CHAPTER III

SEMINEAR-FIELDS

Definition 3.1. A seminear-ring $(K,+,.)$ is said to be a seminear-field iff there exists an elemelnt a in K such that $a^2 = a$ and $(K\setminus\{a\},.)$ is a group.

It is clear that any near-field is a seminear-field.

Example 3.2. $\mathbb{Q}^+ \cup \{0\}$ and $\mathbb{R}^+ \cup \{0\}$ with the usual addition and multiplication are seminear-fields.

Example 3.3. Let $(G,.)$ be a group with zero element ∞. We can define $+$ on G so that $(G,+,.)$ is a seminear-field by

(1) $x + y = \infty$ for all $x, y \in G$,

(2) $x + y = \infty$ if $x \neq y$ and $x + y = x$ if $x = y$ for all x, y.

Example 3.4. Let $(G,.)$ be a group and a be a symbol not representing an element of G. Let $G^* = G \cup \{a\}$. We can define $+$ on G^* and extend $.$ to G^* by

(1) $a.x = x.a = a$ and $x + y = x$ for all $x, y \in G^*$,

(2) $a.x = x.a = a$ and $x + y = y$ for all $x, y \in G^*$,

(3) $a.x = x.a = x$ and $x + y = x$ for all $x, y \in G^*$ and

(4) $a.x = x.a = x$ and $x + y = y$ for all $x, y \in G^*$,

then $(G^*,+,.)$ is a seminear-field.

Example 3.5. Let D be a division seminear-ring. Let a be a symbol not representing an element of D. We can extend $+$ and $.$ to $D^* = D \cup \{a\}$ by

(1) $a.x = x.a = a$ and $a + x = x + a = x$ for all $x \in D^*$,

(2) $a.x = x.a = a$ and $a + x = x + a = a$ for all $x \in D^*$ and

(3) $a.x = x.a = x$ and $x + a = x + 1$, $a + x = 1 + x$ for all $x \in D^*$.

Then $(D^*,+,.)$ is a seminear-field.

Example 3.6. Let $K = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a,\ c \in Q^+,\ b \in Q \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ with the usual addition and multiplication. Then $(K,+,.)$ is a seminear-field.

Remark. Let $K = \{a, x\}$. Define $.$ and $+$ on K by $a.x = x.a = a$, $a.a = a$, $x.x = x$, and $a + a = x + a = a$, $x + x = a + x = x$. Then $(K,+,.)$ is a seminear-field. In this case there does not exist a unique element a in K such that $(K \setminus \{a\},.)$ is a group and $a^2 = a$. However, if $\|K\| > 2$, we do get the uniqueness as the following shows.

Theorem 3.7. Let $(K,+,.)$ be a seminear-field of order > 2. Let $a \in K$ be such that $a^2 = a$ and $(K \setminus \{a\},.)$ is a group. If there is an element b in K such that $b^2 = b$ and $(K \setminus \{b\},.)$ is a group, then $b = a$.

Proof. Let 1 denote the identity of $K \setminus \{a\}$. Suppose $b \neq a$. Since $a^2 = a \in K \setminus \{b\}$, a is the identity of $(K \setminus \{b\},.)$. Since $b^2 = b$ is in $K \setminus \{a\}$, $b = 1$. Hence $(K \setminus \{1\},.)$ is a group with the identity a. Let $x \in K \setminus \{1, a\}$. Then there exists a $y \in K \setminus \{1\}$ such that $xy = a$. If $y = a$, then $x = xa = xy = a$. Thus $x = a$, a contradiction. If $y \neq a$, then we have that $x \neq a$ and $y \neq a$ but $xy = a$ which contradicts $(K \setminus \{a\},.)$ is a group. Hence $b = a$.#

Definition 3.8. Let $(K,+,.)$ be a seminear-field and $L \subseteq K$. L is a subseminear-field of K iff $(L,+,.)$ is a seminear-field.

Theorem 3.9. Let K be a seminear-field. Then there exists a smallest subseminear-field contained in K.

Proof. Let $a \in K$ be such that $a^2 = a$ and $(K \setminus \{a\}, \cdot)$ is a group. Let $1 \in K \setminus \{a\}$ be the multiplicative identity. Then a and 1 are the only two idempotents in K. Let L be a subseminear-field of K. Then $L \subseteq K$ and L has exactly two idempotents. Hence a, $1 \in L$.

Case 1. K contains a subseminear-field L of order 2. Then $L = \{a, 1\}$. Clearly, L is the smallest subseminear-field of K. So done.

Case 2. Every subseminear-field of K has order > 2. Let L be a subseminear-field of K. There exists an $a_1 \in L$ such that $(L \setminus \{a_1\}, \cdot)$ is a group and $a_1^2 = a_1$. Let e be the identity of $(L \setminus \{a_1\}, \cdot)$. Claim that $a_1 = a$ and $e = 1$. Since $e^2 = e$, either $e = a$ or $e = 1$. If $e = a$, then $a_1 = 1$ (since a_1 is an idempotent). Let $x \in L \setminus \{a, 1\}$. Then there exists a $y \in L \setminus \{1\}$ such that $xy = a$. If $y = a$, then $x = xe = xa = xy = a$. Thus $x = a$, a contradiction. If $y \neq a$, then we have that $x \neq a$, $y \neq a$ and $xy = a$ which contradicts $(K \setminus \{a\}, \cdot)$ is a group. Hence $e = 1$, so $a_1 = a$.

Let $\{L_\alpha\}_{\alpha \in I}$ be the set of all subseminear-fields of K. Then $(L_\alpha \setminus \{a\}, \cdot)$ is a group for all $\alpha \in I$. Let $M = \bigcap_{\alpha \in I} L_\alpha$. Clearly, M is a subseminear-ring of K and 1, $a \in M$. $M \setminus \{a\} = (\bigcap_{\alpha \in I} L_\alpha) \setminus \{a\} = \bigcap_{\alpha \in I} (L_\alpha \setminus \{a\})$ is an intersection of subgroups of $(K \setminus \{a\}, \cdot)$. Thus $(M \setminus \{a\}, \cdot)$ is a group. Hence M is a subseminear-field of K. Clearly, M is the smallest subseminear-field of K. #

Definition 3. 10. Let K be a seminear-field. Then the prime seminear-field of K is the smallest subseminear-field of K (which must exist by Theorem 3.9).

<u>Theorem 3.11.</u> Let $(K,+,.)$ be a seminear-field and a an element in K such that $a^2 = a$ and $(K\setminus\{a\},.)$ is a group. Then ($a.x = a$ for all $x \in K$ or $a.x = x$ for all $x \in K$) and ($x.a = a$ for all $x \in K$ or $x.a = x$ for all $x \in K$).

<u>Proof.</u> Consider a.1.

<u>Case 1.</u> $a.1 = a$. Claim that $a.x = a$ for all $x \in K$. Let $x \in K\setminus\{a\}$. Suppose $a.x \neq a$. Thus $a.x \in K\setminus\{a\}$ which is a group, so there exists a $y \in K\setminus\{a\}$ such that $(a.x).y = 1$. Thus $a = a.1 = a.((a.x).y) = a.(a.(x.y)) = (a.a).(x.y) = a.(x.y) = (a.x).y = 1$. Thus $a = 1$, a contradiction. Hence $a.x = a$ for all $x \in K$.

<u>Case 2.</u> $a.1 \neq a$. Thus $(a.1)^2 = (a.1).(a.1) = a.(1.(a.1)) = a.(a.1) = (a.a).1 = a.1$, so $a.1 = 1$. Let $x \in K\setminus\{a\}$. Thus $a.x = a.(1.x) = (a.1).x = 1.x = x$. Hence $a.x = x$ for all $x \in K$.

Therefore $a.x = a$ for all $x \in K$ or $a.x = x$ for all $x \in K$. Similarly, we can show that $x.a = a$ for all $x \in K$ or $x.a = x$ for all $x \in K$. #

From Theorem 3.11, we see that there are four types of seminear-fields:

(1) Seminear-fields with $ax = xa = a$ for all x.

(2) Seminear-fields with $ax = xa = x$ for all x.

(3) Seminear-fields with $ax = a$ and $xa = x$ for all x.

(4) Seminear-fields with $ax = x$ and $xa = a$ for all x.

We call (1) category I seminear-fields, (2) category II seminear-fields, (3) category III seminear-fields and (4) category IV seminear-fields.*

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*See page 57.

Note that Example 3.2, Example 3.3, Example 3.4(1), Example 3.4(2), Example 3.5(1) and Example 3.5(2) are category I seminear-fields, Example 3.4(3), Example 3.4(4) and Example 3.5(3) are category II seminear-fields. In Example 3.4(1) if $|G| = 1$, define $\cdot$ by $a \cdot x = a$ and $x \cdot a = x$ for all $x \in G^*$, then $(G^*, +, \cdot)$ is a category III seminear-field and if we define $\cdot$ by $a \cdot x = x$ and $x \cdot a = a$ for all $x \in G^*$, then $(G^*, +, \cdot)$ is a category IV seminear-field.

