CHAPTER II

LINEAR TRANSFORMATION SEMIGROUPS WHICH HAVE PROPER DENSE SUBSEMIGROUPS

Certain transformation semigroups having proper dense subsemigroups were characterized in [1] and [2]. The purpose of this chapter is to characterize some well-known linear transformation semigroups having proper dense subsemigroups. We are interested in the following linear transformation semigroups on a vector space V :

(1) the multiplicative semigroup of all linear transformations of V (LT_V),

(2) the multiplicative group of all 1 - 1 onto linear transformations of V (LG_V),

(3) the multiplicative semigroup of all 1 - 1 linear transformations of V (LM_V) and

(4) the multiplicative semigroup of all onto linear transformations of V (LE_V).

We introduce necessary and sufficient conditions on V and its field for each of the linear transformation semigroups (1), (2), (3) and (4) to have a proper dense subsemigroup. As consequences of these results, the following standard matrix semigroups which have proper dense subsemigroups are characterized :

(1)′ the multiplicative semigroup of all $n \times n$ matrices over a field F ($M_n(F)$) where n is a positive integer, (2)′ the multiplicative group of all $n \times n$ nonsingular matrices over a field F ($G_n(F)$) where n is a positive integer.

To characterize the multiplicative semigroup of all linear transformations of a vector space V (LT_V) having a proper dense subsemigroup, the following five lemmas are required.

Lemma 2.1. Let G be an abelian group. If G has an element of infinite order, then G has a proper dense subsemigroup.

Proof : Assume that G has an element g which is of infinite order. Then $g^n \neq 1$ for every $n \in \mathbb{N}$ where 1 is the identity of G. Thus $(g^{-1})^n \neq 1$ for every $n \in \mathbb{N}$. Set

$$H = \{x \in G \mid x \text{ has finite order}\}.$$

Then $g \notin H$ and $g^{-1} \notin H$. It is clearly seen that H is a subgroup of G. Next, let

$$K = \{hg^n \mid h \in H,\ n \in \mathbb{N} \cup \{0\}\},$$

that is, $K = \bigcup_{n=0}^{\infty} Hg^n = H \cup Hg \cup Hg^2 \cup \ldots$. Then K is a subsemigroup of G containing H. Claim that $g^{-1} \notin K$. To prove the claim, suppose $g^{-1} \in K$. Then there exists a positive integer m such that $g^{-1} \in Hg^m$ since $g^{-1} \notin H$. It follows that $g^{-1-m} \in H$, so $(g^{-1})^{m+1} \in H$. By the property of H, $(g^{-1})^{m+1}$ has finite order which implies that g^{-1} has finite order. Therefore $g^{-1} \in H$ which is a contradiction. Hence we have the claim, that is, $g^{-1} \notin K$.

This cannot occur for all possibilities of k_1^*, k_2^* because of the following reasons :

(1) If $k_1^*-k_2^* = 0$, then $g^{-1} = g d_1^* d_2^* \varepsilon D$, which is a contradiction.

(2) If $k_1^*-k_2^* > 0$, then $0 < k_1^* - k_2^* < k_1^*$ which is contrary to the property of k_1^* since $g^{-1} = (g d_1^* d_2^*) x^{k_1^* - k_2^*}$ and $g d_1^* d_2^* \varepsilon D$.

(3) If $k_1^*-k_2^* < 0$, then $0 < k_2^*-k_1^* < k_2^*$ which is contrary to the property of k_2^* since $g^{-1} = (g d_1^* d_2^*) x^{-(k_2^*-k_1^*)}$ and $g d_1^* d_2^* \varepsilon D$.

Hence $x^{-1} \varepsilon D$. This proves that $D \cup D^{-1} = G$. Then $< D \cup D^{-1} > = G$. By Theorem 1.4, D is dense in G.

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Lemma 2.2. For any field F, if char(F) = 0, then $(1+1)^n \neq 1$ for every positive integer n where 1 is the identity of F, and hence F has a nonzero element of infinite order under multiplication.

Proof : To prove that $(1+1)^n \neq 1$ for every positive integer n, suppose that $(1+1)^m = 1$ for some positive integer m. From the binomial expansion of $(1+1)^m$, we have that $1 = (1+1)^m = 1 + \binom{m}{1} + \ldots + \binom{m}{m-1} + 1$ where for positive integers n, r and $r \leq n$, $\binom{n}{r}$ denotes the number of combinations of n different things taken r at a times. It follows from the expansion that $1 + 1 + \ldots + 1$ ($2^m - 1$ times) is 0. This is a contradiction since char(F) = 0. #

Lemma 2.3. Let F be a field such that every nonzero element of F has finite order under multiplication. Then the subfield of F generated by a finite subset of F is finite.

