhypermodules of the appropriate hypermodules. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $\{0\} \neq ID \subseteq N$.

Case 1 $ID \subseteq K$. Then $\{0\} \neq ID \subseteq K$. Since K is weakly prime, $I \subseteq (K : M)$ or $D \subseteq K$. Thus $I \subseteq (N : M)$ or $D \subseteq N$ since $K \subseteq N$.

Case 2 $ID \nsubseteq K$. Then $\{K\} \neq I(D/K) \subseteq N/K$. Since N/K is weakly prime, $I \subseteq (N/K : M/K)$ or $D/K \subseteq N/K$. Thus $IM \subseteq N$ or $D \subseteq N$, so $I \subseteq (N : M)$ or $D \subseteq N$. Therefore N is a weakly prime subhypermodule of M. □

By Proposition 2.1.13, we know that if N is a prime subhypermodule, then $(N : M)$ is a prime hyperideal. But if we change "prime" to "weakly prime", this property may not hold. Thus we study conditions that imply this property.

Proposition 2.2.20. *Let M be an R-hypermodule such that $\{0\}$ is a prime subhypermodule and N a subhypermodule of M such that $\varnothing \neq (N : M) \neq R$. If N is a weakly prime subhypermodule of M, then $(N : M)$ is a weakly prime hyperideal. (This R-hypermodule is also called a prime R-hypermodule.)*

Proof. Assaume that N is a weakly prime subhypermodule of M. Let A and B be hyperideals of R such that $\{0\} \neq AB \subseteq (N : M)$. Suppose that $B \nsubseteq (N : M)$. There exist $b \in B$ and $m \in M$ such that $bm \nsubseteq N$. Note that

$$A\langle bm\rangle \subseteq A(Rbm + \mathbb{Z}bm) \subseteq N.$$

If $A\langle bm\rangle = \{0\}$, then $A \subseteq \big(\{0\} : M\big)$ or $\langle bm\rangle \subseteq \{0\}$. Thus $A \subseteq (\{0\} : M) \subseteq (N : M)$. If $A\langle bm\rangle \neq \{0\}$, then $A \subseteq (N : M)$ since N is weakly prime. □

Corollary 2.2.21. *Let M be a prime hypermodule such that $M = RM$ and N a subhypermodule of M such that $(N : M) \neq \varnothing$. If N is a weakly prime subhypermodule of M, then $(N : M)$ is a weakly prime hyperideal.*

Proof. Assume that N is a weakly prime subhypermodule of M. Then $N \neq M$. Suppose that $(N : M) = R$. Then $M = RM \subseteq N$, a contradiction. Hence $(N : M) \neq R$. The result now follows from Proposition 2.2.20. □

Finally, we determine the relations between prime and weakly prime subhypermodules. It is clear that prime subhypermodules are weakly prime subhypermodules. Therefore we give a condition which implies that weakly prime subhypermodules are prime subhypermodules.

Proposition 2.2.22. *Let N be a weakly prime subhypermodule of an R-hypermodule M. If $(N : M)N \neq \{0\}$, then N is a prime subhypermodule of M.*

Proof. Assume that $(N : M)N \neq \{0\}$ and let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. If $ID \neq \{0\}$, then we are done. Assume that $ID = \{0\}$.

Case 1 $IN \neq \{0\}$. Then $IN \subseteq I(D + N) \subseteq ID + IN = \{0\} + IN = IN$. Therefore $\{0\} \neq I(D + N) \subseteq N$. Since N is weakly prime, $I \subseteq (N : M)$ or $D + N \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$.

Case 2 $IN = \{0\}$.

Case 2.1 $(N : M)D \neq \{0\}$. Then $(N : M)D \subseteq \big(I + (N : M)\big)D \subseteq ID + (N : M)D = (N : M)D$. Hence $\{0\} \neq \big(I + (N : M)\big)D \subseteq N$. Since N is weakly prime, $I + (N : M) \subseteq (N : M)$ or $D \subseteq N$. Thus $I \subseteq (N : M)$ or $D \subseteq N$.

Case 2.2 $(N : M)D = \{0\}$. Then $(N : M)N \subseteq \big(I + (N : M)\big)(D + N) \subseteq ID + IN + (N : M)D + (N : M)N = (N : M)N \subseteq N$. Hence $\{0\} \neq \big(I + (N : M)\big)(D + N) \subseteq N$. Since N is weakly prime, $I + (N : M) \subseteq (N : M)$ or $D + N \subseteq N$. Therefore $I \subseteq (N : M)$ or $D \subseteq N$. □

Corollary 2.2.23. *Let N be a weakly prime subhypermodule of an R-hypermodule M which is not prime. If I is a hyperideal of R such that $I \subseteq (N : M)$, then $IN = \{0\}$. In particular, $(N : M)N = \{0\}$.*

Proof. By the previous proposition, we have $(N : M)N = \{0\}$. Assume that I is a hyperideal of R such that $I \subseteq (N : M)$. Then $IN \subseteq (N : M)N = \{0\}$. Hence $IN = \{0\}$. □

CHAPTER III

PRIME AND WEAKLY PRIME SUBHYPERMODULES OF MULTIPLICATION HYPERMODULES

In this chapter, we introduce multiplication hypermodules and give some properties of prime and weakly prime subhypermodules of multiplication hypermodules.

3.1 Multiplication Hypermodules

First, we give a definition of multiplication hypermodules.

Definition 3.1.1. Let M be an R-hypermodule. Then M is called a *multiplication R-hypermodule* if for every subhypermodule N of M, $N = [IM]$ for some hyperideal I of R.

Recall that, in general, if N is a subhypermodule of an R-hypermodule M, then $(N : M)$ may be empty. We show that if M is a multiplication R-hypermodule, then $(N : M)$ is always nonempty.

Proposition 3.1.2. *Let M be a multiplication R-hypermodule. Then $(N : M)$ is nonempty for every subhypermodule N of M.*

Proof. Let N be a subhypermodule of M. Then there exists a hyperideal I such that $N = [IM]$. Hence $IM \subseteq N$, i.e., $I \subseteq (N : M)$. Since I is nonempty, $(N : M)$ is nonempty. □

Next, we determine an explicit form for a subhypermodule of a multiplication hypermodule.

Proposition 3.1.3. *Let M be a multiplication R-hypermodule. If N is a subhypermodule of M, then $N = [(N : M)M]$.*

Proof. Let N be a subhypermodule of M. Then $N = [IM]$ for some hyperideal I of R. Thus $IM \subseteq N$, so $I \subseteq (N : M)$. Hence $N = [IM] \subseteq [(N : M)M]$. Conversely, $[(N : M)M] \subseteq N$ since $(N : M)M \subseteq N$. Therefore $N = [(N : M)M]$. □

Corollary 3.1.4. *Let M be a multiplication R-hypermodule and N a subhypermodule of M. Then $M = [RM]$. Moreover, $N = M$ if and only if $(N : M) = R$.*

Proof. By the above proposition, $M = [(M : M)M] = [RM]$. Assume $N = M$. Then $(N : M) = (M : M) = R$. Conversely, assume that $(N : M) = R$. Then $N = [(N : M)M] = [RM] = M$. □

Recall from Chapter I that $N = \sum_{n\in N}\langle n\rangle$ for any subhypermodules N of an R-hypermodule M. If M is also a multiplication R-hypermodule, then the following proposition is obtained.

Proposition 3.1.5. *Let N be a subhypermodule of a multiplication R-hypermodule M. Then $N = \sum_{n\in N}[I_nM] = \left[\left(\sum_{n\in N} I_n\right)M\right]$ where for each $n \in N$, I_n is any hyperideal of R such that $\langle n\rangle = I_nM$.*

Proof. This is straightforward. □

The following proposition gives a characterization of multiplication hypermodules.

Proposition 3.1.6. *An R-hypermodule M is a multiplication R-hypermodule if and only if for each $m \in M$, there exists an hyperideal I of R such that $\langle m\rangle = [IM]$.*

Proof. First, assume that M is a multiplication R-hypermodule. Let $m \in M$. Since $\langle m\rangle$ is a subhypermodule of M, there exists a hyperideal I of R such that $\langle m\rangle = [IM]$.

Conversely, assume that for each $m \in M$, there exists an hyperideal I of R such that $\langle m\rangle = [IM]$. Let N be a subhypermodule of M. Then for each $x \in N$ there exists a hyperideal I_x of R such that $\langle x\rangle = [I_xM]$. We claim that $N = \left[\left(\sum_{x\in N} I_x\right)M\right]$. For each $x \in N$, it follows that $x \in \langle x\rangle = [I_xM] \subseteq$

$\big(\sum_{x\in N} I_x\big)M$, so $N \subseteq \Big[\big(\sum_{x\in N} I_x\big)M\Big]$. Now, let $m \in \Big[\big(\sum_{x\in N} I_x\big)M\Big]$. Then $m \in \Big[\big(\sum_{x\in N} I_x\big)M\Big] \subseteq \sum_{x\in N}[I_xM] = \sum_{x\in N}\langle x\rangle$. It follows that $m \in N$ since $\langle x\rangle \subseteq N$ for each $x \in N$. □

Corollary 3.1.7. *Let M be an R-hypermodule such that $x \in Rx$ for all $x \in M$. Then M is a multiplication R-hypermodule if and only if for each $m \in M$, there exists an hyperideal I of R such that $[Rm] = [IM]$.*

Proof. This follows from Proposition 3.1.6. □

Corollary 3.1.8. *Let M be an R-hypermodule such that $N = RN$ for every subhypermodule N of M. Then M is a multiplication R-hypermodule if and only if for each $m \in M$, there exists a hyperideal I of R such that $[Rm] = [IM]$.*

Proof. This follows from Proposition 3.1.6. □

Next, we study some properties of multiplication hypermodules. First, we give a lemma which is related to homomorphisms of hypermodules.