<u>Theorem 3.12.</u> If K is a category III or a category IV seminear-field then $\|K\| = 2$.

<u>Proof.</u> Let K be a category III seminear-field. Thus $ax = a$ and $xa = x$ for all $x \in K$. Suppose $\|K\| > 2$. Let $x \in K \setminus \{a, 1\}$. Then $x^2 = xx = (xa)x = x(ax) = xa = x$. Thus $x = 1$ or a, a contradiction. Hence $\|K\| = 2$.

Let K be a category IV seminear-field. Thus $ax = x$ and $xa = a$ for all $x \in K$. Suppose $\|K\| > 2$. Let $x \in K \setminus \{a, 1\}$. Then $x^2 = xx = x(ax) = (xa)x = ax = x$. Thus $x = 1$ or a, a contradiction. #

From Theorem 3.12, we can easily find all category III and category IV seminear-fields. Since $\|K\| = 2$, $1 + 1 = 1$ or $a + a = a$. For category III seminear-fields we have 12 cases to consider. They are:

(1)

+	a	1
a	a	a
1	a	a

(2)

+	a	1
a	a	a
1	a	1

(3)

+	a	1
a	a	a
1	1	a

(4)

+	a	1
a	a	1
1	a	a

(5)

+	a	1
a	a	a
1	1	1

(6)

+	a	1
a	a	1
1	a	1

(7)

+	a	1
a	a	1
1	1	a

(8)

+	a	1
a	a	1
1	1	1

(9)

+	a	1
a	1	1
1	1	1

(10)

+	a	1
a	1	1
1	a	1

(11)

+	a	1
a	1	a
1	a	1

(12)

+	a	1
a	1	a
1	1	1

(3), (4), (10) and (12) cannot be seminear-fields since:

For (3), $1 + (a + 1) = 1 + a = 1$ but $(1 + a) + 1 = 1 + 1 = a$,

for (4), $1 + (a + 1) = 1 + 1 = a$ but $(1 + a) + 1 = a + 1 = 1$,

for (10), $a + (1 + a) = a + a = 1$ but $(a + 1) + a = 1 + a = a$,

for (12), $a + (1 + a) = a + 1 = a$ but $(a + 1) + a = a + a = 1$.

To show (1), (2), (5), (6), (7), (8), (9) and (11) are seminear-fields, let $x, y, z \in K$.

For (5), $(x + y) + z = x + z = x + (y + z)$ and $(x + y)z = xz = xz + yz$, for (6), $(x + y) + z = z = y + z = x + (y + z)$ and $(x + y)z = yz = xz + yz$, for (1), $(x + y) + z = a = x + (y + z)$ and $(x + y)z = az = a = xz + yz$, for (9), $(x + y) + z = 1 = x + (y + z)$ and $(x + y)z = 1z = 1 = xz + yz$.

For (2), $(a + a) + a = a + a = a + (a + a)$,

$(a + a) + 1 = a + 1 = a + a = a + (a + 1)$,

$(a + 1) + a = a + a = a + (1 + a)$,

$(1 + a) + a = a + a = 1 + a = 1 + (a + a)$,

$(1 + 1) + a = 1 + a = 1 + (1 + a)$,

$(1 + a) + 1 = a + 1 = 1 + a = 1 + (a + 1)$,

$(a + 1) + 1 = a + 1 = a + (1 + 1)$

$(1 + 1) + 1 = 1 + 1 = 1 + (1 + 1)$

$(a + a)a = aa = a = a + a = aa + aa$,

$(a + a)1 = a1 = a = a + a = a1 + a1$,

$(a + 1)a = aa = a = a + 1 = aa + 1a$,

$(1 + a)a = aa = a = 1 + a = 1a + aa$,

$(1 + 1)a = 1a = 1 = 1 + 1 = 1a + 1a,$

$(1 + a)1 = a1 = a = 1 + a = 11 + a1,$

$(a + 1)1 = a1 = a = a + 1 = a1 + 11,$

$(1 + 1)1 = 11 = 1 = 1 + 1 = 11 + 11.$

For (7), $(a + a) + a = a + a = a + (a + a),$

$(a + a) + 1 = a + 1 = a + (a + 1),$

$(a + 1) + a = 1 + a = a + 1 = a + (1 + a),$

$(1 + a) + a = 1 + a = 1 + (a + a),$

$(1 + 1) + a = a + a = 1 + 1 = 1 + (1 + a),$

$(1 + a) + 1 = 1 + 1 = 1 + (a + 1),$

$(a + 1) + 1 = 1 + 1 = a + a = a + (1 + 1),$

$(1 + 1) + 1 = a + 1 = 1 + a = 1 + (1 + 1),$

$(a + a)a = aa = a = a + a = aa + aa,$

$(a + a)1 = a1 = a = a + a = a1 + a1,$

$(a + 1)a = 1a = 1 = a + 1 = aa + 1a,$

$(1 + a)a = 1a = 1 = 1 + a = 1a + aa,$

$(1 + 1)a = aa = a = 1 + 1 = 1a + 1a,$

$(1 + a)1 = 11 = 1 = 1 + a = 11 + a1,$

$(a + 1)1 = 11 = 1 = a + 1 = a1 + 11,$

$(1 + 1)1 = a1 = a = 1 + 1 = 11 + 11.$

For (8), $(a + a) + a = a + a = a + (a + a),$

$(a + a) + 1 = a + 1 = a + (a + 1),$

$(a + 1) + a = 1 + a = a + 1 = a + (1 + a),$

$(1 + a) + a = 1 + a = 1 + (a + a),$

$(1 + 1) + a = 1 + a = 1 + 1 = 1 + (1 + a),$

$(1 + a) + 1 = 1 + 1 = 1 + (a + 1),$

$(a + 1) + 1 = 1 + 1 = a + 1 = a + (1 + 1),$

$(1 + 1) + 1 = 1 + 1 = 1 + (1 + 1),$

$(a + a)a = aa = a = a + a = aa + aa,$

$(a + a)1 = a1 = a = a + a = a1 + a1,$

$(a + 1)a = 1a = 1 = a + 1 = aa + 1a,$

$(1 + a)a = 1a = 1 = 1 + a = 1a + aa,$

$(1 + 1)a = 1a = 1 = 1 + 1 = 1a + 1a,$

$(1 + a)1 = 11 = 1 = 1 + a = 11 + a1,$

$(a + 1)1 = 11 = 1 = a + 1 = a1 + 11,$

$(1 + 1)1 = 11 = 1 = 1 + 1 = 11 + 11,$

For (11), $(a + a) + a = 1 + a = a + 1 = a + (a + a),$

$(a + a) + 1 = 1 + 1 = a + a = a + (a + 1),$

$(a + 1) + a = a + a = a + (1 + a),$

$(1 + a) + a = a + a = 1 + 1 = 1 + (a + a),$

$(1 + 1) + a = 1 + a = 1 + (1 + a),$

$(1 + a) + 1 = a + 1 = 1 + a = 1 + (a + 1),$

$(a + 1) + 1 = a + 1 = a + (1 + 1),$

$(1 + 1) + 1 = 1 + 1 = 1 + (1 + 1),$

$(a + a)a = 1a = 1 = a + a = aa + aa,$

$(a + a)1 = 11 = 1 = a + a = a1 + a1,$

$(a + 1)a = aa = a = a + 1 = aa + 1a,$

$(1 + a)a = aa = a = 1 + a = 1a + aa,$

$(1 + 1)a = 1a = 1 = 1 + 1 = 1a + 1a,$

$(1 + a)1 = a1 = a = 1 + a = 11 + a1,$

$(a + 1)1 = a1 = a = a + 1 = a1 + 11,$

$(1 + 1)1 = 11 = 1 = 1 + 1 = 11 + 11.$

By defining $f(a) = 1$ and $f(1) = a$, (1) $\cong$ (9), (2) $\cong$ (8) and (7) $\cong$ (11). Therefore, up to isomorphism, there are 5 category III seminear-fields.

For category IV seminear-fields, we have $ax = x$ and $xa = a$ for all x. Thus $a + a = 1a + 1a = (1 + 1)a = a$ and $1 + 1 = a1 + a1 =$

$(a + a)1 = a1 = 1$. Thus $a + a = a$ and $1 + 1 = 1$. Hence they are four cases to consider. They are

(1)

+	a	1
a	a	a
1	a	1

(2)

+	a	1
a	a	a
1	1	1

(3)

+	a	1
a	a	1
1	a	1

(4)

+	a	1
a	a	1
1	1	1

Claim that (1), (2), (3) and (4) are all seminear-fields. Let x, y, z be in K. For (2), $(x + y) + z = x + z = x = x + (y + z)$ and $(x + y)z = xz = xz + yz$. For (3), $(x + y) + z = y + z = x + (y + z)$ and $(x + y)z = yz = xz + yz$.