Proof : Since every nonzero element of F has finite order under multiplication, we have by Lemma 2.2 that char(F) = p for some prime p. Let A be a finite subset of F. If $A = \emptyset$ or $A = \{0\}$, then the subfield of F generated by A is the prime subfield of F, so it is isomorphic to $\mathbb{Z}_p$, the field of integers modulo p and hence the subfield of F generated by A is finite. Assume that $A \neq \emptyset$ and $A \neq \{0\}$. Let S be the multiplicative subsemigroup of F generated by A. Then $|S| \leq (\prod_{x \in A \setminus \{0\}} \text{ord}(x)) + 1 < \infty$ where for $x \in A \setminus \{0\}$, ord(x) denotes the order of x under multiplication. Let K be the additive subsemigroup of F generated by S. Then $|K| \leq p^{|S|} < \infty$. Due to the fact that every finite subsemigroup of a group is a group, we have that K is a field. It is clearly seen that K is the smallest subfield of F generated by A. Therefore the lemma is proved.

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Lemma 2.4. For any field F and any positive integer n, if $M_n(F)$ has a proper dense subsemigroup, then $G_n(F)$ has a proper dense subsemigroup.

Proof : Let U be a proper dense subsemigroup of $M_n(F)$. Claim that $G_n(F) \not\subseteq U$. Suppose that $G_n(F) \subseteq U$. Since $G_n(F)$ is a subgroup of $M_n(F)$, $\text{Dom}(G_n(F), M_n(F)) = G_n(F) \neq M_n(F)$. Therefore $G_n(F) \subsetneq U$. Let $A \in M_n(F)$ be such that rank A = n-1. Since U is dense in $M_n(F)$, $A \in \text{Dom}(U, M_n(F))$. By Theorem 1.1, there exist

$B_o, B_1, \ldots, B_{2m} \in U, C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m \in M_n(F)$ such that

$$A = B_oD_1, \quad B_o = C_1B_1$$

$$C_iB_{2i} = C_{i+1}B_{2i+1}, \; B_{2i-1}D_i = B_{2i}D_{i+1}, \; i = 1,2,\ldots,m-1,$$

$$B_{2m-1}D_m = B_{2m}.$$

From the remark on page 8, we get that

$$A = B_oD_1$$

$$A = C_iB_{2i-1}D_i = C_iB_{2i}D_{i+1} = C_{i+1}B_{2(i+1)-1}D_{i+1}, \; i = 1,2, \ldots, m-1$$

and

$$A = C_mB_{2m}.$$

Then we have from the above equalities that rank $A \leq$ rank B_i for $i = 0,1, \ldots, 2m$. Thus rank $B_i \geq n-1$ for $i = 0,1, \ldots, 2m$. By Theorem 1.7, rank $B_i = n$ for $i = 0,1, \ldots, 2m$ since $G_n(F) \subseteq U$. Since $C_1B_1 = B_o$, $C_1 = B_oB_1^{-1}$, so rank $C_1 = n$. From $C_iB_{2i} = C_{i+1}B_{2i+1}$ for $i = 1,2, \ldots, m-1$, we have that $C_{i+1} = C_iB_{2i}B_{2i+1}^{-1}$ for $i = 1,2, \ldots, m-1$. Since rank $C_1 = n$, it follows inductively that rank $C_i = n$ for $i = 1,2, \ldots, m$. Then rank $C_m = n$. Since rank B_{2m} = rank $C_m = n$ and $A = C_mB_{2m}$, we have that rank $A = n$. This is a contradiction. Thus $G_n(F) \not\subseteq U$, so $G_n(F) \cap U \neq G_n(F)$. It is known that $M_n(F) \setminus G_n(F)$ is an ideal of $M_n(F)$. Then $\text{Dom}(M_n(F) \setminus G_n(F), M_n(F)) = M_n(F) \setminus G_n(F)$. If $G_n(F) \cap U = \emptyset$, then $U \subseteq M_n(F) \setminus G_n(F)$ which implies that $M_n(F) = \text{Dom}(U, M_n(F)) \subseteq \text{Dom}(M_n(F) \setminus G_n(F), M_n(F)) = M_n(F) \setminus G_n(F)$, a contradiction. Therefore $G_n(F) \cap U \neq \emptyset$. Hence $G_n(F) \cap U$ is a proper subsemigroup of $G_n(F)$.