Lemma 3.1.9. *Let M and M' be R-hypermodules. If $f : M \to M'$ is a homomorphism, then $f\big(\langle x\rangle\big) = \big\langle f(x)\big\rangle$ for every $x \in M$.*

Proof. Assume that $f : M \to M'$ is a homomorphism. By Proposition 2.1.21, $f(ax) = af(x)$ for all $a \in \mathbb{Z}$ and $x \in M$. Let $x \in M$. First, let $t \in f\big(\langle x\rangle\big)$. Then there exists $l \in \langle x\rangle = [Rx]+[\mathbb{Z}x]$ such that $t = f(l)$. Thus $l \in \sum_{i=1}^{n} r_ix+\sum_{i=1}^{k} a_ix$ where $n, k \in \mathbb{N}$, $r_i \in R$ and $a_i \in \mathbb{Z}$ for all i. Hence

$$t = f(l) \in f\left(\sum_{i=1}^{n} r_ix + \sum_{i=1}^{k} a_ix\right) = \sum_{i=1}^{n} r_if(x) + \sum_{i=1}^{k} a_if(x)$$
$$\subseteq \big[Rf(x)\big] + \big[\mathbb{Z}f(x)\big] = \big\langle f(x)\big\rangle.$$

This shows that $f\big(\langle x\rangle\big) \subseteq \big\langle f(x)\big\rangle$.

Next, let $t \in \big\langle f(x)\big\rangle$. Then $t \in \big[Rf(x)\big] + \big[\mathbb{Z}f(x)\big]$, i.e., $t \in \sum_{i=1}^{n} r_if(x) + \sum_{i=1}^{k} a_if(x) = \sum_{i=1}^{n} f(r_ix) + \sum_{i=1}^{k} f(a_ix)$ where $n, k \in \mathbb{N}$, $r_i \in R$ and $a_i \in \mathbb{Z}$ for all i. Thus

$$t \in f\left(\sum_{i=1}^{n} r_i x\right) + f\left(\sum_{i=1}^{k} a_i x\right) = f\left(\sum_{i=1}^{n} r_i x + \sum_{i=1}^{k} a_i x\right)$$
$$\subseteq f\big([Rx] + [\mathbb{Z}x]\big) = f\big(\langle x\rangle\big).$$

This shows that $\big\langle f(x)\big\rangle \subseteq f\big(\langle x\rangle\big)$. □

Proposition 3.1.10. *Every homomorphic image of a multiplication R-hypermodule is a multiplication R-hypermodule.*

Proof. Let M and M' be R-hypermodules, M a multiplication R-hypermodule and $f : M \to M'$ a surjective homomorphism. Let $x' \in M'$. Then there exists $x \in M$ such that $f(x) = x'$. Since M is a multiplication R-hypermodule, $\langle x\rangle = [IM]$ for some hyperideal I of R. We claim that $\langle x'\rangle = [IM']$. Note that by Lemma 3.1.9, we obtain that

$$[IM'] = \big[If(M)\big] = f\big([IM]\big) = f\big(\langle x\rangle\big) = \big\langle f(x)\big\rangle = \langle x'\rangle.$$

Hence M' is a multiplication hypermodule. □

Corollary 3.1.11. *Let M be a multiplication R-hypermodule and N a subhypermodule of M. Then M/N is a multiplication R-hypermodule.*

Proof. Define $f : M \to M/N$ by $f(m) = m + N$ for all $m \in M$. It is clear that f is surjective and $f(0) = 0 + N$. It is easy to check that f is a surjective homomorphism. It, then, follows from Proposition 3.1.10 that M/N is a multiplication R-hypermodule. □

3.2 Prime and Weakly Prime Subhypermodules

The main results of this chapter are given in this section. Our aim is to characterize prime and weakly prime subhypermodules of a multiplication hypermodule.

In general, we know that if N is a prime subhypermodule of an R-hypermodule M such that $(N : M)$ is not empty, then $(N : M)$ is a prime hyperideal. In the next proposition, we consider this statement under the assumption that M is a multiplication R-hypermodule.

Proposition 3.2.1. *Let M be a multiplication R-hypermodule and N a subhypermodule of M. Then N is a prime subhypermodule of M if and only if $(N : M)$ is a prime hyperideal of R.*

Proof. The necessary part follows from Propositions 2.1.13 and 3.1.2 and Corollary 3.1.4.

Next, assume that $(N : M)$ is a prime hyperideal of R. Then $(N : M) \neq R$, so that $N \neq M$ by Corollary 3.1.4. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq N$. Since M is a multiplication R-hypermodule, $D = [JM]$ for some hyperideal J of R. Thus

$$(IJ)M = I(JM) \subseteq I[JM] = ID \subseteq N.$$

This shows that $IJ \subseteq (N : M)$. Since $(N : M)$ is a prime hyperideal of R, either $I \subseteq (N : M)$ or $J \subseteq (N : M)$. Then $I \subseteq (N : M)$ or $JM \subseteq N$, so that $I \subseteq (N : M)$ or $D \subseteq N$. Hence N is a prime subhypermodule of M. □

Next, we define the product of subhypermodules of a multiplication hypermodule.

Definition 3.2.2. Let R be a commutative hyperring and M be a multiplication R-hypermodule. For subhypermodules N and K of M, we define *the product of N and K* as follows :

$$NK = \big[[IJ]M\big]$$

where $N = [IM]$ and $K = [JM]$ for some hyperideals I and J of R.

Note that products of subhypermodules of a multiplication R-hypermodule require the hyperring R to be commutative. As a result, for the rest of this chapter, we let R be a commutative hyperring.

Proposition 3.2.3. *The product of subhypermodules is well-defined.*

Proof. Let N and K be subhypermodules of a multiplication R-hypermodule. Suppose that $N = [I_1M] = [I_2M]$ and $K = [J_1M] = [J_2M]$ for some hyperideals I_1, I_2, J_1 and J_2 of R. Then $\big[[I_1J_1]M\big] = \big[I_1[J_1M]\big] = \big[I_1[J_2M]\big] = \big[[I_1J_2]M\big] = \big[[J_2I_1]M\big] = \big[J_2[I_1M]\big] = \big[J_2[I_2M]\big] = \big[[J_2I_2]M\big] = \big[[I_2J_2]M\big]$. □

The next proposition gives a characterization of prime subhypermodules which is analogous to the definition of prime hyperideals.

Proposition 3.2.4. *Let M be a multiplication R-hypermodule and N a proper subhypermodule of M. Then N is prime if and only if $PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules P and K of M.*

Proof. First, assume that N is a prime subhypermodule of M. Let P and K be subhypermodules of M such that $PK \subseteq N$. Suppose $P = [IM]$ and $K = [JM]$ for some hyperideals I and J of R. Then $I[JM] \subseteq \big[(IJ)M\big] \subseteq \big[[IJ]M\big] = PK \subseteq N$. Since N is prime, $I \subseteq (N : M)$ or $[JM] \subseteq N$. Hence $IM \subseteq N$ or $K \subseteq N$. Thus $P = [IM] \subseteq N$ or $K \subseteq N$.

Conversely, assume that $PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules P and K of M. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $[ID] \subseteq N$. Suppose $D = [JM]$ for some hyperideal J of R. Set $P = [IM]$. Then P and D are subhypermodules of M such that $PD = \big[[IJ]M\big] = \big[I[JM]\big] = [ID] \subseteq N$. By assumption, $P \subseteq N$ or $D \subseteq N$. Thus $IM \subseteq [IM] = P \subseteq N$ or $D \subseteq N$. Therefore $I \subseteq (N : M)$ or $D \subseteq N$, which shows that N is prime. $\square$

In fact, we can define the product of two elements of a multiplication hypermodule.

Definition 3.2.5. Let M be a multiplication R-hypermodule M. For $m, m' \in M$ and a subhypermodule N of M, we define $mm' = \langle m\rangle\langle m'\rangle$, $mN = \langle m\rangle N$ and $Nm = N\langle m\rangle$.

Lemma 3.2.6. *Let M be a multiplication R-hypermodule, N a canonical subhypergroup of $(M, +)$ and P and K subhypermodules of M. Then the following hold.*

(i) *$PK \subseteq N$ if and only if $pK \subseteq N$ for all $p \in P$.*

(ii) *$PK \subseteq N$ if and only if $Pk \subseteq N$ for all $k \in K$.*

(iii) $PK \subseteq N$ *if and only if* $pk \subseteq N$ *for all* $p \in P$ *and* $k \in K$.

Proof. (i) First, assume that $PK \subseteq N$. Then $\big[[(P : M)(K : M)]M\big] \subseteq N$. Let $p \in P$. Then $\langle p \rangle = [I_p M]$ for some hyperideal I_p of R. Note that $I_p M \subseteq P$ so that $I_p \subseteq (P : M)$. Hence $pK = \langle p \rangle K = \big[[I_p(K : M)]M\big] \subseteq \big[[(P : M)(K : M)]M\big] \subseteq N$.

Conversely, assume that $pK \subseteq N$ for all $p \in P$. Since $P = \sum_{p \in P} \langle p \rangle$, we obtain that $PK = \left(\sum_{p\in P} \langle p \rangle\right) K$. Note that for each $p \in P$, $\langle p \rangle = [I_p M]$ for some hyperideal I_p of R. By assumption, $\big[[I_p(K : M)]M\big] = pK \subseteq N$ for all $p \in P$. Hence

$$PK = \left(\sum_{p\in P}[I_pM]\right) K = \left(\Big[(\sum_{p\in P} I_p)M)\Big]\right)\Big([(K : M)M]\Big)$$

$$= \left[\Big[(\sum_{p\in P} I_p)(K : M)\Big]M\right] = \sum_{p\in P}\Big[\big[I_p(K : M)\big]M\Big] \subseteq N.$$

(ii) The proof is similar to (i).

