CHAPTER III

LINEAR TRANSFORMATION SEMIGROUPS WHICH ARE CLOSED IN SOME STANDARD EXTENSIONS

Transformation semigroups which are closed in every extension were studied in [1], [4], [5], [6], [7], [8], [9] and [10]. The purpose of this chapter is to show that the linear transformation semigroups LT_V, LG_V, LM_V and LE_V are closed in their standard extensions T_V, P_V and B_V where V is any vector space and T_V, P_V and B_V are the full transformation semigroup on V, the partial transformation semigroup on V and the semigroup of binary relations on V, respectively. We note that LG_V is closed in every extension since it is a group (see Theorem 1.2).

The following lemma is used to prove that LT_V is closed in T_V, P_V and B_V where V is any vector space.

<u>Lemma 3.1</u>. Let S be a basis of a vector space V. If $\rho \in B_V$ and $\alpha, \beta, \gamma \in LT_V$ are such that $\alpha\beta \cap (S \times V) \subseteq \sigma\beta \cap (S \times V)$ and $\sigma\beta = \rho\gamma$ for some $\sigma \in B_V$, then there exists $\lambda \in LT_V$ such that

(i) $\alpha\beta = \lambda\gamma$ and

(ii) $\lambda\mu \cap (S \times V) \subseteq \rho\mu \cap (S \times V)$ for all $\mu \in LT_V$.

<u>Proof</u> : Let $v \in S$. Since $\alpha\beta \cap (S \times V) \subseteq \sigma\beta \cap (S \times V)$, $(v, v(\alpha\beta)) \in \sigma\beta$. Since $\sigma\beta = \rho\gamma$, $(v, v(\alpha\beta)) \in \rho\gamma$. Choose $v' \in V$ such that $(v, v') \in \rho$

and $(v', v(\alpha\beta)) \in \gamma$. Then $v'\gamma = v(\alpha\beta)$. Therefore for each $w \in S$, we can choose $w' \in V$ such that $(w, w') \in \rho$ and $(w', w(\alpha\beta)) \in \gamma$. Thus $w'\gamma = w(\alpha\beta)$ for all $w \in S$. Let λ be the element of LT_V determined by $w\lambda = w'$ for all $w \in S$. Then for $w \in S$, $w(\lambda\gamma) = (w\lambda)\gamma = w'\gamma = w(\alpha\beta)$. This shows that $\alpha\beta = \lambda\gamma$. For $\mu \in LT_V$ and for $w \in S$, $(w, w\lambda) = (w, w') \in \rho$ and $(w', w'\mu) \in \mu$, so $(w, w(\lambda\mu)) \in \rho\mu$. This proves that $\lambda\mu \cap (S \times V) \subseteq \rho\mu \cap (S \times V)$ for all $\mu \in LT_V$.
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Corollary 3.2. Let S be a basis of a vector space V. If $\rho \in B_V$ and $\alpha, \gamma \in LT_V$ are such that $\alpha = \rho\gamma$, then there exists $\lambda \in LT_V$ such that

(i) $\alpha = \lambda\gamma$ and

(ii) $\lambda\mu \cap (S \times V) \subseteq \rho\mu \cap (S \times V)$ for all $\mu \in LT_V$.

Proof : It follows directly from Lemma 3.1 by using $\beta = 1_V$ and $\sigma = \alpha$.
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Theorem 3.3. For any vector space V, LT_V is closed in T_V, P_V and B_V.

Proof : Since $LT_V \subseteq T_V \subseteq P_V \subseteq B_V$, it follows that if LT_V is closed in B_V, then LT_V is also closed in T_V and P_V (see Chapter I, page 7). Therefore to prove the theorem, it suffices to prove that LT_V is closed in B_V. Let $\delta \in \mathrm{Dom}(LT_V, B_V)$. Then there exist $\alpha_o, \alpha_1, \ldots, \alpha_{2m} \in LT_V$, $\rho_1, \rho_2, \ldots, \rho_m$, $\sigma_1, \sigma_2, \ldots, \sigma_m \in B_V$ such that

$$\delta = \alpha_o\sigma_1 , \ \alpha_o = \rho_1\alpha_1$$

$$\rho_i\alpha_{2i} = \rho_{i+1}\alpha_{2i+1} , \ \alpha_{2i-1}\sigma_i = \alpha_{2i}\sigma_{i+1} , \ i = 1,2,\ldots,m-1 ,$$