Next, we shall show that $G_n^-(F) \cap U$ is dense in $G_n(F)$. Let $A \in G(F) \setminus (G_n(F) \cap U)$. Then $A \in G_n(F) \setminus U$. Since U is dense in $M_n(F)$, there exist $B_o, B_1, \ldots, B_{2m} \in U$, $C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m \in M_n(F)$ such that

$$A = B_o D_1, \quad B_o = C_1 B_1$$

$$C_i B_{2i} = C_{i+1} B_{2i+1}, \quad B_{2i-1} D_i = B_{2i} D_{i+1}, \quad i = 1,2,\ldots,m-1,$$

$$B_{2m-1} D_m = B_{2m} .$$

From the remark on page 8, we have that

$$A = B_o D_1$$

$$A = C_i B_{2i-1} D_i = C_i B_{2i} D_{i+1} = C_{i+1} B_{2(i+1)-1} D_{i+1}, \quad i = 1,2,\ldots,m-1$$

and

$$A = C_m B_{2m} .$$

Since rank $A = n$, we have that rank $B_i = n$ for $i = 0,1,\ldots,2m$, rank C_i = rank $D_i = n$ for $i = 1,2,\ldots,m$. Thus $B_i \in G_n(F) \cap U$ for $i = 0,1,\ldots,2m$ and $C_i, D_i \in G_n(F)$ for $i = 1,2,\ldots,m$. By Theorem 1.1, $A \in \mathrm{Dom}\,(G_n(F) \cap U, G_n(F))$. Hence $\mathrm{Dom}\,(G_n(F) \cap U, G_n(F)) = G_n(F)$.

Therefore $G_n(F) \cap U$ is a proper dense subsemigroup of $G_n(F)$.

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<u>Lemma 2.5</u>. For any field F and any positive integer n, if $M_n(F)$ has a proper dense subsemigroup, then F has a nonzero element of infinite order under multiplication.

<u>Proof</u> : To prove that F has a nonzero element of infinite order under multiplication, suppose on the contrary that every

nonzero element of F has finite order under multiplication. Then by Lemma 2.2, char(F) = p for some prime p. Since $M_n(F)$ has a proper dense subsemigroup, we have by Lemma 2.4 that $G_n(F)$ has a proper dense subsemigroup, say D. Let I be the $n \times n$ identity matrix over F. If $D \cup \{I\} = G_n(F)$, then $D = G_n(F) \setminus \{I\}$ which is impossible since $G_n(F) \setminus \{I\}$ is not a subsemigroup of $G_n(F)$. Then $D \cup \{I\} \subsetneq G_n(F)$. Then $D \cup \{I\}$ is a proper dense subsemigroup of $G_n(F)$. Let $A \in G_n(F) \setminus (D \cup \{I\})$. By Theorem 1.4, $G_n(F) = \langle D \cup D^{-1} \rangle$. Then $A \in \langle D \cup D^{-1} \rangle$. Therefore $A = A_1 A_2 \ldots A_k$ for some $A_1, A_2, \ldots, A_k \in G_n(F) \setminus \{I\}$ such that $A_i \in D$ or $A_i^{-1} \in D$ for all $i = 1,2,\ldots,k$. Since $A \notin D$ and D is a subsemigroup of $G_n(F)$, it follows that $A_j \notin D$ for some $j \in \{1,2,\ldots,k\}$. Then $A_j^{-1} \in D$. Let F_1 be the subfield of F generated by all elements (entries) of A_j. Since every nonzero element of F has finite order under multiplication, by Lemma 2.3, we have that F_1 is a finite field. Then $G_n(F_1)$ is a finite group and $A_j^{-1} \in G_n(F_1)$. Thus there exists a positive integer $m > 1$ such that $(A_j^{-1})^m = I$. Hence $A_j = (A_j^{-1})^{m-1}$. Since $A_j^{-1} \in D$ and $m-1 \geq 1$, we have that $(A_j^{-1})^{m-1} \in D$ which implies that $A_j \in D$. This is a contradiction. Hence F has a nonzero element of infinite order under multiplication.