For (1), $(a + a) + a = a + a = a + (a + a)$,

$(a + a) + 1 = a + 1 = a + a = a + (a + 1)$,

$(a + 1) + a = a + a = a + (1 + a)$,

$(1 + a) + a = a + a = 1 + a = 1 + (a + a)$,

$(1 + 1) + a = 1 + a = 1 + (1 + a)$,

$(1 + a) + 1 = a + 1 = 1 + a = 1 + (a + 1)$,

$(a + 1) + 1 = a + 1 = a + (1 + 1)$,

$(1 + 1) + 1 = 1 + 1 = 1 + (1 + 1)$.

$(a + a)a = aa = a = a + a = aa + aa$,

$(a + a)1 = a1 = 1 = 1 + 1 = a1 + a1$,

$(a + 1)a = aa = a = a + a = aa + 1a$,

$(1 + a)a = aa = a = a + a = 1a + aa$,

$(1 + 1)a = 1a = a = a + a = 1a + 1a$,

$(1 + a)1 = a1 = 1 = 1 + 1 = 11 + a1$,

$(a + 1)1 = a1 = 1 = 1 + 1 = a1 + 11$,

$(1 + 1)1 = 11 = 1 = 1 + 1 = 11 + 11$.

For (4), $(a + a) + a = a + a = a + (a + a)$,

$(a + a) + 1 = a + 1 = a + (a + 1)$,

$$(a + 1) + a = 1 + a = a + 1 = a + (1 + a),$$

$$(1 + a) + a = 1 + a = 1 + (a + a),$$

$$(1 + 1) + a = 1 + a = 1 + 1 = 1 + (1 + a),$$

$$(1 + a) + 1 = 1 + 1 = 1 + (a + 1),$$

$$(a + 1) + 1 = 1 + 1 = a + 1 = a + (1 + 1),$$

$$(1 + 1) + 1 = 1 + 1 = 1 + (1 + 1),$$

$$(a + a)a = aa = a = a + a = aa + aa,$$

$$(a + a)1 = a1 = 1 = 1 + 1 = a1 + a1,$$

$$(a + 1)a = 1a = a = a + a = aa + 1a,$$

$$(1 + a)a = 1a = a = a + a = 1a + aa,$$

$$(1 + 1)a = 1a = a = a + a = 1a + 1a,$$

$$(1 + a)1 = 11 = 1 = 1 + 1 = 11 + a1,$$

$$(a + 1)1 = 11 = 1 = 1 + 1 = a1 + 11,$$

$$(1 + 1)1 = 11 = 1 = 1 + 1 = 11 + 11.$$

By defining $f(a) = 1$ and $f(1) = a$, we have that $(1) \cong (4)$.

Therefore, up to isomorphism, there are three category IV seminear-fields.

From now on we shall study category I and category II seminear-fields. First we shall study category I seminear-fields and from now on the word "seminear-field" will mean a category I seminear-field. If we wish to study category II seminear-fields, we shall say "category II seminear-fields".

Remark. Note that if K is a category I seminear-field, then $K \times K$ is never a seminear-field since $(a,1)(1,a) = (a,a)$ so $K \times K \setminus \{(a,a)\}$ is not a group under multiplication.

<u>Theorem 3.13</u>. Let K be a seminear-field and $a \in K$ be such that $a^2 = a$ and $(K \sim \{a\}, \cdot)$ is a group. Then either $a + x = a$ for all $x \in K$ or $a + x = x$ for all $x \in K$ and either $x + a = a$ for all $x \in K$ or $x + a = x$ for all $x \in K$.

<u>Proof</u>. First we shall show that $a + a = a$. $a + a = aa + aa = (a + a)a = a$. Now consider $a + x$.

<u>Case 1</u>. There exists an $x \in K \sim \{a\}$ such that $a + x = a$. Let $u \in K$. Then $a + u = ax^{-1}u + xx^{-1}u = (a + x)x^{-1}u = ax^{-1}u = a$. Thus $a + u = a$ for all $u \in K$.

<u>Case 2</u>. $a + x \neq a$ for all $x \in K \sim \{a\}$. Then $a + 1 \neq a$. Let $y = a + 1$. Then $y \neq a$ and $a + y = a + (a + 1) = (a + a) + 1 = a + 1 = y$. Let $u \in K$. Then $a + u = ay^{-1}u + yy^{-1}u = (a + y)y^{-1}u = yy^{-1}u = u$. Thus $a + u = u$ for all $u \in K$.

Therefore either $a + x = a$ for all $x \in K$ or $a + x = x$ for all $x \in K$. Similarly, we can show that either $x + a = a$ for all $x \in K$ or $x + a = x$ for all $x \in K$.#

Theorem 3.13 indicates that there are four types of seminear-fields.

(1) $a + x = x + a = x$ for all x. In this case, a behaves as an additive identity (which is usually denoted by 0) we call this type a <u>seminear-field of zero type</u> or a <u>0-seminear-field</u>. We shall denote the zero element of this type by 0.

Note that Example 3.2 and example 3.6 are 0-seminear-fields.

(2) $a + x = x + a = a$ for all x. In this case, we call a <u>seminear-field of infinity type</u> or <u>∞-seminear-field</u> and denote the zero element of this type by ∞.

Note that Example 3.5(2) is an ∞-seminear-field.

(3) $a + x = x$ and $x + a = a$ for all x. Then for all x, y

$$x + y = x + (a + y) = (x + a) + y = a + y = y.$$

Thus $(K,+)$ is a right zero semigroup, so we call this type a right zero seminear-field.

Note that Example 3.4(2) is a right zero seminear-field.

(4) $a + x = a$ and $x + a = x$ for all x. Then for all x, y

$$x + y = (x + a) + y = x + (a + y) = x + a = x.$$

Thus $(K,+)$ is a left zero semigroup, so we call this type a left zero seminear-field.

Note that Example 3.4(1) is a left zero seminear-field.

Remark. Note that right zero and left zero seminear-fields are also left distributive and they all come from division seminear-rings by adjoining a multiplicative zero.

Theorem 3.14. Let K be a 0-seminear-field. Then either every nonzero element of K has an additive inverse (in which case K is a near-field) or no nonzero element of K has an additive inverse.

Proof. Suppose that there exists an $x \in K - \{0\}$ such that x has an additive inverse y. Thus $x + y = y + x = 0$. Let $z \in K$. Then

$$z + yx^{-1}z = xx^{-1}z + yx^{-1}z = (x + y)x^{-1}z = 0x^{-1}z = 0 \text{ and}$$
$$yx^{-1}z + z = yx^{-1}z + xx^{-1}z = (y + x)x^{-1}z = 0x^{-1}z = 0.$$

Thus z has an additive inverse. Hence we have the theorem. #

Definition 3.15. Let S be a seminear-ring with ∞. Let $y \in S$. Then $z \in S$ is said to be a right complement of y iff $y + z = \infty$. A left complement of y is similarly defined. A complement of y is an element of S which is both a right and a left complement of y.

<u>Definition 3.16</u>. Let S be a seminear-ring with ∞. Let $y \in S$. Then y is said to be <u>right limited</u> iff the only right complement of y is ∞. <u>Left limited</u> is similarly defined. y is <u>limited</u> iff it is both right and left limited. If every noninfinity element of S is right limited then S is <u>right limited</u>. <u>Left limited</u> and <u>limited</u> seminear-rings are similarly defined.

<u>Definition 3.17</u>. Let K be an ∞-seminear-field and let $x \in K$. The <u>left core of x</u>, denoted by LCor(x), $= \{y \in K \mid y + x = \infty\}$. The <u>right core of x</u>, denoted by RCor(x), $= \{y \in K \mid x + y = \infty\}$. The <u>core of x</u>, denoted by Cor(x), $=$ LCor(x) $\cap$ RCor(x).

<u>Theorem 3.18</u>. Let K be an ∞-seminear-field. Then

(1) $\infty \in$ Cor(x) for all $x \in K$.

(2) For all $x \in K$ ($y \in$ LCor(x) and $z \in K$ imply that $z + y$ is in LCor(x)) and ($y \in$ RCor(x) and $z \in K$ imply that $y + z \in$ RCor(x)).

(3) For all x, $y \in K \sim \{\infty\}$ ($y \in$ LCor(x) iff $yx^{-1} \in$ LCor(1) and $xy^{-1} \in$ RCor(1)) and ($y \in$ RCor(x) iff $yx^{-1} \in$ RCor(1) and $xy^{-1} \in$ LCor(1)). (Therefore $y \in$ Cor(x) iff $yx^{-1} \in$ Cor(1) and $xy^{-1} \in$ Cor(1).)