(iii) First, assume that $PK \subseteq N$. By (i), we have that $pK \subseteq N$ for all $p \in P$, i.e., $\langle p \rangle K \subseteq N$ for all $p \in P$. Thus by (ii) $\langle p \rangle k \subseteq N$ for all $p \in P$ and $k \in K$. By definition, $\langle p \rangle k = \langle p \rangle \langle k \rangle = pk$, so $pk \subseteq N$ for all $p \in P$ and $k \in K$.

Conversely, assume that $pk \subseteq N$ for all $p \in P$ and $k \in K$. Then $p\langle k \rangle \subseteq N$ for all $p \in P$ and $k \in K$, so that $P\langle k \rangle \subseteq N$ for all $k \in K$ by (i). Thus $Pk \subseteq N$ for all $k \in K$. By (ii), $PK \subseteq N$. □

The following result is analogous to Proposition 3.2.4. Its proof makes use of Lemma 3.2.6.

Proposition 3.2.7. *Let N be a proper subhypermodule of a multiplication R-hypermodule M. Then N is prime if and only if $mm' \subseteq N$ implies $m \in N$* or *$m' \in N$ for all $m, m' \in M$.*

Proof. The necessary part is obtained from Proposition 3.2.4 and the definition of the product mm'.

Conversely, assume that $mm' \subseteq N$ implies $m \in N$ or $m' \in N$ for all $m, m' \in M$. Let P and K be subhypermodules of M such that $PK \subseteq N$ and $K \nsubseteq N$. Then there exists $k \in K \setminus N$. To show that $P \subseteq N$, let $p \in P$. Since $PK \subseteq N$, we have $pk \subseteq N$ by Lemma 3.2.6. By assumption, $p \in N$ or $k \in N$. Thus $p \in N$. This shows $P \subseteq N$. We conclude that N is a prime subhypermodule of M. □

By considering the multiplication of elements in a multiplication hypermodule, zero divisors can be defined.

Definition 3.2.8. Let M be a multiplication R-hypermodule. An element $m \in M \setminus \{0\}$ is called a *zero divisor* if there exists $m' \in M \setminus \{0\}$ such that $mm' = \{0\}$.

The following result gives a characterization of prime subhypermodules in terms of zero divisors.

Proposition 3.2.9. *Let N be a proper subhypermodule of a multiplication R-hypermodule M. Then N is prime of M if and only if M/N has no zero divisors.*

Proof. First, assume that N is a prime subhypermodule of M. Suppose that there exist $\overline{m}, \overline{m}' \in M/N$ such that $\overline{m}\,\overline{m}' = \{\overline{0}\}$. Let $\langle \overline{m} \rangle = \big[I(M/N)\big]$ and $\langle \overline{m}' \rangle = \big[J(M/N)\big]$ for some hyperideals I and J of R. Then $\big[[IJ](M/N)\big] = \langle \overline{m} \rangle \langle \overline{m}' \rangle = \overline{m}\,\overline{m}' = \{\overline{0}\}$. Thus $I[JM] \subseteq \big[I[JM]\big] = \big[[IJ]M\big] \subseteq N$. Since N is prime, $I \subseteq (N : M)$ or $[JM] \subseteq N$. Then $[IM] \subseteq N$ or $[JM] \subseteq N$. Thus $\langle \overline{m} \rangle = \{\overline{0}\}$ or $\langle \overline{m}' \rangle = \{\overline{0}\}$. Hence $\overline{m} = \overline{0}$ or $\overline{m}' = \overline{0}$.

Conversely, assume that M/N has no zero divisors. Let $m, m' \in M$ be such that $mm' \subseteq N$. Then $\overline{m}, \overline{m}' \in M/N$ with $\overline{m}\,\overline{m}' = \{\overline{0}\}$. Therefore, $\overline{m} = \overline{0}$ or $\overline{m}' = \overline{0}$. Hence $m \in N$ or $m' \in N$. Thus N is prime by Proposition 3.2.7. □

The above proposition gives another characterization of prime subhypermodules. Finally, we characterize weakly prime subhypermodules of multiplication R-hypermodules.

Lemma 3.2.10. *Let P and K be subhypermodules of a multiplication R-hypermodule M. Then the following hold.*

(i) $PK = \{0\}$ *if and only if* $pK = \{0\}$ *for all* $p \in P$.

(ii) $PK = \{0\}$ *if and only if* $Pk = \{0\}$ *for all* $k \in K$.

(iii) $PK = \{0\}$ *if and only if* $pk = \{0\}$ *for all* $p \in P$ *and* $k \in K$.

Proof. This is an immediate consequence of Lemma 3.2.6 by using $N = \{0\}$. $\square$

We finish with a characterization of weakly prime subhypermodules of a multiplication R-hypermodule in the following proposition.

Proposition 3.2.11. *Let N be a proper subhypermodule of a multiplication R-hypermodule M. Then N is weakly prime if and only if $\{0\} \neq PK \subseteq N$ implies $P \subseteq N$* or *$K \subseteq N$ for all subhypermodules P and K of M.*

Proof. First, assume that N is a weakly prime subhypermodule of M. If N is prime, then we are done. Suppose that N is not prime. Let P and K be subhypermodules of M such that $\{0\} \neq PK \subseteq N$ and assume for a contradiction that $P \nsubseteq N$ and $K \nsubseteq N$. We show that $PK = \{0\}$ by applying Lemma 3.2.10.

We claim that $pK = \{0\}$ for all $p \in P \setminus N$. It follows from $P = \sum_{l \in P} \langle l \rangle$ that $P = \Big[\big(\sum_{l \in P} I_l\big) M\Big]$ where for each $l \in P$, I_l is a hyperideal of R such that $\langle l \rangle = [I_l M]$. Let $p \in P \setminus N$ and let $\langle p \rangle = [I_p M]$ and $K = [IM]$ for some hyperideals I_p and I of R. Then $\Big[\big[\big(\sum_{l \in P} I_l\big) I\big] M\Big] = PK \subseteq N$. Thus $pK = \langle p \rangle K = \big[[I_p I] M\big] \subseteq \Big[\big[\big(\sum_{l \in P} I_l\big) I\big] M\Big] \subseteq N$. Hence $I\langle p \rangle = I[I_p M] \subseteq \big[[I I_p] M\big] = \big[[I_p I] M\big] \subseteq N$, i.e., $I \subseteq \big(N : \langle p \rangle\big)$. By Corollary 2.2.16 and Proposition 1.1.16, $I \subseteq (N : M)$ or $I \subseteq \big(\{0\} : \langle p \rangle\big)$. Thus $IM \subseteq N$ or $I \subseteq \big(\{0\} : \langle p \rangle\big)$ so that $K = [IM] \subseteq N$ or $I \subseteq \big(\{0\} : \langle p \rangle\big)$. Since $K \nsubseteq N$, we must have $I \subseteq \big(\{0\} : \langle p \rangle\big)$, i.e., $I\langle p \rangle = \{0\}$. Therefore $pK = \langle p \rangle K = \big[[I_p I] M\big] = \big[[I I_p] M\big] = \big[I[I_p M]\big] = [I\langle p \rangle] = \{0\}$.

Similarly, we have $Pk = \{0\}$ for all $k \in K \setminus N$. It remains to show that $pk = \{0\}$ for all $p \in P \cap N$ and $k \in K \cap N$. Let $p \in P \cap N$ and $k \in K \cap N$. By Corollary 2.2.23, $pk = \langle p \rangle \langle k \rangle \subseteq NN = [(N : M)N] = \{0\}$.

Thus $PK = \{0\}$, which is a contradiction. Hence $P \subseteq N$ or $K \subseteq N$.

Conversely, assume that $\{0\} \neq PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules P and K of M. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $\{0\} \neq ID \subseteq N$. Set $K = [IM]$ so that K is a subhypermodule of M. Then $KD = \big[[I(D : M)] M\big] = \Big[I\big[(D : M)M\big]\Big] = [ID] \subseteq N$. Thus $\{0\} \neq KD \subseteq N$. By assumption, $K \subseteq N$ or $D \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$. This shows that N is weakly prime. $\square$

CHAPTER IV
FUZZY HYPERIDEALS AND FUZZY SUBHYPERMODULES

Fuzzy sets is an interesting area for doing research, see [1], [3], [4], [5], [11], [14] and [19]. In [14], J.N. Mordeson and D.S. Malik gathered together many concepts related to fuzzy sets, for example, L-subgroup, L-ideals and L-submodules. Moreover, there has been much work done on fuzzy sets of hyperstructures, see [3], [5], [11] and [19]. In [5], R. Ameri and R. Mahjoob investigated some properties of fuzzy hyperideals and prime fuzzy hyperideals.

In this chapter, we study fuzzy subsets of hyperrings and hypermodules, inspired by [5]. Then we extend these to fuzzy subhypermodules. Basic notations related to fuzzy subsets follow from [5] and are given below.

A *fuzzy subset* of a nonempty set X is a function μ from X to $[0,1]$. Denote by F^X the collection of all fuzzy subsets of X. A fuzzy subset μ of X is called *non-constant* if there exist $x, y \in X$ such that $\mu(x) \neq \mu(y)$. For each subset A of X and $a \in [0,1]$, define $a_A \in F^X$ as follows:

$$a_A(x) = \begin{cases} a, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in X$. Moreover, we let $a_x = a_{\{x\}}$ for all $x \in X$.

For $\mu \in F^X$ and $a \in [0,1]$, define μ_a by

$$\mu_a = \{x \in X \mid \mu(x) \geq a\},$$

then μ_a is called the *a-cut* or *a-level* subset of μ.