$$\alpha_{2m-1}\sigma_m = \alpha_{2m} .$$

Let S be a basis of V. Since $\alpha_o = \rho_1\alpha_1$, by Corollary 3.2, there exists $\lambda_1 \in LT_V$ such that $\alpha_o = \lambda_1\alpha_1$ and $\lambda_1\alpha_2 \cap (S \times V) \subseteq \rho_1\alpha_2 \cap (S \times V)$. Since $\lambda_1\alpha_2 \cap (S \times V) \subseteq \rho_1\alpha_2 \cap (S \times V)$ and $\rho_1\alpha_2 = \rho_2\alpha_3$, it follows by Lemma 3.1 that there exists $\lambda_2 \in LT_V$ such that $\lambda_1\alpha_2 = \lambda_2\alpha_3$ and $\lambda_2\alpha_4 \cap (S \times V) \subseteq \rho_2\alpha_4 \cap (S \times V)$. Suppose that k is an integer such that $1 \leq k \leq m-2$, $\lambda_k\alpha_{2k} = \lambda_{k+1}\alpha_{2k+1}$ and $\lambda_{k+1}\alpha_{2k+2} \cap (S \times V) \subseteq \rho_{k+1}\alpha_{2k+2} \cap (S \times V)$. It follows from $\lambda_{k+1}\alpha_{2k+2} \cap (S \times V) \subseteq \rho_{k+1}\alpha_{2k+2} \cap (S \times V)$, $\rho_{k+1}\alpha_{2k+2} = \rho_{k+2}\alpha_{2k+3}$ and Lemma 3.1 that there exists $\lambda_{k+2} \in LT_V$ such that $\lambda_{k+1}\alpha_{2k+2} = \lambda_{k+2}\alpha_{2k+3}$ and $\lambda_{k+2}\alpha_{2k+4} \cap (S \times V) \subseteq \rho_{k+2}\alpha_{2k+4} \cap (S \times V)$. This proves that $\lambda_i\alpha_{2i} = \lambda_{i+1}\alpha_{2i+1}$ and $\lambda_{i+1}\alpha_{2i+2} \cap (S \times V) \subseteq \rho_{i+1}\alpha_{2i+2} \cap (S \times V)$ for all $i = 1,2,\ldots,m-1$. Thus $\lambda_{m-1}\alpha_{2m-2} = \lambda_m\alpha_{2m-1}$ and $\lambda_m\alpha_{2m} \cap (S \times V) \subseteq \rho_m\alpha_{2m} \cap (S \times V)$. Now we have the equalities :

$$\delta = \alpha_o\sigma_1 , \ \alpha_o = \lambda_1\alpha_1$$

$$\lambda_i\alpha_{2i} = \lambda_{i+1}\alpha_{2i+1} , \ \alpha_{2i-1}\sigma_i = \alpha_{2i}\sigma_{i+1} , \ i = 1,2,\ldots,m-1 ,$$

$$\alpha_{2m-1}\sigma_m = \alpha_{2m}$$

and $\alpha_o, \alpha_1, \ldots, \alpha_{2m}, \lambda_1, \lambda_2, \ldots, \lambda_m \in LT_V$, $\sigma_1, \sigma_2, \ldots, \sigma_m$ B_V. Then by the remark on page 8, $\delta = \lambda_m\alpha_{2m}$ which implies that $\delta \in LT_V$ since $\lambda_m, \alpha_{2m} \in LT_V$. Hence $Dom(LT_V, B_V) = LT_V$, so LT_V is closed in B_V, as required.

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To prove that LM_V is closed in T_V, P_V and B_V where V is a vector space, the following lemma is required.

Lemma 3.4. Let V be a vector space. If $\alpha \in B_V$ is such that $\alpha\beta \in LM_V$ for some $\beta \in LM_V$, then $\alpha \in LM_V$.

Proof : Let $\alpha \in B_V$ and $\beta \in LM_V$ be such that $\alpha\beta \in LM_V$. Then $\alpha\beta = \gamma$ for some $\gamma \in LM_V$. Since $\beta \in LM_V$, the inverse map of β, $\beta^{-1} : \nabla\beta \to V$ is 1-1 and linear. Now we have that $\alpha\beta : V \to \nabla\alpha\beta \subseteq \nabla\beta$ and $\beta^{-1} : \nabla\beta \to V$ are 1-1 and linear. These imply that $\alpha = (\alpha\beta)\beta^{-1}$ is 1-1 and linear and $\Delta\alpha = V$, so $\alpha \in LM_V$. #

Theorem 3.5. For any vector space V, LM_V is closed in T_V, P_V and B_V.