(4) For all x, $y \in K$ ($x \in$ LCor(y) iff $y \in$ RCor(x)) (Therefore for all x, $y \in K$ $x \in$ Cor(y) iff $y \in$ Cor(x).)

(5) For all $x \in K \sim \{\infty\}$ LCor(x) = LCor(1).x, RCor(x) = RCor(1).x and Cor(x) = Cor(1).x.

(6) For all $x \in K$ ($x \in$ LCor(y) implies $xz \in$ LCor(yz) for all z in K) and ($x \in$ RCor(y) implies $xz \in$ RCor(yz) for all z in K). (Hence for all $x \in K$ ($x \in$ Cor(y) implies $xz \in$ Cor(yz) for all $z \in K$.) The converse is true for $z \in K \sim \{\infty\}$.

(7) For all x, y, $z \in K$ ($x \in$ LCor(y + z) iff $x + y \in$ LCor(z)) and ($x \in$ RCor(y + z) iff $z + x \in$ RCor(y)).

<u>Proof.</u> (1) Since $x + \infty = \infty + x = \infty$ for all $x \in K$, $\infty \in Cor(x)$ for all $x \in K$.

(2) Let $x \in K$. Let $y \in LCor(x)$ and $z \in K$. Then $y + x = \infty$ and so $(z + y) + x = z + (y + x) = z + \infty = \infty$. Hence $z + y \in LCor(x)$. Thus for all $x \in K$ $y \in LCor(x)$ and $z \in K$ imply that $z + y \in LCor(x)$. Similarly, we can prove that for all $x \in K$ $y \in RCor(x)$ and $z \in K$ imply that $y + z \in RCor(x)$.

(3) Let $x, y \in K - \{\infty\}$. Assume that $y \in LCor(x)$. Thus $y + x = \infty$. Then $yx^{-1} + 1 = yx^{-1} + xx^{-1} = (y + x)x^{-1} = \infty x^{-1} = \infty$ and $1 + xy^{-1} = yy^{-1} + xy^{-1} = (y + x)y^{-1} = \infty y^{-1} = \infty$. Thus $yx^{-1} \in LCor(1)$ and $xy^{-1} \in RCor(1)$.

Conversely, assume that $yx^{-1} \in Lcor(1)$ and $xy^{-1} \in RCor(1)$. Then $1 + xy^{-1} = yx^{-1} + 1 = \infty$. Thus $y + x = (1 + xy^{-1})y = \infty y = \infty$, so $y \in LCor(x)$.

Therefore $y \in LCor(x)$ iff $yx^{-1} \in LCor(1)$ and $xy^{-1} \in RCor(1)$.

By similarly proof, we have that $y \in RCor(x)$ iff $yx^{-1} \in RCor(1)$ and $xy^{-1} \in LCor(1)$.

(4) Let $x, y \in K$. Thus $x \in LCor(y) \Leftrightarrow x + y = \infty \Leftrightarrow y \in RCor(x)$.

(5) Let $x \in K - \{\infty\}$. To show $LCor(x) \subseteq LCor(1).x$, let $y \in LCor(x)$. By (3), $yx^{-1} \in LCor(1)$. Thus $y = (yx^{-1})x \in LCor(1).x$. Conversely, let $z \in LCor(1)$. Thus $z + 1 = \infty$. Then $zx + x = (z + 1)x = \infty x = \infty$, so $zx \in LCor(x)$. Hence $LCor(x) = LCor(1).x$.

By similarly proof, $RCor(x) = RCor(1).x$ and $Cor(x) = Cor(1).x$.

(6) Let $x \in K$. Let $y \in K$ be such that $x \in LCor(y)$ and let $z \in K$. Thus $x + y = \infty$, so $xz + yz = (x + y)z = \infty z = \infty$. Thus $xz \in LCor(yz)$. By similarly proof, $x \in RCor(y)$ and $z \in K$ imply

$xz \in RCor(yz)$.

Conversely, assume that $x, y \in K$, $z \in K \smallsetminus \{\infty\}$ and $xz \in LCor(yz)$. Thus $xz + yz = \infty$, so $x + y = (x + y)zz^{-1} = (xz + yz)z^{-1} = \infty z^{-1} = \infty$. Thus $x \in LCor(y)$. By similarly proof, for all $x, y \in K$, $z \in K \smallsetminus \{\infty\}$ $xz \in RCor(yz)$ implies $x \in RCor(y)$.

(7) Let $x, y, z \in K$. $x \in LCor(y + z) \Leftrightarrow x + (y + z) = \infty$ $\Leftrightarrow (x + y) + z = \infty \Leftrightarrow x + y \in LCor(z)$.

$x \in RCor(y + z) \Leftrightarrow (y + z) + x = \infty \Leftrightarrow y + (z + x) = \infty \Leftrightarrow$ $z + x \in RCor(y)$.#

Theorem 3.19. Let K be an ∞-seminear-field and let $x, y \in K \smallsetminus \{\infty\}$. Then

(1) The cardinality of $LCor(x)$ equals the cardinality of $LCor(y)$ and each one is a right multiplicative translate of the other.

(2) The cardinality of $RCor(x)$ equals the cardinality of $RCor(y)$ and each one is a right multiplicative translate of the other.

(3) The cardinality of $Cor(x)$ equals the cardinality of $Cor(y)$ and each one is a right multiplicative translate of the other.

Proof. (1) For $z \in LCor(x)$, by Theorem 3.18(5), there is a $u \in LCor(1)$ such that $z = ux$. Define $f: LCor(x) \to LCor(y)$ by $f(z) = uy$. By Theorem 3.18(5), $uy \in LCor(y)$. To show that f is well-defined, let $z_1 = z_2 \in LCor(x)$. Let $u_1, u_2 \in LCor(1)$ be such that $z_1 = u_1x$ and $z_2 = u_2x$. Thus $u_1x = u_2x$. Since $x \neq \infty$, so $u_1 = u_2$. Thus $u_1y = u_2y$. To show f is one-to-one, let $z_1, z_2 \in LCor(x)$ be such that $f(z_1) = f(z_2)$. Let $u_1, u_2 \in LCor(1)$ be such that $z_1 = u_1x$ and $z_2 = u_2x$. Thus $u_1y = u_2y$. Since $y \neq \infty$, so $u_1 = u_2$. Thus $z_1 = z_2$. To show that f is onto, let $w \in LCor(y)$. Let $v \in LCor(1)$ be such that $w = vy$. Then $vx \in LCor(x)$. Thus $f(vx) = vy = w$. Therefore f is one-to-one and onto. To show $LCor(x)$ is a right multiplicative translate of $LCor(y)$, let $z \in LCor(x)$.

By Theorem 3.18(3), $zx^{-1} \in LCor(1)$. By Theorem 3.18(5), $zx^{-1}y \in LCor(y)$. Thus $z = (zx^{-1}y)y^{-1}x \in LCor(y).y^{-1}x$, so $LCor(x) \subseteq LCor(y).y^{-1}x$. Now let $w \in LCor(y)$. By Theorem 3.18(3), $wy^{-1} \in LCor(1)$. By Theorem 3.18(5), $wy^{-1}x \in LCor(x)$. Thus $LCor(y).y^{-1}x \subseteq LCor(x)$. Therefore $LCor(x) = LCor(y).y^{-1}x$.

By similarly proof, we have (2) and (3).#

Corollary. If one noninfinity element of an ∞-seminear-field is left limited (right limited, limited), then all noninfinity elements are left limited (right limited, limited).

Proof. Follows from an argument similar to the one given in Theorem 3.19.#

Definition 3.20. Let K be a seminear-field and a the zero of $(K, \cdot)$. Define $A_L = \{ x \in K \mid x + y = a \text{ for all } y \in K \}$,

$A_R = \{ x \in K \mid y + x = a \text{ for all } y \in K \}$ and $A = A_L \cap A_R$.

Theorem 3.21. Let K be a seminear-field. Then

(1) If K is a 0-seminear-field, then $A = A_L = A_R = \emptyset$.

(2) If K is an ∞-seminear-field, then $A = \{\infty\}$ or $A = K$, $A_L = \{\infty\}$ or $A_L = K$ and $A_R = \{\infty\}$ or $A_R = K$.

(3) If K is a right zero seminear-field, then $A = A_L = \emptyset$ and $A_R = \{a\}$.

(4) If K is a left zero seminear-field, then $A = A_R = \emptyset$ and $A_L = \{a\}$.

Proof.