For $\mu, \nu \in F^X$, we say that *μ is contained in ν* if $\mu(x) \leq \nu(x)$ for all $x \in X$, and denote this by $\mu \subseteq \nu$. For $a, b \in [0,1]$, we define $a \vee b$ and $a \wedge b$ by

$$a \vee b = \max\{a, b\} \quad \text{and} \quad a \wedge b = \min\{a, b\}.$$

For a nonempty subset A of X and a fuzzy subset μ of X, we define $\bigvee_{x\in A}\mu(x)$ and $\bigwedge_{x\in A}\mu(x)$ by

$$\bigvee_{x\in A}\mu(x)=\sup\{\mu(x)\mid x\in A\}\quad\text{and}\quad\bigwedge_{x\in A}\mu(x)=\inf\{\mu(x)\mid x\in A\}.$$

For $\mu,\nu\in F^X$, we define fuzzy subsets $\mu\cup\nu$ and $\mu\cap\nu$ of X by

$$(\mu\cup\nu)(x)=\mu(x)\vee\nu(x)\quad\text{and}\quad(\mu\cap\nu)(x)=\mu(x)\wedge\nu(x)$$

for all $x\in X$.

Let μ_i be a fuzzy subset of X for all $i\in\lambda$ and $x\in X$. Then define $\bigvee_{i\in\lambda}\mu_i(x)$ and $\bigwedge_{i\in\lambda}\mu_i(x)$ by

$$\bigvee_{i\in\lambda}\mu_i(x)=\sup\{\mu_i(x)\mid i\in\lambda\}\quad\text{and}\quad\bigwedge_{i\in\lambda}\mu_i(x)=\inf\{\mu_i(x)\mid i\in\lambda\}.$$

Moreover, we define fuzzy subsets $\bigcup_{i\in\lambda}\mu_i$ and $\bigcap_{i\in\lambda}\mu_i$ of X by

$$(\bigcup_{i\in\lambda}\mu_i)(x)=\bigvee_{i\in\lambda}\mu_i(x)\quad\text{and}\quad(\bigcap_{i\in\lambda}\mu_i)(x)=\bigwedge_{i\in\lambda}\mu_i(x).$$

for all $x\in X$.

We seperate this chapter into three parts, namely, fuzzy hyperideals of hyperrings, fuzzy subhypermodules of hypermodules and prime fuzzy subhypermodules.

Example Let μ be the fuzzy subset of $\mathbb{N}$ defined by $\mu(x)=\frac{1}{x}$. Then the $\frac{1}{2}$-level subset of μ is $\mu_{\frac{1}{2}}=\{n\in\mathbb{N}\mid\mu(n)\geq\frac{1}{2}\}=\{1,2\}$, $\bigvee_{x\in 2\mathbb{N}}\mu(x)=\sup\{\mu(x)\mid x\in 2\mathbb{N}\}=\frac{1}{2}$ and $\bigwedge_{x\in 2\mathbb{N}}\mu(x)=\inf\{\mu(x)\mid x\in 2\mathbb{N}\}=0$.

Let ν be the fuzzy subset of $\mathbb{N}$ defined by $\nu(x)=\frac{1}{x+1}$. Then $\nu\subseteq\mu$ and $\mu\cup\nu$ and $\mu\cap\nu$ are the fuzzy subsets of $\mathbb{N}$ defined by

$$\begin{aligned}(\mu\cup\nu)(x)&=\mu(x)\vee\nu(x)=\frac{1}{x}\vee\frac{1}{x+1}=\frac{1}{x}\quad\text{and}\\(\mu\cap\nu)(x)&=\mu(x)\wedge\nu(x)=\frac{1}{x}\wedge\frac{1}{x+1}=\frac{1}{x+1}\end{aligned}$$

for all $x\in\mathbb{N}$. Hence we see that $\mu\cup\nu=\mu$ and $\mu\cap\nu=\nu$, as we would expect when $\nu\subseteq\mu$.

4.1 Fuzzy Hyperideals of Hyperrings

We recall the definition and properties of fuzzy hyperideals from the work of R. Ameri and R. Mahjoob. They defined and investigated, in [5], fuzzy hyperideals and prime fuzzy hyperideals. We would like to gather some of their results here in order to obtain ideas that we can extend to fuzzy subhypermodules.

Definition 4.1.1. [5] A fuzzy subset α of a hyperring R is called a *fuzzy hyperideal* of R if for every $x, y \in R$,

(i) $\bigwedge_{z\in x+y} \alpha(z) \geq \alpha(x) \wedge \alpha(y)$,

(ii) $\alpha(-x) \geq \alpha(x)$, and

(iii) $\bigwedge_{z\in xy} \alpha(z) \geq \alpha(x) \vee \alpha(y)$.

It is easy to show that if α is a fuzzy hyperideal of a hyperring R, then $\alpha(x) = \alpha(-x)$ for all $x \in R$.

Example 4.1.2. Let I be a hyperideal of a hyperring R and $c \in [0, 1]$. Then c_I is a fuzzy hyperideal of R.

We show that, in fact, condition (ii) in the definition of a fuzzy hyperideal of a hyperring R can be omitted if the hyperring R satisfies $a \in Ra$ (or $a \in aR$) for all $a \in R$.

Proposition 4.1.3. *Let R be a hyperring such that $a \in Ra$ (or $a \in aR$) for all $a \in R$ and α a fuzzy subset of R. Then α is a fuzzy hyperideal of R if and only if*

(i) $\bigwedge_{z\in x+y} \alpha(z) \geq \alpha(x) \wedge \alpha(y)$, *and*

(ii) $\bigwedge_{z\in xy} \alpha(z) \geq \alpha(x) \vee \alpha(y)$,

for all $x, y \in R$.

Proof. Without loss of generality, we assume that R satisfies $a \in Ra$ for all $a \in R$. The proof of the necessary part is clear. Conversely, assume that (i) and (ii) hold. It suffices to show that $\alpha(-x) \geq \alpha(x)$ for all $x \in R$. Let $x \in R$. Then $-x \in$

$R(-x)$, so that there exists $r \in R$ such that $-x \in r(-x) = (-r)x$. Moreover, $\alpha(-x) \geq \bigwedge_{z\in(-r)x} \alpha(z) \geq \alpha(-r) \vee \alpha(x) \geq \alpha(x)$. Hence α is a fuzzy hyperideal of R. □

The following are interesting results from [5] serving as guidelines for the next section.

Proposition 4.1.4. [5] *Let α be a fuzzy hyperideal of a hyperring R. Then $\alpha(0) \geq \alpha(x)$ for all $x \in R$.*

Proposition 4.1.5. [5] *Let α be a fuzzy subset of a hyperideal R. Then α is a fuzzy hyperideal of R if and only if every nonempty a-level subset of α is a hyperideal of R.*

Let α be a fuzzy subset of a hyperring R. Define α_* as follows:

$$\alpha_* = \{x \in R \mid \alpha(x) = \alpha(0)\}.$$

Note that α_* is the nonempty $\alpha(0)$-level subset of α. By Proposition 4.1.5, α_* is a hyperideal of R if α is a fuzzy hyperideal of R.

Definition 4.1.6. [5] Let R be a hyperring and α a fuzzy subset of R. Define $\langle\alpha\rangle$ to be the smallest fuzzy hyperideal of R containing α.

Proposition 4.1.7. [5] *Let R be a hyperring and α_i be a fuzzy hyperideal of R for all $i \in \lambda$. Then $\bigcap_{i\in\lambda} \alpha_i$ is a fuzzy hyperideal of R. Moreover, $\langle\beta\rangle = \bigcap\{\alpha \mid \alpha$ is a fuzzy hyperideal such that $\beta \subseteq \alpha\}$ for all fuzzy subsets β of R.*

Proposition 4.1.8. [5] *Let R be a hyperring, A a nonempty subset of R and $a \in [0,1]$. Then $\langle a_A\rangle = a_{\langle A\rangle}$.*

4.2 Fuzzy Subhypermodules of Hypermodules

In this section, fuzzy subsets of hypermodules are investigated. We give a definition of a fuzzy subhypermodule of a hypermodule. This notion is derived from fuzzy hyperideals of hyperrings. The idea for constructing this definition arises from [5] and [14].

Definition 4.2.1. Let M be an R-hypermodule. A fuzzy subset μ of M is called a *fuzzy subhypermodule* of M if for all $r \in R$ and $x, y \in M$,

(i) $\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(-x) \geq \mu(x)$, and

(iii) $\bigwedge_{z \in rx} \mu(z) \geq \mu(x)$.

We can see that a fuzzy hyperideal of a hyperring R is a fuzzy subhypermodule of the R-hypermodule R. Moreover, it is clear that if μ is a fuzzy subhypermodule of an R-hypermodule M, then $\mu(x) = \mu(-x)$ for all $x \in M$.

Some properties of fuzzy subhypermodules that parallel those of fuzzy hyperideals can be obtained.

Proposition 4.2.2. *Let M be an R-hypermodule such that $m \in Rm$ for all $m \in M$ and μ a fuzzy subset of M. Then μ is a fuzzy subhypermodule of M if and only if*

(i) $\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y)$, *and*

(ii) $\bigwedge_{z \in rx} \mu(z) \geq \mu(x)$,

for all $x, y \in M$.

Proof. This can be proved similarly to Proposition 4.1.3. □

Proposition 4.2.3. *Let μ be a fuzzy subhypermodule of an R-hypermodule M. Then $\mu(0) \geq \mu(x)$ for all $x \in M$.*

Proof. For each $x \in M$, it follows that $\mu(0) \geq \bigwedge_{z \in x+(-x)} \mu(z) \geq \mu(x) \wedge \mu(-x) = \mu(x)$, since $0 \in x + (-x)$. □

Fuzzy subhypermodules and subhypermodules are related as seen in the following proposition.

