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QUASINORMAL MODES OF BLACK HOLES IN VARIOUS DIMENSIONS

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โหมดกึ่งปกติคือโหมดของคลื่นสสารที่พลวัตไปในบริเวณกาลอวกาศของหลุมดำ หรือในอีกแง่หนึ่งก็คือผลเฉลยของสมการคลื่นภายใต้เงื่อนไขขอบเขตเฉพาะตัวคือ ที่บริเวณขอบฟ้าเหตุการณ์ของหลุมดำจะปรากฏเฉพาะคลื่นในโหมดที่เคลื่อนที่เข้าหาหลุมดำเท่านั้น และที่อินฟินิตี้มีเฉพาะโหมดของคลื่นที่เคลื่อนที่ออก เงื่อนไขขอบเขตดังกล่าวทำให้ ความถี่กึ่งปกติที่สอดคล้องกับโหมดกึ่งปกตินั้นมีค่าไม่ต่อเนื่องและมีค่าเป็นจำนวนจินตภาพติดลบ ซึ่งแสดงถึงโหมดการสลายตัวของผลเฉลยของสมการคลื่นดังกล่าว ในรายงานฉบับนี้เราทำการทบทวนวิธีคำนวณเชิงวิเคราะห์สำหรับโหมดกึ่งปกติของหลุมดำที่อยู่ในมิติต่างๆ สำหรับหลุมดำในสามมิติ เราทบทวนถึงกรณีของหลุมดำบีทีซี ทั้งแบบปกติและแบบที่มีการหมุนอีกทั้งยังทบทวนถึงหลุมดำชวาร์ซชิลด์ในปริภูมิแอนไทเดอสิสเตอร์ ในกรณีของหลุมดำสี่มิติ เราทบทวนงานวิจัยของหลุมดำชวาร์ซชิลด์โดยใช้วิธีการคำนวณเชิงตัวเลข และสำหรับหลุมดำในห้ามิติ เราทบทวนหลุมดำชวาร์ซชิลด์ในปริภูมิแอนไทเดอสิสเตอร์โดยใช้วิธีการรบกวนอันดับที่หนึ่ง อีกทั้งได้เสนอวิธีการคำนวณเชิงวิเคราะห์สำหรับ โหมดกึ่งปกติของหลุมดำคาลูซา-ไคลน์แบบที่มีการหมุนและมีขอบฟ้าเหตุการณ์แบบถูกบีบ โดยทำการคำนวณสมการไคลน์-กอร์ดอนในกาลอวกาศโค้ง และทำการประมาณถึงผลเฉลยของสมการดังกล่าวใน บริเวณขอบฟ้าเหตุการณ์และอินฟินิตี้ จากนั้นทำการเปรียบเทียบผลเฉลยทั้งสองภายใต้เงื่อนไขขอบเหตุข้างต้นและคำนวณความถี่กึ่งปกติ ผลการคำนวณพบว่าความถี่กึ่งปกติของหลุมดำดังกล่าวมีค่าน้อยลงเมื่อหลุมดำหมุนเร็วขึ้น และมีค่าขึ้นอยู่กับเลขควอนตัมพื้นฐาน

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Quasinormal modes are the modes of a wave propagating in the spacetime around the black hole. The definition is the wave solution that satisfies certain boundary conditions, i.e. only ingoing at the horizon and the outgoing at the infinity. According to these boundary conditions, the corresponding frequencies namely, quasinormal frequencies, are allowed to be a discrete set of complex number. These yield damping modes to the wave solution. Practically, quasinormal frequencies can be numerically obtained by solving the Schrödinger-like equation under particular boundary conditions. However, many previous works have suggested the possibility to determine these frequencies analytically. Therefore in this thesis, we mainly aim to investigate quasinormal modes of black holes in various dimensions by using analytical method. For three dimensional cases, quasinormal frequencies of BTZ and rotating BTZ solution are calculated. A large three dimensional AdS Schwarzschild black hole are explored and its quasinormal modes are also obtained. Then, a massive scalar perturbation on four dimensional Schwarzschild metric are numerically investigated. For five dimensions, first order perturbation is applied for the study of quasinormal modes of a five dimensional AdS Schwarzschild background. Ultimately, we have proposed semi-analytic calculation for the quasinormal frequencies of a rotating Kaluza-Klein black hole with squashed horizons.

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Chapter I

INTRODUCTION

"Space-time does not claim existence in its own right, but only as a structural quality of the (gravitational) field"

-A. Einstein

One of the most mysterious objects in the universe is black holes. Black holes were firstly explored as a solution of the Einstein field equations [1, 2, 3, 4, 5]. So, they are purely gravitational objects. Their essential singularities hidden inside the black holes are very bizarre. It is the place where gravity becomes infinite and all the physics law are broken down. Additionally, this puzzling end of physics is expected to be resolved by the presence of quantum gravity effects. On the other hand, black holes could be regarded as a final stage of exhausted massive stars which make them exist as real astrophysical objects. The first attempt to evaluate whether black holes are physical objects was initiated by Regge and Wheeler in 1957 [6]. They perturbed black hole solution and observed its respond. Surprisingly, the results show that after black holes were disturbed they will be under a small oscillation before returning to a static state and producing gravitational waves which radiate energy away to infinity. Based on this work, Viveshwara 1970 has proved that oscillation frequencies of external field outside black hole must be decaying [7, 8]. These frequencies have been termed as "quasinormal frequencies" and corresponding modes are so-called "quasinormal modes". More precisely, quasinormal modes are the modes of oscillation which have their own characteristics. They are not truly stationary but damped quite rapidly, some parts of wave are absorbed into the black holes during decay process. Also, its frequencies depend on black hole intrinsic parameters which are mass, charge and angular momentum.

Nowadays, quasinormal modes have been studied widely for various black hole types and also different kinds of external field, i.e., scalar and, vector fields.

Most of these studies have been done by numerical calculation. The most popular technique would be a continued fraction which is developed by Leaver [9] and was improved further by Nollert [10]. However, one found that for three dimensional black holes (BTZ for example), it turns out there is a possibility to obtain quasinormal frequencies analytically [11, 12] by transforming the Regge-Wheeler equation into hypergeometric differential equation. This fact motivated us to apply the same technique for others black hole system and try to calculate quasinormal frequencies. Hence, in this thesis, we aim to investigate many black hole models which are able to calculate quasinormal modes analytically.

This thesis is organized as follows. In Chapter 2, we introduce briefly some idea about relativity theory both for special and general version. Then, four well-known black hole solutions are reviewed in the following order, (*i*) Schwarzschild black hole, (*ii*) Reissner-Nordström black hole, (*iii*) Kerr black hole and finally (*iv*) BTZ black hole. In Chapter 3, the definition of quasinormal modes is stated and Regge-Wheeler equation is derived. We also discuss the application of quasinormal modes in this chapter. In Chapter 4, we reviewed quasinormal modes of three dimensional black holes. First, we consider the massless scalar field perturbation around BTZ black hole and calculate its quasinormal frequencies analytically following the work done by Vitor Cardoso [11]. Then, a massive scalar field evolves in the rotating BTZ background has been investigated which had been done by Danny Birmingham [12]. Also, an analytical formula of quasinormal frequencies is obtained. In the end of this chapter, we determine quasinormal modes of massless scalar field in a large AdS three dimensional Schwarzschild black hole [13]. Then in Chapter 5, we evaluate the quasinormal frequencies of four dimensional Schwarzschild black holes by using a continued fraction method which has been done by Zhidenko et.al [40]. In Chapter 6, quasinormal frequencies of five dimensional black holes are reviewed. We first begin with a large AdS five dimensional Schwarzschild black hole and determine its quasinormal frequencies by using first order perturbation method which had been proposed by Siopsis et.al [13]. Finally, we follow the matching solution technique [14, 15] to calculate quasinormal frequencies of a rotating Kaluza-Klein black hole with squashed horizons and observe the effect of compactified extra dimensions to those frequencies. Lastly, we summarize all the results of our study in Chapter 7.

For the sake of simplicity throughout this thesis, we shall assume geometrized unit $G = c = 1$ unless otherwise stated.

Chapter II

RELATIVITY AND BLACK HOLES

"It is always pleasant to have exact solutions in simple form at your disposal"

-K. Schwarzschild

It is well known that, the advanced theory of gravity is general theory of relativity (GR) proposed by Albert Einstein in 1915 [3]. GR has revolutionized our idea about space and time. It suggests that time itself is not an absolute quantity but relative; time flowing could be affected by the presence of the gravitational field. Moreover, in the GR paradigm, gravity should no longer be considered as a force but only a curvature of the spacetime caused by matter. There are many situations which confirm the correction of the GR, i.e., the perihelion precession of Mercury's orbit, the deflection of light near a massive object. One of the most astonishing about GR is, it predicts the existence of compact objects which even light cannot escape from, later-called *black holes*. More specifically, black holes occur as solutions of the Einstein field equations. Nowadays, there are many black hole solutions emerged from GR. In this chapter, we will briefly review both the special and general relativity theory. Later, the most four well-known exact solutions are discussed mathematically in the following order. First, *Schwarzschild solution*: the first non-trivial exact solution. Second, *Reissner-Nordström solution*: charged black hole. Third, *Kerr solution*: spinning black hole. Finally, *BTZ solution*: (2+1) dimensional black hole.

2.1 Special Relativity

1905, the *Annus Mirabillis* (miracle year) of Albert Einstein. He wrote four fundamental papers in that year. These four articles contributed widely to the modern physics and also revolutionized our concept about space, time and matter. His third paper in that year was about the reconcilement between Maxwell's Equations and the law of classical mechanics; by re-considering the Newtonian mechanics in the speed of light regime. This theory became known later as special theory of relativity (SR) [16]. SR is based on two important postulates. *1) All inertial observers are equivalent. 2) The speed of light c in vacuum is the same in all inertial systems.* By applying these two postulates, one can obtain *Lorentz transformation* which connecting two different inertial frames with relative velocity v,

$$
\begin{aligned}
ct' &= \gamma(ct - \beta x), \\
x' &= \gamma(x - \beta ct), \\
y' &= y, \\
z' &= z.
\end{aligned}
$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. This Lorentz transformation is also known as a boost in x-direction. From this transformation, we see that time and spaces coordinates are mixed. In addition, the interval (squared) between two events in an inertial frame S can be written as

$$
\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2. \tag{2.1}
$$

Clearly, (2.1) is invariant under Lorentz transformation. By observing the above equation, we are lead to consider space and time as a four-dimensional object called *spacetime*. Moreover, as a consequence of those postulates, there are two strange phenomenon occur when speed of the object approaches to the speed of light— Time-dilation and Length-contraction.

By using the four vector notation inspired by the Lorentz transformation, Einstein published his fourth paper [17] in 1905. He proposed equivalence between mass and energy. This became the most well-known physics formula—i.e. $E = mc^2$. Then, one can say that this equivalence is a consequence of special relativity. After publishing SR, Einstein continued his work to a more "general" theory of relativity which will be discussed in the next section.

2.2 General Relativity

In SR, Einstein considered only an inertial frame of reference where acceleration was neglected. Then, in order to extent his SR to a more general theory, he needs to consider a general frame of reference including the effect of acceleration. He tried to accomplish his new theory with a thought experiment about the free-falling elevator[*]. Finally, Einstein proposed the equivalence principle,

> *In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.* [2]

The principle of equivalence explains that if we are in the free-falling frame under a gravitational field, locally gravity seems to disappear so we recover special relativity. From this argument, Einstein can imply that gravity may not act as a force but pseudo force. Afterwards, he came up with an important concept about gravitation,

> *gravity should no longer be regarded as a force but a manifestation of the spacetime curvature which curved by the presence of matter.* [2]

This statement is the most essential point of a new theory of gravity which later called general theory of relativity (GR) [3]. To describe the curvature of spacetime quantitatively, Einstein needs a mathematical tool that has coordinate independence property. After many years of trial and error, Einstein successfully formulated a tensorial equation for his new theory (GR) in 1915. *Einstein field equations* takes the form[†]

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.2}$$

where G and c is Newton constant and the speed of light in vacuum respectively. $G_{\mu\nu}$ is an *Einstein tensor* which contains all the information about spacetime geometry. $T_{\mu\nu}$ is called *energy-momentum tensor* which can be considered as a source for gravitational field. The divergence-less of Einstein tensor suggests the conservation of energy—i.e.$\nabla_\mu T^{\mu\nu} = 0$. Einstein tensor can be expressed mathematically as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.3}$$

where $R_{\mu\nu}$ is *Ricci tensor* which is a contraction of Riemann tensor—i.e. $R^{\alpha}_{\mu\alpha\nu} = R_{\mu\nu}$. R is *Ricci scalar* or a curvature scalar. $g_{\mu\nu}$ is metric tensor which defines concept of distance on a manifold.

[*]For more detail, see [1].

[†]See Appendix A.1 for the derivation of Einstein field equation.

In fact, (2.2) is not the full form of Einstein field equations yet. The complete form is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.4}$$

where Λ is a *cosmological constant*. It was originally introduced by Einstein to produce static universe. But the observation tells us that our universe is expanding not static as Einstein would assume. So Λ was removed†. But recently, physicists have discovered the cosmic acceleration. This has renewed an interest of cosmological constant as a source of the acceleration.

Solution of Einstein field equation normally written in a form called *line-element or metric*. They describe an invariant interval in any spacetime. Let's consider (2.1) as an infinitesimal quantities. It reads

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{2.5}$$

This line-element shows the geometry of 4 dimensional flat spacetime which expressed by Cartesian coordinates*. It can be rewritten in a tensorial form

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu, \tag{2.6}$$

and

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

where $\eta_{\mu\nu}$ is *Minkowski metric tensor* or *Minkowski metric*. This solution is a special case of GR since it explains geometry of a flat spacetime. In general, we obtain the general solution by replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$.Then (2.6) becomes

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{2.8}$$

There are three possible values for ds^2 as follows:

$$ds^2 > 0 \text{ is spacelike interval,}$$
$$ds^2 = 0 \text{ is lightlike or null interval,}$$
$$ds^2 < 0 \text{ is timelike interval.}$$

†Einstein calling this as "the biggest blunder he ever made of my life".

*Due to the coordinate-independent, the flat metric can be described by the other coordinate system, e.g., in spherical coordinates it reads $ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$.

Note that, these definition will interchange between spacelike and timelike if the metric signature becomes $(+,-,-,-)$. Spacelike interval represents non-physically related region. Null interval express light trajectory in spacetime, it forms the light cone structure in the spacetime diagram. For timelike interval, non-zero massive particle must be contained within the light cone and shows a massive particle path in the spacetime. To describes the motion of particle in curved spacetime, we define equation of motion in GR as,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{2.9}$$

This is geodesic equation. It explains "free falling" motions of a particle that means there is no external force acts on the object. Therefore, RHS of (2.9) is zero. λ is an affine parameter practically chosen to be the proper time "τ". The effect of curved spacetime to equation of motion comes from Christoffel connection $\Gamma^\mu_{\nu\sigma}$ given by

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\alpha}(\partial_\nu g_{\sigma\alpha} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma}). \tag{2.10}$$

In flat space the metric tensor in (2.10) is replaced by $\eta_{\mu\nu}$. So, geodesic equation reduces to $\frac{d^2x^\mu}{d\lambda^2} = 0$, it represents particle's motion with constant velocity in flat spacetime.

2.3 Black Holes

[*]The concept of black holes was originally proposed by John Michell in 1784. He discussed classical objects which their escape velocities exceed the speed of light. Later in 1795, Pierre-Simon Laplace also pointed out that there could be massive stars whose gravity is so strong that not even light can escape from it[†]. In 1916, Karl Schwarzschild was able to solves the Einstein field equations and obtained the simplest well known exact solution [18], *the Schwarzschild solution.* By investigating the solution's structure, its mathematical singularity is emerged which implies the existence of black holes geometry called *Schwarzschild black hole.* Hence, the ideas about black holes were theoretically supported for the first time from the Schwarzschild solution. Einstein was surprised by this result. Since, he did not expect the exact solution will be found so soon. However, Einstein had never accepted the ideas about black holes until his death in 1955.

[*]See [23] for more detail about timeline in black holes research.

[†]See [4] for the detail of his calculation.

Schwarzschild's work had widely opened the study of black holes physics. In 1918, Hans Reissner and Gunnar Nordström successfully solved the Einstein-Maxwell equations for charged spherical-symmetric object [19, 20] called *Reissner-Nordstöm black hole*. Five years later, George D. Birkhoff proved the uniqueness of the Schwarzschild solution. It stated that the spacetime outside a spherical symmetric object always governed by the Schwarzschild metric. At that time, physicists believe that black holes were originated from the exhausted stars via gravitational collapse process. Thus in 1939, J. Robert Oppenheimer and Hartland Snyder calculated the pressure-free homogeneous fluid sphere that collapses under the influence of gravity [21]. The result shows that the object will cut itself from the outside universe. This was the first evidence which confirms the existence of the black holes as an astrophysical object. For uncharged axial-symmetric rotating system, Roy Kerr was able to solve Einstein equations in vacuum for such a system in 1963 [22]. This solution is referred as a *Kerr black hole*. Nowadays, there are many branches of black holes physics research both theoretical and observational. For example, thermodynamic properties of black holes originated by Stephen Hawking, the perturbation of black holes which later leads to the study of quasinormal modes, the search for gravitational waves that occur due to a massive astrophysical object including black holes. Black holes are also used for testing many theories about the modified gravity and quantum gravity.

Theoretically, black holes are considered to be the solution of Einstein field equations. Many black hole metrics come from solving the Einstein equations under particular assumptions. In the following subsection, we will discuss some of the well-known black hole metrics.

2.3.1 Schwarzschild Black Hole

Karl Schwarzschild was the first one who was able to solve the Einstein field equations analytically. He reduces the complexity of the equations by assuming the spherical symmetry and solved for the vacuum solutions. His solution represents spacetime geometry outside a spherically symmetric matter distribution. To obtain such a solution, Schwarzschild needs to seek out for the most general form of the static spatially isotropic metric.

The word *static* imposes two properties for the metric: (i) all the metric component $g_{\mu\nu}$ must be independent of time coordinate (say x^0); (ii) line-element ds^2 are invariant under $x^0 \longrightarrow -x^0$ transformation. A spacetime that satisfies only

(i) condition is called *stationary* [2]. We will encounter such a spacetime again when consider the spinning black hole. *Spatially isotropic* means that the metric looks the same from all directions. This condition implies spherical symmetric property to the Schwarzschild metric. Starting from the most general form of spatially isotropic metric [2],

$$ds^2 \;=\; -A(t,r)dt^2 + B(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.11)$$

It is trivial to obtain the *static* property. By requiring that the metric components are independent of x^0. Thus, the above metric reduces to

$$ds^2 \;=\; -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.12)$$

This is the most general form of the metric with static spatially isotropic properties. We can find the exact form of the function $A(r), B(r)$ by solving this line-element in vacuum-space field equations.

$$G_{\mu\nu} \;=\; 0. \qquad (2.13)$$

or in equivalent form (see Appendix A.2)

$$R_{\mu\nu} \;=\; 0. \qquad (2.14)$$

From the metric (2.12), we can calculate all the non-vanishing Christoffel connection via (2.10). Thus, the Ricci curvature tensor can be constructed by

$$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu} \;=\; \partial_\rho\Gamma^{\rho}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu}. \qquad (2.15)$$

Finally, the *Schwarzschild solution* is obtained [2]

$$ds^2 \;=\; -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \qquad (2.16)$$

This metric describes the spacetime geometry outside the spherically symmetric matter. The Schwarzschild spacetime has asymptotically flat property which means as r$\longrightarrow\infty$ the metric becomes flat $g_{\mu\nu} \longrightarrow \eta_{\mu\nu}$. At first glance, the Schwarzschild metric seems to have two gravitational singularity at the surface $r = 2M$ and $r = 0$. The first one is called *Schwarzschild radius* which defines the radius of the Schwarzschild black hole. It also acts as a boundary of the black hole called an *event horizon*, once anything come inside this boundary then it is impossible to be seen from the outside observer. The other singularity lies at the black hole's center. It is the place where the curvature (gravity) becomes infinite. In fact, the surface $r = 2M$ is only coordinates singularity which can be removed out by choosing new coordinates properly. For example, if we introduce new coordinate as

$$t' \;=\; t + 2M \ln\left|\frac{r}{2M} - 1\right|. \tag{2.17}$$

Using the above relation, thus we can transform (2.16) to

$$ds^2 \;=\; -\Big(1 - \frac{2M}{r}\Big)dt'^2 + \frac{4M}{r}dt'dr + \Big(1 + \frac{2M}{r}\Big)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.18}$$

This is Schwarzschild metric but written in *Eddington-Finkelstein* coordinates (t', r, θ, ϕ) instead of the Schwarzschild coordinates (t, r, θ, ϕ). We see that $r = 2M$ is no longer problem, our metric (2.18) is still finite at this surface. The metric (2.16) explains geometry of the spacetime outside the spherically symmetric object which being seen by the distant observers. From their perspective, every particle which tries to approach an event horizon will never get inside the black hole since our metric becomes divergent there. Instead, the Eddington-Finkelstein coordinates represent the view point of an approaching particle. Thus from the particle's frame of reference, they can safely move through an event horizon. Inside the event horizon, coordinate t and coordinate r which is timelike and spacelike respectively at the outside will swap their role, coordinate t will be spacelike while coordinate r becomes timelike. So inside the Schwarzschild black hole, every motion will be forced to move in the reducing r coordinate direction and hitting the singularity unavoidably.

So far, we have shown the properties of a static spherically symmetric metric. In the next subsection, we will further investigate about the metric which describe the spacetime outside a static spherically symmetric charged object.

2.3.2 Reissner-Nordström Black Hole

In the last section, we discuss about the static spherically symmetric object and obtain the Schwarzschild solution. We will now further investigate more about a metric outside static spherically symmetric charged matter. The exterior of such an object is filled with a static electric field. Therefore, we need to solve Einstein field equations for a static spherically symmetric with the existence of energy-momentum tensor for a pure electromagnetic field [2]. The Einstein-Maxwell field equations take the form

$$G_{\mu\nu} \;=\; 8\pi T_{\mu\nu}. \tag{2.19}$$

$T_{\mu\nu}$ is the Maxwell energy-momentum tensor. It is defined as [1]

$$T_{\mu\nu} \;=\; \frac{1}{4\pi}(-g^{\sigma\rho}F_{\mu\sigma}F_{\nu\rho} + \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}). \tag{2.20}$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$. $F_{\mu\nu}$ and A_μ is Maxwell strength tensor and 4-vector potential respectively. Thus, it is obvious that the Maxwell tensor is anti-symmetric which implies the trace-free of the Maxwell energy momentum tensor. Then, the Einstein-Maxwell field equations become

$$R_{\mu\nu} \;=\; 8\pi T_{\mu\nu}. \tag{2.21}$$

Moreover, the Maxwell tensor must satisfy the source-free Maxwell's equations [1]

$$\begin{aligned} \nabla_\nu F^{\mu\nu} &= 0, \\ \partial_{[\mu}F_{\nu\sigma]} &= 0. \end{aligned}$$

We make an assumption that our point charged particle settles at the origin of the coordinates system. As a consequence of spherically symmetric, the 4-vector potential takes the simple form

$$[A^\mu] \;=\; \Big(\phi(r), a(r), 0, 0\Big),$$

where $\phi(r)$ and $a(r)$ may be interpreted as electrostatic potential and the radial component of the 3-vector potential[2]. Hence, the field(Maxwell)-strength tensor has the form

$$[F^{\mu\nu}] \;=\; E(r)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

In this case, we may interprets $E(r)$ as the radial component of electric field as $r \longrightarrow \infty$. Now our task is to write down the static spherically symmetric metric and insert them into field equations (2.21) together with the Maxwell's equations. Then determine the unknown function of the metric. Besides the metric defined in (2.12), we can use another form of such a metric which is defined as [1]

$$ds^2 \;=\; -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.23}$$

Finally, we obtain the *Reissner-Nordström metric*

$$ds^2 \;=\; -\Big(1 - \frac{2M}{r} + \frac{k^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{k^2}{r^2}\Big)^{-1}dr^2 + r^2d\Omega^2, \tag{2.24}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. As $k \longrightarrow 0$, we recover the Schwarzschild metric therefore we may interpret k as a total charge of the particle. This solution shows the spacetime geometry around the non-rotating point charged particle. One can obtain the coordinate singularity of the metric (2.24) by setting $g^{11} = 0$. This gives us a result

$$r_{\pm} \;\; = \;\; M \pm \sqrt{M^2 - k^2}. \tag{2.25}$$

The Reissner-Nortström has double event horizon, i.e., *inner and outer horizon*. Clearly, there exist three different cases to determine the values of the Reissner-Nordström's radius.

Case I $M^2 > k^2$: In this case, there exist both singularities at $r = r_{\pm}$. These surfaces represent the event horizon of the Reissener-Nordström black hole. In the region $r_- < r < r_+$, the situation is identically with the Schwarzschild. Once inside this region, the t and r will change their characters from timelike to spacelike and vice versa respectively. Hence the particles are forced to move in the direction of decreasing r until they reach the next surface $r = r_-$. Afterward, when they approach the r_- surface, the coordinate t and coordinate r will return to their usual timelike and spacelike properties. Consequently, inside the inner horizon $r < r_-$, particles can manage themselves to avoid the black hole's singularity which is different from the Schwarzschild case. Moreover, if we do *maximally analytic extension* of the Reissner-Nordström geometry, one finds that there exist such a hypothetical solution in which looks like a time reversal black hole namely, *white hole*. In the white hole, a particle inside the inner horizon will be ejected from the hole and nothing can across back inside the hole. In fact, the Schwarzschild solution can give such a white hole solution by extending the Eddinton-Finkelstein coordinates further. However, we will not discuss more about the white hole since it is beyond our scope here.

Case II $M^2 < k^2$: In this case, it appears that both $r_{\pm}$ become imaginary. As a result, g^{11} is regular everywhere except $r = 0$ thus there are no coordinate singularities anymore. Both event horizons now disappear, only the intrinsic singularity is left nakedly; the absence of the event horizons lead to the fact that coordinate t and r remain their own properties which are timelike and spacelike. Thus, the naked singularity can also be avoided by the particles. However the physical situation that comes after the existence of the naked singularity is; for an example, there exist the closed timelike curves which allow the possible of the time traveling. Since such an extreme unphysical scenario emerges, Roger Penrose has proposed *cosmic censorship conjecture* in 1969. This conjecture stated that the singularities must be covered by an event horizon to prevent the formation of a naked singularity. However, nowadays there are many theoretical evidences which give a contradiction to this conjecture. Thus, the naked singularities may not be necessarily covered and could possibly exist in the real universe.

Case III $M^2 = k^2$: This one is called *extreme* Reissner-Nordström black hole. In this case, the outer hoizon r_+ and inner horizon r_- will coincide at $r = M$. The coordinate r is always spacelike except at $r = M$ it becomes null. Hence, the singularity $r = 0$ is a timelike as in the other cases. Thus for this black hole, it is again possible to avoid the singularity. Moreover, an extremal black hole is practically used as a toy model; they are usually investigated the role of black hole in quantum gravity. Also in supersymmetric theories, extremal black hole can leave the symmetries unbroken, which is considerable aid in calculations. [5]

Charged black hole would hardly be observed in nature since it is rapidly discharged by the surrounding opposite charges in the accretion disk. However, it is worth to study this solution because its mathematical structure is similar to that of the more complicated solution; Kerr solution which describe the rotating black hole which we will discuss in the next section [1]. On the other hand it also helps us to understand better about the spacetime geometry.

2.3.3 Kerr Black Hole

In the last two sections, we discuss charged and un-charged spherical black hole. But, most astrophysical objects are indeed rotating. So, we need to construct the metric that describes such an object. It appears that we cannot apply a static isotropic metric in this case because the rotation axis needs a special direction. Hence, the isotropic property is destroyed. In order to represent steadily rotating matter, we must introduce a stationary axisymmetric metric [2]

$$ds^2 = -A(r,\theta)dt^2 + B(r,\theta)\left(d\phi - \omega(r,\theta)dt\right)^2 + C(r,\theta)dr^2 + D(r,\theta)d\theta^2. \tag{2.26}$$

By considering 4-momentum of the particle approaching this metric (2.26), one can prove that the function $\omega(r,\theta) = \frac{d\phi}{dt}$. The rate of change of coordinate ϕ with respect to the coordinate time shows that the spacetime itself is rotating. Any particle reach at this neighborhood will has been dragged by the effect of pure gravitational field. This effect is called *frame dragging effect*.