(1) Let K be a 0-seminear-field. Thus $x + 0 = 0 + x = x$ for all $x \in K$. Suppose that $A_L \neq \emptyset$. Thus there exists an $x \in K$ such that

$x + y = 0$ for all $y \in K$. Hence $x = x + 0 = 0$. Thus $0 + y = 0$ for all $y \in K$, so $y = 0 + y = 0$ for all $y \in K$. Hence $K = \{0\}$, a contradiction. Therefore $A_L = \emptyset$. Similarly, we can show that $A_R = \emptyset$. Thus $A = \emptyset$.

(2) Let K be an ∞-seminear-field. Thus $x + \infty = \infty + x = \infty$ for all $x \in K$. Thus $A \neq \emptyset$, $A_L \neq \emptyset$ and $A_R \neq \emptyset$. Assume that $A_L \neq \{\infty\}$. Thus there exists an $x \in K \smallsetminus \{\infty\}$ such that $x + y = \infty$ for all $y \in K$. Thus $1 + yx^{-1} = xx^{-1} + yx^{-1} = (x + y)x^{-1} = \infty x^{-1} = \infty$ for all $y \in K$. Let $z \in K \smallsetminus \{\infty\}$. Let $w \in K$. Then $1 + wz^{-1} = 1 + (wz^{-1}x)x^{-1} = \infty$. Thus $z + w = (1 + wz^{-1})z = \infty z = \infty$. Thus $z \in A_L$. Hence $A_L = K$. Therefore $A_L = \{\infty\}$ or $A_L = K$. By similarly proof, $A_R = \{\infty\}$ or $A_R = K$. Hence $A = \{\infty\}$ or $A = K$.

(3) Let K be a right zero seminear-field. Thus $x + y = y$ for all $x, y \in K$. Suppose $A_L \neq \emptyset$. Thus there exists an $x \in K$ such that $x + y = a$ for all $y \in K$. Then $y = x + y = a$ for all $y \in K$. Thus $K = \{a\}$, a contradiction. Hence $A_L = \emptyset$.

Since $x + a = a$ for all $x \in K$, $a \in A_R$. Let $x \in A_R$. Thus $y + x = a$ for all $y \in K$. Then $x = y + x = a$, so $x = a$. Hence $A_R = \{a\}$. Therefore $A = \emptyset$.

(4) Let K be a left zero seminear-field. Thus $x + y = x$ for all $x, y \in K$. Suppose $A_R \neq \emptyset$. Thus there exists an $x \in K$ such that $y + x = a$ for all $y \in K$. Thus $y = y + x = a$ for all $y \in K$. Thus $K = \{a\}$, a contradiction. Hence $A_R = \emptyset$.

Since $a + x = a$ for all $x \in K$, $A_L \neq \emptyset$. Let $x \in A_L$. Thus $x + y = a$ for all $y \in K$. Then $x = x + y = a$, so $x = a$. Hence $A_L = \{a\}$. Therefore $A = \emptyset$.#

<u>Theorem 3.22.</u> Let K be a 0-seminear-field. If there exists an a_0 in $K \smallsetminus \{0\}$ such that for all $x, y \in K$ ($x + a_0 = y + a_0$ implies $x = y$), then for all $z \in K$ we get that ($x + z = y + z$ implies $x = y$).

Proof. Let $z \in K$. Let $x, y \in K$ be such that $x + z = y + z$. If $z = 0$, then $x = y$. Assume $z \neq 0$. Then

$$xz^{-1}a_0 + a_0 = (x + z)z^{-1}a_0 = (y + z)z^{-1}a_0 = yz^{-1}a_0 + a_0.$$

By assumption, $xz^{-1}a_0 = yz^{-1}a_0$. Since $z^{-1}a_0 \neq 0$, $x = y$. #

In a 0-seminear-field of order 2 such that $1 + 1 = 1$, we have that $1 + 0 = 1$, $0 + 1 = 1$ and $1 + 1 = 1$. Thus 1 is an additive zero and $1 \neq 0$. This cannot occur in a seminear-field of order > 2.

Theorem 3.23. Let K be a seminear-field of order > 2. Let a be the zero of K. If K has an additive zero e, then $e = a$.

Proof. Suppose $e \neq a$. Since $x + e = e + x = e$ for all $x \in K$, $xe^{-1} + 1 = 1 + xe^{-1} = 1$ for all $x \in K$. Let $C = \{xe^{-1} \mid x \in K\}$. Thus $C = K$. Then 1 is also an additive zero. Hence $e = e + 1 = 1$. Let $x \in K \sim \{0, 1\}$. Thus $x + 1 = 1$, so $1 + x^{-1} = x^{-1}$. Since $1 + x^{-1} = 1$, $x^{-1} = 1$. Thus $x = 1$, a contradiction. Hence $e = a$. #

In an ∞-seminear-field of order 2 such that $1 + 1 = 1$, we have that 1 is an additive identity and $1 \neq \infty$. This cannot occur in a seminear-field of order > 2.

Theorem 3.24. Let K be a seminear-field of order > 2. Let a be the zero of K. If K has an additive identity e, then $e = a$.

Proof. Suppose $e \neq a$. Since $x + e = e + x = x$ for all $x \in K$, $xe^{-1} + 1 = 1 + xe^{-1} = xe^{-1}$ for all $x \in K$. Let $C = \{xe^{-1} \mid x \in K\}$. Thus $C = K$. Then 1 is also an additive identity. Hence $e = e + 1 = 1$. Let $x \in K \sim \{0, 1\}$. Thus $x + 1 = x$, so $1 + x^{-1} = 1$. Since $1 + x^{-1} = x^{-1}$, $x^{-1} = 1$. Thus $x = 1$, a contradiction. Hence $e = a$. #

Theorem 3.25. If K is a seminear-field such that + and . are equal, then $\|K\| = 2$.

Proof. Suppose $\|K\| > 2$. Let $x \in K \sim \{0, 1\}$. Thus $x^2 = xx = x + x = (1 + 1)x = (1.1)x = 1.x = x$. Thus x = 1 or 0, a contradiction. Hence $\|K\| = 2$.#

If a seminear-ring S of order > 1 contains an additive infinity ($x + \infty = \infty + x = \infty$ for all $x \in S$), then ∞ is not left and right additively cancellative. However, we can give the following definition.

Definition 3.26. Let S be a seminear-ring with additive infinity ∞. Then S is said to be infinity left additively cancellative (∞-L.A.C.) iff for all x, y, $z \in S$ ($x + y = x + z$ and $x \neq \infty$ imply that $y = z$). Infinity right additive cancellativity(∞-R.A.C.) and infinity additive cancellativity(∞-A.C.) are similarly defined.

If a seminear-ring S of order > 1 contains a multiplicative zero 0, then 0 is not left and right multiplicatively cancellative. However, we can give the following definition.

Definition 3.27. Let S be a seminear-ring with multiplicative zero 0. Then S is said to be zero left multiplicatively cancellative (0-L.M.C.) iff for all x, y, $z \in S$ ($xy = xz$ and $x \neq 0$ imply $y = z$). Zero right multiplicative cancellativity(0-R.M.C.) and zero multiplicative cancellativity(0-M.C.) are similarly defined.

Definition 3.28. Let S be a seminear-ring. Define

$B_L = \{x \in S \mid x \text{ is L.A.C.}\}$, $B_R = \{x \in S \mid x \text{ is R.A.C.}\}$, $B = B_L \cap B_R$,

$M_L = \{x \in S \mid x \text{ is L.M.C.}\}$, $M_R = \{x \in S \mid x \text{ is R.M.C.}\}$ and $M = M_L \cap M_R$.

Theorem 3.29. Let S be a seminear-ring. Then

(1) $B_L = \emptyset$ or B_L is an additive subsemigroup of S.

(2) $B_R = \emptyset$ or B_R is an additive subsemigroup of S.

(Therefore $B = \emptyset$ or B is an additive subsemigroup of S.)

(3) $M_L = \emptyset$ or M_L is a multiplicative subsemigroup of S.

(4) $M_R = \emptyset$ or M_R is a multiplicative subsemigroup of S.

(Therefore $M = \emptyset$ or M is a multiplicative subsemigroup of S.)

Proof.

(1) Assume that $B_L \neq \emptyset$. Let $x, y \in B_L$ and $z_1, z_2 \in S$ be such that $(x + y) + z_1 = (x + y) + z_2$. Then $x + (y + z_1) = x + (y + z_2)$. Thus $y + z_1 = y + z_2$ because $x \in B_L$. Since $y \in B_L$, $z_1 = z_2$. Thus $x + y \in B_L$. Hence B_L is an additive subsemigroup of S. Therefore $B_L \neq \emptyset$ or B_L is an additive subsemigroup of S.

(2) By similarly proof as (1).