Proposition 4.2.4. *Let μ be a fuzzy subset of an R-hypermodule M. Then μ is a fuzzy subhypermodule of M if and only if every nonempty a-level subset of μ is a subhypermodule of M.*

Proof. First, assume that μ is a fuzzy subhypermodule of M. Let μ_a be a nonempty a-level subset of μ. Let $x, y \in \mu_a$ and $r \in R$. Then $\mu(x) \geq a$ and $\mu(y) \geq a$. To show that $x - y \subseteq \mu_a$, let $z \in x - y$. Then $\mu(z) \geq \bigwedge_{t \in x-y} \mu(t) \geq \mu(x) \wedge \mu(-y) = \mu(x) \wedge \mu(y) \geq a$. Thus $z \in \mu_a$. Hence $x - y \subseteq \mu_a$. Next, we show that $rx \subseteq \mu_a$. Let $z \in rx$. Then $\mu(z) \geq \bigwedge_{t \in rx} \mu(t) \geq \mu(x) \geq a$, i.e., $z \in \mu_a$. Hence $rx \subseteq \mu_a$. Therefore, μ_a is a subhypermodule of M.

Conversely, assume that every nonempty a-level subset of μ is a subhypermodule of M. Let $r \in R$ and $x, y \in M$. To show that $\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y)$, let $a = \mu(x) \wedge \mu(y)$. Hence μ_a is nonempty since $x, y \in \mu_a$. By assumption, μ_a is a subhypermodule of M. Then $z \in \mu_a$, i.e., $\mu(z) \geq a$, for all $z \in x + y$. Thus $\bigwedge_{z \in x+y} \mu(z) \geq a = \mu(x) \wedge \mu(y)$. To show that $\mu(-x) \geq \mu(x)$ and $\bigwedge_{z \in rx} \mu(z) \geq \mu(x)$, let $a = \mu(x)$. Then μ_a is nonempty since $x \in \mu_a$. By assumption, μ_a is a subhypermodule of M. Then $-x \in \mu_a$ and $z \in \mu_a$ for all $z \in rx$, i.e., $\mu(-x) \geq a = \mu(x)$ and $\bigwedge_{z \in rx} \mu(z) \geq \bigwedge_{z \in rx} a = a = \mu(x)$. Therefore, μ is a fuzzy subhypermodule of M. □

Let μ be a fuzzy subset of an R-hypermodule M. Similar to the previous section, we define μ_* as follows:

$$\mu_* = \big\{x \in M \mid \mu(x) = \mu(0)\big\}.$$

Then μ_* is the nonempty $\mu(0)$-level subset of μ. Moreover, μ_* is a subhypermodule of M if μ is a fuzzy subhypermodule of M.

Proposition 4.2.5. *Let μ be a fuzzy subhypermodule of an R-hypermodule M and $x, y \in M$. If $\bigwedge_{z \in x+y} \mu(z) = \mu(0)$, then $\mu(x) = \mu(y)$.*

Proof. Assume that $\bigwedge_{z \in x+y} \mu(z) = \mu(0)$. Then $\mu(z) \geq \mu(0)$ for all $z \in x + y$. Thus $\mu(z) = \mu(0)$ for all $z \in x + y$ by Proposition 4.2.3. Let $z \in x + y$. Then $x \in z + (-y)$ and

$$\mu(x) \geq \bigwedge_{t \in z+(-y)} \mu(t) \geq \mu(z) \wedge \mu(-y) = \mu(z) \wedge \mu(y) = \mu(0) \wedge \mu(y) = \mu(y).$$

Hence $\mu(x) \geq \mu(y)$. Since $x + y = y + x$, we also have $z \in y + x$, and the same argument shows $\mu(y) \geq \mu(x)$. Therefore, $\mu(x) = \mu(y)$. □

Corollary 4.2.6. *Let μ be a fuzzy subhypermodule of an R-hypermodule M and $x, y \in M$. If $\bigwedge_{z \in x-y} \mu(z) = \mu(0)$, then $\mu(x) = \mu(y)$.*

The following proposition shows one simple way to construct fuzzy subhypermodules.

Proposition 4.2.7. *Let N be a subhypermodule of an R-hypermodule M and $c \in [0,1]$. Then c_N is a fuzzy subhypermodule of M.*

Proof. We apply Proposition 4.2.4 to show that c_N is a fuzzy subhypermodule of M. Let $a \in R$. If $a = 0$, then $(c_N)_a = M$. If $0 < a \leq c$, then $(c_N)_a = N$. Otherwise, $(c_N)_a = \varnothing$. This shows that there are only two possibilities for the nonempty a-level subsets of c_N, namely, N and M. Thus every nonempty a-level subset of c_N is a subhypermodule of M. Thus c_N is a fuzzy subhypermodule of M. □

We define the product of fuzzy subsets of a hyperring and the product of a fuzzy subset of a hyperring and a fuzzy subset of a hypermodule.

Definition 4.2.8. Let M an R-hypermodule, α, β fuzzy subsets of R and μ a fuzzy subset of M. The *product of α and β*, denoted by $\alpha\beta$, is defined as follows: for all $z \in R$,

$$(\alpha\beta)(z) = \begin{cases} \displaystyle\bigvee_{\substack{x,y \in R, \\ z \in xy}} \big(\alpha(x) \wedge \beta(y)\big), & \text{if } z \in R^2 \text{ (where } R^2 := RR), \\ 0, & \text{otherwise.} \end{cases}$$

The *product of α and μ*, denoted by $\alpha\mu$, is defined similarly, as follows: for all $m \in M$,

$$(\alpha\mu)(m) = \begin{cases} \displaystyle\bigvee_{\substack{a \in R, n \in M, \\ m \in an}} \big(\alpha(a) \wedge \mu(n)\big), & \text{if } m \in RM, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.2.9. *Let M be an R-hypermodule, A and B nonempty subsets of R, X a nonempty subset of M and $a, b \in [0,1]$. Then $a_A b_B = (a \wedge b)_{AB}$ and $a_A b_X = (a \wedge b)_{AX}$*

Proof. To show that $a_A b_B = (a \wedge b)_{AB}$, let $r \in R$. If $r \notin R^2$, then $r \notin AB$ so that $(a_A b_B)(r) = 0 = (a \wedge b)_{AB}(r)$. Assume that $r \in R^2$. If $r \in AB$, then there exist $x \in A$ and $y \in B$ such that $r \in xy$ and

$$(a_A b_B)(r) = \bigvee_{\substack{x',y' \in R,\\ z \in x'y'}} \big(a_A(x') \wedge b_B(y')\big) = a_A(x) \wedge b_B(y) = a \wedge b = (a \wedge b)_{AB}(r).$$

Assume that $r \notin AB$, then $r_1 \notin A$ or $r_2 \notin B$ for all $r_1, r_2 \in R$ such that $r \in r_1 r_2$. Then

$$(a_A b_B)(r) = \bigvee_{\substack{x,y \in R,\\ r \in xy}} \big(a_A(x) \wedge b_B(y)\big) = 0 = (a \wedge b)_{AB}(r).$$

Hence $a_A b_B = (a \wedge b)_{AB}$. Similarly, $a_A b_X = (a \wedge b)_{AX}$. □

Lemma 4.2.10. *Let M be an R-hypermodule A, B nonempty subsets of R, X a nonempty subset of M and $a, b, c \in [0, 1]$. Then $(a_A b_B)c_X = a_A(b_B c_X)$. In fact, $a_A b_B c_X$ is well-defined.*

Proof. By above proposition, $(a_A b_B)c_X = (a \wedge b)_{AB} c_X = (a \wedge b \wedge c)_{(AB)X} = (a \wedge b \wedge c)_{A(BX)} = a_A(b \wedge c)_{BX} = a_A(b_B c_X)$. Thus $(a_A b_B)c_X = a_A(b_B c_X)$. □

Recall that if I is a hyperideal of a hyperring R, then $RI \subseteq I$ and $IR \subseteq I$; moreover, if N is a subhypermodule of an R-hypermodule M, then $RN \subseteq N$. Next, we prove some analogous properties of fuzzy hyperideals and fuzzy sub-hyprmodules.

Proposition 4.2.11. *Let M be an R-hypermodule.*

(i) *If α is a fuzzy hyperideal of R, then $\alpha\beta \subseteq \alpha$ and $\beta\alpha \subseteq \alpha$ for any fuzzy subsets β of R.*

(ii) *If μ is a fuzzy subhypermodule of M, then $\alpha\mu \subseteq \mu$ for any fuzzy subsets α of R.*

Proof. (i) Assume that α is a fuzzy hyperideal of R. Let $r \in R$. If $r \notin R^2$, it follows that

$$(\alpha\beta)(r) = 0 \leq \alpha(r).$$

Assume, now, that $r \in R^2$. We claim that $\alpha(r) \geq \alpha(x)$ for all $x, y \in R$ such that $r \in xy$. Let $x, y \in R$ be such that $r \in xy$. Since α is a fuzzy hyperideal,

$$\alpha(r) \geq \bigwedge_{t \in xy} \alpha(t) \geq \alpha(x) \vee \alpha(y) \geq \alpha(x)$$

as claimed. Thus

$$(\alpha\beta)(r) = \bigvee_{\substack{x,y \in R, \\ r \in xy}} \big(\alpha(x) \wedge \beta(y)\big) \leq \bigvee_{\substack{x,y \in R, \\ r \in xy}} \alpha(x) \leq \alpha(r).$$

Hence $\alpha\beta \subseteq \alpha$. Similarly, $\beta\alpha \subseteq \alpha$.