In order to obtain the Kerr metric, we must insert the stationary metric (2.26) into (2.14) and then solve for the unknown function $A(r,\theta)$ and $B(r, theta)$. Hence, we get the *Kerr metric* (written in *Boyer-Lindquist coordinates*)[2]

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}d\phi^2. \tag{2.27}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2Mr + a^2, \\ \Sigma &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned}$$

In the limit $a \longrightarrow 0$, the Kerr solution tends to the Schwarzschild case so it makes sense to interpret parameter a as spin parameter. In fact, a relates to total angular momentum via $J = Ma$. To find the event horizons, we must solve $\Delta = 0$. Then, the event horizons of Kerr solution take the form

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{2.28}$$

The Kerr solution (2.27) describes the spacetime around a massive rotating body. It has stationary and axisymmetric properties. Like Reissner-Norström, the Kerr metric has two event horizons, i.e., *inner and outer horizon* which cause from the quadratic factor in Δ. There is another special surface for Kerr solution, if we set $g_{tt} = 0$

$$r_{s_{\pm}} = M \pm \sqrt{M^2 - a^2\cos^2\theta}. \tag{2.29}$$

These surfaces show another interesting character for a stationary axisymmetric metric except from the frame-dragging. Inside these surfaces nothing can remain stationary it will be forced to move in the same direction with the black hole's rotation. These surfaces are called *stationary limit surfaces.* The region between the outer static surface and the outer horizon is called the *Ergosphere.* It is possible to extract the black hole's energy from this region of the rotating black hole which was proposed by Roger Penrose. Although (2.28) expresses the coordinate singularity of the Kerr solution but we could obtain the intrinsic singularity by setting $\rho^2 = r^2 + a^2\cos^2\theta = 0$. These yields

$$r = 0, \quad \theta = \frac{\pi}{2}.$$

By using a proper coordinate transformation, one can transform the above condition to $x^2 + y^2 = a^2$. Hence, surprisingly, Kerr's singularity has ring-shaped with the radius "a". Let us investigate more on the event horizon of Kerr solution (2.28), it is clear that there exist three possible cases relevant to the relative values of M^2 and a^2. This is in many way similar to that of the Reissner-Nordström black hole.

Case I $M^2 > a^2$: In this case, both outer and inner horizon is real so two coordinate singularities exist. But they can be removed by changing the coordinates as the Schwarzschild metric. We can divide radial coordinate into three regions. $r_+ < r$ this region lies at the outermost of the Kerr solution. In this region, t is timelike while r is spacelike. $r_- < r < r_+$ is the region where space and time interchange their role. This is similar to region inside the Schwarzschild's radius. More specifically, particle must move in the decreasing direction of coordinate r. The last one is the region $r < r_-$, again time and radial coordinate regain their properties at the outside of the outer horizon. Therefore once again, the intrinsic singularity of Kerr black hole can be avoided since it has the timelike singularity which is similar to that of the Reissner-Nordström solution. We expect this situation would occur in the real gravitational collapse since this case seems to be the most physical than the other cases.

Case II $M^2 < a^2$ There is nothing different from the Charged black hole. Both coordinate singularities are now disappeared then naked singularity is appearing.

Case III $M^2 = a^2$ An extreme Kerr black hole is also the same as Reissner-Nordström. Two event horizons coincide at the surface $r = M$ which is a null hypersurface. However, it may be possible that near-extremal Kerr black hole could emerge in the real astrophysical situation. As the matters forming in the accretion disk around Kerr black hole fall inside, they also increase mass and angular momentum to the hole. While the matters are falling in, they create a radiation in which carries away the angular momentum. In addition, a detailed calculation shows that the limiting value is $a \approx 0.998M$ [2].

In conclusion, we see that the stationary axisymmetric metric leads to two fascinating phenomena, i.e., frame-dragging effect and stationary limit surfaces. These are the special characters of such a metric which differ from the Schwarzschild and Reissner-Nordström black hole. On the other hand, while the intrinsic singularity of the Schwarzschild solution are spacelike, Reissner-Nordström and Kerr singularity is timelike. It is worthy to note that, although Einstein equation itself has non-linearity property but all of the exaction solutions can be derived by choosing appropriate assumptions and symmetries. In the next section, we will introduce another interesting vacuum solution with the presence of cosmological constant in (2+1) dimensional spacetime namely, a BTZ black hole.

2.3.4 BTZ Black Hole

From the definition of Riemann tensor, a vacuum solution of (2+1) dimensional spacetime is essentially flat. Thus in the past, there was no anticipation for the existence of the black hole solutions in (2+1) dimensional gravity. However, strikingly, in 1992 Bañados, Teitelboim, and Zanelli discovered [24] the vacuum solution in (2+1) dimensions with negative cosmological constant; it is called BTZ solution. This solution is used as a toy model for the study of (2+1) dimensional quantum gravity and also supergravity theories. To derive this solution, we begin with the most general form of the static isotropic metric in "Schwarzschild coordinates",

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2, \tag{2.30}$$

where $-\infty < t < \infty$, $0 < r < \infty$, $0 \leq \phi \leq 2\pi$. In fact, since this solution lives in three dimensions the topology of angular part becomes a disk instead of 2-sphere as in four dimensional case. Now go back to the Einstein field equations with negative cosmological constant $\Lambda = -1/l^2$ (2.4). We shall consider only the vacuum solution, thus field equations become (See Appendix A.2)

$$R_{\mu\nu} = -\frac{1}{l^2} g_{\mu\nu}. \tag{2.31}$$

Then, plug in (2.30) into the above equation we get

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{2.32}$$

This solution asymptotically becomes ($r \longrightarrow \infty$) anti de sitter spacetime. Note that, if $\frac{r^2}{l^2}$ is vanished the solution reduces to flat metric. The event horizon is given by

$$r_{\pm} = \pm l\sqrt{M}. \tag{2.33}$$

Moreover, we can construct spinning BTZ black hole solution as we have done in Kerr black hole. The rotating BTZ black hole takes the form

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2,$$

where $J = Ma$ is total angular momentum. One easily sees that this metric reduces to spinless BTZ when $J \longrightarrow 0$. The hole's radius is given by

$$r_{\pm} = l\left[\frac{M}{2}\left(1 \pm \sqrt{1-\left(\frac{J}{Ml}\right)^2}\right)\right]^{1/2}. \qquad (2.34)$$

As we expect, event horizons split into two surfaces, i.e., outer and inner horizon. If l grows very large the black hole exterior is pushed away to infinity and one is left just with the inside [24]. Moreover, if we set $M = -1$ and $J \longrightarrow 0$ the BTZ metric becomes ordinary anti-de sitter space. So far, we have shown the basic knowledge of the well-known black hole solution. Hence, the concept of quasinormal modes of black holes will be introduced in the next chapter.

Chapter III

BLACK HOLE PERTURBATIONS

"Black hole shows only three characteristics to the outside world (mass, charge, spin) and comparing the situation to a room full of bald-pated people who had one characteristic in common, but no differences in hair length, style or color for individual variations."(no hair theorem)

-J. A. Wheeler

After a brief review on GR and black holes in the previous chapter, we now ready to make a discussion on black hole perturbation which later leads to the study of quasinormal modes. In this chapter, we begin with a brief introduction about the beginning of the study on black hole perturbation. Then, the equation of perturbation around black hole metric is derived. This leads us to the Schrödinger-like equation which is the major equation throughout our study. After that, we introduce the definition of quasinormal modes and quasinormal frequencies both mathematically and physically. Some noteworthy literature reviews are discussed. Finally, we denote the contribution of quasinormal modes to other fundamental problem in physics.

3.1 Literature Review

In the mid 1950's, physicists wondered whether the black hole could be regarded as an astronomical object. In order to answer this question, they started to study the perturbation of black holes. More specifically, they studied the evolution of the physical fields outside black holes which has been proved later by Vishveshwara [7] that the fields can be treated as a perturbation in the black hole spacetime.

The physical fields are always assumed to be weak thus there is no effect of their energy-momentum tensor on the black hole metric [26]. Even though Einstein theory itself is a nonlinear theory, it turns out that the only linear perturbation is well suited. In fact, the study of perturbed black holes were first pioneered by Regge and Wheeler in 1957. Their question was whether the Schwarzschild black holes become unstable under small perturbations on black holes which are assumed in the linearized Einstein's equations. If black holes cannot stand against a small perturbation and turn into an unbounded state, then, black holes could not be determined as an astrophysical object. In their original paper [6], they studied the perturbations of the metric directly

$$g_{\mu\nu} \;=\; g_{\mu\nu}^{background} + h_{\mu\nu}, \tag{3.1}$$

where $h_{\mu\nu}$ is sufficiently small. Hence, the remaining terms in the calculation are linear terms in $h_{\mu\nu}$. They obtained the wave equation with an effective potential. This equation is now known as the *Regge-Wheeler equation*

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right]\Psi(r) \;=\; 0, \tag{3.2}$$

where effective potential can be written as

$$V(r) \;=\; \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{3.3}$$

In (3.2), we have used the *tortoise coordinate* r_* that was first introduced by Wheeler in 1995. For example, it is related to the standard Schwarzschild radial coordinate by $dr_* = \left(1 - \frac{2M}{r}\right)^{-1} dr$. Their results suggest that when disturbed, black hole will experience a small oscillation and later regain its stable state once again.

Thereafter, their work was extended to a Reissner-Nordström by Zerilli 1974, but one found that for a rotating black hole the problem was far more complicated. However, Teukolsky (1972) was successfully able to reduce the wave equation into a single equation by using Newman-Penrose formalism. Hence, the stability of Kerr black hole is explored by following Teukolsky's work. The detail on black hole perturbations both mathematically and physically can be found in Chandrasekhar's book (1973) [25] and Frolov,Novikov [26].

3.2 Wave Equations Near Black Holes

To describe dynamical system in general relativity, let's first consider the Einstein-Hillbert action

$$S = \int \sqrt{-g}\Big(R + \mathcal{L}_M\Big)d^4x. \tag{3.4}$$

Where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_M$ is the matter Lagrangian which describes the matter field ϕ. By varying the above action with respect to the metric tensor, one obtains the Einstein field equations (For more detail see Appendix A.1)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3.5}$$

Where we have set $G = c = 1$. The energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{3.6}$$

The field equations in curved spacetime

$$\frac{\delta\mathcal{L}_M}{\delta\phi_a} = 0. \tag{3.7}$$

Then, let's consider the metric and fields perturbation as

$$g_{\mu\nu} \longrightarrow g^0_{\mu\nu} + \delta g_{\mu\nu}, \tag{3.8}$$

$$\phi \longrightarrow \phi^0_a + \delta\phi_a. \tag{3.9}$$

We shall assume the perturbations are weak, therefore $\mathcal{O}(\delta g_{\mu\nu})^2$, $\mathcal{O}(\delta\phi_a)$, $\mathcal{O}(\delta\phi_a)^2$ and higher are negligible. After inserting perturbed metric and fields into (3.5) and (3.7), one found that the unperturbed metric and fields satisfy (3.5),(3.7) as usual. Moreover, we also obtain linear equation for perturbation of $\delta g_{\mu\nu}$ and $\delta\phi_a$. In our scope of study, matter fields are assumed to contribute negligibly to the curvature of the black hole spacetime. Now, let's consider the Schwarzschild-like metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.10}$$

For the sake of simplicity, we describe massless scalar field in such a background by using *Klein-Gordon equation* in curved background which takes the form (See Appendix A.3)

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi(x)) = 0. \tag{3.11}$$

We now using separation of variable method by the following ansatz

$$\Phi(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\Psi(t,r)}{r}Y_{l,m}(\theta,\phi), \tag{3.12}$$

where $Y_{l,m}(\theta,\phi)$ is the spherical harmonics, the integer $l \geq 0$ and $|m| \leq l$ are called the multipole number and the azimuthal number respectively. Spherical harmonics are also satisfied the following eigenvalue equation

$$\Delta_{\theta,\phi}Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi), \tag{3.13}$$

where the angular part of Laplasian $\Delta_{\theta,\phi}$ is defined as

$$\Delta_{\theta,\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{3.14}$$

Then, substitute the ansatz into the Klein-Gordon equation (3.11), we get the wave-like equation with the effective potential

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_l(r)\right)\Psi(t,r) = 0, \tag{3.15}$$

where

$$V_l(r) = f(r)\left(\frac{f'(r)}{r} + \frac{l(l+1)}{r^2}\right). \tag{3.16}$$

Note that if we choose $\Psi(t,r) = e^{-i\omega t}\psi(r)$ this wave-like equation becomes *Regge-Wheeler equation*

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l(r)\right]\psi(r) = 0. \tag{3.17}$$

This equation will play as an important role when dealing with quasinormal modes of black holes. It describes the scalar field in curved background which can be treated as a perturbation in the black hole metric. Beside scalar particle, we can also consider other fields in the black hole metric. Practically, in most cases, the equation of motion of the fields can be reduced to the wave-like equation (3.15).

One could possibly find that for the massless Dirac field, the effective potential takes the form [28]

$$V_{D_\pm} = f(r)\frac{\kappa_\pm^2}{r^2} \pm \frac{d}{dr_*}\frac{\kappa_\pm\sqrt{f(r)}}{r}, \qquad \kappa_\pm = 1,2,3,... \tag{3.18}$$

and for the electromagnetic field, the effective potential is defined as [29]

$$V_{EM} = f(r)\frac{l(l+1)}{r^2}, \qquad l = 1,2,3,.... \tag{3.19}$$

Note that, these potential are only valid for the spacetime metric which is defined by (3.10). If we replace the Schwarzschild-like by other metrics, such as the axisymmetric metric, then we must re-derive the corresponding effective potential.

3.3 Quasinormal Modes

Everyone should be familiar with the concept of "normal modes". When we first studied about wave theory, we usually assume that there is no energy loss in the oscillating system, for an example, violin's strings vibration. By perturbing such a system, it will respond by choosing a discrete set of real frequencies which produces a "characteristic sound" [30]. The corresponding modes which are the superposition of stationary modes are so-called normal modes. Surprisingly, black holes can provide such an individual sound too. By perturbing the fields near the black holes, one finds that black holes will respond by producing a characteristic oscillation. Vishveshwara [7] was the first one who discovered this fact by investigation of a gaussian gravitational wavepacket developed in the Schwarzschild geometry. He found that the signal exists in a finite time interval and later disappears by the damped oscillating frequencies. These damped frequencies also depend on the parameters which describe the black holes, i.e., mass, charge and angular momentum. More specifically, these frequencies are not dependent of any initial configuration. This shows a unique character of the vibrating system involving black holes. These oscillations are called as *quasinormal modes* and the corresponding damped frequencies *quasinormal frequencies*. The word "quasi-" shows the deviation from the "normal modes": quasinormal modes are in fact not stationary modes because they are governed by the exponentially damped frequencies.

As we mentioned earlier, when we discuss about the resonant system such as a violin, we usually make an ideally well-suited study assumption, that there is no decaying modes exists. This fact leads to the normal modes and corresponding normal frequencies. These stationary modes also indicate that there is no energy loss from our system. However, if we turn to the more realistic case, no such idealized situation would occur in the real physical situation. Nevertheless, it is impossible to prohibit the dissipative process in the black hole schematic. There always exist "non-stationary" modes in the perturbed black hole metric. Since, black holes are just only the pure gravitational object; any spacetime perturbation generates the gravitational waves which radiate energy away to infinity. The deeper mathematical detail on quasinormal modes can be found in a very neat review by Kokkotas and Schmidt [31].

3.3.1 Mathematical Definition of Quasinormal Modes

In the previous discussion, we investigate the concept of quasinormal modes physically. Now, we shall further discuss about its mathematical structure. As previously mentioned, we can formulate the partial differential equation that describes the physical field's evolution in curved spacetime by using a background metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{3.20}$$

Then, substitute into the Klein-Gordon equation (3.11) (for a scalar field). Finally, we obtain the wave-like equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V_l(r)\right)\Psi(t,x) = 0. \tag{3.21}$$

Where x is spatial coordinate which normally varies from $-\infty$ to ∞. It is usually to place an event horizon at $-\infty$ and for the rest of this note we shall assume so unless otherwise stated. We shall decompose the field into time dependence and spatial dependence,

$$\Psi(t,x) = e^{-i\omega t}\psi(x). \tag{3.22}$$

Yet again, we have derived the Regge-Wheeler equation

$$\left[\frac{d^2}{dx^2} + \omega^2 - V_l(x)\right]\psi(x) = 0. \tag{3.23}$$

In this section, we will restrict ourselves to asymptotically flat spacetime. Thus, the effective potential satisfies

$$x \longrightarrow \pm\infty, \qquad V(x) \longrightarrow 0.$$

Hence, at the horizon and infinity the solution of (3.21) is just an ordinary plane wave solution

$$\begin{aligned} \Psi(t,r) &\sim e^{-i\omega t}e^{\pm i\omega x} \qquad : x \longrightarrow -\infty, \\ \Psi(t,r) &\sim e^{-i\omega t}e^{\pm i\omega x} \qquad : x \longrightarrow \infty. \end{aligned}$$

The boundary conditions for the quasinormal modes are the purely ingoing modes at an event horizon and outgoing at infinity. This means that the physical field can radiate away to both asymptotic regions and disappear from the study region.

$$\begin{aligned} \text{Ingoing} &: \psi(x) \sim e^{-i\omega x}, x \longrightarrow -\infty, \\ \text{Outgoing} &: \psi(x) \sim e^{i\omega x} \;\;, x \longrightarrow \infty. \end{aligned}$$

These boundary conditions imply an allowance of energy loss assumption. In addition, there is only a discrete set of complex frequencies which satisfy these boundary conditions [30]. These frequencies are the quasinormal frequencies and the wave function $\phi(x)$ are defined to be the quasinormal modes.

It has been proved by Vishveshwara [8] that for the Schwarzschild black hole, quasinormal frequencies must be negative imaginary. This reflects the fact that quasinormal frequencies are exponentially decay in time, more physically, that means the black hole geometry is losing its energy via the gravitational waves. For other geometry such as Schwarzschild de-Sitter/anti-de-Sitter or Kerr, one finds that the quasinormal frequencies has also been verified as a negative imaginary. It is very difficult to solve the Regge-Wheeler equation (3.21) and obtain numerical values for the quasinormal frequencies since they have imaginary parts. However, Chandrasekhar and Detweiler [32] successfully solved some of the quasinormal modes for Schwarzschild geometry in 1975. Their work had inspired many numerical techniques which had been developed since then. One of the most popular techniques was first done by Leaver [9] in 1985. He calculated quasinormal modes for Schwarzschild and Kerr black hole by using the Frobenius or continued fraction method (this will be discussed again in chapter5). Based on Leaver's method, Nollert [10] was able to improve the convergence of an infinite continued fraction which made the higher modes more accurate. In 2000, Horowitz and Hubeny [33] were the first who calculate quasinormal modes for Schwarzschild anti-de sitter spacetime. This work becomes very important since it can be used for testing the AdS/CFT correspondence qualitatively. Despite that, it is a non-trivial task to calculate quasinormal modes but for a less complicated system like non-rotating BTZ black hole, it turns out that one can arrange the field equation (3.21) into a known differential equation and analytical solution can be obtained. This was first proved by Vitor et.al [11] in 2001. After that, Siopsis et.al [13] was able to derive an analytic approximate formula for quasinormal frequencies of a large AdS Schwarzschild in five dimensions. For more detail on various numerical techniques see [27, 31] and references therein.

In this note, we will only focus on the two following computational methods. First, the continued fraction will be used in chapter 4 when we deal with quasinormal modes of four dimensional Schwarzschild black holes. For the rest, we will apply an analytical method to obtain quasinormal frequencies of black holes in three and five dimensions.

3.4 Application of Quasinormal Modes

Black holes are nontrivial solutions of the general relativity. They let us investigate many rich physics in a strong gravitational regime. They have also been called a “hydrogen atom” in general relativity [30]. Like hydrogen atom in quantum mechanics, black holes are governed by a few parameters (mass, charge, angular momentum) which made them easy to study. All of the general relativistic properties are embedded in black holes. These give some notably importance of black holes in fundamental physics theory. Now, we shall investigate further about the application of their unique vibration.

There are three main reasons for studying the physics of quasinormal modes which are discussed as follows.

3.4.1 AdS/CFT Correspondence

At present, standard model is the best theory in describing the microscopic world. It has made many important predictions of the existence of the fundamental particles/anti-particles. Some of its predictions were experimentally confirmed at the high level of accuracy. The standard model based on prior theory called quantum field theory (QFT). In QFT’s paradigm, particles were represented as a quantization of a particular field. Thus, in QFT, fundamental particles were replaced by fields. However, up until now, we have not ever been able to incorporate QFT with gravity. Since, they are contradictory both in philosophical level and mathematic structure.

So far, there are two candidates for the theory which combines QFT and gravity together, they are string theory and loop quantum gravity theory. In this context, we shall only discuss the string theory. In string theory, we always assume that all the fundamental particles can be represented as strings. Since string theory attempts to describe gravity at small scale, we might say that string theory contains gravity. On the other hand, quantum chromodynamics (QCD) is a theory which describes matters in the nuclear level namely, quarks and gluons. QCD is based on gauge group $SU(3)$. This may be interpreted by saying that quarks have three colors [30]. Surprisingly, t’ Hooft suggested that QCD has asymptotic freedom. That means as energy decreases the effective coupling constant increases and vice versa. This is a crucial point, at low energies; QCD becomes nonperturbative theory since the coupling constant is becomes strong. Hence, perturbative calculation of QCD at low energies cannot be performed. Additionally, as we expand to higher order of perturbation, the higher order contributions cannot be neglected. This causes a major problem for the development of QCD. Fortunately, there exists a gauge-gravity duality suggested by Maldacena namely, AdS/CFT correspondence. It states that physics in a bulk d-dimensional anti-de sitter (AdS) space is dual to physics at the AdS (d-1)-dimensional boundary.

AdS/CFT was originally motivated by the duality between type IIB string theory in $AdS_5 \times S^5$ (gravity side) and four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (non-gravity side). This duality allows us to perform a calculation for a thermalization timescale in CFT by investigating the physics in AdS space. According to AdS/CFT conjecture, large static black holes in AdS can be approximately compared to the thermal equilibrium in dual CFT, perturbing the black holes means perturbing that thermal state. The decay of oscillation (quasinormal frequencies) is equivalently to the relaxation of the system back to thermal equilibrium [36]. Thus, we can obtain the thermalization timescale in CFT by calculating the quasinormal frequencies of black holes in the anti-de sitter space. Consequently, quasinormal modes were used as a mathematical tool which allows us to avoid many difficulties of the calculation in the CFT. The relation between quasinormal modes of black holes in the bulk and the conformal field theory in the boundary was first suggested by Horowitz and Hubeny [33].

3.4.2 Black Hole Area Quantization

In order to reconcile gravity and quantum mechanics, besides the string theory which we have already discussed, there is another attempt to develop such a theory namely, the loop quantum gravity (LQG). LQG proposes the quantum mechanics of spacetime or quantum geometry that means in LQG gravitational field (spacetime) can be quantized. The strong point over string theory is that LQG needs no existence of the higher dimensions since it has never been observed before. On the other hand, black holes play an ideal model of gravity, thus the quantization of black holes might gives us an initial footstep to the theory of quantum gravity. The first attempt of quantized black hole was pioneered by Bekenstein [37]. His concept was originally based on the idea about the (horizon) area of non-extremal black hole act as a classical *adiabatic invariant* which correlates with the discrete quantum spectrum. Therefore, Bekenstein proposed that the area of non-extremal quantum black hole should have a discrete spectrum. Inspired by Christodoulou's reversible processes and Heisenberg uncertainty principle, Bekenstein conjectured the area quantization formula [37] of quantum black hole

$$A_n = \gamma l_p^2 n, \qquad n = 1, 2, ..., \tag{3.24}$$

where γ is a dimensionless parameter and $l_p = \sqrt{\frac{\hbar G}{c^3}}$ is the Planck length. This formula can be implied that horizon area is consisted of small pieces of equal area γl_p^2. Each piece can be considered as degrees of freedom in which quantum mechanically being referred as distinct quantum states. By assumption that each patch is properly equivalent, thus the total number of quantum states is

$$\Omega = k^n = k^{\frac{A}{\gamma l_p^2}}, \tag{3.25}$$

where k is represented a number of quantum states. As a result of statistical physics, the entropy will relate with the black hole area horizon by $\ln \Omega$ (Boltzmann constant sets to be unity)

$$S_{BH} = \frac{A}{\gamma l_p^2} \ln k. \tag{3.26}$$

But we know the Hawking's entropy formula

$$S_{BH} = \frac{A}{4}. \tag{3.27}$$

By comparing these two formulas, we get

$$\gamma = 4 \ln k. \tag{3.28}$$

Hence, in order to complete an equation of area quantization, we need to determined the constant k. The proposal to evaluate this parameter was provided by Hod [38] in 1998. Hod's idea is based on the Bohr's correspondence principle: *transition frequencies at large quantum numbers should equal classical oscillation frequencies* [30]. The word large quantum numbers can be inferred to the asymptotic region of the classical oscillating quasinormal frequencies. This statement are consistent with the phrase *quantum transition does not take time* which implicitly associated these oscillations with the highly damped quasinormal frequencies [30]. As we already mentioned, Nollert's technique allows us to compute the highly damped quasinormal frequencies of Schwarzschild black hole that take the form [30]

$$M\omega_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + \mathcal{O}\left[(n+1)^{-1/2}\right]. \tag{3.29}$$

Thus, Hod made further observed that the numerical value of 0.0437123 agrees with $\frac{\ln 3}{8\pi}$. Thus Hod found the asymptotic structure ($n \longrightarrow \infty$) of the above equation

$$M\omega_n = \frac{\ln 3}{8\pi} - \frac{i}{4}\left(n + \frac{1}{2}\right). \tag{3.30}$$

In addition, let's consider the area of the Schwarzschild black hole

$$\begin{aligned} A &= 4\pi r_s^2, \\ &= 16\pi M^2, \end{aligned} \qquad (3.31)$$

where $r_s = 2M$ is Schwarzschild radius. Using relation $dM = E = \hbar\omega$ and from (3.30) $Re(M\omega_n) = \frac{\ln 3}{8\pi}$. Therefore we conclude that

$$\gamma = 4\ln 3. \qquad (3.32)$$

Finally, the black hole area quantization formula is obtained

$$A_n = 4\; n\; l_p^2 \ln 3, \qquad n = 1, 2, 3, ... \quad . \qquad (3.33)$$

We see that with the help of the highly damped quasinormal frequencies of Schwarzschild black hole, the complete equation for area quantization was obtained. However, there is no a physical insight to relate those two subjects consistently, quasinormal modes and LQG. It is just a numerical curiosity of Hod which resulted in this prediction. There are still a lot of works to do in order to confirm the deep connection between those two theories. Nevertheless, motivated by Hod's idea, Dreyer followed the same argument to fix a *Barbero-Immirzi parameter* which emerged in loop quantum gravity [30].