(3) Assume that $M_L \neq \emptyset$. Let $x, y \in M_L$ and $z_1, z_2 \in S$ be such that $(xy)z_1 = (xy)z_2$. Then $x(yz_1) = x(yz_2)$. Thus $yz_1 = yz_2$ because $x \in M_L$. Since $y \in M_L$, $z_1 = z_2$. Thus $xy \in M_L$. Hence M_L is a multiplicative subsemigroup of S. Therefore $M_L = \emptyset$ or M_L is a multiplicative subsemigroup of S.

(4) By similarly proof as (3).#

Theorem 3.30. Let K be a 0-seminear-field. Then B_L, B_R and B are right idels of $(K,\cdot)$.

Proof. Since $0 \in B = B_L \cap B_R$, $B \neq \emptyset$, $B_L \neq \emptyset$ and $B_R \neq \emptyset$. To show that B_L is a right ideal of $(K,\cdot)$, let $x \in K$ and $z \in B_L$. Let $z_1, z_2 \in K$ be such that $zx + z_1 = zx + z_2$. If $x = 0$, then $z_1 = z_2$. Assume that $x \neq 0$. Thus $z + z_1x^{-1} = (zx + z_1)x^{-1} = (zx + z_2)x^{-1} =$

$z + z_2x^{-1}$. Since $z \in B_L$, $z_1x^{-1} = z_2x^{-1}$. Thus $z_1 = z_2$, so $zx \in B_L$. Thus $B_LK \subseteq B_L$. Therefore B_L is a right ideal of $(K,.)$.

Similarly, we can show that B_R is a right ideal of $(K,.)$. Since $B = B_L \cap B_R$, B is a right ideal of $(K,.)$.#

Theorem 3.31. Let K be an ∞-seminear-field. Then

(1) If $x \in B_L$ and $y \in K \setminus \{\infty\}$, then $xy \in B_L$.

(2) If $x \in B_R$ and $y \in K \setminus \{\infty\}$, then $xy \in B_R$.

(Therefore if $x \in B$ and $y \in K \setminus \{\infty\}$, then $xy \in B$.)

Proof.

(1) Let $x \in B_L$ and $y \in K \setminus \{\infty\}$. Let $z_1, z_2 \in K$ be such that $xy + z_1 = xy + z_2$. Then $x + z_1y^{-1} = (xy + z_1)y^{-1} = (xy + z_2)y^{-1} = x + z_2y^{-1}$. Since $x \in B_L$, $z_1y^{-1} = z_2y^{-1}$. Thus $z_1 = z_2$, so $xy \in B_L$.

(2) By similarly proof as (1).#

Theorem 3.32. A seminear-field can be left additively cancellative only if it is a 0-seminear-field or a right zero seminear-field. Furthermore, a right zero seminear-field must be left additively cancellative.

Proof. Let K be an ∞-seminear-field. Then ∞ is not left additively cancellative. Hence K cannot be left additively cancellative.

Let K be a left zero seminear-field and a the zero of K. Then 1 is not left additively cancellative since $1 + 1 = 1 = 1 + a$ but $1 \neq a$.

Let K be a right zero seminear-field. Let $x \in K$ and $y, z \in K$ be such that $x + y = x + z$. Thus $y = x + y = x + z = z$, so x is left additively cancellative. Hence K is left additively cancellative.#

Theorem 3.33. A seminear-field can be right additively cancellative only if it is a 0-seminear-field or a left zero seminear-field. Furthermore, a left zero seminear-field must be right additively cancellative.

Proof. The proof is similar to Theorem 3.32.#

Corollary. A seminear-field can be additively cancellative only if it is a 0-seminear-field.

Proof. Follows directly from Theorem 3.32 and Theorem 3.33.#

Theorem 3.34. Let K be an ∞-seminear-field. Then

(1) If K is ∞-L.A.C., then K is right limited.

(2) If K is ∞-R.A.C., then K is left limited.

(Therefore if K is ∞-A.C., then K is limited.)

Proof.

(1) Let $x \in K - \{\infty\}$ and y be a right complement of x. Thus $x + y = \infty$. Then $x + y = x + \infty$, so $y = \infty$ since K is ∞-L.A.C. Hence x is right limited. Thus K is right limited.

(2) The proof is similar to (1).#

Theorem 3.35. Let K be a seminear-field. Then

(1) If one nonzero element of K is left additively cancellative, then all nonzero elements are left additively cancellative.

(2) If one nonzero element of K is right additively cancellative, then all nonzero elements are right additively cancellative.

(Therefore if one nonzero of K is additively cancellative, then all nonzero elements are additively cancellative.)

Proof. Let a be the zero of K.

(1) Let $x \in K \sim \{a\}$ be left additively cancellative. Let y be an element in $K \sim \{a\}$ and $z_1, z_2 \in K$ be such that $y + z_1 = y + z_2$. Then $1 + z_1 y^{-1} = (y + z_1)y^{-1} = (y + z_2)y^{-1} = 1 + z_2 y^{-1}$, so $x + z_1 y^{-1} x = x + z_2 y^{-1} x$. Since x is left additively cancellative, $z_1 y^{-1} x = z_2 y^{-1} x$. Thus $z_1 = z_2$, so y is left additively cancellative. Therefore all nonzero elements are left additively cancellative.

(2) The proof is simlar to (1).#

Remark. (1) Let K be a 0-seminear-field which is not a near-field. By theorem 3.15, $x + y \neq 0$ if $x, y \in K \sim \{0\}$. Thus $(K \sim \{0\}, +, \cdot)$ is a division seminear-ring.

(2) Let K be a limted ∞-seminear-field. Then $x + y \neq \infty$ if $x, y \in K \sim \{\infty\}$. So again $(K \sim \{\infty\}, +, \cdot)$ is a division seminear-ring.

Theorem 3.36. If S is a finite seminear-ring with multiplicative zero 0 which is 0-M.C., then S must be a seminear-field.

Proof. Since S is 0-M.C., $(S \sim \{0\}, \cdot)$ is a finite cancellative semigroup. Thus $(S \sim \{0\}, \cdot)$ is a group by Theorem 1.18. Since 0 is a multiplicative zero of S, $(S, \cdot)$ is a group with zero. Hence S is a seminear-field.#

Theorem 3.37. Let K be a right zero seminear-field and K' the prime seminear-field of K, Then $K' \cong \{0, 1\}$ with the structure

$\cdot$	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	0	1

Proof. Since K is a right zero seminear-field, $x + y = y$ for all x, y $\in$ K. Thus $0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 1$ and $1 + 0 = 0$. where 0 is the zero of K and 1 is the identity of $(K \setminus \{0\}, \cdot)$. Thus we have the theorem.#

<u>Theorem 3.38</u>. Let K be a left zero seminear-field and K' the prime seminear-field of K. Then $K' \cong \{0, 1\}$ with the structure

$\cdot$	0	1
0	0	0
1	0	1

+	0	1
0	0	0
1	1	1

<u>Proof</u>. Since K is a left zero seminear-field, $x + y = x$ for all x, y $\in$ K. Thus $0 + 0 = 0$, $0 + 1 = 0$, $1 + 0 = 1$ and $1 + 1 = 1$ where 0 is the zero of K and 1 is the identity of $(K \setminus \{0\}, \cdot)$. Thus we have the theorem.#

<u>Theorem 3.39</u>. Let K be a finite 0-seminear-field and K' the prime seminear-field of K. Then $K' \cong \mathbb{Z}_p$, p a prime or $K' \cong \{0, 1\}$ with the structure

$\cdot$	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	1

<u>Proof</u>. By Theorem 3.14, K is a near-field or no nonzero of K has an additive inverse.

<u>Case. No nonzero of K has an additive inverse</u>. Thus $(K - \{0\}, +)$ is a finite semigroup, so there exists an $x \in K - \{0\}$ such that $x + x = x$. Thus $1 + 1 = 1$. Thus $K' \cong \{0, 1\}$ with the structure

$\cdot$	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	1

<u>Case K is a near-field.</u> Define $f: \mathbb{Z} \to K$ by

$$f(n) = \begin{cases} 1 + \dots + 1 \text{ (n times)} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ (-1) + \dots + (-1) \text{ (|n| times)} & \text{if } n < 0. \end{cases}$$

Claim that $f(mn) = f(m)f(n)$ and $f(m + n) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}$.

<u>Case m > 0, n > 0.</u> Thus $mn > 0$ and $m + n > 0$. Then

$$f(m)f(n) = (1 \overset{m \text{ times}}{+ \dots +} 1)(1 \overset{n \text{ times}}{+ \dots +} 1)$$

$$= 1(1 \overset{n \text{ times}}{+ \dots +} 1) + \dots + 1(1 \overset{n \text{ times}}{+ \dots +} 1) \quad \text{(m times)}$$

$$= (1 \overset{n \text{ times}}{+ \dots +} 1) + \dots + (1 \overset{n \text{ times}}{+ \dots +} 1) \quad \text{(m times)}$$

$$= 1 + \dots + 1 \quad \text{(mn times)}$$

$$= f(mn)$$

$$f(m + n) = 1 + \dots + 1 \quad \text{(m + n times)}$$

$$= (1 \overset{m \text{ times}}{+ \dots +} 1) + (1 \overset{n \text{ times}}{+ \dots +} 1)$$

$$= f(m) + f(n)$$

<u>Case m > 0, n < 0.</u> Thus $mn < 0$ and $|n| = -n$.