The proof of (ii) parallels the proof of (i). □

Proposition 4.2.12. *Let M be an R-hypermodule and μ_i be a fuzzy subhypermodule of M for all $i \in \lambda$. Then $\bigcap_{i \in \lambda} \mu_i$ is a fuzzy subhypermodule of M.*

Proof. Let $x, y \in M$. Since μ_i is a fuzzy subhypermodule of M for all $i \in \lambda$,

$$\bigwedge_{z \in x+y} \mu_i(z) \geq \mu_i(x) \wedge \mu_i(y) \geq \big(\bigwedge_{j \in \lambda} \mu_j(x)\big) \wedge \big(\bigwedge_{j \in \lambda} \mu_j(y)\big) = \big(\bigcap_{j \in \lambda} \mu_j\big)(x) \wedge \big(\bigcap_{j \in \lambda} \mu_j\big)(y)$$

for all $i \in \lambda$. Hence $\bigwedge_{i \in \lambda}\big(\bigwedge_{z \in x+y} \mu_i(z)\big) \geq \big(\bigcap_{i \in \lambda} \mu_i\big)(x) \wedge \big(\bigcap_{i \in \lambda} \mu_i\big)(y)$. Thus $\bigwedge_{z \in x+y}\big(\bigwedge_{i \in \lambda} \mu_i(z)\big) = \bigwedge_{i \in \lambda}\big(\bigwedge_{z \in x+y} \mu_i(z)\big) \geq \big(\bigcap_{i \in \lambda} \mu_i\big)(x) \wedge \big(\bigcap_{i \in \lambda} \mu_i\big)(y)$, i.e., $\bigwedge_{z \in x+y}\big((\bigcap_{i \in \lambda} \mu_i)(z)\big) \geq \big(\bigcap_{i \in \lambda} \mu_i\big)(x) \wedge \big(\bigcap_{i \in \lambda} \mu_i\big)(y)$.

It can be shown similarly that for all $r \in R$ and $x \in M$, $\big(\bigcap_{i \in \lambda} \mu_i\big)(-x) \geq \big(\bigcap_{i \in \lambda} \mu_i\big)(x)$ and $\bigwedge_{z \in rx}\big((\bigcap_{i \in \lambda} \mu_i)(z)\big) \geq \big(\bigcap_{i \in \lambda} \mu_i\big)(x)$. Therefore, $\bigcap_{i \in \lambda} \mu_i$ is a fuzzy subhypermodule of M. □

Definition 4.2.13. Let M be an R-hypermodule and μ a fuzzy subset of M. Define $\langle \mu \rangle$ to be the smallest fuzzy subhypermodule of M containing μ and call this $\langle \mu \rangle$ the *fuzzy subhypermodule of M generated by μ* .

Proposition 4.2.14. *Let M be an R-hypermodule and μ a fuzzy subset of M. Then* $\langle \mu \rangle = \bigcap\{\nu \mid \nu$ is a fuzzy subhypermodule of M such that $\mu \subseteq \nu\}$.

Proof. This proof is straightforward. □

Proposition 4.2.15. *Let M be an R-hypermodule, A and B nonempty subsets of R, X a nonempty subset of M and $a, b, c \in [0, 1]$. Then*

(i) $\langle a_A \rangle = a_{\langle A \rangle}$ *and* $\langle a_X \rangle = a_{\langle X \rangle}$,

(ii) $\langle a_A b_B \rangle = (a \wedge b)_{\langle AB \rangle}$ *and* $\langle a_A b_X \rangle = (a \wedge b)_{\langle AX \rangle}$, *and*

(iii) $\langle a_A b_B c_X \rangle = (a \wedge b \wedge c)_{\langle ABX \rangle}$.

Proof. (i) We obtain that $\langle a_A \rangle = a_{\langle A \rangle}$ by [5]. Next, we show that $\langle a_X \rangle = a_{\langle X \rangle}$. Since $a_X \subseteq a_{\langle X \rangle}$ and $a_{\langle X \rangle}$ is a fuzzy subhypermodule of M, $\langle a_X \rangle \subseteq a_{\langle X \rangle}$. Let μ be a fuzzy subhypermodule of M such that $a_X \subseteq \mu$. We show that $a_{\langle X \rangle} \subseteq \mu$. Let $m \in M$. If $m \notin \langle X \rangle$, then $a_{\langle X \rangle}(m) = 0 \leq \mu(m)$. Assume that $m \in \langle X \rangle$. Then $m \in \langle X \rangle = [RX] + [\mathbb{Z}X]$ so that $m \in m_1 + m_2$ for some $m_1 \in [RX]$ and $m_2 \in [\mathbb{Z}X]$. Hence $\mu(m) \geq \mu(m_1) \wedge \mu(m_2)$. Since $m_1 \in [RX]$, $m_1 \in r_1x_1 + r_2x_2 + \cdots + r_nx_n$ where $r_i \in R$ and $x_i \in X$ for all $i \in \{1, 2, 3, \ldots, n\}$. Then

$$\mu(m_1) \geq \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n) \geq a_X(x_1) \wedge a_X(x_2) \wedge \cdots \wedge a_X(x_n) = a.$$

Similarly, $\mu(m_2) \geq a$. Hence $\mu(m) \geq a = a_{\langle X \rangle}(m)$. This shows $a_{\langle X \rangle} \subseteq \mu$ for all fuzzy subhypermodules μ such that $a_X \subseteq \mu$. Thus $a_{\langle X \rangle} \subseteq \langle a_X \rangle$. Therefore, $\langle a_X \rangle = a_{\langle X \rangle}$.

(ii) By (i) and Proposition 4.2.9, we obtain that $\langle a_A b_B \rangle = \langle (a \wedge b)_{AB} \rangle = (a \wedge b)_{\langle AB \rangle}$ and $\langle a_A b_X \rangle = \langle (a \wedge b)_{AX} \rangle = (a \wedge b)_{\langle AX \rangle}$.

(iii) This proof follows easily from (ii). □

4.3 Prime Fuzzy Subhypermodules

We start off this section by defining $(\nu : \mu)$ where μ and ν are fuzzy subsets of an R-hypermodule M. This definition is inspired by that of $(X : Y)$ from Chapter I where X and Y are nonempty subsets of M.

Definition 4.3.1. Let μ and ν be fuzzy subsets of an R-hypermodule M. Define $(\nu : \mu)$ by

$$(\nu : \mu) = \bigcup \{\alpha \in F^R \mid \alpha\mu \subseteq \nu\}$$

Proposition 4.3.2. *Let* $c \in [0, 1]$ *and* N *a subhypermodule of an* R*-hypermodule* M*. Then*

$$(1_N \cup c_M : 1_M) = 1_{(N:M)} \cup c_R.$$

Proof. First, we show that $1_{(N:M)} \cup c_R \subseteq \big(1_N \cup c_M : 1_M\big)$, i.e., we prove that $\big(1_{(N:M)} \cup c_R\big)1_M \subseteq 1_N \cup c_M$. Let $x \in M$. If $x \notin RM$, we are done. Assume that $x \in RM$.

Case 1 $x \in N$. We obtain that $(1_N \cup c_M)(x) = 1$ and the result follows.

Case 2 $x \notin N$. We have $(1_N \cup c_M)(x) = c$. Since $x \notin N$, it follows that $x \notin (N : M)M$. Then $r \notin (N : M)$ for all $r \in R$ and $n \in M$ with $x \in rn$. Thus

$$\begin{aligned}\big(1_{(N:M)} \cup c_R\big)1_M(x) &= \bigvee_{\substack{r\in R, n\in M,\\ x\in rn}} \Big(\big(1_{(N:M)} \cup c_R\big)(r) \wedge 1_M(n)\Big)\\ &= \bigvee_{\substack{r\in R, n\in M,\\ x\in rn}} c = c = (1_N \cup c_M)(x).\end{aligned}$$

We conclude that $1_{(N:M)} \cup c_R \subseteq \big(1_N \cup c_M : 1_M\big)$.

Next, we show that $\alpha \subseteq 1_{(N:M)} \cup c_R$ for all fuzzy subsets α of R such that $\alpha 1_M \subseteq 1_N \cup c_M$. Let α be a fuzzy subset of R such that $\alpha 1_M \subseteq 1_N \cup c_M$ and $a \in R$.

Case 1 $a \in (N : M)$. We obtain that $(1_{(N:M)} \cup c_R)(a) = 1$ and thus $\alpha(a) \leq (1_{(N:M)} \cup c_R)(a)$.

Case 2 $a \notin (N : M)$. There exists $m \in M$ such that $am \nsubseteq N$. Thus there exists $t \in am$ such that $t \notin N$. Since $\alpha 1_M \subseteq 1_N \cup c_M$, it follows that $(\alpha 1_M)(t) \leq (1_N \cup c_M)(t) = c$. Then $\alpha(a) \wedge 1_M(m) \leq c$. Hence $\alpha(a) = \alpha(a) \wedge 1_M(m) \leq c = (1_{(N:M)} \cup c_R)(a)$.

This shows that $\alpha \subseteq 1_{(N:M)} \cup c_R$ for all α such that $\alpha 1_M \subseteq 1_N \cup c_M$. Hence $\big((1_N \cup c_M) : 1_M\big) \subseteq 1_{(N:M)} \cup c_R$. Therefore, $\big((1_N \cup c_M) : 1_M\big) = 1_{(N:M)} \cup c_R$. □

Our aim for this section is to study prime fuzzy subhypermodules. Prime fuzzy subhypermodules are defined using an idea similar to the one used to define prime subhypermodules of a hypermodule. In fact, prime fuzzy subhypermodules are an extension of prime fuzzy hyperideals.

We recall the definition of prime fuzzy hyperideals from [5].

Definition 4.3.3. [5] A fuzzy hyperideal p of a hyperring R is called a *prime fuzzy hyperideal* if p is non-constant and for all fuzzy hyperideals α, β of R if $\alpha\beta \subseteq p$, then $\alpha \subseteq p$ or $\beta \subseteq p$.

In [5], the authors gave a characterization of prime fuzzy hyperideals of hyperrings.

Theorem 4.3.4. [5] *Let p be a fuzzy hyperideal. Then p is a prime fuzzy hyperideal of R if and only if $p(0) = 1$, p_* is a prime hyperideal of R and $p = 1_{p_*} \cup c_R$ for some $c \in [0, 1)$.*

Now, we define a fuzzy subhypermodule of a hypermodule. The idea for this definition comes from the book [14] and the paper [4].