Proof : Let $\delta \in Dom(LM_V, B_V)$. Then there exist $\alpha_o, \alpha_1, \dots, \alpha_{2m} \in LM_V$, $\rho_1, \rho_2, \dots, \rho_m$, $\sigma_1, \sigma_2, \dots, \sigma_m \in B_V$ such that

$$\delta = \alpha_o\sigma_1, \quad \alpha_o = \rho_1\alpha_1$$

$$\rho_i\alpha_{2i} = \rho_{i+1}\alpha_{2i+1}, \quad \alpha_{2i-1}\sigma_i = \alpha_{2i}\sigma_{i+1}, \quad i = 1,2,\dots,m-1,$$

$$\alpha_{2m-1}\sigma_m = \alpha_{2m}.$$

Since $\alpha_1 \in LM_V$ and $\rho_1\alpha_1 = \alpha_o \in LM_V$, by Lemma 3.4, $\rho_1 \in LM_V$. Since $\rho_2\alpha_3 = \rho_1\alpha_2 \in LM_V$ and $\alpha_3 \in LM_V$, by Lemma 3.4, $\rho_2 \in LM_V$. From $\rho_i\alpha_{2i} = \rho_{i+1}\alpha_{2i+1}$ for $i = 1,2,\dots,m-1$ and $\rho_1 \in LM_V$, it follows by Lemma 3.4 inductively that $\rho_i \in LM_V$ for $i = 1,2,\dots,m$. Thus $\rho_m \in LM_V$. But from the remark on page 8, $\delta = \rho_m\alpha_{2m}$. Hence $\delta \in LM_V$.

This proves that $Dom(LM_V, B_V) = \bar{LM_V}$. Therefore LM_V is closed in B_V. Since $LM_V \subseteq T_V \subseteq P_V \subseteq B_V$, it follows that LM_V is also closed in T_V, P_V and B_V.

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We prove in the last theorem that for any vector space V, the linear transformation semigroup LE_V is closed in T_V, P_V, and B_V.

Theorem 3.6. For any vector space V, LE_V is closed in T_V, P_V and B_V.

Proof : Let $\delta \in Dom(LE_V, B_V)$. Since $LE_V \subseteq LT_V$, $Dom(LE_V, B_V) \subseteq Dom(LT_V, B_V)$ (see Chapter I page 7), by Theorem 3.3, $Dom(LT_V, B_V) = LT_V$. Thus $\delta \in LT_V$. Since $\delta \in Dom(LE_V, B_V)$, there exist $\alpha_o, \alpha_1, \ldots, \alpha_{2m} \in LE_V$, $\rho_1, \rho_2, \ldots, \rho_m$, $\sigma_1, \sigma_2, \ldots, \sigma_m \in B_V$ such that

$$\delta = \alpha_o\sigma_1, \quad \alpha_o = \rho_1\alpha_1$$

$$\rho_i\alpha_{2i} = \rho_{i+1}\alpha_{2i+1}, \quad \alpha_{2i-1}\sigma_i = \alpha_{2i}\sigma_{i+1}, \quad i = 1,2,\ldots,m-1,$$

$$\alpha_{2m-1}\sigma_m = \alpha_{2m}.$$

Since $\alpha_{2m-1}\sigma_m = \alpha_{2m}$ and $\nabla\alpha_{2m} = V$, it follows that $\nabla\sigma_m = V$. Therefore $\nabla(\alpha_{2m-2}\sigma_m) = V$. From $\alpha_{2m-3}\sigma_{m-1} = \alpha_{2m-2}\sigma_m$ and $\nabla(\alpha_{2m-2}\sigma_m) = V$, we get $\nabla\sigma_{m-1} = V$. From $\alpha_{2i-1}\sigma_i = \alpha_{2i}\sigma_{i+1}$ for $i = 1,2,\ldots,m-1$ and $\nabla\sigma_m = V$, it follows inductively that $\nabla\sigma_i = V$ for $i = 1,2,\ldots,m$. Thus $\nabla\sigma_1 = V$. Since $\delta = \alpha_o\sigma_1$, we get $\nabla\delta = \nabla(\alpha_o\sigma_1) = V$. Now we have $\delta \in LT_V$ and $\nabla\delta = V$, it follows that $\delta \in LE_V$. This proves that $Dom(LE_V, B_V) = LE_V$. Hence LE_V is closed in B_V. From $LE_V \subseteq T_V \subseteq P_V \subseteq B_V$, we have that LE_V is also closed in T_V and P_V.

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