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<u>Theorem 2.6.</u> For any vector space V over a field F, LT_V has a proper dense subsemigroup if and only if one of the following statements holds :

(1) $\dim V = \infty$.

(2) F has a nonzero element of infinite order under multiplication.

Proof : Assume that LT_V has a proper dense subsemigroup. To prove that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication, suppose that $\dim V < \infty$. Then $LT_V \cong M_n(F)$ where $n = \dim V$. Thus $M_n(F)$ has a proper dense subsemigroup. It follows from Lemma 2.5 that F has a nonzero element of infinite order under multiplication. This proves that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication.

For the converse, assume that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication. Suppose that $\dim V < \infty$ and F has a nonzero element of infinite order under multiplication. Then $LT_V \cong M_n(F)$ where $n = \dim V$. By Lemma 2.1, $F \setminus \{0\}$ has a multiplicative proper dense subsemigroup. It follows that F has a multiplicative proper dense subsemigroup. Thus by Theorem 1.6, $M_n(F)$ has a proper dense subsemigroup. Hence LT_V has a proper dense subsemigroup.

It remains to show that if $\dim V = \infty$, then LT_V has a proper dense subsemigroup. To prove this, assume that $\dim V = \infty$. Set

$$D = \{\alpha \in LT_V \mid \dim (V / \nabla\alpha) = \infty\} \cup \{1_V\}$$

where 1_V is the identity map on V. Since for $\alpha \in LG_V$, $\nabla\alpha = V$, we have that $\alpha \notin D$ for all $\alpha \in LG_V \setminus \{1_V\}$. Then $D \neq LT_V$. Claim that $D \setminus \{1_V\}$ is a left ideal of LT_V. To prove this, let $\alpha, \beta \in LT_V$ and $\dim (V/\nabla\alpha) = \infty$. Since $\nabla\beta\alpha \subseteq \nabla\alpha$, we obtain $\dim (V/\nabla\beta\alpha) \geq \dim (V/\nabla\alpha)$, so $\dim (V/\nabla\beta\alpha) = \infty$. This proves that $D \setminus \{1_V\}$ is a subsemigroup of LT_V. Hence D is a proper subsemigroup of LT_V.

Let B be a basis of V. Then $|B| = \infty$. Therefore there exists an countably infinite subset B' of B such that $|B| = |B \setminus B'|$. This implies that there exists a map $\beta : B \to B \setminus B'$ which is 1-1 and onto. Let $\lambda : B \to B$ be defined by

$$v\lambda = \begin{cases} v\beta^{-1} & \text{if } v \in B \setminus B' \\ v & \text{if } v \in B' \end{cases}$$

Then $\beta\lambda = 1_B$. Let $\overline{\beta}$ and $\overline{\lambda}$ be the linear transformations of V extended linearly from β and λ, respectively. Then $\overline{\beta}\overline{\lambda} = 1_V$. Since $V\beta = B \setminus B'$, we get $V\overline{\beta} = \langle B \setminus B' \rangle$. We have consequently that $\{v + V\overline{\beta} \mid v \in B'\}$ is a basis of $V/V\overline{\beta}$. Hence $\dim (V/V\overline{\beta}) = |B'| = \infty$. Therefore $\overline{\beta} \in D$ and $\overline{\beta} \neq 1_V$.

To prove that $\mathrm{Dom}(D, LT_V) = LT_V$, let $\alpha \in LT_V \setminus D$. Since $D \setminus \{1_V\}$ is a left ideal of LT_V, we get $\alpha\overline{\beta} \in D$. Now we have the following equalities :

$$\begin{aligned} \alpha &= (\alpha\overline{\beta})\overline{\lambda} \quad , \quad \alpha\overline{\beta} \in D \\ &= \alpha\overline{\beta}\overline{\lambda} \quad , \quad \overline{\beta} \in D \\ &= \alpha 1_V \quad , \quad \overline{\beta}\overline{\lambda} = 1_V \in D \end{aligned}$$

which is a zigzag in D over LT_V with value α. By Theorem 1.1, $\alpha \in \mathrm{Dom}(D, LT_V)$. This proves that D is dense in LT_V, as required.

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The following corollary is obtained directly from Theorem 2.6 and the fact that for any field F and any positive integer n, $M_n(F) \cong LT_{F^n}$ and $\dim (F^n) = n$ where F^n is a vector space $F \times \ldots \times F$ (n times) over F.