Definition 4.3.5. A non-constant fuzzy subhypermodule μ of an R-hypermodule M is said to be *prime* if for all fuzzy hyperideals α of R and fuzzy subhypermodules ν of M, if $\alpha\nu \subseteq \mu$, then $\alpha \subseteq (\mu : 1_M)$ or $\nu \subseteq \mu$.

We determine some properties related to prime fuzzy subhypermodules.

Proposition 4.3.6. *Let M be an R-hypermodule. If μ is a prime fuzzy subhypermodule of M, then μ_* is a prime subhypermodule of M.*

Proof. Assume that μ is a prime fuzzy subhypermodule. Recall that μ_* is a subhypermodule of M. Since μ is prime, μ is non-constant. Thus $\mu_* \neq M$. Let I and D be a hyperideal of R and a subhypermodule of M, respectively, such that $ID \subseteq \mu_*$. Let $\alpha = 1_I$ and $\nu = 1_D$. Then $\alpha\nu = 1_I 1_D = 1_{ID} \subseteq 1_{\mu_*} \subseteq \mu$. Since μ is prime, $\alpha \subseteq (\mu : 1_M)$ or $\nu \subseteq \mu$, i.e., $1_I \subseteq (\mu : 1_M)$ or $1_D \subseteq \mu$. If $1_I \subseteq (\mu : 1_M)$, then $1_{IM} = 1_I 1_M \subseteq \mu$. Thus $IM \subseteq \mu_*$ or $D \subseteq \mu_*$. Therefore $I \subseteq (\mu_* : M)$ or $D \subseteq \mu_*$. □

The following results give a characterization of prime fuzzy subhypermodules. This result is similar to Theoerm 4.3.4. However, the conditions $M = RM$ and $(\mu_* : M) \neq \varnothing$ are required.

Theorem 4.3.7. *Let μ be a fuzzy subhypermodule of an R-hypermodule M such that $M = RM$ and $(\mu_* : M) \neq \varnothing$. If μ is a prime fuzzy subhypermodule of M, then μ_* is a prime subhypermodule of M, $\mu(0) = 1$ and $\mu = 1_{\mu_*} \cup c_M$ for some $c \in [0, 1)$.*

Proof. Assume that μ is a prime fuzzy subhypermodule of M. By Proposition 4.3.6, μ_* is a prime subhypermodule of M. Let us show that $\mu(0) = 1$. Since μ is non-constant, there exists $x \in M$ such that $\mu(x) < \mu(0)$. Define a fuzzy subset α of R by $\alpha = 1_{(\mu_*:M)}$ and a fuzzy subset ν of M by $\nu = \mu(0)_M$. Then α is a fuzzy hyperideal of R and ν is a fuzzy subhypermodule of M such that $\alpha\nu = 1_{(\mu_*:M)}\mu(0)_M = \mu(0)_{(\mu_*:M)M} \subseteq \mu(0)_{\mu_*} \subseteq \mu$. Since μ is prime and $\nu \nsubseteq \mu$, we have $\alpha \in (\mu : 1_M)$, i.e., $1_{(\mu_*:M)}1_M \subseteq \mu$. Since $(\mu_* : M) \neq \varnothing$, let $y \in (\mu_* : M)M$. Then $y \in \mu_*$, i.e., $\mu(y) = \mu(0)$, and $1_{(\mu_*:M)}1_M(y) = 1$. Thus $1 = 1_{(\mu_*:M)}1_M(y) \leq \mu(y) = \mu(0)$. Hence $\mu(0) = 1$.

Next, we show that $\mu = 1_{\mu_*} \cup c_M$ for some $c \in [0, 1)$. Note that it is sufficient to show there exists $c \in [0, 1)$ such that $\mu(x) = c$ for all $x \notin \mu_*$. Since $\mu(0) = 1$ and μ is non-constant, there exists $c \in [0, 1)$ and $x \notin \mu_*$ such that $\mu(x) = c < 1$. To show that $\mu(y) = \mu(x)$ for all $y \notin \mu_*$, let $y \notin \mu_*$. Then $c_x \subseteq \mu$. By Proposition 4.2.15, $c_{\langle x\rangle} = \langle c_x\rangle \subseteq \mu$. We have $c_R1_{\langle x\rangle} = c_{R\langle x\rangle} \subseteq c_{\langle x\rangle} \subseteq \mu$. Since μ is prime and $1_{\langle x\rangle} \nsubseteq \mu$, $c_R \subseteq (\mu : 1_M)$, which implies $c_R1_M \subseteq \mu$. Since $M = RM$, it follows that $c_M = c_{RM} = c_R1_M \subseteq \mu$. Thus $\mu(x) = c = c_M(y) \leq \mu(y)$. Similarly, $\mu(y) \leq \mu(x)$. Therefore $\mu(x) = \mu(y)$. We conclude that $\mu = 1_{\mu_*} \cup c_M$. □

Theorem 4.3.8. *Let μ be a fuzzy subhypermodule of an R-hypermodule M. If μ_* is a prime subhypermodule of M, $\mu(0) = 1$ and $\mu = 1_{\mu_*} \cup c_M$ for some $c \in [0, 1)$, then μ is a prime fuzzy subhypermodule of M.*

Proof. Suppose that μ_* is a prime subhypermodule, $\mu(0) = 1$ and $\mu = 1_{\mu_*} \cup c_M$ for some $c \in [0, 1)$. Since $\mu_* \neq M$ and $\mu = 1_{\mu_*} \cup c_M$, we obtain that μ is non-constant. Let α and ν be a fuzzy hyperideal of R and a fuzzy subhypermodule of M, respectively, such that $\alpha\nu \subseteq \mu$. Suppose that $\alpha \nsubseteq (\mu : 1_M)$ and $\nu \nsubseteq \mu$. Since $\nu \nsubseteq \mu$, there exists $x \in M$ such that $\nu(x) > \mu(x)$. Then $\mu(x) \neq 1$, i.e., $x \notin \mu_*$. Thus $\langle x\rangle \nsubseteq \mu_*$ and $\nu(x) > \mu(x) = c$. Since $\alpha \nsubseteq (\mu : 1_M)$, i.e., $\alpha 1_M \nsubseteq \mu$, there exists $t \in M$ such that $\alpha 1_M(t) > \mu(t)$. There exist $r \in R$ and $y \in M$ such that $t \in ry$ and $\alpha(r) = \alpha(r) \wedge 1_M(y) > \mu(t)$. Hence $\mu(t) \neq 1$, i.e., $t \notin \mu_*$ and $\alpha(r) > \mu(t) = c$. Since $t \in ry \subseteq rM$ and $t \notin \mu_*$, $rM \nsubseteq \mu_*$. Hence $r \notin (\mu_* : M)$. Thus $\langle r\rangle \nsubseteq (\mu_* : M)$.

Now, we have $\langle x\rangle \nsubseteq \mu_*$, $\langle r\rangle \nsubseteq (\mu_* : M)$, $\nu(x) > c$ and $\alpha(r) > c$. Since μ_* is prime, $\langle r\rangle \nsubseteq (\mu_* : M)$ and $\langle x\rangle \nsubseteq \mu_*$, we conclude that $\langle r\rangle\langle x\rangle \nsubseteq \mu_*$. Then there exists $l \in \langle r\rangle\langle x\rangle$ such that $l \notin \mu_*$. Note that

$$
\begin{aligned}
l \in \langle r\rangle\langle x\rangle &= \big([Rr] + [rR] + [RrR] + [\mathbb{Z}r]\big)\big([Rx] + [\mathbb{Z}x]\big)\\
&= [RrRx] + [Rrx] + [rRx] + [\mathbb{Z}rx].
\end{aligned}
$$

Then $l \in l_1+l_2+l_3+l_4$ for some $l_1 \in [RrRx]$, $l_2 \in [Rrx]$, $l_3 \in [rRx]$ and $l_4 \in [\mathbb{Z}rx]$. Since $l \notin \mu_*$, there exists $i \in \{1, 2, 3, 4\}$ such that $l_i \notin \mu_*$.

Assume that $l_1 \notin \mu_*$. Since $l_1 \in [RrRx]$, there exist $n \in \mathbb{N}$ and $s_i, t_i \in R$ for all $i \in \{1, 2, \ldots, n\}$ such that $l_1 \in s_1rt_1x + s_2rt_2x + \ldots + s_nrt_1x$. Then $l_1 \in l_{11}+l_{22}+\cdots+l_{nn}$ where $l_{ii} \in s_irt_ix$. Since $l_1 \notin \mu_*$, there exists $l_{jj} \notin \mu_*$ for some $j \in \{1, 2, 3, \ldots, n\}$. Then $\mu(l_{jj}) = c$. Since $l_{jj} \in s_jrt_jx$, it follows that $l_{jj} \in s_jz$ for some $z \in rt_jx$. Then $c = \mu(l_{jj}) \geq \mu(z)$. Since $z \in rt_jx$, we obtain that $z \in rz_1$ for some $z_1 \in t_jx$. Thus $c = \mu(l_{jj}) \geq \mu(z) \geq (\alpha\nu)(z) \geq \alpha(r) \wedge \nu(z_1) \geq \alpha(r) \wedge \nu(x)$. That is $c \geq \alpha(r) \wedge \nu(x)$. Therefore, $\alpha(r) \leq c$ or $\nu(x) \leq c$, a contradiction.

Similarly, we obtain a contradiction if $l_i \notin \mu_*$ for $i = 2, 3$ or 4. Therefore, we conclude that μ is a prime fuzzy subhypermodule of M. □

This is the immediate consequences of Theorem 4.3.7 and Theorem 4.3.8.

Corollary 4.3.9. *Let μ be a fuzzy subhypermodule of an R-hypermodule M such that $M = RM$ and $(\mu_* : M) \neq \varnothing$. Then μ is a prime fuzzy subhypermodule of M if and only if μ_* is a prime subhypermodule of M, $\mu(0) = 1$ and $\mu = 1_{\mu_*} \cup c_M$ for some $c \in [0, 1)$.*

The previous results give one way to construct examples of prime fuzzy subhypermodules which we make use of in the following example.