3.4.3 Black Hole Parameters Estimation

In astronomy, it is very important to observe many astrophysical objects and their phenomena. Besides, astronomers must look for information from those objects as well. Practically, they can gather several data by observing stellar or solar oscillations. Such data can tell us information about the internal structure of a star. One would expect that, in the similar way, it is possible to extract individual information from black holes. As an astronomical object, black holes are believed to be a supernova remnant. After supernova explosion, a left-out compact object will violently oscillate for a short period of time. Then gravitational radiation will carry away the energy to infinity and the initial oscillation will exponentially damp out. These gravitational waves also carry out information about the compact object too. According to quasinormal modes theory, this information could be referred as black hole parameters. However, gravitational waves have not been detected yet, but the indirect effect has been accurately investigated by Hulse and Taylor for a binary pulsar system [30]. Nowadays, many gravitational waves detectors are now operating and starting to collect the data, for instance, LIGO and LISA. However, the weakness of the gravitational signal makes it hard to detect. In order to increase the chance, we have to extend the sensitivity of our detector and look for a very strong source of gravitational waves such as, black holes collision and massive binary stellar system. The quasinormal modes process take place in the final stage of the gravitational signal. On the other hand, since quasinormal modes are decaying with time thus only the fundamental modes (lowest imaginary part) can reach us. As we already mentioned earlier, quasinormal frequencies depend on the black hole intrinsic parameters. To illustrate this point, for Kerr black holes, the quasinormal frequencies depend only on black hole mass and its angular momentum. Therefore, one can approximately extract the black hole parameters solely from the observation of frequencies and damping time. For a rotating black hole, one can approximate an analytic formula which relates the frequencies and damping time with black hole parameters. These may take the form [34, 35]

$$\begin{aligned} M\omega &\approx \left[1 - \frac{63}{100}(1-a)^{3/10}\right] \approx (0.37 + 0.19a), \\ \tau &\approx \frac{4M}{(1-a)^{9/10}}\left[1 - \frac{63}{100}(1-a)^{3/10}\right]^{-1} \approx M(1.48 + 2.09a). \end{aligned}$$

From these two equations, if we are able to measure the ringing frequencies and damping time, thus, we could in principle obtain the black hole parameters by inverting both formulae.

So far, we have discussed the definition of quasinormal modes and investigated the major equation governing the perturbations in the black hole geometry. At the end, we have investigated some applications of the quasinormal modes of black holes. In the following chapter, we will begin to investigate the quasinormal modes and quasinormal frequencies of the black holes in three dimensional spacetime.

Chapter IV

QUASINORMAL MODES OF THREE DIMENSIONAL BLACK HOLES

"God used beautiful mathematics in creating the world."

-P. Dirac

In the previous section, we discussed the definition of quasinormal modes and the application of quasinormal modes to other fundamental physics. Now, we are in a good position to investigate the calculation of quasinormal modes in detail. We review the study of quasinormal modes of three dimensional black hole solutions. Despite that, there is no black hole solution exists in three dimensional spacetime with asymptotical flatness. It turns out that one could construct the black hole metric by introducing the cosmological constant to the Einstein field equations. Hence, it is very interesting to study the quasinormal modes of the three dimensional black holes. We shall first investigate the scalar perturbation in BTZ black hole and compute its quasinormal frequencies. Subsequently, quasinormal modes of a massive scalar field in rotating BTZ metric are calculated. Finally, we determine quasinormal modes of three dimensional AdS-Schwarzschild black hole. It appears that all of three cases provide us some possibility to obtain quasinormal frequencies analytically. This is a crucial point of the study of quasinormal modes of black hole in three dimensional spacetime.

4.1 Quasinormal Modes of BTZ Black Hole

So far, we have only discussed the quasinormal modes of asymptotically flat black holes. Now, the quasinormal modes of BTZ black hole will be investigated. As already mentioned in Chapter 2, BTZ black hole is an exact solution incorporative with negative cosmological constant that emerges in (2+1) dimensional spacetime. The BTZ solution has asymptotically anti-de sitter spacetime. According to AdS/CFT conjecture, BTZ black hole allows us to compute thermalization timescale in the dual 2d conformal field theory which was verified by Birmingham [36]. Quasinormal modes of AdS black holes were first discussed by Horowitz and Hubeny [33] but they concerned only black holes which live in four, five and seven dimensions. However, quasinormal modes of BTZ black hole were successfully computed by Cardoso and Lemos [11]. Surprisingly, they could manage to reduce the wave equation (3.2) to the known differential equation and finally obtained an analytical formula for BTZ quasinormal frequencies. This made BTZ an important example because it shows us the possibility to study quasinormal modes analytically.

In this section, we review the study of quasinormal modes of BTZ black hole as follows. First, we reconsider the BTZ solution and determine its metric tensor. Second, we construct the wave equation by using the Klein-Gordon equation in curved background (3.11). Then, the wave equation will be transformed to the known equation called *hypergeometric equation*. Finally, quasinormal frequencies are obtained. Most of this section is based on Vitor Cardoso's work [11].

Let us recall the BTZ metric as defined in Chapter2 (opposite in signature)

$$ds^2 = \left(-M + \frac{r^2}{l^2}\right) dt^2 - \left(-M + \frac{r^2}{l^2}\right)^{-1} dr^2 - r^2 d\phi^2, \tag{4.1}$$

where we choose the cosmological constant to be $\Lambda = -1/l^2$ and $-\infty < t < \infty,\ 0 < r < \infty,\ 0 \leq \phi \leq 2\pi$. Then both event horizons are determined

$$r_\pm = \pm l\sqrt{M}. \tag{4.2}$$

In our study, we normally regard the horizon as the outer horizon r_+. From the metric above, we can compute the metric tensor and its inverse as

$$g_{\mu\nu} = \begin{pmatrix} -M + \frac{r^2}{l^2} & 0 & 0 \\ 0 & -\left(-M + \frac{r^2}{l^2}\right)^{-1} & 0 \\ 0 & 0 & -r^2 \end{pmatrix},$$

$$g^{\mu\nu} = \begin{pmatrix} \left(-M + \frac{r^2}{l^2}\right)^{-1} & 0 & 0 \\ 0 & -\left(-M + \frac{r^2}{l^2}\right) & 0 \\ 0 & 0 & -\frac{1}{r^2} \end{pmatrix}.$$

The determinant of metric tensor is $\sqrt{-g} = r$. Now, all the prescription we need to construct the wave equation is obtained. We therefore formulate the Schrödinger-like equation in the next section.

4.1.1 Scalar Perturbation Around The BTZ Black Hole

The dynamics of a real massless scalar field in curved background is described by

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi(x)) \;=\; 0. \tag{4.3}$$

Then, we can write down all of non-vanishing components as

$$\frac{1}{\sqrt{-g}}\left[\partial_0(g^{00}\sqrt{-g}\partial_0\Phi)+\partial_1(g^{11}\sqrt{-g}\partial_1\Phi)+\partial_2(g^{22}\sqrt{-g}\partial_2\Phi)\right] \;=\; 0, \tag{4.4}$$

where we choose $(0,1,2)=(t,r,\phi)$. To separate each variable, we use the following ansatz

$$\Phi(x) \;=\; \frac{\Psi(r)}{\sqrt{r}}e^{-i\omega t}e^{im\phi}, \tag{4.5}$$

where m is angular quantum number. Substitute all of the results into the (4.4), we get

$$\begin{aligned}
g^{00}\partial_0^2\Phi &= \frac{-\omega^2\Phi}{A(r)},\\
g^{22}\partial_2^2\Phi &= \frac{m^2\Phi}{r^2},\\
\partial_1(\sqrt{-g}g^{11}\partial_1\Phi) &= -\left(\frac{A(r)\Psi''}{\sqrt{r}}+\frac{2\sqrt{r}\Psi'}{l^2}+\frac{3A(r)\Psi}{4r^{5/2}}-\frac{A(r)\Psi}{2r^{5/2}}-\frac{\Psi}{\sqrt{r}l^2}\right),
\end{aligned}$$

where we have used $A(r)\equiv -M+\frac{r^2}{l^2}$ and "prime" denotes derivative with respect to the radial coordinate r. Thus, we obtain the wave equation which takes the following form

$$A(r)^2\Psi''(r)+\frac{2rA(r)}{l^2}\Psi'(r)-\left(-\omega^2+\frac{m^2A(r)}{r^2}-\frac{A(r)^2}{4r^2}+\frac{A(r)}{l^2}\right)\Psi(r) \;=\; 0. \tag{4.6}$$

Note that, from the three variables equation, it reduces to only a radial equation since other variables can be factored out. Additionally, lets introduce the tortoise coordinate r_* which is defined by

$$dr_* \;=\; \left(-M+\frac{r^2}{l^2}\right)^{-1}dr,$$

or,

$$r \;=\; -\sqrt{M}l\coth\left(\frac{\sqrt{M}r_*}{l}\right).$$

Here, we see that as $r \longrightarrow r_+$ corresponds to $r_* \longrightarrow \infty$ and $r \longrightarrow \infty$ corresponds to $r_* \longrightarrow 0$. Under the tortoise coordinate, we use the following chain rules

$$\frac{d}{dr}\left(\frac{d\Psi}{dr}\frac{dr}{dr_*}\right)\frac{dr}{dr_*} = A^2\Psi'' + \frac{2rA}{l^2}\Psi',$$

then using the above relation to transform (4.6) into the Regge-Wheeler equation.

$$\frac{d^2\Psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\Psi = 0, \tag{4.7}$$

where (See appendix C.1 for the plotting of the effective potential.)

$$V(r) = \frac{3r^2}{4} - \frac{Mm^2}{r^2} - \frac{M^2}{4r^2} - \frac{M}{2} + m^2, \tag{4.8}$$

and we have set $l = 1$. Thus, we need to solve the above equation with the appropriate boundary condition to obtain the quasinormal frequencies.

4.1.2 The Exact Solution

In order to obtain an analytical formula for quasinormal frequencies, we first rewrite the effective potential in the tortoise coordinate

$$\frac{d^2\Psi}{dr_*^2} + \left[\omega^2 - \frac{3M}{4\sinh^2(\sqrt{M}r_*)} - \frac{M}{4\cosh^2(\sqrt{M}r_*)} - \frac{m^2}{\cosh^2(\sqrt{M}r_*)}\right]\Psi = 0. \tag{4.9}$$

We introduce a new variable $x = \frac{1}{\cosh^2(\sqrt{M}r_*)}$. Hence, x has a range from $[0,1]$ these correspond to horizon and infinity respectively. Let's consider,

$$\frac{dx}{dr_*} = -2\sqrt{M}\text{sech}^2(\sqrt{M}r_*)\tanh(\sqrt{M}r_*),$$
$$\frac{d^2x}{dr_*^2} = -2M\text{sech}^4(\sqrt{M}r_*) + 4M\text{sech}^2(\sqrt{M}r_*)\tanh^2(\sqrt{M}r_*).$$

Hence,

$$\frac{d}{dx}\left(\frac{d\Psi}{dx}\frac{dx}{dr_*}\right)\frac{dx}{dr_*} = 4Mx^2(1-x)\frac{d^2\Psi}{dx^2} + 2M(2x-3x^2)\frac{d\Psi}{dx}.$$

Thus, we substitute this result back to (4.9) and obtain the canonical form of 2nd-order differential equation with variable x

$$4x^2(1-x)\frac{d^2\Psi}{dx^2} + (4x-6x^2)\frac{d\Psi}{dx} + \tilde{V}(x)\Psi = 0, \tag{4.10}$$

where effective potential is defined as

$$\tilde{V}(x) = \frac{1}{4x(1-x)}\left[\frac{4\omega^2(1-x)}{M} - 3M - x(1-x) - \frac{4m^2x(1-x)}{M}\right]. \tag{4.11}$$

In the final step, we replace the wave function Ψ with

$$\Psi = \frac{(x-1)^{3/4}}{x^{\frac{i\omega}{2M^{1/2}}}} y. \tag{4.12}$$

Finally, we obtain

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0, \tag{4.13}$$

and,

$$\begin{aligned} a &= 1 + \frac{im}{2\sqrt{M}} - \frac{i\omega}{2\sqrt{M}}, \\ b &= 1 - \frac{im}{2\sqrt{M}} - \frac{i\omega}{2\sqrt{M}}, \\ c &= 1 - \frac{i\omega}{\sqrt{M}}. \end{aligned}$$

This is the canonical form of hypergeometric differential equation whose solution is a *hypergeometric function*. Now, we need to solve (4.13) with the appropriate boundary conditions. Normally, boundary conditions of quasinormal modes are defined as follows, *(i)* ingoing modes are only allowed at the black hole's horizon $e^{i\omega r_*}$, *(ii)* near infinity, only outgoing modes exists $e^{-i\omega r_*}$. However, from the effective potential (4.8), one can see that as $r \longrightarrow \infty$ then $V(r)$ becomes divergent. Therefore, the field at infinity must be vanished. One of the solutions of (4.13) takes a following form [39]

$$y = (1-x)^{c-a-b} \; {}_2F_1(c-a, c-b, c; x), \tag{4.14}$$

here ${}_2F_1$ is the standard hypergeometric function of the second kind. This solution must satisfy the boundary condition $x = 1$,$y = 0$. Since, $c-a-b = -1$ we therefore find that F must be zero at $x = 1$. Let's consider the following identity

$${}_2F_1(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Hence,

$${}_2F_1(c-a, c-b, c; 1) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

Note that, from now on we shall denote the standard form of hypergeometric function of the second kind with $F(x)$. To satisfy the boundary conditions, we finally get

$$a = -n \quad \text{or} \quad b = -n, \tag{4.15}$$

where $n = 0, 1, 2, ...$, these also yield

$$\omega = \pm m - 2i\sqrt{M}(n+1). \tag{4.16}$$

4.1.3 Results and Discussion

So far, the scalar perturbation of non-rotating BTZ black hole has been reviewed. Using an analytical method, one obtains the exact solution for the quasinormal frequencies of such a black hole. These frequencies are obtained as shown in (4.16). As promised, they depend on black hole intrinsic parameter (mass) and also have negative imaginary part. Some of the result is demonstrated by Table 4.1 . One sees that the real part depends only on the angular quantum number m whereas the imaginary part scale with the horizon or black hole's radius (recall $r_+ = \sqrt{M}l$). This result also agrees well with the numerical calculation which had been done by

m=0,n=0		
$\sqrt{M}$	ω_R	$-\omega_I$
1	0	2
2	0	4
3	0	6
4	0	8
5	0	10

Table 4.1: Fundamental modes (n=0) of quasinormal frequencies of a BTZ black hole

Vitor's work [11]. One can use this result to further investigate on AdS/CFT as a toy model. Also, this calculation technique has opened many later works which attempt to determine quasinormal frequencies analytically.

4.2 Quasinormal Modes of Rotating BTZ Black Hole

In the previous section, we have discussed the quasinormal modes of non-rotating BTZ black hole. Now, we will move to a little bit more complicated system. Rotating BTZ black hole, it is also an exact solution in three dimensional stationary spacetime with the presence of a negative cosmological constant. The first study on quasinormal modes of rotating BTZ black hole was investigated by Birmingham [12]. He suggested the relation between quasinormal modes and Choptuik scaling parameter. Via AdS/CFT, he found that one can interpret Choptuik parameter as timescale for the returning to thermal equilibrium in dual conformal field theory. Moreover, Birmingham also found the analytical way to compute the quasinormal frequencies of this black hole. We shall therefore follow the argument in [12] for this section. Let's recall rotating BTZ metric in the following form

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2, \tag{4.17}$$

again $J = Ma$ is black hole total angular momentum. From the metric above, one can calculate event horizons by $g_{00} = 0$

$$r_\pm = l \left[\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right)\right]^{1/2}, \tag{4.18}$$

where $r_\pm$ denote outer/inner horizon and we choose $\Lambda = -1/l^2$. Moreover, one can easily prove

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2r_+ r_-}{l}. \tag{4.19}$$

The components of metric tensor are calculated to be

$$g_{\mu\nu} = \begin{pmatrix} -\left(-M + \frac{r^2}{l^2}\right) & 0 & -\frac{J}{2} \\ 0 & \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} & 0 \\ -\frac{J}{2} & 0 & r^2 \end{pmatrix}.$$

It is non-trivial to compute the inverse metric. We use the following relation; $g_{\mu\nu} g^{\mu\lambda} = \delta^\lambda_\nu$ to determine the inverse metric

$$g^{\mu\nu} = \begin{pmatrix} \frac{r^2}{\left(Mr^2 - \frac{J^2}{4} - \frac{r^4}{l^2}\right)} & 0 & \frac{J}{2\left(Mr^2 - \frac{J^2}{4} - \frac{r^4}{l^2}\right)} \\ 0 & \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) & 0 \\ \frac{J}{2\left(Mr^2 - \frac{J^2}{4} - \frac{r^4}{l^2}\right)} & 0 & \frac{\left(M - \frac{r^2}{l^2}\right)}{\left(Mr^2 - \frac{J^2}{4} - \frac{r^4}{l^2}\right)} \end{pmatrix}.$$

Finally, the determinant of the metric tensor is $\sqrt{-g} = r$.

4.2.1 Wave Equation of Massive Scalar Field

The equation of motion for a real massive scalar field in curved spacetime is described by

$$\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi(x)) - \frac{\mu}{l^2} \Phi(x) = 0, \tag{4.20}$$

where mass of scalar field is denoted by μ. As before, we use $(t,r,\phi) \longrightarrow (0,1,2)$. We can write down all the non-vanishing components of (4.20)

$$\begin{aligned}\frac{1}{\sqrt{-g}}[\partial_0(g^{00}\sqrt{-g}\partial_0\Phi + g^{20}\sqrt{-g}\partial_2\Phi) + \partial_1(g^{11}\sqrt{-g}\partial_1\Phi)& \\ +\partial_2(g^{02}\sqrt{-g}\partial_0\Phi + g^{22}\sqrt{-g}\partial_2\Phi)] - \frac{\mu}{l^2}\Phi &= 0 \qquad (4.21)\end{aligned}$$

Using the following ansatz

$$\Phi = R(r)e^{-i\omega t}e^{im\phi}, \qquad (4.22)$$

where again m is angular quantum number. Then substitute the ansatz into (4.21),

$$\begin{aligned}\partial_1(g^{11}\sqrt{-g}\partial_1\Phi) &= \left[\left(Ar + \frac{J^2}{4r}\right)R'' + \left(A + \frac{2r^2}{l^2} - \frac{J^2}{4r^2}\right)R'\right]e^{-i\omega t}e^{im\phi}, \\ \partial_0(g^{00}\sqrt{-g}\partial_0\Phi + g^{20}\sqrt{-g}\partial_2\Phi) &= \left[\frac{\omega^2 r}{\left(\frac{J^2}{4r^2}+A\right)} - \frac{Jm\omega}{2r\left(\frac{J^2}{4r^2}+A\right)}\right]R(r)e^{-i\omega t}e^{im\phi}, \\ \partial_2(g^{02}\sqrt{-g}\partial_0\Phi + g^{22}\sqrt{-g}\partial_2\Phi) &= -\left[\frac{Jm\omega}{2r\left(\frac{J^2}{4r^2}+A\right)} + \frac{Am^2}{r\left(\frac{J^2}{4r^2}+A\right)}\right]R(r)e^{-i\omega t}e^{im\phi},\end{aligned}$$

where $A \equiv \left(-M + \frac{r^2}{l^2}\right)$ and prime denotes derivative with respect to the radial coordinate r. Eventually, here comes the Schrödinger-like equation in rotating BTZ background

$$\begin{aligned}\left(A + \frac{J^2}{4r^2}\right)R'' + \left(\frac{A}{r} + \frac{2r}{l^2} - \frac{J^2}{4r^3}\right)R' +& \\ \left(\frac{\omega^2}{(A+\frac{J^2}{4r^2})} - \frac{Jm\omega}{r^2(A+\frac{J^2}{4r^2})} - \frac{Am^2}{r^2(A+\frac{J^2}{4r^2})} - \frac{\mu}{l^2}\right)R &= 0.\end{aligned}$$

Then, we introduce a new variable

$$\begin{aligned}z &= \frac{r^2 - r_+^2}{r^2 - r_-^2}, \qquad (4.23) \\ r &= \pm\frac{\sqrt{zr_-^2 - r_+^2}}{\sqrt{z-1}}.\end{aligned}$$

Note that $r = r_+$ corresponds with $z = 0$ and $r = \infty$ corresponds with $z = 1$. Now consider,

$$\begin{aligned}\frac{dr}{dz} &= \frac{1}{2}\left[\frac{r_+^2 - r_-^2}{(z-1)^{3/2}(zr_-^2 - r_+^2)^{1/2}}\right], \\ \frac{d^2r}{dz^2} &= \frac{r_+^2 - r_-^2}{4}\left[\frac{3r_+^2 + r_-^2(1-4z)}{(z-1)^{5/2}(zr_-^2 - r_+^2)^{3/2}}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dR}{dz} &= \frac{R'}{2}\left[\frac{r_+^2 - r_-^2}{(z-1)^{3/2}(zr_-^2 - r_+^2)^{1/2}}\right], \\ \frac{d^2R}{dz^2} &= \frac{R''}{4}\left[\frac{r_+^2 - r_-^2}{(z-1)^{3/2}(zr_-^2 - r_+^2)^{1/2}}\right]^2 + \frac{R'(r_+^2 - r_-^2)}{4}\left[\frac{3r_+^2 + r_-^2(1-4z)}{(z-1)^{5/2}(zr_-^2 - r_+^2)^{3/2}}\right].\end{aligned}$$

Now, we transform the following term into z variable

$$
\begin{aligned}
\left(A+\frac{J^2}{4r^2}\right) &\longrightarrow \frac{z(r_+^2-r_-^2)^2}{l^2(z-1)(zr_-^2-r_+^2)},\\
\left(\frac{A}{r}+\frac{2r}{l^2}-\frac{J^2}{4r^3}\right) &\longrightarrow \frac{\left(r_+^2-r_-^2\right)\left[r_+^2(2+z)-r_-^2z(1+2z)\right]}{l^2(z-1)^{1/2}(zr_-^2-r_+^2)^{3/2}},\\
\left(\frac{\omega^2}{(A+\frac{J^2}{4r^2})}-\frac{Jm\omega}{r^2(A+\frac{J^2}{4r^2})}-\frac{Am^2}{r^2(A+\frac{J^2}{4r^2})}\right) &\longrightarrow \frac{(z-1)}{z\,(r_+^2)-r_-^2)^2}\big[z(r_-^2-r_+^2)l^2\omega^2-\\
&\quad 2m\omega r_-r_+(z-1)l-m^2(r_-^2-zr_+^2)\big],
\end{aligned}
$$

Ultimately, we obtain the radial equation

$$
z(1-z)\frac{d^2R}{dz^2}+(1-z)\frac{dR}{dz}+\left(\frac{\tilde{A}}{z}+B+\frac{C}{1-z}\right)R = 0. \quad (4.24)
$$

where,

$$
\begin{aligned}
\tilde{A} &= \frac{l^4}{4(r_+^2-r_-^2)^2}(\omega r_+-\frac{m}{l}r_-)^2,\\
B &= -\frac{l^4}{4(r_+^2-r_-^2)^2}(\omega r_--\frac{m}{l}r_+)^2,\\
C &= -\frac{\mu}{4}.
\end{aligned}
$$

Let's define a new radial variable

$$
R(z) = z^{\alpha}(1-z)^{\beta}F(z). \quad (4.25)
$$

Then, the radial equation turns into the hypergeometric differential equation [39]

$$
z(1-z)F''+[c-(a+b+1)z]F'-abF = 0, \quad (4.26)
$$

where prime represents derivative with respect to the variable z. Here,

$$
\begin{aligned}
c &= 2\alpha+1,\\
a+b &= 2\alpha+2\beta,\\
ab &= (\alpha+\beta)^2-B,\\
\alpha^2 &= -\tilde{A},\\
\beta &= \frac{1}{2}(1\pm\sqrt{1+\mu}).
\end{aligned} \quad (4.27)
$$

We choose $\alpha=-i\sqrt{\tilde{A}}$ and $\beta=\frac{1}{2}(1-\sqrt{1+\mu})$. At horizon $z=0$, there are two linearly independent solutions of (4.26) which take the form $F(a,b,c,z)$ and $z^{1-c}F(a-c+1,b-c+1,2-c,z)$ [39]. According to the definition of the quasinormal modes, the purely ingoing modes at horizon are given by

$$
R(z) = z^{\alpha}(1-z)^{\beta}F(a,b,c,z). \quad (4.28)
$$

Then, using the linear transformation which is given by [39]. Therefore, the above solution can be transformed to solution at infinity $z = 1$

$$R(z) = z^{\alpha}(1-z)^{\beta}(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1,1-z) \\ +z^{\alpha}(1-z)^{\beta}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z). \tag{4.29}$$

Obviously, the first term vanishes when $z = 1$ since $\beta + c - a - b$ is always positive. But we need the boundary condition that the field must completely vanish at infinity. So, this gives us a constraint on the second term

$$c - a = -n, \qquad c - b = -n, \tag{4.30}$$

where ($n = 0, 1, 2, ...$). Consequently, we obtain the quasinormal frequencies of rotating BTZ black hole

$$\begin{aligned} \omega_L &= \frac{m}{l} - \frac{2i(r_+ + r_-)}{l^2}\left[n + \frac{1}{2} + \frac{\sqrt{1+\mu}}{2}\right], \\ \omega_R &= -\frac{m}{l} - \frac{2i(r_+ + r_-)}{l^2}\left[n + \frac{1}{2} + \frac{\sqrt{1+\mu}}{2}\right]. \end{aligned} \tag{4.31}$$

4.2.2 Results and Discussion

These two results have negative imaginary part and also depend on the black hole parameters which are mass and spin parameter (hidden in r_+, r_-). These results also agree with the previous case (non-rotating) by setting $l = 1, J = 0, \mu = 0$, The above relations will reduce to (4.16). In the original paper [12], they further investigate the relation of quasinormal modes and Choptuik parameter but we will not discuss it since it is beyond our scope here. However, this example also shows us an important property of the quasinormal modes of three dimensional black holes in which one could possibly obtain an analytical formula of quasinormal frequencies.

4.3 Quasinormal Modes of 3d AdS-Schwarzschild Black Hole

So far, we have seen two examples of the computation about the quasinormal modes of three dimensional black holes. In those cases, one can manage to reduce the wave equation to the hypergeometric differential equation and determine the quasinormal frequencies. Similarly, in this section, we shall further study on three dimensional black hole namely, three dimensional AdS-Schwarzschild black hole. This work was pioneered by Siopsis and Musiri [13]. They proposed the perturbative calculation to obtain the quasinormal frequencies of AdS-Schwarzschild black hole in three and five dimensions. Hence, this section is covered by their work. Let's begin by introducing d-dimensional AdS Schwarzschild metric

$$ds^2 = -\left(\frac{r^2}{R^2} + 1 - \frac{\omega_{d-1}M}{r^{d-3}}\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{R^2} + 1 - \frac{\omega_{d-1}M}{r^{d-3}}\right)} + r^2 d\Omega^2_{d-2}, \quad (4.32)$$

where R is AdS radius and ω_{d-1} is gravitational constant. We consider in a large black hole, then, the metric takes the form

$$ds^2 = -\left(\frac{r^2}{R^2} - \frac{\omega_{d-1}M}{r^{d-3}}\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{R^2} - \frac{\omega_{d-1}M}{r^{d-3}}\right)} + r^2 ds^2(E_{d-2}). \quad (4.33)$$

So, we can derived black hole's horizon by setting $g_{00} = 0$

$$r_+ = R\left[\frac{\omega_{d-1}M}{R^{d-3}}\right]^{1/(d-1)}. \quad (4.34)$$

For now we consider in three dimensional ($d = 3$) case, hence, the metric becomes

$$ds^2 = -\frac{1}{R^2}\left(r^2 - \omega_2 M R^2\right)dt^2 + \frac{1}{R^2}\left(r^2 - \omega_2 M R^2\right)^{-1} dr^2 + r^2 dx^2. \quad (4.35)$$

The event horizon takes the form $r_+^2 = \omega_2 M R^2$. Therefore, we can rearrange the above metric into

$$ds^2 = -\frac{1}{R^2}\left(r^2 - r_+^2\right)dt^2 + \frac{R^2 dr^2}{\left(r^2 - r_+^2\right)}dr^2 + r^2 dx^2. \quad (4.36)$$

Then, the metric tensor and its inverse are

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\left(r^2 - r_+^2\right)}{R^2} & 0 & 0 \\ 0 & \frac{R^2}{\left(r^2 - r_+^2\right)} & 0 \\ 0 & 0 & r^2 \end{pmatrix},$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{R^2}{\left(r^2 - r_+^2\right)} & 0 & 0 \\ 0 & \frac{\left(r^2 - r_+^2\right)}{R^2} & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}.$$

The determinant of metric tensor is defined by $\sqrt{-g} = r$.