Corollary 2.7. For any field F and any positive integer n, $M_n(F)$ has a proper dense subsemigroup if and only if F has a nonzero element of infinite order under multiplication.

Corollary 2.8. (1) If V is a vector space over a field of characteristic 0, then LT_V has a proper dense subsemigroup.

(2) If F is a field of characteristic 0, then for any positive integer n, $M_n(F)$ has a proper dense subsemigroup.

Proof : (1) follows from Lemma 2.2 and Theorem 2.6 and (2) follows from Lemma 2.2 and Corollary 2.7.

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The following three lemmas will be used to characterize those vector spaces V for which the semigroups LG_V have proper dense subsemigroups.

Lemma 2.9. Let U be a subsemigroup of a semigroup S such that $S \setminus U$ is an ideal of S. If U has a proper dense subsemigroup, then S has a proper dense subsemigroup.

Proof : Let D be a proper dense subsemigroup of U. Let $\overline{D} = D \cup (S \setminus U)$. Since $S \setminus U$ is an ideal of S and $D \subsetneq U$, we have that $\overline{D}$ is a proper subsemigroup of S. To show that $\overline{D}$ is dense in S, that is, Dom $(\overline{D}, S) = S$, let $x \in S$. If $x \in \overline{D}$, then $x \in$ Dom $(\overline{D}, S)$. Assume that $x \notin \overline{D}$. Then $x \in U \setminus D$. Since D is dense in U (that is, $\mathrm{Dom}(D, U) = U$) and $x \in U \setminus D$, by Theorem 1.1, there exist $u_o, u_1, \ldots, u_{2m} \in D$, $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m \in U$ such that

$$x = u_o y_1, \quad u_o = x_1 u_1$$

$$x_i u_{2i} = x_{i+1} u_{2i+1}, \; u_{2i-1} y_i = u_{2i} y_{i+1}, \; i = 1,2,\ldots,m-1,$$

$$u_{2m-1} y_m = u_{2m}.$$

Because of $D \subseteq \overline{D}$ and $U \subseteq S$, we obtain by Theorem 1.1 that $x \in \mathrm{Dom}(\overline{D}, S)$. This proves that $\mathrm{Dom}(\overline{D}, S) = S$. Hence D is a proper dense subsemigroup of S. #

Lemma 2.10. For any field F and any positive integer n, $M_n(F)$ has a proper dense subsemigroup if and only if $G_n(F)$ has a proper dense subsemigroup.

Proof : Assume that $M_n(F)$ has a proper dense subsemigroup. By Lemma 2.4, $G_n(F)$ has a proper dense subsemigroup.

Conversely, assume that $G_n(F)$ has a proper dense subsemigroup. Since $M_n(F) \smallsetminus G_n(F)$ is an ideal of $M_n(F)$ and $G_n(F)$ has a proper dense subsemigroup, it follows from Lemma 2.9 that $M_n(F)$ has a proper dense subsemigroup.

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Lemma 2.11. If W and Z are subspaces of a vector space V, then there exists a basis B of V such that $B \cap W$ is a basis of W and $B \cap Z$ is a basis of Z.

Proof : Let B_o be a basis of the subspace $W \cap Z$ of V and let B_1 be a basis of W containing B_o. Set

$$\mathcal{A} = \{ A \subseteq Z \mid B_o \subseteq A \text{ and } B_1 \cup A \text{ is linearly independent} \}$$

Then $B_o \in \mathcal{A}$. Partially order $\mathcal{A}$ by inclusion.