Example 4.3.10. Let $R = [0, 1]$. Then $\big(R, \oplus_{\max}, \cdot\big)$ is a Krasner hyperring, see [18], where $\oplus_{\max} : R \times R \to \wp^*(R)$ is a multi-valued function defined by

$$
x \oplus_{\max} y = \begin{cases} \big\{\max\{x, y\}\big\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y, \end{cases}
$$

and $\cdot$ is the usual multiplication on real numbers. Let $K = [0, 0.5]$. Then K is a hyperideal of R. It follows from Example 1.2.38 that $(R, \oplus_{max}, \circ)$ is a hyperring, where $\circ$ is defined as in that example, with $H = K$. Then R is an R-hypermodule. Choose $L = [0, 1)$. It is easy to check that L is a maximal subhypermodule of R, and thus is prime by Proposition 2.1.18. We have $1_L \cup c_R$ is a prime fuzzy subhypermodule of R for all $c \in [0, 1)$ by Theorem 4.3.7.

In Chapter II, we characterized prime subhypermodules under three different conditions: R is commutative, $a \in aR$ for all $a \in R$ and $m \in Rm$ for all $m \in M$, where M is an R-hypermodule. The rest of this chapter is devoted to providing some characterizations of prime fuzzy subhypermodules undr these three conditions in the context of fuzzy subsets. First, we consider the condition that the hyperring is commutative.

Theorem 4.3.11. *Let R be a commutative hyperring and μ a fuzzy subhypermodule of an R-hypermodule M. Then μ is a prime fuzzy subhypermodule of M if and only if*

(i) *μ_* is a prime subhypermodule of M, and*

(ii) *for all $r \in R$, $x \in M$ and $a, b \in [0, 1]$, if $a_r b_x \subseteq \mu$, then $b_x \subseteq \mu$ or $a_r \subseteq (\mu : 1_M)$.*

Proof. First, assume that μ is a prime fuzzy subhypermodule of M. By Proposition 4.3.6, it remains only to prove (ii). Let $r \in R$, $x \in M$ and $a, b \in [0, 1]$. Assume that $a_r b_x \subseteq \mu$. By Proposition 4.2.15 (iii), it follows that $(a \wedge b)_{\langle rx \rangle} = \langle a_r b_x \rangle \subseteq \mu$. We claim that $a_{\langle r \rangle} b_{\langle x \rangle} \subseteq (a \wedge b)_{\langle rx \rangle}$. Let $m \in M$. If $m \notin \langle r \rangle \langle x \rangle$, then $a_{\langle r \rangle} b_{\langle x \rangle}(m) = 0 \leq (a \wedge b)_{\langle rx \rangle}(m)$. Assume that $m \in \langle r \rangle \langle x \rangle$. Since R is commutative,

$$m \in \langle r \rangle \langle x \rangle = \big([Rr] + [\mathbb{Z}r]\big)\big([Rx] + [\mathbb{Z}x]\big) \subseteq [Rrx] + [\mathbb{Z}rx] = \langle rx \rangle.$$

Thus $m \in \langle rx \rangle$ and $a_{\langle r \rangle} b_{\langle x \rangle}(m) = a \wedge b = (a \wedge b)_{\langle rx \rangle}(m)$. Hence $a_{\langle r \rangle} b_{\langle x \rangle} \subseteq (a \wedge b)_{\langle rx \rangle} = \langle a_r b_x \rangle \subseteq \mu$. Since μ is prime, $a_{\langle r \rangle} \subseteq (\mu : 1_M)$ or $b_{\langle x \rangle} \subseteq \mu$. Thus $a_r \subseteq (\mu : 1_M)$ or $b_x \subseteq \mu$.

Conversely, assume that (i) and (ii) hold. Since μ_* is a prime subhypermodule of M, $\mu_* \neq M$. There exists $m' \in M$ such that $m' \notin \mu_*$. Then $\mu(m') < \mu(0)$. Thus μ is non-constant. Let α and ν be a fuzzy hyperideal of R and a fuzzy subhypermodule of M, respectively, such that $\alpha\nu \subseteq \mu$. Assume that $\nu \nsubseteq \mu$. There exists $m \in M$ such that $\nu(m) > \mu(m)$. To show that $\alpha \subseteq (\mu : 1_M)$. Let $r \in R$. Since $\alpha(r)_r \subseteq \alpha$ and $\nu(m)_m \subseteq \nu$, it follows that $\alpha(r)_r\nu(m)_m \subseteq \alpha\nu \subseteq \mu$. By (ii), $\alpha(r)_r \subseteq (\mu : 1_M)$ or $\nu(m)_m \subseteq \mu$. Thus $\alpha(r)_r \subseteq (\mu : 1_M)$, i.e., $\alpha(r) \leq (\mu : 1_M)(r)$. This shows that $\alpha(r) \leq (\mu : 1_M)(r)$ for all $r \in R$, i.e., $\alpha \subseteq (\mu : 1_M)$. We conclude that μ is a prime fuzzy subhypermodule of M. □

For the second characterization, we are interested in the condition $a \in aR$ for all $a \in R$.

Theorem 4.3.12. *Let R be a hyperring such that $a \in aR$ for all $a \in R$ and μ a fuzzy subhypermodule of an R-hypermodule M. Then μ is a prime fuzzy subhypermodule of M if and only if*

(i) *μ_* is a prime subhypermodule of M, and*

(ii) *for all $r \in R$, $x \in M$ and $a, b \in [0,1]$, if $a_r 1_R b_x \subseteq \mu$, then $b_x \subseteq \mu$ or $a_r \subseteq (\mu : 1_M)$.*

Proof. First, assume that μ is a prime fuzzy subhypermodule of M. Again, it remains only to prove (ii). Let $r \in R$, $x \in M$ and $a, b \in [0,1]$. Assume that $a_r 1_R b_x \subseteq \mu$. By Proposition 4.2.15 (iv), $(a \wedge b)_{\langle rRx\rangle} = \langle a_r 1_R b_x\rangle \subseteq \mu$. We claim that $a_{\langle rR\rangle} b_{\langle x\rangle} \subseteq (a \wedge b)_{\langle rRx\rangle}$. Let $m \in M$. If $m \notin \langle rR\rangle\langle x\rangle$, then $a_{\langle rR\rangle} b_{\langle x\rangle}(m) = 0 \leq (a \wedge b)_{\langle rRx\rangle}(m)$. Assume that $m \in \langle rR\rangle\langle x\rangle$. Since $a \in aR$ for all $a \in R$,

$$m \in \langle rR\rangle\langle x\rangle = \big([rR] + [RrR]\big)\big([Rx] + [\mathbb{Z}x]\big) \subseteq [RrRx] + [\mathbb{Z}rRx] = \langle rRx\rangle.$$

Thus $m \in \langle rRx\rangle$ and $a_{\langle rR\rangle} b_{\langle x\rangle}(m) = a \wedge b = (a \wedge b)_{\langle rRx\rangle}(m)$. Hence $a_{\langle rR\rangle} b_{\langle x\rangle} \subseteq (a \wedge b)_{\langle rRx\rangle} = \langle a_r 1_R b_x\rangle \subseteq \mu$. Since μ is prime, $a_{\langle rR\rangle} \subseteq (\mu : 1_M)$ or $b_{\langle x\rangle} \subseteq \mu$. Thus $a_r \subseteq (\mu : 1_M)$ or $b_x \subseteq \mu$.

Conversely, assume that (i) and (ii) hold. Since μ_* is a prime subhypermodule of M, there exists $m' \in M$ such that $m' \notin \mu_*$. Then $\mu(m') < \mu(0)$ and μ is non-constant. Let α and ν be a fuzzy hyperideal of R and a fuzzy subhypermodule of M, respectively, such that $\alpha\nu \subseteq \mu$. Assume that $\nu \nsubseteq \mu$. There exists $m \in M$ such that $\nu(m) > \mu(m)$. To show that $\alpha \subseteq (\mu : 1_M)$, let $r \in R$. Since $\alpha(r)_r 1_R \subseteq \alpha 1_R \subseteq \alpha$ and $\nu(m)_m \subseteq \nu$, we obtain that $\alpha(r)_r 1_R \nu(m)_m \subseteq \alpha\nu \subseteq \mu$. Then $\alpha(r)_r \subseteq (\mu : 1_M)$ or $\nu(m)_m \subseteq \mu$ by (ii). Thus $\alpha(r)_r \subseteq (\mu : 1_M)$, i.e., $\alpha(r) \leq (\mu : 1_M)(r)$. This shows that $\alpha(r) \leq (\mu : 1_M)(r)$ for all $r \in R$. Thus $\alpha \subseteq (\mu : 1_M)$. We conclude that μ is a prime fuzzy subhypermodule of M. □

For the last characterization, we obtain the same characterization as above under the condition $m \in Rm$ for all $m \in M$.

Theorem 4.3.13. *Let M be an R-hypermodule such that $m \in Rm$ for all $m \in M$ and μ a fuzzy subhypermodule of M. Then μ is a prime fuzzy subhypermodule of M if and only if*

(i) *μ_* is a prime subhypermodule of M, and*

(ii) *for all $r \in R$, $x \in M$ and $a, b \in [0, 1]$, if $a_r 1_R b_x \subseteq \mu$, then $b_x \subseteq \mu$ or $a_r \subseteq (\mu : 1_M)$.*

Proof. This proof is similar to the proof of Theorem 4.3.12. □

By comparing the characterizations of prime subhypermodules and prime fuzzy subhypermodules under the same conditions, we observe that the results are similar, by replacing r, m, R and M by a_r, b_m, 1_R and 1_M, respectively.

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