4.3.1 The Wave Equation

We now consider massive Klein-Gordon equation in curved spacetime

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi(x)) - m^2\Phi(x) \;=\; 0. \tag{4.37}$$

Then, the spacetime indices are denoted by $(t, r, x) \longrightarrow (0, 1, 2)$. Let us write down all the non-zero components of the above equation

$$\frac{1}{\sqrt{-g}}\left[\partial_0(g^{00}\sqrt{-g}\partial_0\Phi) + \partial_1(g^{11}\sqrt{-g}\partial_1\Phi) + \partial_2(g^{22}\sqrt{-g}\partial_2\Phi)\right] - m^2\Phi(x) \;=\; 0.$$

The wave equation becomes

$$-\frac{R^2}{(r^2-r_+^2)}\partial_0^2\Phi + \frac{1}{R^2}\left[2r + \left(\frac{r^2-r_+^2}{R^2}\right)\right]\partial_1\Phi + \left(\frac{r^2-r_+^2}{r^2}\right)\partial_1^2\Phi + \frac{1}{r^2}\partial_2^2\Phi - m^2\Phi \;=\; 0. \tag{4.38}$$

The solution may be written as

$$\Phi \;=\; \psi(y)e^{-i\omega t}e^{-ipx}, \tag{4.39}$$

where we introduce new variable $y = \frac{r_+^2}{r^2}$. Hence, one can see that y is restricted within the range $0 < y < 1$. $y = 0$ corresponds with $r \longrightarrow \infty$ and $y = 1$ corresponds to $r \longrightarrow r_+$. Now using the following chain rule

$$\begin{aligned}
\frac{\partial\Phi}{\partial r} &= -\left(\frac{2r_+^2}{r^3}\right)\psi' e^{-i(\omega t+px)}, \\
\frac{\partial^2\Phi}{\partial r^2} &= \left(\frac{4r_+^4}{r^6}\right)\psi'' e^{-i(\omega t+px)} + \left(\frac{6r_+^2}{r^4}\right)\psi' e^{-i(\omega t+px)}.
\end{aligned}$$

Each component is calculated as follow

$$\begin{aligned}
\frac{1}{r^2}\partial_2^2\Phi &= -\frac{p^2}{r^2}\psi e^{-i(\omega t+px)}, \\
-\frac{R^2}{(r^2-r_+^2)}\partial_0^2\Phi &= \frac{\omega^2R^2}{(r^2-r_+^2)}\psi e^{-i(\omega t+px)}, \\
\left(\frac{r^2-r_+^2}{r^2}\right)\partial_1^2\Phi &= \left(\frac{r^2-r_+^2}{R^2}\right)e^{-i(\omega t+px)}\left[\left(\frac{4r_+^4}{r^6}\right)\psi'' + \left(\frac{6r_+^2}{r^4}\right)\psi'\right], \\
\frac{1}{R^2}\left[2r + \left(\frac{r^2-r_+^2}{R^2}\right)\right]\partial_1\Phi &= -\left(\frac{2r_+^2}{R^2r^3}\right)\left[2r + \frac{(r^2-r_+^2)}{r}\right]\psi' e^{-i(\omega t+px)}.
\end{aligned}$$

Finally, we obtain the radial equation

$$(y-1)y^2\left[(y-1)\psi'\right]' + \hat{\omega}^2 y\psi + \hat{p}^2 y(y-1)\psi + \frac{\hat{m}^2}{4}(y-1)\psi \;=\; 0, \tag{4.40}$$

where,

$$\hat{\omega} = \frac{\omega R^2}{2r_+}, \qquad \hat{p} = \frac{pR}{2r_+}, \qquad \hat{m} = mR. \tag{4.41}$$

Now, we shall investigate the solution on the near-horizon region. Let's consider

$$x \equiv (1-y) \simeq 0, \qquad (y \longrightarrow 1).$$

Then our equation takes the form

$$(1-x)^2 x^2 \frac{d^2\psi}{dx^2} + (1-x)^2 x \frac{d\psi}{dx} + \hat{\omega}^2 (1-x)\psi - \hat{p}^2 (1-x) x\psi - \frac{\hat{m}^2}{4} x\psi = 0,$$

as $x \longrightarrow 0$,

$$x^2 \frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx} + \hat{\omega}^2 \psi - \hat{p}^2 x\psi - \frac{\hat{m}^2}{4} x\psi = 0,$$

The solution may be written in the power series form

$$\begin{aligned} \psi &= \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \\ &= x^{\alpha}(a_0 + a_1 x + a_2 x + \ldots), \\ &\simeq x^{\alpha} a_0. \end{aligned}$$

Substitute this result into the above equation and keep only lowest order in x, it yields

$$\begin{aligned} \alpha^2 &= -\hat{\omega}^2, \\ \therefore \alpha &= \pm i\hat{\omega}. \end{aligned}$$

Hence, we obtain the solution of (4.40) in the near horizon limit $(y \longrightarrow 1)$

$$\psi \sim (1-y)^{\pm i\hat{\omega}}. \tag{4.42}$$

The asymptotic behavior at infinity $(y \longrightarrow 0)$ can be determined in the similar way. Then, the solution at infinity is defined as

$$\psi \sim y^{\beta_\pm}, \qquad \beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1+\hat{m}^2}\right). \tag{4.43}$$

Note for a massless case $(m = 0)$, $\beta_+ = 1, \beta_- = 0$. We now write the solution of (4.40)

$$\psi = y(1-y)^{i\hat{\omega}} F(y). \tag{4.44}$$

Finally, the radial equation become the standard hypergeometric equation (for a massless case) [39]

$$y(1-y)F'' + [2 - (3 + 2i\hat{\omega})y]F' - [\hat{p}^2 - 2i\hat{\omega} - \hat{\omega}^2 + 1]F = 0, \tag{4.45}$$

comparing with (4.26), we get

$$\begin{aligned} a &= 1 + i(\hat{p} + \hat{\omega}), \\ b &= 1 + i(\hat{p} - \hat{\omega}), \\ c &= 2. \end{aligned}$$

Then, the solution at infinity ($y \longrightarrow 0$) can be written as [39]

$$\psi = y(1-y)^{i\hat{\omega}} F(1 + i(\hat{p} + \hat{\omega}), 1 + i(\hat{p} - \hat{\omega}), 2; y). \tag{4.46}$$

Again, we use the following identity to transform the solution in the near horizon limit

$$\begin{aligned} F(a,b,c;y) = (1-y)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b, c-a-b+1; 1-y) \\ + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b-c+1; 1-y). \end{aligned} \tag{4.47}$$

Then, the above equation becomes

$$\begin{aligned} F(a,b,2,y) &= y(1-y)^{i\hat{\omega}} \frac{\Gamma(-2i\hat{\omega})}{\Gamma(1-i(\hat{\omega}+\hat{p}))\Gamma(1-i(\hat{\omega}-\hat{p})} F \\ & \quad + y(1-y)^{-i\hat{\omega}} \frac{\Gamma(2i\hat{\omega})}{\Gamma(1+i(\hat{\omega}+\hat{p})\Gamma(1+i(\hat{\omega}-\hat{p})} F \end{aligned} \tag{4.48}$$

For quasinormal modes, we need that only ingoing modes exist at the horizon. Then, the 2nd-term must vanish. We therefore obtain an analytical formula of quasinormal frequencies

$$\hat{\omega} = \pm\hat{p} - in, \qquad n = 1, 2, ... \tag{4.49}$$

4.3.2 Results and Discussion

Quasinormal frequencies of massless scalar field in three-dimensional AdS Schwarzschild black hole are investigated. As shown by (4.49), these frequencies also have negative imaginary part and depend only on black hole intrinsic parameter (hidden in $\hat{p}$). From the result, one sees that as the black hole radius (r_+) decrease the real part of quasinormal frequencies increase while the imaginary part depends on integer n only. However, Schwarzschild metric was originally discovered in four dimensional spacetime but the study of quasinormal modes in three dimensions shows the possibility to transform the wave equation into hypergeometric function and obtain an analytical formula for the quasinormal frequencies. This means that the lowest modes ($n = 0$) indicate the smallest damping rate. As we already mentioned in the previous section, the gravitational signals that reaches us are in these fundamental modes.

In the next chapter, we will continue the study on quasinormal modes of black hole in four dimensions by attempting to use another approach to determine the quasinormal frequencies.

Chapter V

QUASINORMAL MODES OF FOUR DIMENSIONAL BLACK HOLES

"What we observe is not nature itself, but nature exposed to our method of questioning."

-W. Heisenberg

In the previous section, we investigate the quasinormal modes of three dimensional black holes which one found that their wave equation can be reduced to the hypergeometric differential equation and obtain the quasinormal frequencies analytically. However, there is another way to determine those frequencies numerically by the method of continued fraction. This technique was first pioneered by Leaver in 1985 [9] and becomes very popular method in calculating the quasinormal frequencies. Hence in this section, we will study the massive scalar field in four dimensional Schwarzschild background and obtain its quasinormal modes by using this technique.

5.1 Quasinormal Modes of Schwarzschild Black Hole

The study of quasinormal modes of Schwarzschild black hole was firstly investigated by Chandrasekhar in 1975 [32]. He has succeeded in finding some of the Schwarzschild quasinormal frequencies. His work had open widely many calculation techniques since then. Schwarzschild black hole is the most often used in the study of quasinormal modes because it is the simplest solution of Einstein field equations. Another interesting work has been suggested by A. Starobinskii and I. Novikov; they studied a massive scalar field in Schwarzschild background. As a result they found that the massive modes will decay more slowly than the massless cases. So, it is very interesting to investigate how the scalar field's mass will affect the damping rate of the quasinormal frequencies. This question was answered by Konoplya and Zhidenko [40] in 2005. In this section, we shall therefore review their work in detail.

Now, let's recall the Schwarzschild metric

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{5.1}$$

where,

$$f(r) \equiv \left(1 - \frac{2M}{r}\right). \tag{5.2}$$

One can see that the coordinate singularity lies at the surface

$$r = 2M.$$

Then, we calculate the components of the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -f(r)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

and its inverse

$$g^{\mu\nu} = \begin{pmatrix} f(r)^{-1} & 0 & 0 & 0 \\ 0 & -f(r) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix}.$$

Finally, the determinant of metric tensor is determined to be

$$\sqrt{-g} = r^2\sin\theta.$$

Here, all the prescription we need is calculated. Next, we are going to formulate the Schrödinger-like equation for a massive scalar field evolve in the four dimensional Schwarzschild background.

5.1.1 Scalar Perturbation Near Schwarzschild Black Hole

The dynamical of a massive scalar field in the curved spacetime is described by the Klein-Gordon equation (4.20)

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi(x)) + m^2\Phi(x) \;=\; 0 \qquad (5.3)$$

According to the Schwarzschild metric (5.1), all the non-vanishing components of Klein-Gordon in curved background can be expressed

$$\frac{1}{\sqrt{-g}}\left[\partial_0(g^{00}\sqrt{-g}\partial_0\Phi) + \partial_1(g^{11}\sqrt{-g}\partial_1\Phi) + \partial_2(g^{22}\sqrt{-g}\partial_2\Phi) + \partial_3(g^{33}\sqrt{-g}\partial_3\Phi)\right] + m^2\Phi(x) \;=\; 0.$$

Here, we denote the spacetime indices by $(0,1,2,3) \longrightarrow (t,r,\theta,\phi)$. Using the following ansatz

$$\Phi \;=\; \frac{\psi(r)}{r}e^{-i\omega t}Y(\theta,\phi), \qquad (5.4)$$

where $Y(\theta,\phi)$ is *spherical harmonics* [41]. Then each component can be calculated as follows

$$\begin{aligned}
\partial_{00}\Phi &= -\omega^2\Phi,\\
\partial_{22}\Phi &= \partial_{22}Y(\theta,\phi)\frac{\psi(r)}{r}e^{-i\omega t},\\
\partial_{33}\Phi &= \partial_{33}Y(\theta,\phi)\frac{\psi(r)}{r}e^{-i\omega t},\\
\partial_1\Phi &= e^{-i\omega t}Y(\theta,\phi)\left(\frac{\psi'}{r} - \frac{\psi}{r^2}\right),\\
\partial_{11}\Phi &= e^{-i\omega t}Y(\theta,\phi)\left(\frac{\psi''}{r} - \frac{2\psi'}{r^2} + \frac{2\psi}{r^3}\right),
\end{aligned}$$

where "prime" represents derivative with respect to the radial coordinate r. So, the Klein-Gordon equation becomes

$$\Big[-\omega^2 f^{-1}\frac{\psi}{r}e^{-i\omega t}Y - \frac{\psi''}{r}e^{-i\omega t}Yf - Ye^{-i\omega t}f'\left(\frac{\psi'}{r} - \frac{\psi}{r^2}\right) - \frac{\psi}{r}e^{-i\omega t}\partial_{22}Y - \cot\theta\frac{\psi}{r^3}e^{-i\omega t}\partial_2 Y - \frac{\psi}{r^3\sin^2\theta}e^{-i\omega t}\partial_{33}Y\Big] + m^2\frac{\psi}{r}e^{-i\omega t}Y \;=\; 0.$$

Then divide above equation with common factor $\frac{f^{-1}e^{-i\omega t}Y}{r}$

$$\Big[-\omega^2\psi - \psi'' f^2 - ff'\psi' + \frac{\psi}{r}ff' - \frac{\psi f}{Yr^2}\left(\partial_{22}Y + \cot\theta\partial_2 Y + \frac{1}{\sin^2\theta}\partial_{33}Y\right)\Big] + m^2 fY \;=\; 0.$$

Now let's define a tortoise coordinate as

$$dr_* \;=\; f(r)^{-1}dr.$$

Thus we use an ordinary chain rule to transform radial coordinate r to a new one

$$\begin{aligned} \frac{d^2\psi}{dr_*^2} &= \frac{d}{dr}\left(\frac{d\psi}{dr}.\frac{dr}{dr_*}\right)\frac{dr}{dr_*}, \\ &= f^2\psi^{''} + ff^{'}\psi^{'}. \end{aligned}$$

Therefore, we get

$$\frac{d^2\psi}{dr_*^2} + \omega^2\psi + f\big(-m^2 - \frac{f^{'}}{r} + \frac{1}{r^2 Y}\big[\partial_{22}Y + \cot\theta\partial_2 Y + \frac{1}{\sin^2\theta}\partial_{33}Y\big]\big)\psi \;=\; 0.$$

Note that in the bracket [...] is the angular part of Laplacian operator in spherical coordinates. Also the spherical harmonics $Y(\theta,\phi)$ must be satisfied the following eigenvalue equation [41]

$$\nabla^2 Y(\theta,\phi) \;=\; -\frac{l(l+1)}{r^2}Y(\theta,\phi).$$

Finally, we obtain the Schrödinger-like equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right)\psi(r) \;=\; 0, \tag{5.5}$$

where the effective potential can be displayed (See appendix C.2 for the plotting of the effective potential.)

$$V(r) \;=\; f(r)\left(\frac{l(l+1)}{r^2} + \frac{f^{'}(r)}{r} + m^2\right). \tag{5.6}$$

5.1.2 Continued Fraction

After we get (5.5), we now thus determine quasinormal frequencies by the continued fraction method. This technique was first developed by Leaver [9]. By substituting the appropriate power series solution into wave equation, one can manage to obtain the continued fraction of the coefficient of the recurrence relation. Then, one can determine quasinormal frequencies from continued fraction numerically. We now first introduce a proper solution for (5.5) which is defined by [40]

$$\psi(r) \;=\; e^{ir\xi} r^{\left(2iM\xi + \frac{iMm^2}{\xi}\right)}\left(1 - \frac{2M}{r}\right)^{-2iM\omega} N(r), \tag{5.7}$$

where,

$$\xi \equiv \sqrt{\omega^2 - m^2}, \qquad N(r) \equiv \sum_n a_n\left(1 - \frac{2M}{r}\right)^n.$$

In addition, before we substitute this solution into the wave equation, we must re-transform the tortoise coordinate r_* to the former radial coordinate r first. Note that, in this section, we will describe the step of calculation roughly. Anyone who interested in the detail would be recommended to see Appendix B.1 for a mathematica's code which is developed by Alexander Zhidenko. This code helped us to simplify many tedious works that we have to face if we choose the traditional way instead (by hand). However, we attempt to illustrate each step of calculation as much as possible. Hence, after we insert (5.7) into (5.5) and divide by the common factor $e^{ir\xi}r^{\left(2iM\xi+\frac{iMm^2}{\xi}\right)}\left(1-\frac{2M}{r}\right)^{-2iM\omega}$, we get

$$A(r)N^{''}(r)+B(r)N^{'}(r)+C(r)N(r) \;=\; 0,$$

where $A(r), B(r)$ and $C(r)$ are function of radial coordinate r. Then we introduce a new variable z

$$r \;=\; -\frac{2M}{z-1}. \tag{5.8}$$

Hence, the function $N(r)$ becomes $N(z)=\sum_n a_n z^n$. After we change all radial coordinate r to a new one, the previous equation may takes the form

$$A(z)N^{''}(z)+B(z)N^{'}(z)+C(z)N(z) \;=\; 0,$$

Now, we can rearrange the above coefficient into a form

$$\begin{aligned} A(z) &\sim A\left(...z^3+...z^2+...z^1\right), \\ B(z) &\sim B\left(...z^2+...z^1+...z^0\right), \\ C(z) &\sim C\left(...z^1+...z^0\right), \end{aligned}$$

Then, multiply all the coefficients by parameter z

$$A\left(...z^4+...z^3+...z^2\right)\sum_n n(n-1)a_n z^{n-2}+B\left(...z^3+...z^2+...z^1\right)\sum_n n a_n z^{n-1}$$
$$+C\left(...z^2+...z^1\right)\sum_n a_n z^n=0,$$
$$A\left(...z^2+...z^1+...z^0\right)\sum_n n(n-1)a_n z^n+B\left(...z^2+...z^1+...z^0\right)\sum_n n a_n z^n$$
$$+C\left(...z^2+...z^1\right)\sum_n a_n z^n=0,$$

Let us define

$$A\left(\Delta_a z^2+\Box_a z^1+\nabla_a z^0\right)\sum_n n(n-1)a_n z^n+B\left(\Delta_b z^2+\Box_b z^1+\nabla_b z^0\right)\sum_n n a_n z^n$$
$$+C\left(\Delta_c z^2+\Box_c z^1\right)\sum_n a_n z^n=0,$$

Then, we need to expand all the term which takes the following

$$A(\bigtriangledown_a)\sum_n n(n-1)a_n z^{n+0} + A(\Box_a)\sum_n n(n-1)a_n z^{n+1} + A(\Delta_a)\sum_n n(n-1)a_n z^{n+2}$$
$$+B(\bigtriangledown_b)\sum_n n a_n z^{n+0} + B(\Box_b)\sum_n n a_n z^{n+1} + B(\Delta_b)\sum_n n a_n z^{n+2}$$
$$+C(\Box_c)\sum_n a_n z^{n+1} + C(\Delta_c)\sum_n a_n z^{n+2} = 0.$$

Moreover, we can rearrange the above equation into the final form

$$\sum_n \underbrace{\left(n(n-1)A(\bigtriangledown_a) + nB(\bigtriangledown_b)\right)}_{\tilde{\alpha}} a_n z^n$$
$$+\sum_n \underbrace{\left((n-1)(n-2)A(\Box_a + (n-1)B(\Box_b) + C(\Box_c))\right)}_{\tilde{\beta}} a_{n-1} z^n$$
$$+\sum_n \underbrace{\left((n-2)(n-3)A(\Delta_a) + (n-2)B(\Delta_b) + C(\Delta_c)\right)}_{\tilde{\gamma}} a_{n-2} z^n = 0,$$

where on the second term we shift index $(n \longrightarrow n-1)$ while the third term $(n \longrightarrow n-2)$. Then, our equation becomes

$$\sum_n \tilde{\alpha} a_n z^n + \sum_n \tilde{\beta} a_{n-1} z^n + \sum_n \tilde{\gamma} a_{n-2} z^n.$$

We can shift all the indices n that appear in the above equation by $n \longrightarrow n+1$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} \;=\; 0, \tag{5.9}$$

then, if one follows the code in appendix B, one would get the following results

$$\tilde{\alpha} \;\longrightarrow\; \alpha_n = (n+1)(n+1-4Mi\omega),$$
$$\tilde{\beta} \;\longrightarrow\; \beta_n = \frac{M(\omega+\xi)(4M(\omega+\xi)^2 + i(2n+1)(\omega+3\xi))}{\xi} - 2n(n+1) - 1 - l(l+1),$$
$$\tilde{\gamma} \;\longrightarrow\; \gamma_n = \left(n - \frac{Mi(\omega+\xi)^2}{\xi}\right)^2.$$

Thus, from (5.9) one can prove that

$$\frac{a_{n+1}}{a_n} = -\frac{\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+2}/\ldots}} = -\frac{\beta_n}{\alpha_n} + \frac{\gamma_n}{\alpha_n}\left[\frac{\alpha_{n-1}}{\beta_{n-1} - \frac{\gamma_{n-1}\alpha_{n-2}}{\beta_{n-2} - \gamma_{n-2}\alpha_{n-3}/\ldots}}\right].$$

Hence, the final equation can be expressed as

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} - \alpha_{n-3}\gamma_{n-2}/\ldots}} \;=\; \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+3}/\ldots}}. \tag{5.10}$$

So in principle, the above equation can be solved numerically.

5.1.3 Results and Discussion

As mentioned above, one can numerically solve the above equation, by calculating for ω and then increase the depth of continued fraction, then see if ω does not change significantly. Hence, we obtain the correct answer for the quasinormal frequencies. Some of the results are shown in Table 5.1. They have been determined by using the code appears in Appendix B.2. It appears that the real part of quasinormal frequencies decrease as the mode "n" increase. However, during the numerical calculation, one can found that the depth of continued fraction must be increased as the mode n becomes larger in order to obtain the quasinormal frequencies correctly. Fortunately, one can improve the convergence of the continued fraction by following the Nollert's technique as done in [10, 40]. This will also improve the accuracy of the results and less-consumed calculating time. The results in which using the Nollert's method are shown in Fig(5.1). As illustrated, the imaginary parts of quasinormal frequencies are decreasing while the scalar field's mass increases. It is worthy to note that at some particular mass value, imaginary part vanishes. This lead to the existence of long living modes namely, *quasi-resonances*, in which has been discussed firstly by [40].

M=1,m=0,l=0		
n	ω_R	$-\omega_I$
0	0.1104	0.1048
1	0.0861	0.3480
2	0.0756	0.6009

Table 5.1: First three fundamental modes for $M = 1, l = 0$ in massless case ($m = 0$)

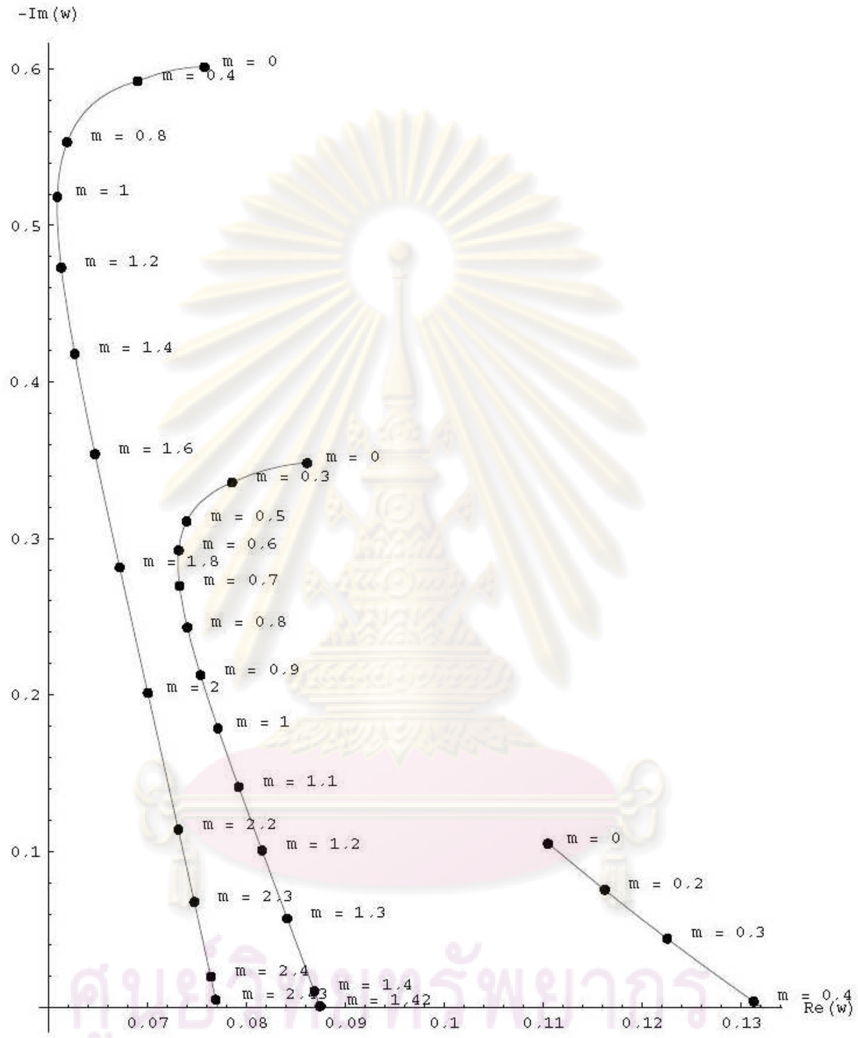

Figure 5.1: This shows the first three fundamental modes at each particular value of the scalar field mass (at fixed $M = 1$) by using the Nollert's improvement. This figure is provided by A. Zhidenko et al, [40].

Chapter VI

QUASINORMAL MODES OF FIVE DIMENSIONAL BLACK HOLES

> *"God not only plays dice, He also sometimes throws the dice where they cannot be seen."*
>
> -S. Hawking

At present, the best candidate for theory of quantum gravity is string theory. In the string theory's paradigm, spacetime are required to have dimensionality more than four. So, it is very interesting to study quasinormal modes of black holes in higher dimensions. Obviously, it makes sense to start with five dimensional cases since its complexity should not be much different from an ordinary four dimensions. Hence, in this chapter, we will investigate quasinormal modes of black holes in five dimensional spacetime. First, we further study a large AdS Schwarzschild black hole and calculate its quasinormal frequencies by using first order perturbation method. Then, we shall introduce a new black hole solution from the Einstein-Maxwell theory namely, rotating Kaluza-Klein black hole with squashed horizons. Finally, its quasinormal frequencies will be determined via an approximation technique.

6.1 Quasinormal Modes of 5d AdS-Schwarzschild Black Hole

As in the Chapter 4, we have already discussed three dimensional AdS Schwarzschild black hole and also compute its quasinormal frequencies. Then for now, we shall further investigate the quasinormal modes of a scalar field in five dimensional AdS Schwarzschild background. By following the work which has been done in Siopsis's work [13], one can manage to transform the Klein-Gordon equation into known differential equation namely, the *Heun equation*. Afterward, the Heun differential equation can be reduced to hypergeometric equation under some proper coordinate transformation. In addition, an obtained equation can be divided into two parts, (i) an ordinary hypergeometric equation, (ii) this part can be regarded as a perturbation term of the main equation. According to this method, we obtain the quasinormal frequencies analytically. The study of quasinormal modes for 5d AdS black holes would be very useful to interpret the thermalization timescales of the thermal system which emerges in four dimensional conformal field theory. Note that, this section was covered by Siopsis's paper [13].

From (4.32), we derive a large AdS Schwarzschild black hole in five dimensions ($d = 5$)

$$ds^2 = -\left(\frac{r^2}{R^2} - \frac{\omega_4 M}{r^2}\right) dt^2 + \left(\frac{r^2}{R^2} - \frac{\omega_4 M}{r^2}\right)^{-1} dr^2 + r^2 ds^2(E_3),$$

where R is AdS radius and ω_4 is gravitational constant. Then, from (4.34), an event horizon can be determined

$$r_+^4 = \omega_4 M R^2.$$

Hence, the metric can be arranged into the following form

$$ds^2 = -\frac{1}{R^2}\left(\frac{r^4 - r_+^4}{r^2}\right) dt^2 + \frac{R^2 r^2}{(r^4 - r_+^4)} dr^2 + r^2 d\vec{x}^2. \tag{6.1}$$

We now determine the components of metric tensor and its inverse

$$g_{\mu\nu} = \begin{pmatrix} -\frac{(r^4 - r_+^4)}{R^2 r^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{R^2 r^2}{(r^4 - r_+^4)} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & r^2 \end{pmatrix},$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{R^2 r^2}{(r^4 - r_+^4)} & 0 & 0 & 0 & 0 \\ 0 & \frac{(r^4 - r_+^4)}{R^2 r^2} & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 & 0 \\ 0 & 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & 0 & r^{-2} \end{pmatrix}.$$

The last one is determinant of metric tensor $\sqrt{-g} = r^3$. We shall denote spacetime indices with $(0, 1, 2, 3, 4) \longrightarrow (t, r, x, y, z)$.