It is clear that if $\{A_\alpha\}_{\alpha \in I}$ is a chain of $\mathcal{A}$, then $\bigcup_{\alpha \in I} A_\alpha$ is an upper bound of $\{A_\alpha\}_{\alpha \in I}$ in $\mathcal{A}$. Then by Zorn's Lemma, $\mathcal{A}$ has a maximal element, say B_2. Therefore $B_1 \cup B_2$ is linearly independent and B_2 is a linearly independent subset of Z containing B_o. Let $v \in Z \setminus B_2$. By the maximality of B_2 in $\mathcal{A}$, $B_1 \cup B_2 \cup \{v\}$ is linearly dependent. But since $B_1 \cup B_2$ is linearly independent, it follows that v is a linear combination of elements in $B_1 \cup B_2$. Then $v \in \langle B_1 \cup B_2 \rangle$. But $\langle B_1 \cup B_2 \rangle = \langle B_1 \rangle + \langle B_2 \rangle = W + \langle B_2 \rangle$, so $v = w + z$ for some $w \in W$ and $z \in \langle B_2 \rangle$. Since $v \in Z$ and $\langle B_2 \rangle \subseteq Z$, we have that $v - z = w \in W \cap Z$. It then follows that $v - z \in \langle B_2 \rangle$ since $B_0 \subseteq B_2$ and $\langle B_o \rangle = W \cap Z$. But $z \in \langle B_2 \rangle$, so we have $v \in \langle B_2 \rangle$. This proves that $Z = \langle B_2 \rangle$. Hence B_2 is a basis of Z.

Let B be a basis of V containing $B_1 \cup B_2$. Then $B \cap W = B_1$ and $B \cap Z = B_2$ which are bases of W and Z, respectively.

Therefore the lemma is proved.

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<u>Theorem 2.12</u>. For any vector space V over a field F, LG_V has a proper dense subsemigroup if and only if one of the following statements holds :

(1) $\dim V = \infty$.

(2) F has a nonzero element of infinite order under multiplication.

<u>Proof</u> : Assume that LG_V has a proper dense subsemigroup. To prove that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication, suppose that $\dim V < \infty$. Then $LG_V \cong G_n(F)$

where $n = \dim V$. Thus $G_n(F)$ has a proper dense subsemigroup. It follows from Lemma 2.10 that $M_n(F)$ has a proper dense subsemigroup. By Corollary 2.7, F has a nonzero element of infinite order under multiplication. This proves that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication.

For the converse, assume that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication. First, assume that $\dim V < \infty$ and F has a nonzero element of infinite order under multiplication. Then $LG_V \cong G_n(F)$ where $n = \dim V$. By Corollary 2.7, $M_n(F)$ has a proper dense subsemigroup which implies by Lemma 2.10 that $G_n(F)$ has a proper dense subsemigroup.

It remains to show that if $\dim V = \infty$, then LG_V has a proper dense subsemigroup. To prove this, assume that $\dim V = \infty$. Let B be a basis of V. Then B is infinite, so there exists a subset B_1 of B such that $|B_1| = |B|$ and $B \setminus B_1$ is countably infinite. Set

$$U = \{\alpha \in LG_V \mid \langle B_1 \rangle \subseteq \langle B_1 \rangle \alpha\} .$$

Then $1_V \in U$. If $\alpha, \beta \in U$, then $\langle B_1 \rangle \subseteq \langle B_1 \rangle\alpha$ and $\langle B_1 \rangle \subseteq \langle B_1 \rangle\beta$ which imply that $\langle B_1 \rangle \subseteq \langle B_1 \rangle\beta \subseteq (\langle B_1 \rangle \alpha)\beta = \langle B_1 \rangle \alpha\beta$ and hence $\alpha\beta \in U$. Let $v \in B \setminus B_1$. Then $|B_1 \cup \{v\}| = |B_1|$ and $|B \setminus (B_1 \cup \{v\})| = |B \setminus B_1|$. Then there exists a 1-1 map φ of B onto itself such that $(B_1 \cup \{v\})\varphi = B_1$ and $(B \setminus (B_1 \cup \{v\}))\varphi = B \setminus B_1$. Let γ be the linear transformation of V such that $\gamma|_B = \varphi$. Then $\gamma \in LG_V$ and $\langle B_1 \cup \{v\} \rangle \gamma \supsetneq \langle B_1 \rangle \gamma$. Hence $\langle B_1 \rangle = \langle B_1 \cup \{v\} \rangle \gamma \supsetneq \langle B_1 \rangle \gamma$ which implies that $\gamma \notin U$. This proves that U is a proper subsemigroup of LG_V containing 1_V.