(1) $m = |n|$. $f(m) + f(n) = (1 \overset{m \text{ times}}{+ \dots +} 1) + ((-1) \overset{|n| \text{ times}}{+ \dots +} (-1))$

$$= 0 = f(0) = f(m + (-|n|)) = f(m + n)$$

(2) $m < |n|$. $f(m + n) = f(m - |n|) = (-1) + \dots + (-1) \ (\ |m - |n||$ times)

$$f(m) + f(n) = (1 \overset{m \text{ times}}{+ \dots +} 1) + ((-1) \overset{|n| \text{ times}}{+ \dots +} (-1))$$

$$= (-1) + \dots + (-1) \quad (|n| - m \text{ times})$$

and $|m - |n|| = -(m - |n|) = |n| - m.$

(3) $m > |n|$. $f(m) + f(n) = (\overbrace{1 + \cdots + 1}^{m \text{ times}}) + (\overbrace{(-1) + \cdots + (-1)}^{n \text{ times}})$

$$= \overbrace{1 + \cdots + 1}^{m - |n| \text{ times}}$$

$$= f(m - |n|) = f(m + n)$$

Thus $f(m + n) = f(m) + f(n)$ and

$$f(m)f(n) = (\overbrace{1 + \cdots + 1}^{m \text{ times}})(\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}})$$

$$= 1(\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}}) + \cdots + 1(\overbrace{(-1) + \cdots (-1)}^{|n| \text{ times}}) \quad (m \text{ times})$$

$$= (\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}}) + \cdots + (\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}}) \quad (m \text{ times})$$

$$= (-1) + \cdots + (-1) \quad (m|n| \text{ times})$$

$$f(mn) = \overbrace{(-1) + \cdots + (-1)}^{|mn| \text{ times}}$$

and $|mn| = -mn = m(-n) = m|n|$. Thus $f(mn) = f(m)f(n)$.

<u>Case $m < 0$, $n < 0$.</u> Thus $|m||n| = (-m)(-n) = mn$ and $|m + n| = -(m + n)$

$= (-m) + (-n) = |m| + |n|$. Then

$f(m + n) = (-1) + \cdots + (-1) \quad (|m + n| \text{ times})$ and

$$f(m) + f(n) = (\overbrace{(-1) + \cdots + (-1)}^{|m| \text{ times}}) + (\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}})$$

$$= (-1) + \cdots + (-1) \quad (|m| + |n| \text{ times}).$$

$$f(m)f(n) = (\overbrace{(-1) + \cdots + (-1)}^{|m| \text{ times}})(\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}})$$

$$= (-1)(\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}}) + \cdots + (-1)(\overbrace{(-1) + \cdots + (-1)}^{|n| \text{ times}})$$

$$(|m| \text{ times})$$

$$= (-((-1) \overset{|n| \text{ times}}{+ \cdots +} (-1))) + \cdots + (-((-1) \overset{|n| \text{ times}}{+ \cdots +} (-1))) \quad (|m| \text{ times})$$

$$= (1 \overset{|n| \text{ times}}{+ \cdots +} 1) + \cdots + (1 \overset{|n| \text{ times}}{+ \cdots +} 1) \quad (|m| \text{ times})$$

$$= 1 + \cdots + 1 \quad (|m||n| \text{ times})$$

$$= 1 + \cdots + 1 \quad (mn \text{ times})$$

$$= f(mn).$$

<u>Case $m < 0, n > 0$.</u> Thus $|mn| = -mn = |m|n$. Then

$$f(m)f(n) = ((-1) \overset{|m| \text{ times}}{+ \cdots +} (-1))(1 \overset{n \text{ times}}{+ \cdots +} 1)$$

$$= (-1)(1 \overset{n \text{ times}}{+ \cdots +} 1) + \cdots + (-1)(1 \overset{n \text{ times}}{+ \cdots +} 1) \quad (|m| \text{ times})$$

$$= (-(1 \overset{n \text{ times}}{+ \cdots +} 1)) + \cdots + (-(1 \overset{n \text{ times}}{+ \cdots +} 1)) \quad (|m| \text{ times})$$

$$= ((-1) \overset{n \text{ times}}{+ \cdots +} (-1)) + \cdots + ((-1) \overset{n \text{ times}}{+ \cdots +} (-1)) \quad (|m| \text{ times})$$

$$= (-1) + \cdots + (-1) \quad (|m|n \text{ times})$$

$$= (-1) + \cdots + (-1) \quad (|mn| \text{ times})$$

$$= f(mn).$$

The proof that $f(m + n) = f(m) + f(n)$ is similar to the case that $m > 0, n < 0$.

<u>Case $m = n = 0$.</u> $f(mn) = f(0) = 0 = 0.0 = f(m)f(n)$ and

$f(m + n) = f(0) = 0 = 0 + 0 = f(m) + f(n)$.

<u>Case $m = 0, n \neq 0$.</u> $f(mn) = f(0) = 0 = 0.f(n) = f(m)f(n)$ and

$f(m + n) = f(n) = 0 + f(n) = f(m) + f(n)$.

Case $m \neq 0$, $n = 0$. $f(mn) = f(0) = 0 = f(m).0 = f(m)f(n)$ and

$$f(m + n) = f(m) = f(m) + 0 = f(m) + f(n).$$

Therefore we have the claim, so f is a homomorphism. Since K is finite, there exist $m \neq n \in \mathbb{Z}^+$ such that $(\overbrace{1 + \dots + 1}^{m \text{ times}}) = (\overbrace{1 + \dots + 1}^{n \text{ times}})$. Thus f is not one-to-one. Hence Ker f $\neq \{0\}$. Since Ker f is an ideal of $\mathbb{Z}$ which is a P.I.D., Ker f $= \langle n \rangle$ for some $n \in \mathbb{Z}^+$. Since $f(1) = 1 \neq 0$, $1 \notin$ Ker f. Thus Ker f $\neq \mathbb{Z}$, so $n \neq 1$. Suppose n is not a prime. Then there exist $1 < r, s < n$, $r, s \in \mathbb{Z}^+$ such that $n = rs$. Since $r, s \notin$ Ker f, $f(r)f(s) \neq 0$. Now $0 \neq f(r)f(s) = f(rs) = f(n) = 0$, a contradiction. Hence n must be a prime. Therefore

$$K \supseteq \text{Im } f \cong \mathbb{Z}/_{\text{Ker } f} = \mathbb{Z}/_{\langle n \rangle} = \mathbb{Z}_n \text{ , n a prime.}$$

Claim that $\mathbb{Z}_n$ contains no proper subseminear-field. Let $F \subseteq \mathbb{Z}_n$ be a subseminear-field of $\mathbb{Z}_n$. Thus $1 \in F$, so $1 + 1$, $1 + 1 + 1$, ... , $1 + \dots + 1$ (n tiems) $\in F$. Thus $\mathbb{Z}_n \subseteq F$, so $F = \mathbb{Z}_n$. Thus we have the claim. Therefore Im f is a subseminear-field of K which contains no proper subseminear-field. Thus $K' = \text{Im } f \cong Z_n$, n a prime.#

Theorem 3.40. Let K be a finite ∞-seminear-field and K' the prime seminear-field of K. Then $K' \cong \{\infty, 1\}$ with the structure

(1)

$\cdot$	∞	1
∞	∞	∞
1	∞	1

+	∞	1
∞	∞	∞
1	∞	1

or (2)

$\cdot$	∞	1
∞	∞	∞
1	∞	1

+	∞	1
∞	∞	∞
1	∞	∞

Proof. For $n \in \mathbb{Z}^+$ define $n1 = 1 + \dots + 1$ (n times) and define $0.1 = \infty$. Note that $(mn)1 = (m1)(n1)$ for all $m, n \in \mathbb{Z}^+$. Since K is finite, there exist $m, n \in \mathbb{Z}_0^+$ such that $m < n$ and $m1 = n1$. Let

$$m' = \min \{ m \in \mathbb{Z}_0^+ \mid \text{there is an } n \in \mathbb{Z}^+ \text{such that } n > m \text{ and } m1 = n1 \}$$

and $\quad n' = \min \{ n \in \mathbb{Z}^+ \mid n > m' \text{ and } m'1 = n1 \}.$

<u>Case $m' = 1$</u>. Claim that $m1 \neq \infty$ for all $m \in \mathbb{Z}^+$. Suppose that there exists an $m \in \mathbb{Z}^+$ such that $m1 = \infty$, then $m1 = 0.1$. Thus $m' = 0$, a contradiction. Hence $m1 \neq \infty$ for all $m \in \mathbb{Z}^+$. Let $C = \{m1 \mid m \in \mathbb{Z}^+\}$. Then $(C,+)$ is a finite semigroup. Thus there exists an $m \in \mathbb{Z}^+$ such that $m1 + m1 = m1$. Since $m1 \neq \infty$ and $(K \smallsetminus \{\infty\},.)$ is a group, $1 + 1 = 1$. Thus $n' = 2$. Hence we have that K' is (1) above.