6.1.1 The Wave Equation

The Klein-Gordon equation in the curved background is defined by

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right]-m^2\Phi \;=\; 0,$$

here m represents mass of scalar field. Thus, non-vanishing components of the above equation may take the form

$$\frac{1}{\sqrt{-g}}\left[\partial_0(g^{00}\sqrt{-g}\partial_0\Phi)+\partial_1(g^{11}\sqrt{-g}\partial_1\Phi)+\partial_2(g^{22}\sqrt{-g}\partial_2\Phi)\right.$$
$$\left.+\partial_3(g^{33}\sqrt{-g}\partial_3\Phi)+\partial_4(g^{44}\sqrt{-g}\partial_4\Phi)\right]+m^2\Phi(x)=0.$$

Hence, we get

$$\begin{aligned}
\sqrt{-g}g^{00}\partial_0^2\Phi &= -\frac{r^5R^2}{(r^4-r_+^4)}\partial_0^2\Phi,\\
\partial_1(\sqrt{-g}g^{11}\partial_1\Phi) &= \frac{r\left(r^4-r_+^4\right)}{R^2}\partial_1^2\Phi+\left(\frac{4r^4}{R^2}+\frac{\left(r^4-r_+^4\right)}{R^2}\right)\partial_1\Phi,\\
\sqrt{-g}\left[g^{22}\partial_2^2\Phi+g^{33}\partial_3^2\Phi+g^{44}\partial_4^2\Phi\right] &= r\vec{\nabla}^2\Phi,
\end{aligned}$$

where, $\vec{\nabla}^2$ is Laplacian operator in Cartesian coordinates. Then, we have derived the wave equation

$$-\frac{R^4}{r^2\left(1-\frac{r_+^4}{r^4}\right)}\partial_0^2\Phi+r^2\left(1-\frac{r_+^4}{r^4}\right)\partial_1^2\Phi+\left(5r-\frac{r_+^4}{r^3}\right)\partial_1\Phi+\frac{R^2}{r^2}\vec{\nabla}^2\Phi \;=\; m^2R^2\Phi.$$

We may introduce the following ansatz for the separation of variable method.

$$\Phi \;=\; \psi(y)e^{-i(\omega t+\vec{p}.\vec{x})}. \tag{6.2}$$

We thus change the radial coordinate r to y

$$y \;=\; \frac{r_+^2}{r^2}. \tag{6.3}$$

So as we approach an event horizon that means $y \longrightarrow 1$ while at infinity y goes to zero. Now let's consider the following chain rule

$$\begin{aligned}
\partial_1\Phi &= -\frac{2r_+^2}{r^3}\psi' e^{-i(\omega t+\vec{p}.\vec{x})},\\
\partial_1^2\Phi &= \frac{4r_+^4}{r^6}\psi'' e^{-i(\omega t+\vec{p}.\vec{x})}+\frac{6r_+^2}{r^4}\psi' e^{-i(\omega t+\vec{p}.\vec{x})},
\end{aligned}$$

here, we denote the prime sign as derivative with respect to y. Therefore each components of the wave equation can be calculated

$$\begin{aligned}
m^2R^2\Phi &= m^2R^2\psi e^{-i(\omega t+\vec{p}.\vec{x})}, \\
\frac{R^2}{r^2}\vec{\nabla}^2\Phi &= -\frac{|\vec{p}|^2R^2}{r^2}\psi e^{-i(\omega t+\vec{p}.\vec{x})}, \\
-\frac{R^4}{r^2\left(1-\frac{r_+^4}{r^4}\right)}\partial_0^2\Phi &= \frac{\omega^2R^4}{r^2(1-y^2)}\psi e^{-i(\omega t+\vec{p}.\vec{x})}, \\
\frac{1}{r^3}\partial_1\left[(r^5-r_+^4 r)\partial_1\Phi\right] &= \left(4(1-y^2)y^2\psi'' + \left[6(1-y^2)y+2y^3-10y\right]\psi'\right)e^{-i(\omega t+\vec{p}.\vec{x})}.
\end{aligned}$$

The exponential factor can be eliminated out then we obtain the radial equation

$$(1-y^2)^2y^2\psi'' - y(1-y^4)\psi' + \left[\frac{\hat{\omega}^2}{4}y - \frac{\hat{p}^2}{4}y(1-y^2) - \frac{\hat{m}^2}{4}(1-y^2)\right]\psi = 0, \tag{6.4}$$

where we define a new parameter as

$$\begin{aligned}
\hat{\omega} &= \frac{\omega R^2}{r_+}, \\
\hat{p} &= \frac{|\vec{p}|R}{r_+}, \\
\hat{m} &= mR.
\end{aligned}$$

Now we determine the solution in the near horizon regime $y \longrightarrow 1$. Let us define a new variable

$$1-y^2 \equiv x \simeq 0.$$

Hence, the radial equation (6.4) in the near horizon limit becomes

$$4x^2(1-x)^2\frac{d^2\psi}{dx^2} + 2x(1-x)(2-x)\frac{d\psi}{dx} + \frac{\hat{\omega}^2}{4}(1-x)^{1/2}\psi - \frac{\hat{p}^2}{4}x(1-x)^{1/2}\psi - \frac{\hat{m}^2}{4}x\psi = 0.$$

however, $x=0$ as we approach the horizon then the above equation can be reduced to

$$4x^2\frac{d^2\psi}{dx^2} + 4x\frac{d\psi}{dx} + \frac{\hat{\omega}^2}{4}\psi - \frac{\hat{p}^2}{4}x\psi - \frac{\hat{m}^2}{4}x\psi = 0. \tag{6.5}$$

We introduce power series solution

$$\begin{aligned}
\psi &= \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \\
&= x^\alpha(a_0 + a_1x + a_2x + ...), \\
&\simeq x^\alpha a_0.
\end{aligned}$$

After substituting the above solution into the reduced wave equation, we thus obtain

$$4\alpha(\alpha-1)x^{\alpha}+4\alpha x^{\alpha}+\frac{\hat{\omega}^2}{4}x^{\alpha} \;=\; 0,$$

where we ignore the higher order term of power of x. Then, we obtain a constraint equation for α

$$\alpha \;=\; \pm\frac{i\hat{\omega}}{4}.$$

Finally, we approximately obtain the solution in the near horizon region

$$\psi_{\pm} \;\sim\; (1-y)^{\pm i\hat{\omega}/4}. \tag{6.6}$$

We use the above argument to obtain the wave solution at infinity $y \longrightarrow 0$. Then, we get

$$\psi_{\pm} \;\sim\; y^{\beta_{\pm}}, \qquad \beta_{\pm}=1\pm\sqrt{1+\frac{\hat{m}^2}{2}}. \tag{6.7}$$

For massless case ($\hat{m}=0$), then $\beta_{+}=2, \beta_{-}=0$. However, there is another singularity occurs at the horizon except $y=1$ which lies at $y=-1$. We therefore need to investigate the solution's behavior in this region too. By defining a new parameter z as

$$y \;=\; z-1.$$

Hence as y approach to another horizon $y=-1$, z becomes zero. We can determine this solution in the similar way as we had done for the others region. Then, we obtain

$$\psi_{\pm} \;\sim\; \left(\frac{1+y}{2}\right)^{\gamma_{\pm}}, \qquad \gamma_{\pm}=\pm\frac{\hat{\omega}}{4}. \tag{6.8}$$

Here, factor 1/2 is put by hand to prevent any contribution from this term when $y \longrightarrow 1$. Ultimately, the general solution of the radial equation (6.4) is written down

$$\psi(y) \;=\; y^2(1-y)^{-i\hat{\omega}/4}\left(\frac{1+y}{2}\right)^{-\hat{\omega}/4}F(y). \tag{6.9}$$

After substituting this solution into the radial wave equation (for massless case), one obtain

$$F''+\left[\frac{3}{y}+\frac{(1-i\hat{\omega}/2)}{y-1}+\frac{(1-i\hat{\omega}/2)}{y+1}\right]F'+\left[\frac{(2-(1+i)\hat{\omega}/4)^2y-q}{y(y^2-1)}\right]F \;=\; 0, \tag{6.10}$$

where $q = \frac{3(-1+i)}{4}\hat{\omega} - \frac{\hat{p}^2}{4} + \frac{\hat{\omega}^2}{4}$. This equation is so-called *Heun differential equation.* Any second order linear differential equation with four singularities can be transformed into this equation. We now again change the variable by $x = y^2$. Let's consider the following relation

$$\begin{aligned} \frac{dF}{dy} &= 2\sqrt{x}\frac{dF}{dx}, \\ \frac{d^2F}{dy^2} &= 4x\frac{d^2F}{dyx^2} + 2\frac{dF}{dx}. \end{aligned}$$

So, Heun equation will take the form

$$\begin{aligned} 4x\frac{d^2F}{dx^2} + 2\frac{dF}{dx} + 2\sqrt{x}\left[\frac{3}{\sqrt{x}} + \frac{(1-i\hat{\omega}/2)}{\sqrt{x}-1} + \frac{(1-i\hat{\omega}/2)}{\sqrt{x}+1}\right]\frac{dF}{dx} & \\ + \left[\frac{(2-(1+i)\hat{\omega}/4)^2\sqrt{x} - q}{\sqrt{x}(x-1)}\right]F &= 0. \end{aligned}$$

Moreover, we can rearrange the above equation into the following form

$$\begin{aligned} x(1-x)\frac{d^2F}{dx^2} + \left[2-(1-i)\frac{\hat{\omega}}{4} - (3-(1+i)\frac{\hat{\omega}}{4})x + (1-\sqrt{x})(1-i)\frac{\hat{\omega}}{4}\right]\frac{dF}{dx} & \\ + \left[-\frac{1}{4}\left(2-(1+i)\frac{\hat{\omega}}{4}\right)^2 + \frac{q}{4\sqrt{x}} + \frac{q}{4} - \frac{q}{4}\right]F &= 0. \end{aligned}$$

Then Heun equation can be regarded as

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0, \quad (6.11)$$

where,

$$\begin{aligned} \mathcal{H}_0 &= x(1-x)\frac{d^2}{dx^2} + \left[2-(1-i)\frac{\hat{\omega}}{4} - (3-(1+i)\frac{\hat{\omega}}{4})x\right]\frac{d}{dx} - \left[\frac{1}{4}\left(2-(1+i)\frac{\hat{\omega}}{4}\right)^2 - q\right], \\ \mathcal{H}_1 &= (1-\sqrt{x})\left[(1-i)\frac{\hat{\omega}}{4}\frac{d}{dx} + \frac{q}{4\sqrt{x}}\right]. \end{aligned}$$

One sees that $\mathcal{H}_0$ is canonical form of hypergeometric differential equation while $\mathcal{H}_1$ can be treated as a perturbation term. We can expand the wave function,

$$F = F_0 + F_1 + ... \quad (6.12)$$

Now, let's discuss the zeroth-order equation

$$\mathcal{H}_0 F_0 = 0. \quad (6.13)$$

The standard hypergeometric equation may take the form [39]

$$x(1-x)\frac{d^2F}{dx^2} + [c-(a+b+1)x]\frac{dF}{dx} - abF = 0.$$

Thus, from (6.13), we determine the following parameters

$$\begin{aligned} c &= 2-(1-i)\frac{\hat{\omega}}{4}, \\ a &= 1+\left(-\frac{(1+i)\hat{\omega}}{8}+\frac{\sqrt{q}}{2}\right), \\ b &= 1+\left(-\frac{(1+i)\hat{\omega}}{8}-\frac{\sqrt{q}}{2}\right). \end{aligned}$$

Then the solution of zeroth-order equation at infinity $x = 0$ is defined by [39]

$$\begin{aligned} F_0 &= F(a,b,c;x), \\ &= F(1+\left(-\frac{(1+i)\hat{\omega}}{8}+\frac{\sqrt{q}}{2}\right), 1+\left(-\frac{(1+i)\hat{\omega}}{8}-\frac{\sqrt{q}}{2}\right), 2-(1-i)\frac{\hat{\omega}}{4}, x). \end{aligned}$$

Using the linear transformation of hypergeometric function, we can transform above relation to the solution at near horizon limit $x = 1$

$$\begin{aligned} F_0 &= \frac{\Gamma(2-(1-i)\frac{\hat{\omega}}{4})\Gamma(i\hat{\omega}/2)}{\Gamma(1-\frac{\hat{\omega}}{8}+\frac{3i\hat{\omega}}{8}-\frac{\sqrt{q}}{2})\Gamma(1-\frac{\hat{\omega}}{8}+\frac{3i\hat{\omega}}{8}+\frac{\sqrt{q}}{2})}F \\ &\quad +(1-x)^{i\hat{\omega}/2}\frac{\Gamma(2-(1-i)\frac{\hat{\omega}}{4})\Gamma(-i\hat{\omega}/2)}{\Gamma(1-(\frac{1+i}{4}\hat{\omega}+\sqrt{q})/2)\Gamma(1-(\frac{1+i}{4}\hat{\omega}-\sqrt{q})/2)}F. \end{aligned}$$

Consequently, wave solution near horizon is expressed

$$F_0 = A_0 + B_0(1-x)^{i\hat{\omega}/2}. \tag{6.14}$$

Boundary condition of quasinormal modes forces us to choose $B_0 = 0$ because at horizon there must no outgoing modes exists. In order to satisfy this condition, we require that

$$2n = \frac{(1+i)}{4}\hat{\omega} \pm \sqrt{q}, \qquad n = 1, 2, ... \tag{6.15}$$

This relation represents the quasinormal frequencies of 5d AdS Schwarzschild black hole but only one that comes from the "non-perturbative" term.

6.1.2 First Order Perturbation

To improve the accuracy of our result, we must include the contribution from the "perturbed" term $\mathcal{H}_1$. We need to expand B_0 for small ω. Now let's recall

$$B_0 = \frac{\Gamma(2-(1-i)\frac{\hat{\omega}}{4})\Gamma(-i\hat{\omega}/2)}{\Gamma(1-(\frac{1+i}{4}\hat{\omega}+\sqrt{q})/2)\Gamma(1-(\frac{1+i}{4}\hat{\omega}-\sqrt{q})/2)} \tag{6.16}$$

For simplicity we may set $\hat{p} = 0$. Hence, parameter q is shown as

$$\sqrt{q} \simeq \sqrt{\frac{3}{4}(-1+i)\hat{\omega}^{1/2}}. \tag{6.17}$$

Now we consider the Taylor's expansion (for small δ)

$$\begin{aligned}\Gamma(z+\delta) &= \Gamma(z)|_{z=1} + \delta\frac{d\Gamma(z)}{dz}|_{z=1} + \delta^2\frac{d^2\Gamma(z)}{dz^2}|_{z=1} + ...,\\ \frac{d\Gamma(z)}{dz} &= -\gamma \simeq 0.577,\\ \frac{d^2\Gamma(z)}{dz^2} &= \int_0^\infty e^{-t}(\ln t)^2 dt \;=\; \gamma^2 + \frac{\pi^2}{6}.\end{aligned}$$

Therefore, let's consider

$$\begin{aligned}\Gamma\left(1-\frac{1}{2}\left(\frac{1+i}{4}\hat{\omega}-\sqrt{\frac{3}{4}(-1+i)}\hat{\omega}^{1/2}\right)\right) &\simeq 1+\frac{1}{2}\left(\frac{1+i}{4}\hat{\omega}+\sqrt{\frac{3}{4}(-1+i)}\hat{\omega}^{1/2}\right)\gamma\\ &\quad +\frac{3}{16}(-1+i)\hat{\omega}\left(\gamma^2+\frac{\pi^2}{6}\right),\\ \Gamma\left(-\frac{i\hat{\omega}}{2}\right) &\simeq \frac{1}{-i\hat{\omega}/2}\Gamma(1-\frac{i\hat{\omega}}{2}) = -\frac{2}{i\hat{\omega}},\\ \Gamma\left(2-\frac{(1-i)}{4}\hat{\omega}\right) &\simeq 1-\frac{(1-i)}{4}\hat{\omega}.\end{aligned}$$

So, B_0 can be completely expanded as

$$B_0 \;\simeq\; -\frac{1}{i\hat{\omega}/2}\left[1+\left(-1+\frac{\pi^2}{8}\right)\frac{(1-i)}{4}\hat{\omega}+...\right]. \tag{6.18}$$

Since, any second order differential equation must have two linearly independent solutions. We now thus construct the second solution G_0 of the zeroth-order equation. We use

$$G_0(x) \;=\; F_0(x)\int^x \frac{W_0(x')dx'}{F_0(x')^2}, \tag{6.19}$$

where $W_0(x)$ is Wronskian which can be calculated via [41]

$$W_0 \;=\; F_0 G_0' - G_0 F_0',$$

or an alternate form, it reads

$$W_0 \;=\; exp\left[-\int_0^x P(x_1)dx_1\right],$$

Here, $P(x_1)$ is the coefficient of the first derivative term. In our case, Wronskian is determined by

$$W_0 \;=\; x^{-2+\frac{(1-i)}{4}\hat{\omega}}(1-x)^{-1+i\hat{\omega}/2}. \tag{6.20}$$

Moreover, the first-order equation is considered

$$\mathcal{H}_0 F_1 \;=\; -\mathcal{H}_1 F_0. \tag{6.21}$$

One can construct the solution for the inhomogeneous differential equation via

$$F_1(x) = G_0(x)\int_0^x \frac{F_0(x')\mathcal{H}_1 F_0(x')dx'}{W_0(x')} - F_0(x)\int_0^x \frac{G_0(x')\mathcal{H}_1 F_0(x')dx'}{W_0(x')}. \tag{6.22}$$

To expand both solutions F_0, G_0, we first transform F_0 by using hypergeometric function identity and then expand by keeping only lowest order in $\hat{\omega}$. Thus, F_0 may take the form

$$F_0(x) = \frac{1-(1-x)^{i\hat{\omega}/2}}{i\hat{\omega}x/2} + ...,$$

whereas for G_0, we may express the second solution as [39]

$$\begin{aligned} G_0(x) &= x^{1-c}F(a-c+1, b-c+1, 2-c; x), \\ &\simeq \frac{1}{x}. \end{aligned}$$

Consequently, one can construct $F_1(x)$ by using (6.22)

$$F_1(x) = C\frac{(1-i)}{4}\hat{\omega}F_0 x + ..., \tag{6.23}$$

where,

$$\begin{aligned} C &\equiv \int_0^x \left[-\frac{1}{x(1+\sqrt{x})} - \frac{1-\sqrt{x}}{x^2}\left(1+\frac{3\sqrt{x}}{4}\right)\ln(1-x)\right]dx, \\ &= 1+\ln 2-\frac{\pi^2}{8} \end{aligned} \tag{6.24}$$

Then to the first order, the solution at the horizon is obtained

$$\begin{aligned} F(x) &\approx F_0(x) + F_1(x), \\ &\approx F_0(x)\left(1+\frac{(1-i)}{4}(1+\ln 2-\frac{\pi^2}{8})\hat{\omega}\right). \end{aligned}$$

Recall that from (6.14) and (6.18), then we have

$$\begin{aligned} B_1 &\equiv B_0\left(1+\frac{(1-i)}{4}(1+\ln 2-\frac{\pi^2}{8})\hat{\omega}\right), \\ &\simeq \frac{1}{-i\hat{\omega}/2}\left(1+\ln 2\frac{1-i}{4}\hat{\omega}\right). \end{aligned} \tag{6.25}$$

Applying the boundary condition at the horizon, so we have to set $B_1 = 0$

$$\begin{aligned} \hat{\omega} &= -\frac{2(1+i)}{\ln 2}, \\ &= -2.89(1+i). \end{aligned} \tag{6.26}$$

We obtain analytical formula of quasinormal frequencies of 5d AdS Schwarzschild black hole in first-order perturbation. It is worthy to note that a positive real part can be derived by replacing the exponent of $\frac{1+y}{2}$ from $-i\hat{\omega}/4$ to $+i\hat{\omega}/4$.

6.1.3 Results and Discussion

So far, we have investigated quasinormal modes of five dimensional AdS Schwarzschild black hole. It appears that one can possibly transform the wave equation (6.4) into the hypergeometric differential equation and its perturbation part. First, we purely determine the zeroth-order part which is standard hypergeometric equation. Hence, one gets the quasinormal frequencies. However, if one needs to improve the results, the first-order perturbation must be included. Then, the first order solution can be constructed; ultimately improved quasinormal frequencies are obtained.

6.2 Quasinormal Modes of Rotating Squashed Kaluza-Klein Black Hole

Einstein's theory of relativity has suggested us that space and time must be combined together as a four dimensional object. Many observations confirm the correctness of events which purely appear only when four dimensional spacetime is considered. This may imply that we might live in four dimensions universe. However, in attempt to formulate theory of quantum gravity, one must further extend dimensionality of the spacetime to higher dimensions. One of the quantum gravity candidates is the string theory. As suggested by string theory, extra dimensions are required to fulfill the completeness and self-consistency of the theory. Some string scenario suggests that the large hardron colliders might be able to create mini black holes if the higher dimensions exist. Therefore, it is of particular interest to study the quasinormal modes of higher dimensional black holes. However, the extra dimensions are expected to be compactified since we have never observed such an effect from them. Such geometry with compactified extra dimension is called a Kaluza-Klein geometry. Unlike the previous section which the black hole was just extended to standard five dimensional spacetime, in this section we will consider a black hole in Kaluza-Klein geometry instead. Higher dimensional black hole equipped with this asymptotic structure is called a Kaluza-Klein black hole.

To construct such a Kaluza-Klein black hole, we equip the asymptotic structure with twisted S^1 bundle over four dimensional flat spacetime. This leads us to the existence of Kaluza-Klein black hole with *squashed horizons*. Such a black hole look like five dimensional squashed black hole near the horizon and explicitly shows the Kaluza-Klein geometry at spatial infinity which is locally $(M^4 \times S^1)$.

The Kaluza-Klein charged black hole with the squashed horizons was successfully constructed by Ishiara and Matsuno [43], and for the rotating case by Wang [44]. Thus, it is very interesting to study how the size of the compactified dimension affects the quasinormal modes of the Kaluza-Klein black hole.

In this section, we first begin with a brief introduction of rotating squashed Kaluza-Klein metric. Then, we will investigate the scalar perturbation around rotating squashed Kaluza-Klein black hole and try to determine its quasinormal frequencies.

6.2.1 Kaluza-Klein Black Hole with Squashed Horizons

To construct the Kaluza-Klein black hole with squashed horizons, we first consider the five dimensional Einstein-Maxwell theory

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}), \tag{6.27}$$

here R is the curvature scalar and $F_{\mu\nu}$ is Maxwell field strength tensor. The solution that satisfies an equation of motion of this action is described by a static electrically charged metric [43]

$$ds^2 = -f(r)dt^2 + \frac{k(r)^2}{r}dr^2 + \frac{r^2}{4}\left[k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2\right].$$

Where the gauge potential is chosen to be

$$A = \pm\frac{\sqrt{3}}{2}\frac{r_+ r_-}{r^2}dt. \tag{6.28}$$

while the others parameters are defined by

$$\begin{aligned}
f(r) &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \quad k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}, \\
\sigma_1 &= -\sin\psi d\theta + \cos\psi\sin\theta d\phi, \\
\sigma_2 &= \cos\psi d\theta + \sin\psi\sin\theta d\phi, \\
\sigma_3 &= d\psi + \cos\theta d\phi,
\end{aligned}$$

and $0 < \theta < \pi$, $\quad 0 < \phi < 2\pi$, $\quad 0 < \psi < 4\pi$. The function $k(r)$ is squash function which characterizes the shape of our horizons. The rest undefined parameters will be termed when considering rotating black hole. This solution is a non-rotating Kaluza-Klein black hole with squashed horizons which derived by Ishihara et.al

[43]. To formulate a rotating version, Wang has suggested "squashing transformation" as follows [44]

$$
\begin{aligned}
dr &\longrightarrow k(r)dr, \\
\sigma_1 &\longrightarrow \sqrt{k(r)}\sigma_1, \\
\sigma_2 &\longrightarrow \sqrt{k(r)}\sigma_2,
\end{aligned}
$$

Thus by following the above argument, one can transform an ordinary five dimensional Kerr black hole with equal angular momentum [45, 46] into rotating Kaluza-Klein black hole with squashed horizons

$$
ds^2 = -dt^2 + \frac{\Sigma}{\Delta}k(r)^2dr^2 + \frac{r^2+a^2}{4}[k(r)(\sigma_1^2+\sigma_2^2)+\sigma_3^2] + \frac{\mu}{r^2+a^2}(dt - \frac{a}{2}\sigma_3)^2, \tag{6.29}
$$

The parameters are defined by

$$
\begin{aligned}
\Sigma &= r^2(r^2+a^2), \\
\Delta &= (r^2+a^2)^2 - \mu r^2.
\end{aligned}
$$

Where μ,a are black hole mass and black hole's spin parameter respectively. The metric (6.29) has three coordinate singularities which are denoted by $r = r_\pm, r = r_\infty$. These will restrict the radial coordinate r within the range $0 < r < r_\infty$. Note that one can obtain the inner and outer horizon by setting $\Delta = 0$

$$
r_\pm = \sqrt{\frac{(\mu - 2a^2) \pm \sqrt{-4a^2\mu + \mu^2}}{2}}.
$$

Moreover, one can easily prove that $r_+^2 + r_-^2 = \mu - 2a^2$ and $(r_+r_-)^2 = a^4$. To determine the shape of horizon, let us recall the function $k(r)$

$$
k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}. \tag{6.30}
$$

Now consider three dimensional surface in which metric takes the form (for $t, r =$ constant)

$$
ds^2 = \frac{r^2+a^2}{4}[k(r)(\sigma_1^2+\sigma_2^2)+\sigma_3^2]. \tag{6.31}
$$

This surface can be regarded as S^1 fiber over base space S^2. Note that, $\sigma_1^2 + \sigma_2^2$ represents S^2 metric. Hence, the ratio between S^1 and base space is characterized by squash function (6.30). As $k(r) \longrightarrow 1$ our metric reduces to five dimensional

Kerr black hole. One can investigate the shape of horizon by considering function $k(r_\pm)$ [43]

$$k(r_\pm) = \frac{r_\infty^2 - r_\mp^2}{r_\infty^2 - r_\pm^2}. \tag{6.32}$$

Since $k(r_+) \geq 1 \geq k(r_-)$, at the outer horizon S^2 is larger than S^1, while at the inner horizon S^2 is smaller than S^1 [43]. These describe the shape of both horizons which is characterized by $k(r_\pm)$. Note that in the case, $r_+ = r_-$, the shape of horizons become perfectly S^3.