To prove that U is dense in LG_V, let $\alpha \in LG_V$. By Lemma 2.11, there exists a basis C of V such that $C \cap \langle B_1 \rangle$ is a basis of $\langle B_1 \rangle$ and $C \cap (\langle B_1 \rangle \alpha)$ is a basis of $\langle B_1 \rangle \alpha$. Let $C_1 = C \cap \langle B_1 \rangle$ and $C_2 = C \cap (\langle B_1 \rangle \alpha)$. Then $|C| = |B| = |B_1| = |C_1|$ and $|C \setminus C_1| = \dim(V/\langle C_1 \rangle)$ $= \dim(V/\langle B_1 \rangle) = |B \setminus B_1|$. Then $C \setminus C_1$ is countably infinite. Let $D = C_2 \setminus C_1$. Then $D = C_2 \cap (C \setminus C_1)$ and $C_1 \cap C_2 = C_2 \setminus D$. Since $C \setminus C_1$ is countably infinite, we have that D is countable.

<u>Case 1</u> : B is uncountable. Then C and C_1 are uncountable. Since $B_1\alpha$ and C_2 are bases of $\langle B_1 \rangle \alpha$ and $B_1\alpha$ is uncountable, we have that C_2 is uncountable. If $C_1 \cap C_2 = \emptyset$, then $C_2 \subseteq C \setminus C_1$ which is impossible since C_2 is uncountable but $C \setminus C_1$ is countably infinite. Thus $C_1 \cap C_2 \neq \emptyset$. Since C_2 and $B_1\alpha$ are both bases of $\langle B_1 \rangle \alpha$, we have that $|C_2| = |B_1\alpha|$. But since $|B_1| = |B_1\alpha|$, so we have $|C_1| = |C_2|$. By the fact that $D \subseteq C_2$, C_2 is uncountable and D is countable, we have that $|C_2| = |(C_2 \setminus D) \cup D| = |C_2 \setminus D| + |D| = |C_2 \setminus D|$. Since $C\alpha$ is a basis of V and $C_1\alpha$ is a basis of $\langle C_1 \rangle \alpha$, we have $\dim(V/\langle C_1 \rangle \alpha) =$ $|C\alpha \setminus C_1\alpha|$. Since C is a basis of V and C_2 is a basis of $\langle B_1 \rangle \alpha$, we obtain that $\dim(V/\langle B_1 \rangle \alpha) = |C \setminus C_2|$. But C_1 is a basis of $\langle B_1 \rangle$, so we have $\dim(V/\langle C_1 \rangle \alpha) = \dim(V/\langle B_1 \rangle \alpha)$. Then $|C\alpha \setminus C_1\alpha| = |C \setminus C_2|$. Hence $|C \setminus C_1| = |C \setminus C_2|$, so $C \setminus C_2$ is countably infinite which implies that $|C \setminus C_2| = |(C \setminus C_2) \cup D| = |C \setminus (C_2 \setminus D)|$ since D is countable and $D \subseteq C_2 \subseteq C$. Hence we get $|C_2 \setminus D| = |C_1|$ and $|C \setminus (C_2 \setminus D)| =$ $|C \setminus C_1|$. Then there exists $\lambda \in LG_V$ such that $(C_2 \setminus D)\lambda = C_1$ and $(C \setminus (C_2 \setminus D))\lambda = C \setminus C_1$. Thus

$$C_1\lambda = ((C_1 \setminus C_2) \cup (C_1 \cap C_2))\lambda$$

$$= (C_1 \setminus C_2)\lambda \cup (C_1 \cap C_2)\lambda$$

$$= (C_1 \setminus C_2)\lambda \cup (C_2 \setminus D)\lambda$$

$$= (C_1 \setminus C_2) \cup C_1$$

which implies that $C_1 \subseteq C_1\lambda$, and hence $<B_1> = <C_1> \subseteq <C_1>\lambda = <B_1>\lambda$. Therefore $\lambda \varepsilon U$. It follows from Theorem 1.3 that $\lambda^{-1} \varepsilon \text{Dom}(U, LG_V)$. Also we have that

$$C_2\lambda = ((C_2 \setminus D) \cup D)\lambda$$

$$= (C_2 \setminus D)\lambda \cup D\lambda$$

$$= C_1 \cup D\lambda$$

which implies that $C_1 \subseteq C_2\lambda$. Therefore $<B_1> = <C_1> \subseteq <C_2>\lambda = <B_1>\alpha\lambda$. Hence $\alpha\lambda \varepsilon U$. From $\lambda^{-1} \varepsilon \text{Dom}(U, LG_V)$ and $\alpha\lambda \varepsilon U \subseteq \text{Dom}(U, LG_V)$, we have $\alpha = (\alpha\lambda)\lambda^{-1} \varepsilon \text{Dom}(U, LG_V)$.