<u>Case $m' \neq 1$</u>. Suppose $m' = 0$. Then $n' \neq 1$ since $1.1 = 1 \neq \infty$. If $m' = 0$ and $n' = 2$, then $1 + 1 = \infty$. So K' is (2) above. Suppose $m' = 0$ and $n' > 2$. It follows from the associativity of addition that for all $k \in \mathbb{Z}^+$ ($k > n'$ implies $k1 = \infty$) and it follows from the property of n' that ($k < n'$ implies $k1 \neq \infty$). Since $n' > 2$, $n'^2 - 2n' + 1 > n'$. Since $n' - 1 > 1$, $n' - 1 \in \mathbb{Z}^+$. Thus $(n' - 1)1 \neq \infty$, so

$$\infty \neq ((n'-1)1)((n'-1)1) = ((n'-1)(n'-1))1 = (n'^2 - 2n' + 1)1$$

$= \infty$, a contradiction. Therefore this case cannot occur. Suppose $m' > 1$. Thus again $m1 \neq \infty$ for all $m \in \mathbb{Z}^+$ by the same argument as the first case. Let $C = \{m1 \mid m \in \mathbb{Z}^+\}$. Then $(C,+)$ is a finite semigroup. As in the first case, $1 + 1 = 1$. Thus $m' = 1$, a contradiction.

Therefore $1 + 1 = 1$ or $1 + 1 = \infty$ we have the theorem.#

Now we shall study category II seminear-fields.

<u>Theorem 3.41</u>. Let K be a category II seminear-field with respect to $a \in K$. Then $(K \smallsetminus \{a\},+,.)$ is a division seminear-ring.

<u>Proof</u>. Let $x, y \in K \smallsetminus \{a\}$. Then $xy \in K \smallsetminus \{a\}$. We must show that $x + y \in K \smallsetminus \{a\}$. Suppose not. Then $x + y = a$. Let 1 be the identity of $(K \smallsetminus \{a\},.)$. Then $1 = a.1 = (x + y)1 = x.1 + y.1 = x + y = a$, a contradiction. Thus $x + y \in K \smallsetminus \{a\}$. Hence $(K \smallsetminus \{a\},+,.)$ is a

division seminear-ring. #

<u>Remark.</u> This theorem shows that every category II seminear-field comes from a division seminear-ring by adding an element.

<u>Theorem 3.42.</u> Let K be a category II seminear-field with respect to $a \in K$ and denote the identity of $(K \setminus \{a\}, \cdot)$ by 1. Then

(1) If $a + a = a$, then $(K,+)$ is a band.

(2) If $a + a \neq a$, then (for all $x, y \in K \setminus \{a\}$ $x + x = y + y$ iff $x = y$) and $a + a = 1 + 1$.

(3) $1 + a = a$ or $1 + a = 1 + 1$.

(4) $a + 1 = a$ or $a + 1 = 1 + 1$.

(5) $x + a = a$ or $x + a = 1 + 1$ for all $x \neq a$.

(6) $a + x = a$ or $a + x = 1 + 1$ for all $x \neq a$.

<u>Proof.</u>

(1) If $a + a = a$, then $x + x = ax + ax = (a + a)x = ax = x$ for all $x \in K$. Thus $(K,+)$ is a band.

(2) If $a + a \neq a$, then $x + x = ax + ax = (a + a)x$ for all x. Thus if $x + x = y + y$, then $(a + a)x = (a + a)y$. Thus $x = y$ since $a + a \neq a$. If $x = 1$, then $1 + 1 = a + a$.

(3) If $1 + a \neq a$, then $1 + a = (1 + a)1 = 1.1 + a.1 = 1 + 1$.

(4) If $a + 1 \neq a$, then $a + 1 = (a + 1)1 = a.1 + 1.1 = 1 + 1$.

(5) Let $x \in K$ be such that $x \neq a$. If $x + a \neq a$, then $x + a = (x + a)1 = x.1 + a.1 = x + 1$.

(6) Let $x \in K$ be such that $x \neq a$. If $a + x \neq a$, then $a + x = (a + x)1 = a.1 + x.1 = 1 + x$. #

Corollary. If D is a division seminear-ring, then for all $x, y \in D$ $x + x = y + y$ iff $x = y$.

Proof. Since D can be embedded in a category II seminear-field as in Example 3.5(3), for all $x, y \in D$ $x + x = y + y$ iff $x = y$ by Theorem 3.42(1) and (2).#

Theorem 3.43. Let K be a finite category II seminear-field and K' the prime seminear-field of K. Then $K' \cong \{a, 1\}$ with $a.1 = 1.a = 1$, $a^2 = a$, $1.1 = 1$ and

(1)

+	a	1
a	a	a
1	a	1

or (2)

+	a	1
a	a	1
1	a	1

or (3)

+	a	1
a	a	a
1	1	1

(4)

+	a	1
a	a	1
1	1	1

or (5)

+	a	1
a	1	a
1	a	1

or (6)

+	a	1
a	1	1
1	1	1

Proof. Let $n1 = 1 + \dots + 1$ (n times) for all $n \in \mathbb{Z}^+$. Since K is finite, there exist $n \in \mathbb{Z}^+$ such that $n1 + n1 = n1$. Let $a \in K$ be such that $a^2 = a$ and $(K \setminus \{a\}, .)$ is a group.

Case $n1 = a$. Thus $a + a = a$. By Theorem 3.42(1), (3) and (4), $1 + 1 = 1$, $1 + a = a$ or $1 + a = 1$, $a + 1 = a$ or $a + 1 = 1$. Thus we have four cases to consider. They are (1), (2), (3) and (4) above. It is easy to check that they are all seminear-fields. Thus $K' \cong (1)$ or $K' \cong (2)$ or $K' \cong (3)$ or $K' \cong (4)$.

Case $n1 \neq a$. Thus $(n1)^{-1}$ exists and so $1 + 1 = 1$. By Theorem 3.42, $a + a = a$ or $a + a = 1$, $a + 1 = a$ or $a + 1 = 1$, $1 + a = a$ or $1 + a = 1$. Thus we have eight cases to consider. They are (1) - (6) above and

(7)

+	a	1
a	1	1
1	a	1

(8)

+	a	1
a	1	a
1	1	1

It is easy to check that (5) and (6) are seminaer-fields but (7) and (8) are not. Thus $K' \cong (1)$ or $K' \cong (2)$ or $K' \cong (3)$ or $K' \cong (4)$ or $K' \cong (5)$ or $K' \cong (6)$.#

<u>Definition 3.44</u>. Let D be a division seminear-ring and $x \in D$. Then x is said to be <u>right standard</u> iff $xy + y = y$ for all $y \in D$ and x is said to be <u>left standard</u> iff $y + xy = y$ for all $y \in D$.

<u>Theorem 3.45</u>. Let K be a category II seminear-field with respect to $a \in K$.

(1) If $x \in K \smallsetminus \{a\}$ has the property that $x + a = a$ then x is right standard in the division seminear-ring $(K \smallsetminus \{a\}, +, \cdot)$.

(2) If $x \in K \smallsetminus \{a\}$ has the property that $a + x = a$ then x is left standard in the division seminear-ring $(K \smallsetminus \{a\}, +, \cdot)$.

<u>Proof</u>.

(1) Let $y \in K$. Thus $xy + y = xy + ay = (x + a)y = ay = y$, so x is right standard.

(2) The proof is similar to (1).#

Footnote.

* If $|K| > 2$, then there exists a unique $a \in K$ such that $a^2 = a$ and $(K \sim \{a\}, .)$ is a group. Therefore the concept of category is well-defined in this case. If $|K| = 2$, then an element $a \in K$ such that $a^2 = a$ and $(K \sim \{a\}, .)$ is a group is not unique. In this case, the category depends on the element. Hence we must say that K is a category I seminear-field with respect to a certain element a. If it is a category I seminear-field with respsct to a, then it is a category II seminear-field with respect to the other element and conversely. However, if it is a category III seminear-field with respect to one element, it will be a category III seminear-field with respect to the other element also. Also, if it is a category IV seminear-field with respect to one element, it will be a category IV seminear-field with respect to the other element.