Hence, near the horizon (6.29) looks like a five-dimensional black hole with squashed horizons. To see the asymptotic structure, we follow Wang's argument [44]. First let's introduce new radial coordinate

$$\rho = \rho_0 \frac{r^2}{r_\infty^2 - r^2}, \tag{6.33}$$

where

$$\begin{aligned} \rho_0^2 &= \frac{k_0}{4}(r_\infty^2 + a^2), \\ k_0 &= \frac{(r_\infty^2 + a^2)^2 - \mu r_\infty^2}{r_\infty^4}. \end{aligned}$$

ρ range from 0 to ∞ while r range from 0 to r_∞. We thus now transform our metric (6.29) to a new coordinate, the new metric may take the form [44]

$$ds^2 = -dt^2 + U d\rho^2 + R^2(\sigma_1^2 + \sigma_2^2) + W^2\sigma_3^2 + V(dt - \frac{a}{2}\sigma_3)^2, \tag{6.34}$$

where

$$\begin{aligned} K^2 &= \frac{\rho + \rho_0}{\rho + \frac{a^2}{r_\infty^2 + a^2}}\rho_0, \\ V &= \frac{\mu}{r_\infty^2 + a^2}K^2, \\ W^2 &= \frac{r_\infty^2 + a^2}{4K^2}, \\ R^2 &= \frac{(\rho + \rho_0)^2}{K^2}, \\ U &= \left(\frac{r_\infty^2}{r_\infty^2 + a^2}\right)^2 \times \frac{\rho_0^2}{W^2 - \frac{\rho r_\infty^2}{4(\rho + \rho_0)}V}. \end{aligned}$$

Then take limit $\rho \longrightarrow \infty$, ultimately the metric approaches [44]

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + \frac{r_\infty^2 + a^2}{4}\sigma_3^2 + \frac{\mu}{r^2 + a^2}(dt - \frac{a}{2}\sigma_3)^2 \tag{6.35}$$

The cross term between dt and σ_3 can be eliminated by introducing new coordinates

$$
\begin{aligned}
\bar{\psi} &= \psi - \frac{2\mu a}{(r_\infty^2 + a^2)^2 + \mu a^2} t, \\
\bar{t} &= \sqrt{\frac{(r_\infty^2 + a^2)^2 - \mu r_\infty^2}{(r_\infty^2 + a^2)^2 + \mu a^2}} t,
\end{aligned}
$$

and define new parameter $\bar{\sigma_3} \equiv d\bar{\psi} + \cos\theta d\phi$. Hence, we obtain rotating Kaluza-Klein black hole with squashed horizon in asymptotic ($\rho \longrightarrow \infty$) limit

$$
ds^2 = -d\bar{t}^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + \frac{(r_\infty^2 + a^2)^2 + \mu a^2}{4(r_\infty^2 + a^2)} \bar{\sigma}_3^2 \tag{6.36}
$$

This metric shows the topology of a twisted S^1 bundle over four dimensional Minkowski spacetime. It obvious that the first four terms describe Minkowski spacetime while the rest is a twisted S^1. Moreover, the size of compactified dimension at infinity is given by

$$
\bar{r}_\infty^2 = \frac{(r_\infty^2 + a^2)^2 + \mu a^2}{r_\infty^2 + a^2}. \tag{6.37}
$$

We see that there are three parameters which control the size of an extra dimension. In addition, if we take $r_\infty \longrightarrow \infty$ the radius of compactified dimension purely depends on r_∞. Thus it is very interesting to see that how this parameter may contribute to the quasinormal frequencies.

We may define a new coordinate time as $dt = Ad\tau$ and $A \equiv \frac{(r_\infty^2+a^2)^2}{2\rho_0 r_\infty^3}$ [14]. Before we end this section let's calculate the components of metric tensor and its inverse

$$
g_{\mu\nu} = \begin{pmatrix}
-\left(1 - \frac{\mu}{r^2+a^2}\right) A^2 & 0 & 0 & -\frac{a\mu\cos\theta}{2(r^2+a^2)} A & -\frac{a\mu}{2(r^2+a^2)} A \\
0 & \frac{\Sigma k^2}{\Delta} & 0 & 0 & 0 \\
0 & 0 & \frac{k(r^2+a^2)}{4} & 0 & 0 \\
-\frac{a\mu\cos\theta}{2(r^2+a^2)} A & 0 & 0 & \frac{((r^2+a^2)^2(k\sin^2\theta + \cos^2\theta) + \mu a^2\cos^2\theta)}{4(r^2+a^2)} & \frac{\cos\theta}{4}\left[\frac{(r^2+a^2)^2+\mu a^2}{4(r^2+a^2)}\right] \\
-\frac{a\mu}{2(r^2+a^2)} A & 0 & 0 & \frac{\cos\theta}{4}\left[\frac{(r^2+a^2)^2+\mu a^2}{r^2+a^2}\right] & \frac{(r^2+a^2)^2+\mu a^2}{4(r^2+a^2)}
\end{pmatrix},
$$

and

$$
g^{\mu\nu} = \begin{pmatrix}
-\frac{(a^2+r^2)^2+a^2\mu}{\Delta A^2} & 0 & 0 & 0 & -\frac{2a\mu}{A\Delta} \\
0 & \frac{\Delta}{k^2\Sigma} & 0 & 0 & 0 \\
0 & 0 & \frac{4}{k(a^2+r^2)} & 0 & 0 \\
0 & 0 & 0 & \frac{4\csc^2\theta}{(a^2+r^2)k} & -\frac{4\cot\theta\csc\theta}{(a^2+r^2)k} \\
-\frac{2a\mu}{A\Delta} & 0 & 0 & -\frac{4\cot\theta\csc\theta}{(a^2+r^2)k} & \frac{4(a^2+r^2-\mu)}{\Delta} + \frac{4\cot^2\theta}{(a^2+r^2)k}
\end{pmatrix},
$$

Finally, the determinant of metric tensor is calculated

$$\sqrt{-g} = \frac{Ak^2 \sin\theta}{8}\sqrt{\Sigma(r^2+a^2)}.$$

We are now ready to formulate an equation of scalar perturbation around rotating Kaluza-Klein black hole with squashed horizons.

6.2.2 Scalar Field Near Rotating Squashed Kaluza-Klein Black Hole

Again, we consider the massless Klein-Gordon in curved spacetime

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0.$$

Hence, all the non-vanishing terms are expressed

$$\frac{1}{\sqrt{-g}}[\partial_0(g^{00}\partial_0\Phi + \sqrt{-g}g^{03}\partial_3\Phi + \sqrt{-g}g^{04}\partial_4\Phi) + \partial_1(\sqrt{-g}g^{11}\partial_1\Phi) + \partial_2(\sqrt{-g}g^{22}\partial_2\Phi) + \partial_3(\sqrt{-g}g^{33}\partial_3\Phi + \sqrt{-g}g^{34}\partial_4\Phi + \sqrt{-g}g^{30}\partial_0\Phi) + \partial_4(\sqrt{-g}g^{44}\partial_4\Phi + \sqrt{-g}g^{40}\partial_0\Phi + \sqrt{-g}g^{43}\partial_3\Phi)] = 0.$$

We shall denote the spacetime indices by $(0,1,2,3,4) \longrightarrow (\tau, r, \theta, \phi, \psi)$. We now substitute the following ansatz into Klein-Gordon equation

$$\Phi = R(\rho)S(\theta)e^{-i(\omega\tau - m\phi - \lambda\psi)}, \tag{6.38}$$

where $|m| \leq l$ and $|2\lambda| \leq 2l$. We also introduce a new radial coordinate ρ as defined by (6.33). Then, let's consider each components of field equation

<u>τ − component</u>

$$\frac{1}{\sqrt{-g}}\partial_0(g^{00}\partial_0\Phi + \sqrt{-g}g^{03}\partial_3\Phi + \sqrt{-g}g^{04}\partial_4\Phi) = \left[((r^2+a^2)^2 + a^2\mu)\frac{\omega^2}{\Delta A^2} - \frac{2a\mu\omega\lambda}{\Delta A}\right]\Phi.$$

<u>r − component</u>

$$\frac{1}{\sqrt{-g}}\partial_1(\sqrt{-g}g^{11}\partial_1\Phi) = \left[\left(\frac{d\rho}{dr}\right)^2 R''(\rho) + \left(\frac{d^2\rho}{dr^2} + \left(\frac{d\rho}{dr}\right)^2\left[\frac{\Delta'}{\Delta} + \frac{\Gamma'}{2\Gamma} - \frac{\Sigma'}{2\Sigma}\right]\right)R'(\rho)\right] \times\frac{\Delta S(\theta)e^{-i(\omega\tau - m\phi - \lambda\psi)}}{k^2\Sigma},$$

where prime denotes derivative with respect to ρ, and $\Gamma \equiv (r^2+a^2)$.

<u>θ − component</u>

$$\frac{1}{\sqrt{-g}}\partial_2(\sqrt{-g}g^{22}\partial_2\Phi) = \frac{4}{\Gamma k}\left[S''(\theta) + \cot\theta S'(\theta)\right]R(\rho)e^{-i(\omega\tau - m\phi - \lambda\psi)}.$$

$\phi-$ component

$$\frac{1}{\sqrt{-g}}\partial_3(\sqrt{-g}g^{33}\partial_3\Phi+\sqrt{-g}g^{34}\partial_4\Phi+\sqrt{-g}g^{30}\partial_0\Phi) \;=\; \frac{4}{\Gamma k\sin^2\theta}\left[m\lambda\cos\theta-m^2\right] \times R(\rho)S(\theta)e^{-i(\omega\tau-m\phi-\lambda\psi)}.$$

$\psi-$ component

$$\frac{1}{\sqrt{-g}}\partial_4(\sqrt{-g}g^{44}\partial_4\Phi+\sqrt{-g}g^{40}\partial_0\Phi+\sqrt{-g}g^{43}\partial_3\Phi)= \left[\frac{4}{\Gamma k}\left(\frac{m\lambda\cos\theta}{\sin^2\theta}-\lambda^2\cot^2\theta\right)- \frac{1}{\Delta}\left(4(\Gamma-\mu)\lambda^2+\frac{2a\mu\omega\lambda}{A}\right)\right]\Phi.$$

In addition, an eigenvalue equation for spheroidal harmonics [41]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\right]S(\theta)-\left[\frac{(m-\lambda\cos\theta)^2}{\sin^2\theta}-E\right]S(\theta) \;=\; 0, \qquad (6.39)$$

where $E=l(l+1)-\lambda^2$ is an eigenvalue of spheroidal harmonics. Hence, our angular part yields

$$(\theta+\phi+\psi) \;=\; \left[\frac{4}{\Gamma k}\left(-l(l+1)+\lambda^2\right)+\frac{1}{\Delta}\left(\frac{(\Gamma^2+a^2\mu)\omega^2}{A^2}-\frac{4a\mu\omega\lambda}{A}-4(\Gamma-\mu)\lambda^2\right)\right]\Phi \qquad (6.40)$$

Now our Klein Gordon equation is in the form

$$\Delta(r)\left(\frac{d\rho}{dr}\right)^2R''+\Delta(r)\left[\left(\frac{d^2\rho}{dr^2}\right)+\left(\frac{d\rho}{dr}\right)^2\left(\frac{\Delta'(\rho)}{\Delta(r)}+\frac{\Gamma'(\rho)}{2\Gamma(r)}-\frac{\Sigma'(\rho)}{2\Sigma(r)}\right)\right]R'$$
$$+\left[\frac{4\Sigma k}{\Gamma}\left(-l(l+1)+\lambda^2\right)+\frac{k^2\Sigma}{\Delta}\left(\frac{(\Gamma^2+a^2\mu)\omega^2}{A^2}-\frac{4a\mu\omega\lambda}{A}-4(\Gamma-\mu)\lambda^2\right)\right]R \;=\; 0.$$

Transforming the following parameter to coordinate ρ

$$\Gamma(r) \longrightarrow \Gamma(\rho)=\frac{\rho r_\infty^2}{\rho+\rho_0}+a^2,$$
$$\Gamma'(\rho)=\frac{\rho_0 r_\infty^2}{(\rho+\rho_0)^2},$$
$$\Sigma(r) \longrightarrow \Sigma(\rho)=\frac{\rho r_\infty^2}{\rho+\rho_0}\left[\frac{\rho r_\infty^2}{\rho+\rho_0}+a^2\right],$$
$$\Sigma'(\rho)=\frac{\rho_0 r_\infty^2}{(\rho+\rho_0)^3}\left[\rho r_\infty^2+a^2(\rho+\rho_0)\right]+\frac{\rho\rho_0 r_\infty^4}{(\rho+\rho_0)^3},$$
$$\Delta(r) \longrightarrow \Delta(\rho)=\frac{(\rho r_\infty^2+a^2(\rho+\rho_0))^2}{(\rho+\rho_0)^2}-\frac{\mu\rho r_\infty^2}{(\rho+\rho_0)}.$$
$$\Delta'(\rho)=\frac{\rho_0 r_\infty^2}{(\rho+\rho_0)^2}\left[\frac{2(\rho r_\infty^2+(\rho+\rho_0)a^2)}{(\rho+\rho_0)}-\mu\right].$$

Let's determine these chain rule formula

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{2r\rho_0 r_\infty^2}{(r_\infty^2 - r^2)^2}, \\ \frac{d^2\rho}{dr^2} &= \frac{2\rho_0 r_\infty^2 (r_\infty^2 + 3r^2)}{(r_\infty^2 - r^2)^3}. \end{aligned}$$

After a bit of tedious work, we ultimately obtain an equation of motion for a massless scalar field in rotating squashed Kaluza-Klein background

$$\Theta \frac{d^2 R(\rho)}{d\rho^2} + \frac{d\Theta}{d\rho}\frac{dR(\rho)}{d\rho} + \left[\frac{\tilde{\mathrm{N}}^2}{\Theta} + \Lambda - l(l+1) + \lambda^2\right] R(\rho) = 0, \tag{6.41}$$

where

$$\begin{aligned} \Theta(\rho) &= \frac{(r_\infty^2 + a^2)}{4 r_\infty^4 \rho_0^2}\left[(\rho r_\infty^2 + a^2(\rho+\rho_0))^2 - \mu\rho(\rho+\rho_0) r_\infty^2\right], \\ \tilde{\mathrm{N}}^2 &= \frac{\mu r_\infty^2 (\rho+\rho_0)^4}{N^4 (r_\infty^2 + a^2)^2}\left[\omega - \frac{\lambda a \mathrm{N}^2 (r_\infty^2 + a^2)}{\rho_0 r_\infty^3}\right]^2, \\ \Lambda &= \frac{4\rho_0^2 r_\infty^6 (\rho+\rho_0)^2 \omega^2}{\mathrm{N}^2 (r_\infty^2 + a^2)^4} - \frac{4\lambda^2 (\rho+\rho_0)^2}{r_\infty^2 + a^2}, \\ \mathrm{N}^2 &= \frac{\rho+\rho_0}{\rho + \frac{a^2}{r_\infty^2 + a^2}\rho_0}. \end{aligned}$$

(See appendix C.3 for the plotting of the effective potential in non-rotating case.) Our next task is to solve the above equation and determine the quasinormal frequencies under some proper boundary conditions.

6.2.3 An Approximative Calculation of Quasinormal Frequencies

In order to obtain the solution of the scalar field equation (6.41) , we will use an approximation method and separate radial equation into two asymptotic regions: for the near horizon $\rho \sim \rho_+$ and far field region $\rho \longrightarrow \infty$. Then, the solution of both regions will be matched in the intermediate region based on Chen [14] and Creek's work [15].

To follow the above argument, we now investigate the solution in the near horizon region ($\rho \sim \rho_+$). In order to formulate a known 2nd-order differential equation namely, hypergeometric equation, we first introduce a new coordinate as defined

$$z = \frac{\Theta}{\left(\rho + \frac{a^2}{r_\infty^2 + a^2}\rho_0\right)^2} \Rightarrow \frac{dz}{d\rho} = (1-z)\frac{B}{\left(\rho + \frac{a^2}{r_\infty^2 + a^2}\rho_0\right)}, \tag{6.42}$$

where $B \equiv 1 - \frac{\rho_0 r_\infty^2 a^2}{\rho(r_\infty^4 - a^4) - a^4\rho_0}$.Thus, for the near horizon limit, the radial equation (6.41) can be expressed in the new variable as

$$z(1-z)\frac{d^2R(z)}{dz^2} + (1 - H_* z)\frac{dR(z)}{dz} + \left[\frac{\mathrm{N}_*^2}{B(\rho_+)^2(1-z)z} - \frac{(l(l+1) - \lambda^2) - \Lambda(\rho_+)}{B(\rho_+)^2(1-z)}\right] R(z) = 0, \quad (6.43)$$

where,

$$\begin{aligned}
\mathrm{N}_*^2 &= \left(1 + \frac{\rho_0}{\rho_+}\right)\left[\rho_+ + \frac{a^2}{r_\infty^2 + a^2}\rho_0\right]^2 \left[\omega - \frac{a\lambda \mathrm{N}(\rho_+)^2(r_\infty^2 + a^2)}{\rho_0 r_\infty^3}\right]^2, \\
\mathrm{H}_* &= 2 - \frac{1}{B(\rho_+)} - \frac{(\rho_0 + \rho_+)}{\mathrm{N}(\rho_+)^2 B(\rho_+)^2}\frac{dB(\rho)}{d\rho}\Big|_{\rho=\rho_+}.
\end{aligned}$$

Notice that, $z \longrightarrow 0$ as r approaches the horizon and $z \longrightarrow 1$ as r closes to the infinity. We redefine the radial solution as $R(z) = z^\alpha(1-z)^\beta P(z)$. Then, we substitute a new radial solution into (6.43). Hence, we obtain the canonical form of the hypergeometric differential equation

$$z(1-z)\frac{d^2P(z)}{dz^2} + \left[c - (1 + a_1 + b)z\right]\frac{dP(z)}{dz} - a_1 b P(z) = 0, \qquad (6.44)$$

with

$$a_1 = \alpha + \beta + H_* - 1, \qquad b = \alpha + \beta, \qquad c = 1 + 2\alpha.$$

While deriving (6.44), it appears that in order to obtain hypergeometric equation there exist two following constraints

$$\begin{aligned}
\alpha_\pm &= \pm\frac{i\mathrm{N}_*}{B(\rho_+)}, \\
\beta_\pm &= \frac{1}{2}\left[(2 - H_*) \pm \sqrt{(H_* - 2)^2 - \frac{4\mathrm{N}_*^2}{B(\rho_+)^2} + \frac{4(l(l+1) - \lambda^2) - \Lambda(\rho_+)}{B(\rho_+)^2}}\right].
\end{aligned}$$

The near horizon limit for the hypergeometric equation is defined by

$$\begin{aligned}
P_{NH}(z) &= A_- z^\alpha (1-z)^\beta F(a_1, b, c; z) \\
&+ A_+ z^{-\alpha}(1-z)^\beta F(a_1 - c + 1, b - c + 1, 2 - c; z), \qquad (6.45)
\end{aligned}$$

Recall that near horizon $z \longrightarrow 0$, then the above equation can be approximated as

$$P_{NH}(z) \sim A_- z^\alpha + A_+ z^{-\alpha},$$

Let's define

$$\begin{aligned}
\aleph &= \sqrt{1 + \frac{\rho_0}{\rho_+}}\left[\rho_+ + \frac{a^2}{r_\infty^2 + a^2}\rho_0\right]^2 \left[\omega - \frac{a\lambda \mathrm{N}(\rho_+)^2(r_\infty^2 + a^2)}{\rho_0 r_\infty^3}\right], \\
y &= \frac{N^2(\rho_+)\ln z}{B(\rho_+ + \rho_0)},
\end{aligned}$$

so,

$$\aleph y = \frac{N_*}{B}\ln z.$$

Hence, we obtain another form of the near horizon solution

$$P_{NH}(z) \sim A_- e^{\pm i\aleph y} + A_+ e^{\mp i\aleph y},$$

The only ingoing modes are allowed at the horizon. This restrict us to $\alpha = \alpha_-$ and we also choose $A_+ = 0$. Moreover, the convergence of the hypergeometric function is also required. Hence, we must choose $\beta = \beta_-$ Finally, we obtain the radial wave solution in the near horizon region

$$P_{NH}(z) = A_- z^{\alpha_-}(1-z)^{\beta_-} F(a_1, b, c; z). \tag{6.46}$$

To match the solution from both sides, we must stretch the solution at near horizon into the intermediate region. Following the process as done in [15], first we transform the argument of the hypergeometric function in the above equation as $z \longrightarrow (1-z)$. It reads

$$P(z)_{NH} = A_- z^{\alpha_-}(1-z)^{\beta_-}\left[\frac{\Gamma(c)\Gamma(c-a_1-b)}{\Gamma(c-a_1)\Gamma(c-b)}F(a_1, b, a_1+b-c+1; 1-z)\right.$$
$$\left. +(1-z)^{c-a_1-b}\frac{\Gamma(c)\Gamma(a_1+b-c)}{\Gamma(a_1)\Gamma(b)}F(c-a_1, c-b, c-a_1-b+1; 1-z)\right]. \tag{6.47}$$

To stretch this solution into a far regime let's consider limit $\rho \longrightarrow \infty$, then function $1-z$ can be written as

$$1 - z \simeq \frac{\mu(r_\infty^2 - a^2)}{4\rho_0 r_\infty^2 \rho}, \tag{6.48}$$

Then, the near horizon solution (6.47) is approximately expressed in a form

$$P_{NH}(z) = A_1 \rho^{-\beta_-} + A_2 \rho^{-\beta_- + N_* - 2}, \tag{6.49}$$

where

$$A_1 = A_-\left[\frac{\mu(r_\infty^2 - a^2)}{4\rho_0 r_\infty^2}\right]^{\beta_-}\frac{\Gamma(c)\Gamma(c-a_1-b)}{\Gamma(c-a_1)\Gamma(a-b)},$$
$$A_2 = A_-\left[\frac{\mu(r_\infty^2 - a^2)}{4\rho_0 r_\infty^2}\right]^{-\beta_- - N_* + 2}\frac{\Gamma(c)\Gamma(a_1+b-c)}{\Gamma(a_1)\Gamma(b)}.$$

On the other hand, we now determine (6.41) in the far field limit ($\rho \longrightarrow \infty$)

$$\Theta \approx \rho^2,$$
$$\Lambda \approx \left[\frac{4\rho_0^2 r_\infty^6}{(r_\infty^2 + a^2)^4}\omega^2 - \frac{4\lambda^2}{r_\infty^2 + a^2}\right]\rho^2,$$
$$\tilde{N}^2 \approx \frac{\mu r_\infty^2}{(r_\infty^2 + a^2)^2}\left[\omega - \frac{a\lambda(r_\infty^2 + a^2)}{\rho_0 r_\infty^3}\right]^2 \rho^4,$$

hence, we approximately obtain an equation in the far field region

$$\rho^2 \frac{d^2 R_{FF}(\rho)}{d\rho^2} + 2\rho \frac{dR_{FF}(\rho)}{d\rho} + \left[\Omega^2\rho^2 - (l(l+1) - \lambda^2)\right] R_{FF}(\rho) \;\; = \;\; 0,$$

where

$$\Omega^2 \;\; = \;\; \frac{\mu r_\infty^2}{(r_\infty^2 + a^2)^2}\left[\omega - \frac{a\lambda(r_\infty^2 + a^2)}{\rho_0 r_\infty^3}\right]^2 + \left[\frac{4\rho_0^2 r_\infty^6 \omega^2}{(r_\infty^2 + a^2)^4} - \frac{4\lambda^2}{(r_\infty^2 + a^2)}\right].$$

Obviously, the above equation is a Bessel equation. Thus, far field solution of the field equation (6.41) can be displayed as the following

$$R_{FF}(\rho) \;\; = \;\; \frac{1}{\sqrt{\rho}}\left[B_1 J_\nu(\Omega\rho) + B_2 Y_\nu(\Omega\rho)\right],$$

where $\nu \equiv \sqrt{(l(l+1) - \lambda^2) + \frac{1}{4}}$, $J_\nu(\Omega\rho)$ and $Y_\nu(\Omega\rho)$ is Bessel of the first and second kind respectively. B_1 and B_2 are the arbitrary constants. If we take limit $\rho \longrightarrow 0$, the above solution can be simplified to

$$R_{FF}(\rho) \;\; \sim \;\; \frac{B_1\left(\frac{\Omega\rho}{2}\right)^\nu}{\sqrt{\rho}\,\Gamma(\nu+1)} - \frac{B_2\Gamma(\nu)}{\pi\sqrt{\rho}\left(\frac{\Omega\rho}{2}\right)^\nu}. \tag{6.50}$$

Now, we ready to match both solution (6.49) and (6.50). However, the different power of ρ prevent us to perform matching technique. We thus follow the method which has been done in [15]. In order to match both solutions, we first need to know an analytic expression of angular eigenvalue $E = l(l+1) - \lambda^2$. From [47], E can be expressed as a power series of $(a\omega)$. Hence, we shall determined up to fifth order

$$\begin{aligned} E = l(l+1) &+ (a\omega)^2 \frac{[2\lambda^2 - 2l(l+1) + 1]}{(2l-1)(2l+3)} \\ &+ (a\omega)^4\left[\frac{2[-3 + 17l(l+1) + l^2(l+1)^2(2l-3)(2l+5)]}{(2l-3)(2l+5)(2l+3)^3(2l-1)^3}\right. \\ &+ \frac{4\lambda^2}{(2l-1)^2(2l+3)^2}\left(\frac{1}{(2l-1)(2l+3)} - \frac{3l(l+1)}{(2l-3)(2l+5)}\right) \\ &+ \left.\frac{2\lambda^4[48 + 5(2l-1)(2l+3)]}{(2l-3)(2l+5)(2l-1)^3(2l+3)^3}\right] + \ldots \end{aligned} \tag{6.51}$$

This form will be used everywhere E appears in our equation. But for E in the power of coefficient, we neglect $(a\omega)^2$ and higher order. Hence,

$$\begin{aligned} -\beta &\simeq l, \\ (\beta + N_* - 2) &\simeq -(l+1), \\ \nu &\simeq \frac{1}{2}(2l+1). \end{aligned}$$

This will restrict the validity of our results to the low black hole's angular momentum. By using the above approximation, we will obtain a constraint on coefficient B_1 and B_2 as follows

$$\frac{B_1}{B_2} = -\frac{\sqrt{\nu}}{\pi}\left[\frac{8\rho_0 r_\infty^2}{\mu\Omega(r_\infty^2 - a^2)}\right]^{2l+1}\frac{\Gamma^2\left(\sqrt{\nu}\right)\Gamma\left(c-a_1-b\right)\Gamma\left(a_1\right)\Gamma\left(b\right)}{\Gamma\left(a_1+b-c\right)\Gamma\left(c-a_1\right)\Gamma\left(c-b\right)}. \quad (6.52)$$

In the far region , the far field solution can be written as

$$R_{FF}(\rho) = A_{in}\frac{e^{-i\Omega\rho}}{\rho} + A_{out}\frac{e^{i\Omega\rho}}{\rho}, \quad (6.53)$$

with A_{in} and A_{out} are defined to be $\frac{B_1 \pm iB_2}{2\sqrt{2\pi\Omega}}$ respectively. From the boundary condition, the solution in asymptotic region contains only the outgoing modes. Thus, we must set $A_{in} = 0$. Finally, we obtain the quasinormal frequencies by solving the following equation

$$B_1 + iB_2 = 0. \quad (6.54)$$

6.2.4 Results and Discussion

To solve (6.54), we use the mathematica's code which is provided in Appendix B.3. Quasinormal frequencies of scalar field propagating in the rotating squashed Kaluza-Klein background are shown in Fig 6.1 and Fig 6.2. In the first figure, we found that as r_∞ increasing the numerical values of the real part of ω also increase while the imaginary part is grows at the first but decreases later on. The right-most line represents the frequencies in the case $a = 0$ while the left-most represents the case $a = 0.3$ Hence, we see that as "a" is getting larger, the numerical values of the quasinormal frequencies get smaller. While in Fig 6.2, we fixed $l = 2, \mu = 1, a = 0.1$ and plot from left to right as $\lambda = 0.5$ and $\lambda = 0$.

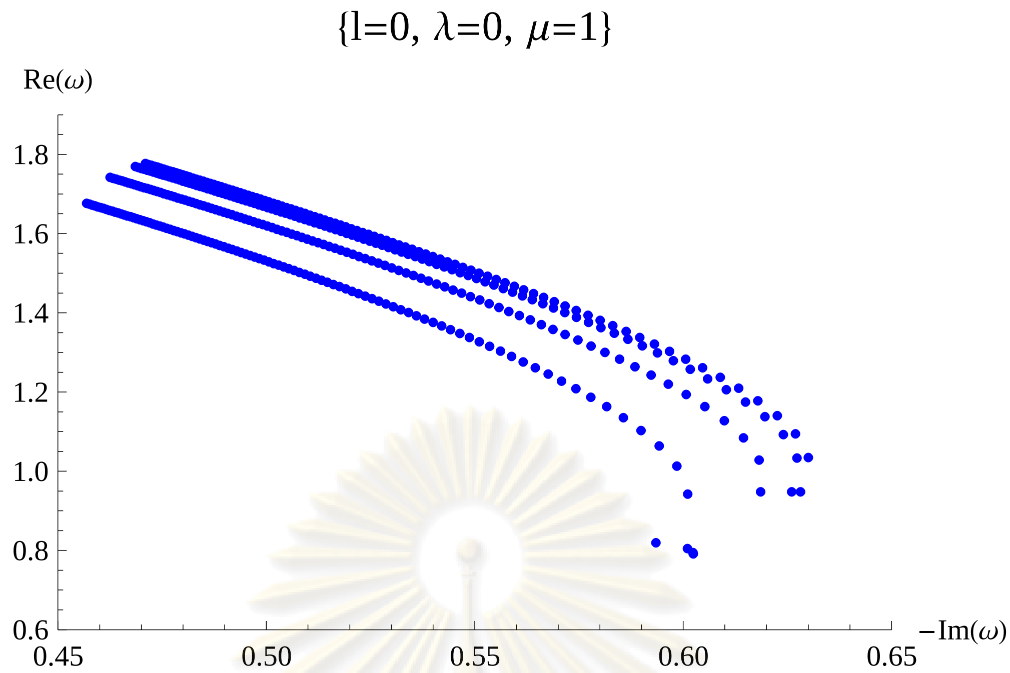

Figure 6.1: Quasinormal frequencies of the scalar field in the rotating squashed Kaluza-Klein black hole spacetime, for fixed $l = 0, \lambda = 0, \mu = 1$. The angular momentum, a, ranges from 0 to 0.3 and r_∞ runs, from 2 to 100. ω corresponds to modes $\nu = \frac{1}{2}$.

In Fig 6.3 and Fig 6.4, we plot the imaginary and real part of the quasinormal frequencies against r_∞ respectively. In these two graphs, we fixed $a = 0.1, l = 2, \mu = 1$ and vary parameter λ from 0 to 1.5 as shown in both figures. As λ is getting larger, the numerical values of quasinormal frequencies both real and imaginary part get bigger. While in Fig 6.5 and Fig 6.6, we fixed $a = 0.1, \lambda = 0, \mu = 1$ and plot the imaginary and real part of the quasinormal frequencies against r_∞ respectively. For Fig 6.5, we observe that at small r_∞ the lowest modes ($l = 0$) seems to dominate over the others however as r_∞ is getting larger the first ($l = 1$) and the second ($l = 2$) fundamental modes dramatically decrease. For Fig 6.6, the real part increases as angular index l increasing. At last, Fig 6.7 and Fig 6.8, we plot the imaginary part and the real part of the quasinormal frequencies against r_∞ for fixed $a = 0.2, \lambda = 0.5, \mu = 1$. Each lines represent each values of the angular index l.

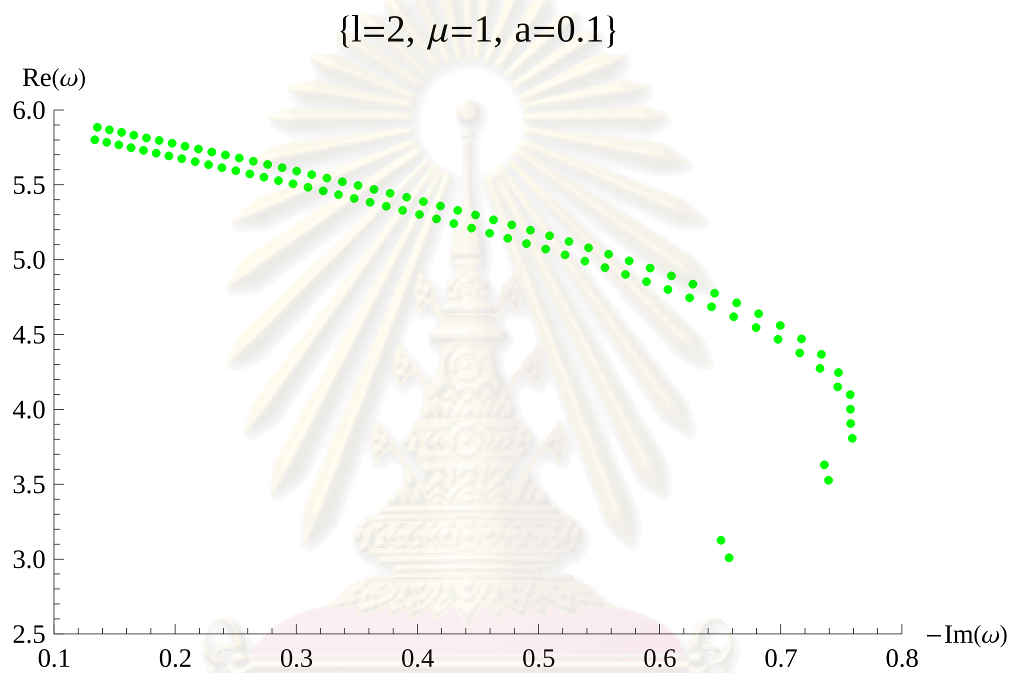

Figure 6.2: Quasinormal frequencies of scalar field in the rotating squashed Kaluza-Klein black hole spacetime, for fixed $l = 2, \mu = 1, a = 0.1$. For $\lambda = 0$ is the right-most line and $\lambda = 0.5$ is on the left-most, whereas r_∞ of each line runs from 2-50. For $\lambda = 0$, ω corresponds to modes $\nu = \frac{5}{2}$ while $\lambda = 0.5$ relates with $\nu = 2.45$

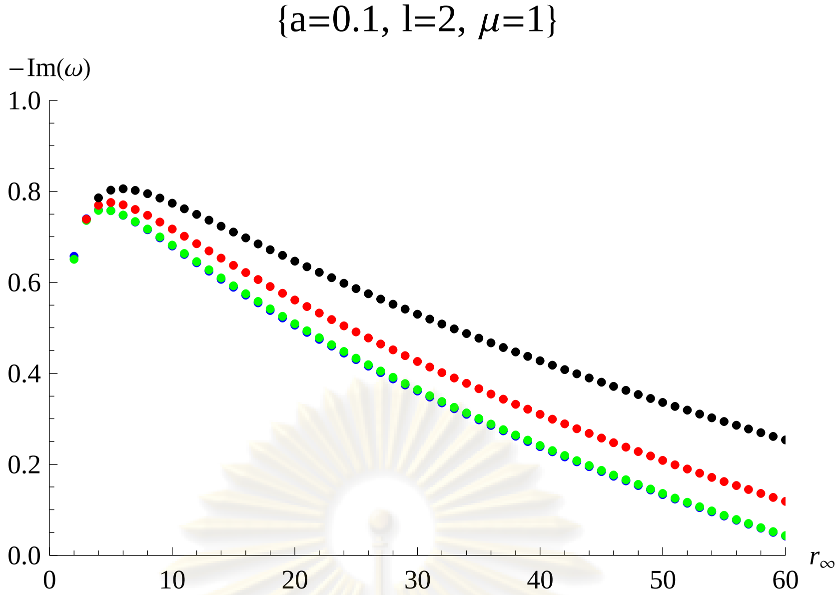

Figure 6.3: Plotting the imaginary part against r_∞, for fixed $a = 0.1, l = 2, \mu = 1$. Each lines show the differences in parameter λ, where $\lambda = 0$(blue), $\lambda = 0.5$(green), $\lambda = 1$(red), $\lambda = 1.5$(black). ω corresponds to modes ν as follow $\frac{5}{2}$(blue), 2.45(green), 2.29(red) and 2(black).

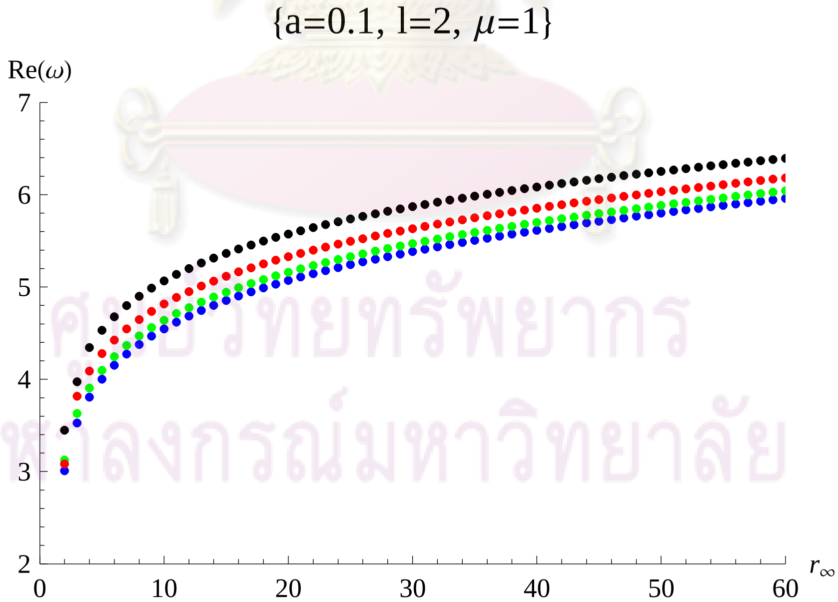

Figure 6.4: Plotting the real part against r_∞, for fixed $a = 0.1, l = 2, \mu = 1$. Each lines show the differences in parameter λ, where $\lambda = 0$(blue), $\lambda = 0.5$(green), $\lambda = 1$(red), $\lambda = 1.5$(black). ω corresponds to modes ν as follow $\frac{5}{2}$(blue), 2.45(green), 2.29(red) and 2(black).

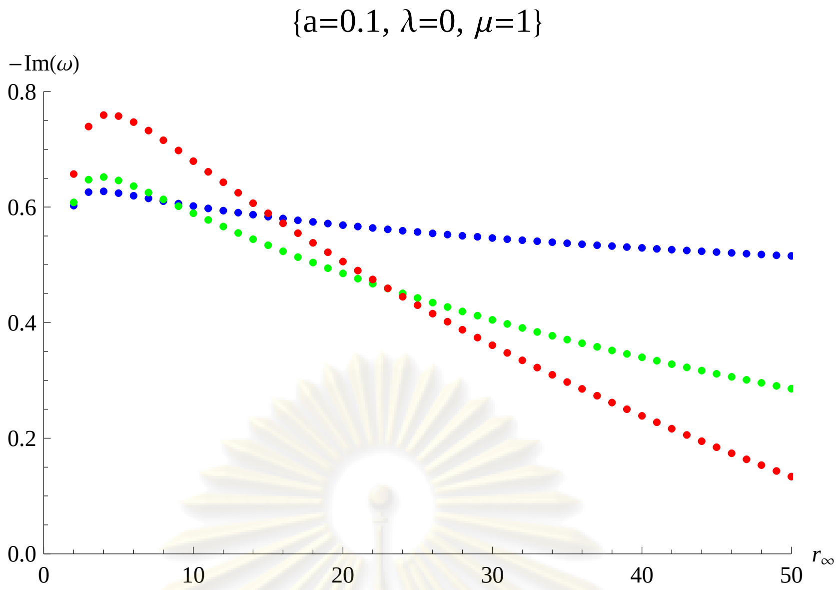

Figure 6.5: Plotting the imaginary part against r_∞, for fixed $a = 0.1, \lambda = 0, \mu = 1$. Each lines show the differences in parameter l, where $l = 0$(blue), $l = 1$(green), $l = 2$(red). ω corresponds to modes ν as follow $\frac{1}{2}$(blue), $\frac{3}{2}$(green) and $\frac{5}{2}$(red).

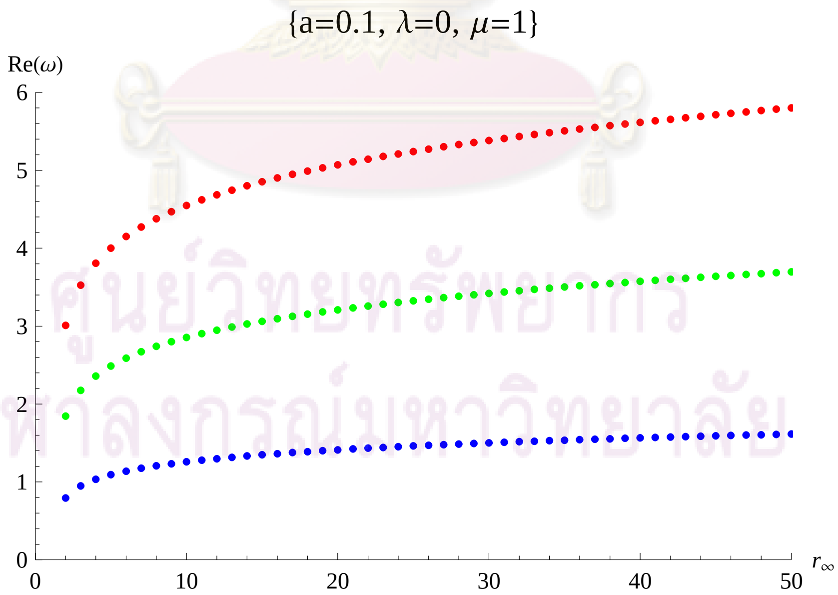

Figure 6.6: Plotting the real part against r_∞, for fixed $a = 0.1, \lambda = 0, \mu = 1$. Each lines show the differences in parameter l, where $l = 0$(blue), $l = 1$(green), $l = 2$(red). ω corresponds to modes ν as follow $\frac{1}{2}$(blue), $\frac{3}{2}$(green) and $\frac{5}{2}$(red).

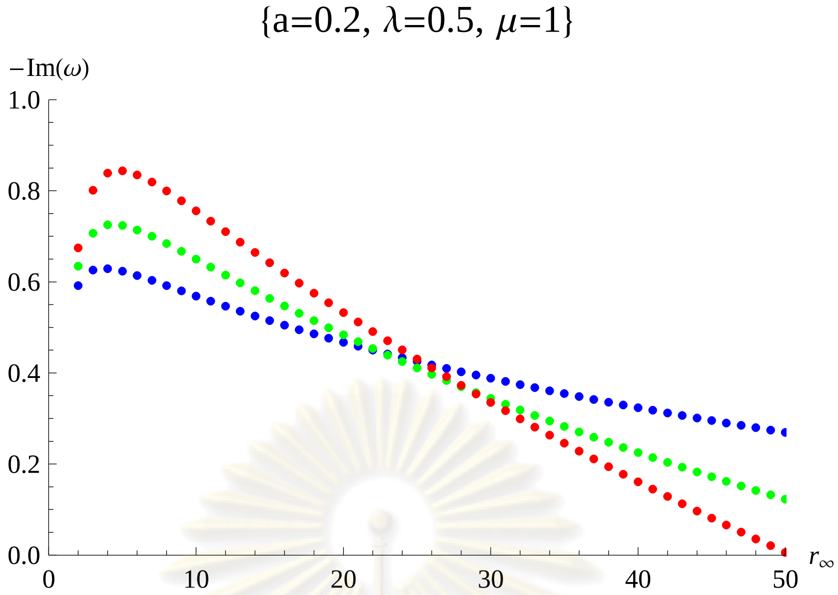

Figure 6.7: Plotting the imagianary part against r_∞, for fixed $a = 0.2, \lambda = 0.5, \mu = 1$. Each lines show the differences in parameter l, where $l = 1$(blue), $l = 2$(green), $l = 3$(red). whereas r_∞ of each line runs from 2-50. ω corresponds to modes ν as follow 1.41(blue), 2.45(green) and 3.46(red).

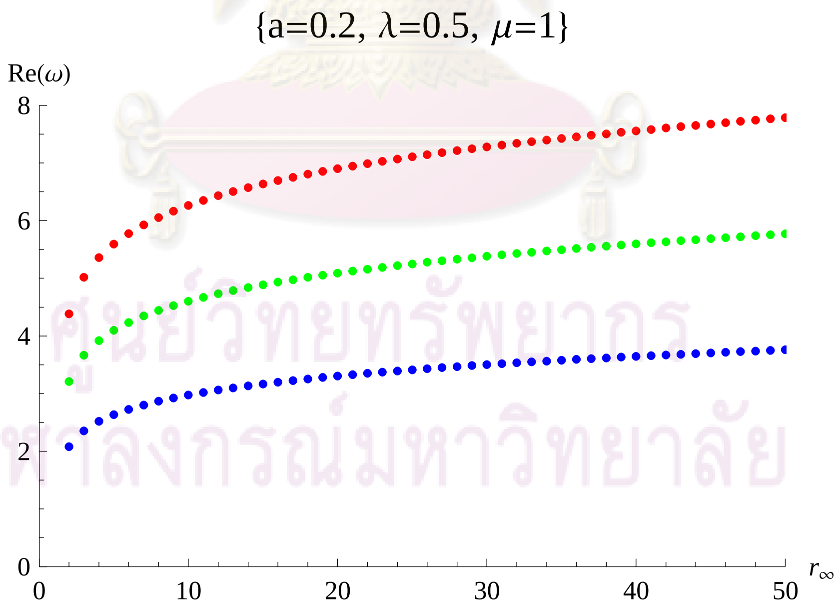

Figure 6.8: Plotting the real part against r_∞, for fixed $a = 0.2, \lambda = 0.5, \mu = 1$. Each lines show the differences in parameter l, where $l = 1$(blue), $l = 2$(green), $l = 3$(red). whereas r_∞ of each line runs from 2-50. ω corresponds to modes ν as follow 1.41(blue), 2.45(green) and 3.46(red).

Chapter VII

SUMMARY

"I was like a boy playing on the sea-shore, and diverting myself now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

-I. Newton

From the beginning of this work, we have briefly reviewed various black hole solutions (Chapter2) to give key concept about each of their properties and their role as a solution of Einstein's theory of gravity. Then, we discussed the definition of quasinormal modes and its application to certain elementary physics area (Chapter 3). The quasinormal modes and quasinormal frequencies of black holes in three, four and five dimensions have been studied in Chapter 4, 5 and 6 respectively.

In Chapter 4, we have considered the quasinormal modes of a BTZ black hole by an analytical method. This method allows us to reduce the Schrödinger-like equation to a hypergeometric differential equation. Hence, by choosing appropriate boundary conditions, the quasinormal frequencies of BTZ black hole are obtained analytically [11]. Its results are shown in Table 4.1. Its real part depends on the angular quantum number of the field while the imaginary part depends on the black hole's mass. Then, the quasinormal frequencies of "rotating" BTZ black hole are investigated. Like the previous case, one can use the same method to obtain its frequencies. As expect, these frequencies are determined by black hole's mass and angular momentum. If one turns off its spinning, the results will perfectly coincide with the non-rotating case [12]. Lastly, a large AdS three dimensional Schwarzschild black hole has been considered. It turns out that, one can compute its quasinormal frequencies by following the same argument as done in two previous cases [13]. In summary, three dimensional black holes have been used as testing models because their quasinormal frequencies can be easily determined analytically. On the other hand, AdS term must be included to construct black hole solutions in three dimensions. Therefore, investigating their quasinormal modes may reveal the interpretation of the conformal field theory in two dimensions.

In Chapter 5, quasinormal modes of Schwarzschild black hole have been reviewed. Unlike chapter 4, for four dimensions it turns out that we cannot reduce the wave equation into the standard hypergeometric differential equation. So, we need to use another approach to investigate quasinormal frequencies of the Schwarzschild black hole. Fortunately, by substitution an appropriate power series solution into the Schrödinger-like equation, one obtains the recurrence relation in a continued fraction form. Hence, the quasinormal frequencies can be determined by solving these relations numerically. Some of the results are shown by Table 5.1. Moreover, from Figure 5.1 which has been done by A. Zhidenko et.al[40], one sees that at particular value of the field's mass a zero imaginary part occurs.

Lastly, in Chapter 6, we have discussed about the quasinormal modes of five dimensional black holes. We begin with the investigation of a large AdS Schwarzschild in five dimensions. It appears that one can obtain an analytical formula for ω by perturbative calculation [13]. To improve the accuracy of the result, first order perturbation must be considered. Finally, the quasinormal modes of rotating squashed Kaluza-Klein black hole have been studied. By matching an asymptotic solution of both sides (near horizon and infinity) [14, 15], we can determine quasinormal frequencies numerically. Some of the results are shown in Figure 6.1 and Figure 6.2. In brief, as black hole's spin parameter "a" increases the numerical value of ω decreases. In addition, we also observe that as λ is getting larger, the numerical values of the quasinormal frequencies get bigger. However, this matching technique restricts the validity of our result to low-angular momenta. So, in the future work we may try to extend this validity further by applying the perturbation method.

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APPENDICES

Appendix A

Useful Calculation

A.1 Einstein-Hilbert action

By the principle of least action, *Hilbert action* or *Einstein-Hilbert action* yields the Einstein field equations. In order to construct the action for GR*, we must define a Lagrangian L which is a scalar under coordinate transformation and depends on the metric tensor $g_{\mu\nu}$. Our dynamical variable is now the metric tensor. From our knowledge, gravity represents as a manifestation of spacetime curvature. Then, we may expect that our Lagrangian should be derived from the Riemann curvature tensor. The only scalar object that can be derived from the curvature tensor is Ricci scalar R. So the simplest plausible action of gravitation is given by

$$S_{EH} = \int R\sqrt{-g}d^4x. \tag{A.1}$$

where $\sqrt{-g}$ is determinant of metric tensor. To construct Einstein field equations, we need to study the behavior of S_{EH} under the small variation of the inverse of metric tensor $g^{\mu\nu}$. By Using $R = g^{\mu\nu}R_{\mu\nu}$ we get,

$$\begin{aligned} \delta S_{EH} &= \int \delta(g^{\mu\nu}R_{\mu\nu}\sqrt{-g})d^4x, \\ &= \int \delta(g^{\mu\nu})R_{\mu\nu}\sqrt{-g}d^4x + \int g^{\mu\nu}\delta(R_{\mu\nu})\sqrt{-g}d^4x + \int g^{\mu\nu}R_{\mu\nu}\delta(\sqrt{-g})d^4x, \\ &\equiv \delta S_1 + \delta S_2 + \delta S_3. \end{aligned} \tag{A.2}$$

Let's examine δS_2 first, The Riemann curvature tensor is given by

$$R^{\rho}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{A.3}$$

Hence,

$$\delta R^{\rho}_{\mu\lambda\nu} = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \delta(\Gamma^{\rho}_{\lambda\sigma})\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}(\delta\Gamma^{\sigma}_{\nu\mu}) - (\delta\Gamma^{\rho}_{\nu\sigma})\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}(\delta\Gamma^{\sigma}_{\lambda\mu}).$$

*This appendix was covered by [5]

The variation of the curvature tensor can be obtained by first varying the connection with respect to the metric. Consider the variation of the Christoffel connection

$$\nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\beta}\delta\Gamma^\beta_{\nu\mu} - \Gamma^\beta_{\lambda\nu}\delta\Gamma^\rho_{\beta\mu} - \Gamma^\beta_{\lambda\mu}\delta\Gamma^\rho_{\beta\nu}, \tag{A.4}$$

$$\nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}) = \partial_\nu(\delta\Gamma^\rho_{\lambda\mu}) + \Gamma^\rho_{\nu\beta}\delta\Gamma^\beta_{\lambda\mu} - \Gamma^\beta_{\nu\lambda}\delta\Gamma^\rho_{\beta\mu} - \Gamma^\beta_{\nu\mu}\delta\Gamma^\rho_{\beta\lambda}. \tag{A.5}$$

By using (A.4)−(A.5), we obtain the variation of the curvature tensor

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}), \tag{A.6}$$

and let $\rho = \lambda$

$$\delta R^\lambda_{\mu\lambda\nu} = \delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}). \tag{A.7}$$

Then, substitute (A.7) into δS_2

$$\begin{aligned}\int\sqrt{-g}d^4x g^{\mu\nu}(\delta R_{\mu\nu}) &= \int\sqrt{-g}d^4x g^{\mu\nu}\Big[\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\Big],\\ &= \int\sqrt{-g}d^4x\nabla_\sigma\Big[g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu})\Big].\end{aligned} \tag{A.8}$$

Then we consider the variation of the Christoffel connection. The connection is given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\Big[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\lambda g_{\mu\nu}\Big]. \tag{A.9}$$

So,

$$\begin{aligned}\delta\Gamma^\sigma_{\mu\nu} &= \frac{1}{2}\delta g^{\sigma\lambda}\Big[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\lambda g_{\mu\nu}\Big] + \frac{1}{2}g^{\sigma\lambda}\Big[\partial_\mu\delta g_{\nu\sigma} + \partial_\nu\delta g_{\mu\sigma} - \partial_\lambda\delta g_{\mu\nu}\Big]\\ &= \Big[\Gamma^\beta_{\mu\nu}g_{\lambda\beta}\delta g^{\sigma\lambda}\Big] - \frac{1}{2}\Big[\nabla_\mu(g_{\nu\lambda}\delta g^{\sigma\lambda}) + \nabla_\nu(g_{\mu\lambda}\delta g^{\sigma\lambda}) + g^{\sigma\lambda}\nabla_\lambda\delta g_{\mu\nu} + 2\Gamma^\beta_{\mu\nu}g_{\lambda\beta}\delta g^{\sigma\lambda}\Big]\\ &= -\frac{1}{2}\Big[g_{\nu\lambda}\nabla_\mu(\delta g^{\sigma\lambda}) + g_{\mu\lambda}\nabla_\nu(\delta g^{\sigma\lambda}) + \nabla^\sigma(\delta g_{\mu\nu})\Big].\end{aligned}$$

Now using $\nabla^\sigma\delta g_{\sigma\nu} = -g_{\mu\nu}g_{\alpha\sigma}\nabla^\sigma\delta g^{\mu\alpha}$ then the above relation become

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\Big[g_{\nu\lambda}\nabla_\mu(\delta g^{\sigma\lambda}) + g_{\mu\lambda}\nabla_\nu(\delta g^{\sigma\lambda}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\Big]. \tag{A.10}$$

Substitute (A.10) back into δS_2, we finally get

$$\delta S_2 = \int\sqrt{-g}d^4x\nabla_\sigma\Big[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\Big]. \tag{A.11}$$

By Stokes's theorem, this is equivalent to the surface term at infinity so it does not contribute anything to the total variation. Then, $\delta S_2 = 0$. In order to deal with δS_3, let's consider the following identity first

$$\begin{aligned}
\ln(\det g_{\mu\nu}) &= Tr(\ln g_{\mu\nu}) \\
\frac{1}{g}\delta g &= g^{\mu\nu}\delta g_{\mu\nu} \\
\delta g &= -g(g_{\mu\nu}\delta g^{\mu\nu}) \\
\therefore \delta(\sqrt{-g}) &= -\frac{1}{2}(-g)^{-\frac{1}{2}}\delta g \\
&= -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu}). \qquad \text{(A.12)}
\end{aligned}$$

Finally, putting all results into (A.2) we get

$$\delta S_{EH} = \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}. \qquad \text{(A.13)}$$

By the principle of least action $\delta S_{EH} = 0$ and since $\delta g^{\mu\nu}$ is arbitrary. We finally get Einstein field equations in vacuum

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \qquad \text{(A.14)}$$

To obtain the Einstein field equations with matter source, we begin with the action

$$\begin{aligned}
S &= \frac{1}{2\kappa}S_{EH} + S_M \\
&= \int (\frac{1}{2\kappa}\mathcal{L}_{EH} + \mathcal{L}_M)d^4x. \qquad \text{(A.15)}
\end{aligned}$$

where $\kappa = \frac{8\pi G}{c^4}$ and $\mathcal{L}_M$ is matter Lagrangian. By varying this action with respect to inverse of the metric tensor

$$\begin{aligned}
\frac{\delta S}{\delta g^{\mu\nu}} &= \int (\frac{1}{2\kappa}\frac{\delta \mathcal{L}_{EH}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}})d^4x = 0 \\
\therefore \frac{1}{2\kappa}\frac{\delta \mathcal{L}_{EH}}{\delta g^{\mu\nu}} &= -\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \\
\frac{\sqrt{-g}}{2\kappa}G_{\mu\nu} &= -\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \qquad \text{(A.16)}
\end{aligned}$$

Let's defines energy-momentum tensor

$$\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} = -T_{\mu\nu}. \qquad \text{(A.17)}$$

and put them back to (A.16). Then we recover the Einstein field equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \qquad \text{(A.18)}$$

A.2 An equivalent formulation of vacuum-space field equations

In empty space the Einstein field equations become

$$\begin{aligned} G_{\mu\nu} &= 0 \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 0 \\ R_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}R. \qquad \text{(A.19)} \end{aligned}$$

contract (A.19) with $g^{\mu\nu}$

$$\begin{aligned} R &= 2R \\ \therefore R &= 0. \end{aligned}$$

Note that $g^{\mu\nu}g_{\mu\nu}$= number of the spacetime dimension (in this case is equal to 4). Substitute this result into (A.19); we'll get field equations in vacuum-space

$$R_{\mu\nu} = 0. \qquad \text{(A.20)}$$

For an empty-space field equations equips with the cosmological constant, it reads

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} \\ R &= 2R - 4\Lambda \\ R &= 4\Lambda. \qquad \text{(A.21)} \end{aligned}$$

Then, we obtain field equations in empty-space with the cosmological constant.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \qquad \text{(A.22)}$$

A.3 Klein-Gordon Equation in Curved Spacetime

We begin with an action for a real scalar field in the curved background

$$S = \int \left[\frac{1}{2}g^{\mu\nu}(\nabla_\mu\Phi)(\nabla_\nu\Phi) - V(\Phi)\right]\sqrt{-g}d^4x. \qquad \text{(A.23)}$$

The first term may be interpreted as a kinetic energy of the field while the second as its potential energy. Since Φ is scalar field, we therefore can replace covariant derivative with ordinary derivative.

$$S = \int \left[\frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - V(\Phi)\right]\sqrt{-g}d^4x. \qquad \text{(A.24)}$$

By plugging the above action into the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial L}{\partial \Phi} = 0, \tag{A.25}$$

we thus obtain an equation of motion for a real scalar field

$$\begin{aligned} \frac{\partial L}{\partial \Phi} &= -\frac{dV(\Phi)}{d\Phi}\sqrt{-g}, \\ \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) &= \partial_\mu \left[g^{\mu\nu} \sqrt{-g} (\partial_\nu \Phi) \right], \end{aligned}$$

so,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[g^{\mu\nu} \sqrt{-g} (\partial_\nu \Phi) \right] + \frac{dV(\Phi)}{d\Phi} = 0. \tag{A.26}$$

It is convenient to choose $V(\Phi) = \frac{1}{2} m^2 \Phi^2$. Thus, the field equation becomes

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[g^{\mu\nu} \sqrt{-g} (\partial_\nu \Phi) \right] + m^2 \Phi = 0. \tag{A.27}$$

This equation is known as the *Klein-Gordon equation* in the curved background which describes dynamics of a massive scalar field in curved spacetime.

Appendix B

Mathematica Codes

B.1 The Determination of Coefficients

```
ClearAll["Global`*"];
(*Copyleft by Alexander Zhidenko (2005). All wrongs reserved!*)


A[r_] = 1 - 2 M / r;

Ψ[r_] = Exp[ⅈ r Sqrt[ω^2 - μ^2]] r^(2 ⅈ M Sqrt[ω^2 - μ^2] + ⅈ M μ^2 / Sqrt[ω^2 - μ^2]) (1 - 2 M / r)^(-2 ⅈ M ω) y[r];


Eq1 = Simplify[
   Expand[((A[r]^2 ∂_{r,r} Ψ[r] + ∂_r A[r] A[r] ∂_r Ψ[r]) + (ω^2 - A[r] (l (l + 1) / r^2 + ∂_r A[r] / r + μ^2)) Ψ[r]) /
     Coefficient[Ψ[r], y[r]]]];


r[z_] = - 2 M / (z - 1);
Eq2 =
  Simplify[Expand[Eq1 /. y''[r] → (y''[z] 4 M^2 / r^4 - y'[z] 4 M / r^3) /. y'[r] → y'[z] 2 M / r^2 /. y[r] → y[z] /.
      r -> r[z] /. ((x_)^a_)^b_ → x^(a b)]];
Eq3 = Numerator[Simplify[Expand[Eq2 / Coefficient[Eq2, y''[z]]]]];

S = Simplify[Coefficient[Eq3, y''[z]]];
T = Simplify[Coefficient[Eq3, y'[z]]];
U = Simplify[Coefficient[Eq3, y[z]]];
Series[S, {z, 0, 6}];
Series[T, {z, 0, 6}];
Series[U, {z, 0, 6}];

(*S must be of order z^2, T and U *)
(*z → 0 correspond wiht r → horizon*)
While[Abs[Limit[S / z^2, z → 0]] == 0, S = S / z; T = T / z; U = U / z;];
While[Abs[Limit[z^2 / S, z → 0]] == 0, S = S * z; T = T * z; U = U * z;];
While[! (Abs[Limit[T, z → 0]] === 0), S = S * z; T = T * z; U = U * z;];
While[! (Abs[Limit[U * z, z → 0]] === 0), S = S * z; T = T * z; U = U * z;];

(* After "while" step S→S*z T→T*z U→U*z *)
(*Simplifying*)
(*This step we divide each term with order of z which come from the derivative of y[z]*)
SE = Expand[S / z^2];
TE = Expand[T / z];
UE = Expand[U];
```

```
Print["Calculating coefficients..."];
st = 0;
While[! ((c[st, n_] = Simplify[(n - st) (n - st - 1) Coefficient[SE, z, st] +
        (n - st) Coefficient[TE, z, st] + Coefficient[UE, z, st]]) === 0), Print[st++]];
(*After the coefficient is zero we expect the all other are zero which usually happens.*)
Print["It seems the other are zeroes, so we have ",
  st, " terms in the reccurence relation."];
```

```
Calculating coefficients...

0

1

2

It seems the other are zeroes, so we have 3 terms in the reccurence relation.
```

```
c[0, n];
c[1, n];
c[2, n];
```

$$\alpha[\texttt{n\_}] = (\texttt{c}[0, \texttt{n}] \;/.\; \texttt{n} \to \texttt{n}+1) \Big/ \left(\left(\sqrt{\omega^2-\mu^2}\right)^3\right)$$

$$\beta[\texttt{n\_}] = \texttt{FullSimplify}\left[\texttt{Expand}\left[(\texttt{c}[1, \texttt{n}] \;/.\; \texttt{n} \to \texttt{n}+1) \Big/ \left(\sqrt{\omega^2-\mu^2}\right)^3\right]\right]$$

$$\gamma[\texttt{n\_}] = \texttt{FullSimplify}\left[\texttt{Expand}\left[(\texttt{c}[2, \texttt{n}] \;/.\; \texttt{n} \to \texttt{n}+1) \Big/ \left(\sqrt{\omega^2-\mu^2}\right)^3\right]\right]$$

$$(1+n)\,(1+n-4\,i\,M\,\omega)$$

$$\frac{1}{\sqrt{-\mu^2+\omega^2}}\left(-\left(1+l+l^2+2\,n\,(1+n)\right)\sqrt{-\mu^2+\omega^2}-\right.$$
$$\left. i\,M\,(1+2\,n)\left(3\,\mu^2-4\,\omega\left(\omega+\sqrt{-\mu^2+\omega^2}\right)\right)-4\,M^2\left(-4\,\omega^2\left(\omega+\sqrt{-\mu^2+\omega^2}\right)+\mu^2\left(3\,\omega+\sqrt{-\mu^2+\omega^2}\right)\right)\right)$$

$$\frac{1}{\left(-\mu^2+\omega^2\right)^{3/2}}\left(n^2\left(-\mu^2+\omega^2\right)^{3/2}-2\,i\,M\,n\left(\mu^2-\omega^2\right)\left(\mu^2-2\,\omega\left(\omega+\sqrt{-\mu^2+\omega^2}\right)\right)-\right.$$
$$\left. M^2\left(8\,\omega^4\left(\omega+\sqrt{-\mu^2+\omega^2}\right)+\mu^4\left(4\,\omega+\sqrt{-\mu^2+\omega^2}\right)-4\,\mu^2\,\omega^2\left(3\,\omega+2\sqrt{-\mu^2+\omega^2}\right)\right)\right)$$

Figure B.1: This figure shows a method to determine the recurrence relation and its coefficients. This code is provided by Alexander Zhidenko.

B.2 Continued Fraction

```
ClearAll["Global`*"];

α[n_] := (1 + n) (1 + n - 4 ⅈ M ω);
```

$$\beta[n\_] := \frac{1}{\sqrt{-\mu^2+\omega^2}}\left(-\left(1+l+l^2+2n(1+n)\right)\sqrt{-\mu^2+\omega^2} - i M (1+2n)\left(3\mu^2-4\omega\left(\omega+\sqrt{-\mu^2+\omega^2}\right)\right) - 4M^2\left(-4\omega^2\left(\omega+\sqrt{-\mu^2+\omega^2}\right)+\mu^2\left(3\omega+\sqrt{-\mu^2+\omega^2}\right)\right)\right);$$

$$\gamma[n\_] := \frac{1}{\left(-\mu^2+\omega^2\right)^{3/2}}\left(n^2\left(-\mu^2+\omega^2\right)^{3/2} - 2 i M n\left(\mu^2-\omega^2\right)\left(\mu^2-2\omega\left(\omega+\sqrt{-\mu^2+\omega^2}\right)\right) - M^2\left(8\omega^4\left(\omega+\sqrt{-\mu^2+\omega^2}\right)+\mu^4\left(4\omega+\sqrt{-\mu^2+\omega^2}\right)-4\mu^2\omega^2\left(3\omega+2\sqrt{-\mu^2+\omega^2}\right)\right)\right);$$

```
FromGeneralContinuedFraction[{a0_, l_List?MatrixQ}] :=
 a0 + Fold[#2[[1]] / (#1 + #2[[2]]) &, 0, Reverse[l]]
FromGeneralContinuedFraction[{a0_, {}}] := a0


n = 0;
M = 1;
l = 0;
μ = 0;
For[j = 0, j ≤ 100, j++, Print[j, FindRoot[
   FromGeneralContinuedFraction[{β[n], Table[{-α[n - i] γ[n - i + 1], β[n - i]}, {i, j}]}] ==
     (-FromGeneralContinuedFraction[{0, Table[{-α[n + i - 1] γ[n + i], β[n + i]}, {i, j}]}]) /.
    x_[y_] :> Subscript[x, y], {ω, 0.1 - 0.1 I}, MaxIterations → 200]]]

Eq4 = FromGeneralContinuedFraction[{β[n], Table[{-α[n - i] γ[n - i + 1], β[n - i]}, {i, j}]}] /.
   x_[y_] :> Subscript[x, y];
Eq5 = -FromGeneralContinuedFraction[{0, Table[{-α[n + i - 1] γ[n + i], β[n + i]}, {i, j}]}] /.
   x_[y_] :> Subscript[x, y];
```

Figure B.2: Mathematica's code for calculation of the continued fraction.

B.3 Solving Omega for Rotating Squashed Kaluza Klein Black Hole

$$\nu = l\,(l+1) + (a\,\omega)^2\left(\frac{2\,\lambda^2 - 2\,l\,(l+1) + 1}{(2\,l-1)\,(2\,l+3)}\right) + (a\,\omega)^4\left(\frac{2\left(-3 + 17\,l\,(l+1) + l^2\,(l+1)^2\,(2\,l-3)\,(2\,l+5)\right)}{(2\,l-3)\,(2\,l+5)\,(2\,l+3)^3\,(2\,l-1)^3} + \frac{4\,\lambda^2}{(2\,l-1)^2\,(2\,l+3)^2}\left(\frac{1}{(2\,l-1)\,(2\,l+3)} - \frac{3\,l\,(l+1)}{(2\,l-3)\,(2\,l+5)}\right) + \frac{2\,\lambda^4\,(48 + 5\,(2\,l-1)\,(2\,l+3))}{(2\,l-3)\,(2\,l+5)\,(2\,l-1)^3\,(2\,l+3)^3}\right) + \frac{1}{4};$$

$$a1 = \alpha[\rho] + \beta[\rho] + d[\rho] - 1;$$

$$b = \alpha[\rho] + \beta[\rho];$$

$$c = 1 + 2\,\alpha[\rho];$$

$$\Omega = \sqrt{\frac{4\,x^2\,Inf^6\,\omega^2}{\left(Inf^2 + a^2\right)^4} - \frac{4\,\lambda^2}{Inf^2 + a^2} + \frac{\mu\,Inf^2}{\left(Inf^2 + a^2\right)^2}\left(\omega - \frac{a\,\lambda\left(Inf^2 + a^2\right)}{x\,Inf^3}\right)^2};$$

$$\rho h = \frac{x\,rh^2}{Inf^2 - rh^2};$$

$$rh = \sqrt{\frac{\left(\mu - 2\,a^2\right) + \sqrt{-4\,a^2\,\mu + \mu^2}}{2}};$$

$$x = \sqrt{\frac{\left(Inf^2 + a^2\right)\left(\left(Inf^2 + a^2\right)^2 - \mu\,Inf^2\right)}{4\,Inf^4}};$$

$$\alpha[\rho\_] := -\frac{i\,k[\rho]}{A[\rho]};$$

$$\beta[\rho\_] := \frac{1}{2}\left((2 - d[\rho]) - \sqrt{(d[\rho] - 2)^2 - \frac{4\,k[\rho]^2}{A[\rho]^2} + \frac{4\left(\left(\nu - \frac{1}{4}\right) - \Lambda[\rho]\right)}{A[\rho]^2}}\right);$$

$$A[\rho] := 1 - \frac{x\,Inf^2\,a^2}{\rho\left(Inf^4 - a^4\right) - a^4\,x};$$

$$K[\rho] := \sqrt{\frac{\rho + x}{\rho + \frac{a^2}{Inf^2 + a^2}\,x}};$$

$$d[\rho\_] := 2 - \frac{1}{A[\rho]} - \frac{(\rho + x)\,D[A[\rho],\,\rho]}{K[\rho]^2\,A[\rho]^2};$$

$$k[\rho\_] := \sqrt{1 + \frac{x}{\rho}}\left(\rho + \frac{a^2}{Inf^2 + a^2}\,x\right)\left(\omega - \frac{a\,\lambda\,K[\rho]^2\left(Inf^2 + a^2\right)}{x\,Inf^3}\right);$$

$$\Lambda[\rho\_] := \frac{4\,x^2\,Inf^6\,(\rho + x)^2}{K[\rho]^2\left(Inf^2 + a^2\right)^4}\,\omega^2 - \frac{4\,\lambda^2\,(\rho + x)^2}{Inf^2 + a^2};$$

```
ClearAll["Global`*"];

l = 0;
λ = 0;
μ = 1;


a = 0.1 ap;

Eq1 = (-√ν/Pi) (8 x Inf^2/(μ Ω (Inf^2 - a^2)))^(2 l+1) ((Gamma[√ν])^2 Gamma[c - a1 - b] Gamma[a1] Gamma[b])/(Gamma[a1 + b - c] Gamma[c - a1] Gamma[c - b]) /. ρ → ρh;
Eq2 = FindRoot[Eq1 == -ⅈ, {ω, 1. - 0.5 I}, MaxIterations → 10 000 000, PrecisionGoal → 10];
For[ap = 0, ap ≤ 3, ap++,
  {Print[For[Inf = 2, Inf ≤ 100, Inf++, {Print[t[Inf, a] = Last[Last[Eq2]], " ",
      Inf, " ", a, " ", rh], Print[Style[(Eq1 /. Eq2), Bold]]}]]}];
```

Figure B.3: This code is used for calculating the quasinormal frequencies of rotating squashed Kaluza Klein black hole at fixed $l = 0, \lambda = 0$ and suppose the black hole mass equal to 1 whereas an outer horizon lies at 1.

Appendix C

Effective Potential

C.1 Effective Potential of the BTZ Black Hole

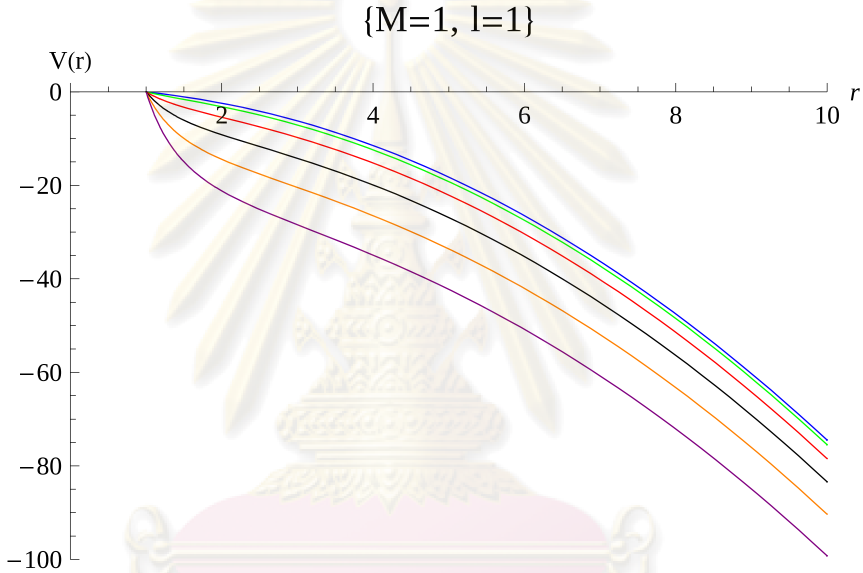

Figure C.1: The effective potential of the BTZ(AdS) black hole, for fixed $M = 1$, the curvature radius $l = 1$ and $m = 0$(Blue), $m = 1$(Green), $m = 2$(Red), $m = 3$(Black), $m = 4$(Orange), $m = 5$(Purple).

C.2 Effective Potential of the Schwarzschild Black Hole

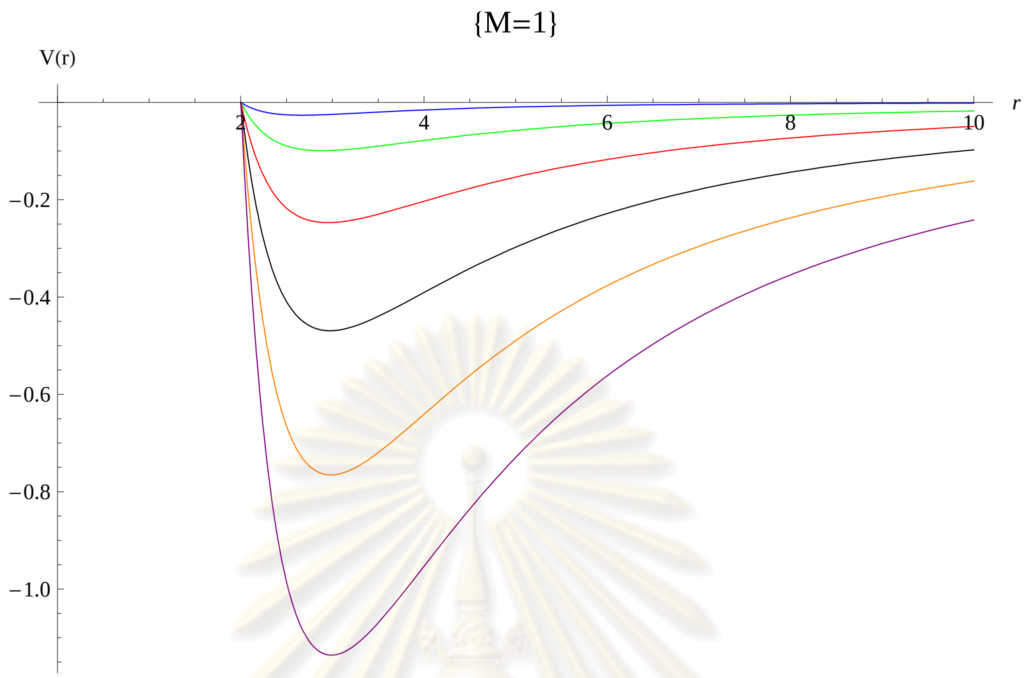

Figure C.2: The effective potential of the 4-d Schwarzschild black hole, for fixed $M = 1$ an angular quantum number $l = 0$(Blue), $l = 1$(Green), $l = 2$(Red), $l = 3$(Black), $l = 4$(Orange), $l = 5$(Purple).

C.3 Effective Potential of the Squashed Kaluza-Klein Black Hole

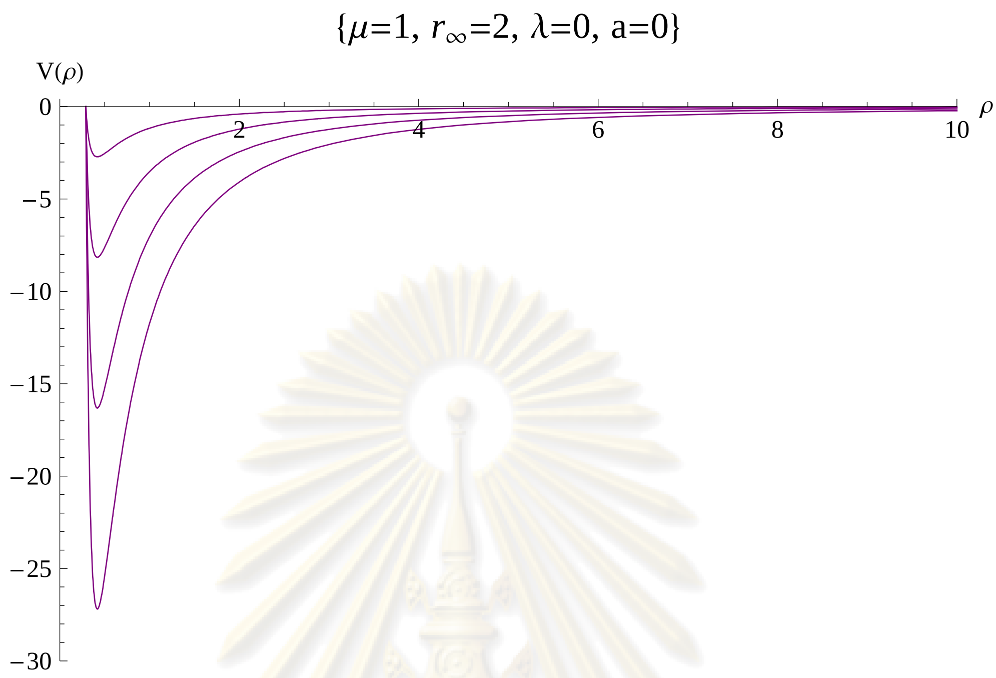

Figure C.3: The effective potential of the Squashed KK black hole, for fixed $\mu = 1$, $r_\infty = 2$, $\lambda = 0$ and parameter l ranges from 1-4. The depth of the potential increases with parameter l.

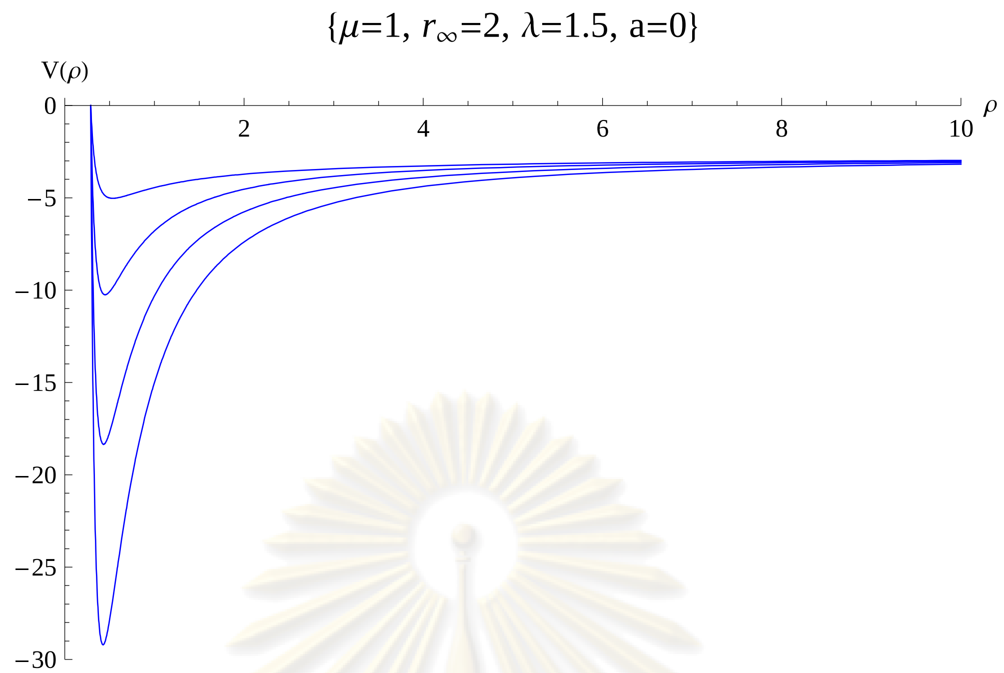

Figure C.4: The effective potential of the Squashed KK black hole, for fixed $\mu = 1$, $r_\infty = 2$, $\lambda = 1.5$ and parameter l ranges from 1-4. The depth of the potential increases with parameter l.

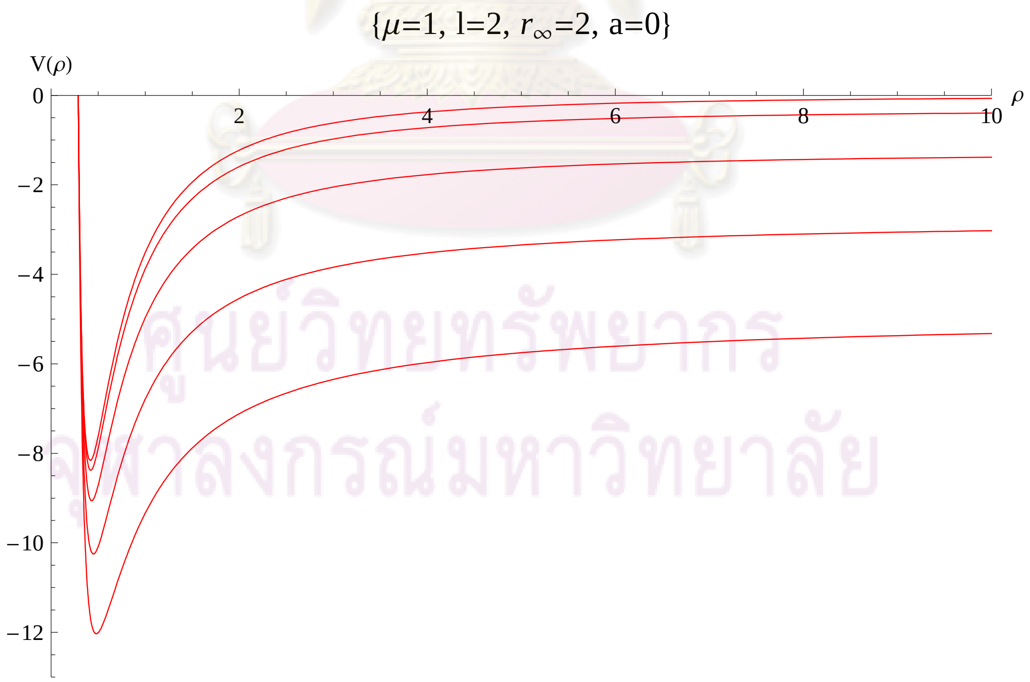

Figure C.5: The effective potential of the Squashed KK black hole, for fixed $\mu = 1$, $r_\infty = 2$, $l = 2$ and parameter λ between 0-2 (0.5 for each step). The depth of the potential increases with parameter λ. The overall potential also increases with λ.

VITAE

Mr. Supakchai Ponglertsakul was born in 5 July 1986 and received his Bachelor's degree in physics from Mahidol University in 2008. He has studied general relativity and quantum field theory for his Master's degree. His research interests are in theoretical physics, particularly in the area of black hole physics both classical and semi-classical level.

Presentations

1. Approximate Calculation for Quasinormal modes of Rotating Squashed Kaluza-Klein Black Hole: The 19^{th} National Graduate Research Conference, Rajabhat Rajanagarindra University, Chachoengsao, Thailand, 23 - 24 December 2010.

International Schools

1. 1^{st} CERN School Thailand, Chulalongkorn University, Bangkok, 4 - 13 October 2010.

2. Pre-CERN School Thailand, Chulalongkorn University, Bangkok, 4 - 6 October 2010.

3. Attended Siam Physics Congress 2010, Sai Yok, Kanchanaburi, Thailand, 25 - 27 March 2010

4. The Siam GR+HEP+COSMO Symposium IV, Naresuan University, Thailand, 26 - 28 July 2009.

5. Attended Siam Physics Congress 2009, Cha-am, Phetchburi, Thailand, 19 - 21 March 2009.