<u>Case 2 :</u> B is countably infinite. Then C and C_1 are countably infinite. Since $B_1\alpha$ is countably infinite and $B_1\alpha$ and C_2 are bases of $<B_1>\alpha$, we have that C_2 is contably infinite.

<u>Subcase 2.1</u> : $C \setminus (C_1 \cup C_2)$ is infinite. It follows from Theorem 1.5 (ii), there exists $\eta \varepsilon LG_V$ such that $C_1\eta \subseteq C_1$ and $C_2\eta \subseteq C_1$. Then $C_1 \subseteq C_1\eta^{-1}$ and $C_2 \subseteq C_1\eta^{-1}$, so $<B_1> = <C_1> \subseteq <C_1>\eta^{-1} = <B_1>\eta^{-1}$ and $<B_1>\alpha = <C_2> \subseteq <C_1>\eta^{-1} = <B_1>\eta^{-1}$. Hence $<B_1> \subseteq <B_1>(\alpha\eta)^{-1}$. Thus η^{-1}, $(\alpha\eta)^{-1} \varepsilon U$. By Theorem 1.3, $\alpha\eta \varepsilon \text{Dom}(U, LG_V)$ which implies that $\alpha = (\alpha\eta)\eta^{-1} \varepsilon \text{Dom}(U, LG_V)$.

Subcase 2.2 : $(C \setminus (C_1 \cup C_2))\check{}$ is finite. First assume that $C_1 \cap C_2$ is infinite. It follows from Theorem 1.5 (i) that there exists $\gamma \in LG_V$ such that $C_1 \subseteq C_1\gamma$ and $C_1 \subseteq C_2\gamma$ and hence $<B_1> = <C_1> \subseteq <C_1>\gamma = <B_1>\gamma$ and $<B_1> = <C_1> \subseteq <C_2>\gamma = <B_1>\alpha\gamma$. Then $\gamma, \alpha\gamma \in U$ which implies by Theorem 1.3 that $\alpha = (\alpha\gamma)\gamma^{-1} \in \mathrm{Dom}(U, LG_V)$.

Next assume that $C_1 \cap C_2$ is finite. It follows by Theorem 1.5 (iii), there exists $\lambda \in LG_V$ such that $C\lambda = C$, $C_1\lambda \subseteq C_1$ and $C_2\lambda \cap C_1$ is infinite. From $C_1\lambda \subseteq C_1$, we get $<B_1> = <C_1> \subseteq <C_1>\lambda^{-1} = <B_1>\lambda^{-1}$, so $\lambda^{-1} \in U$. By Theorem 1.5 (i), we have that there exists $\mu \in LG_V$ such that $C_1 \subseteq C_1\mu$ and $C_1 \subseteq (C_2\lambda)\mu$. Then $<B_1> = <C_1> \subseteq <C_1>\mu = <B_1>\mu$ and $<B_1> = <C_1> \subseteq <C_2\lambda>\mu = <B_1>\alpha\lambda\mu$, so we have $\mu, \alpha\lambda\mu \in U$. Therefore by Theorem 1.3, $\alpha\lambda = (\alpha\lambda\mu)\mu^{-1} \in \mathrm{Dom}(U, LG_V)$. Hence $\alpha = (\alpha\lambda)\lambda^{-1} \in \mathrm{Dom}(U, LG_V)$ since $\lambda^{-1} \in U$.

Therefore U is dense in LG_V, as required.

#

The following corollary is obtained directly from Theorem 2.12 and the fact that for any field F and any positive integer n , $G_n(F) \cong LG_{F^n}$ where $n = \dim(F^n)$.

Corollary 2.13. For any field F and any positive integer n , $G_n(F)$ has a proper dense subsemigroup if and only if F has a nonzero element of infinite order under multiplication.

Corollary 2.14. (1) If V is a vector space over a field of characteristic 0, then LG_V has a proper dense subsemigroup ,

(2) If F is a field of characteristic 0 and n is a positive integer, then $G_n(F)$ has a proper dense subsemigroup.

Proof : (1) follows from Lemma 2.2 and Theorem 2.12 and (2) follows from Lemma 2.2 and Corollary 2.13.

#

Next, to prove that the conditions (1) and (2) of Theorem 2.16 and Theorem 2.19 are also necessary and sufficient conditions for each of the linear transformation semigroups LM_V and LE_V to have a proper dense subsemigroup, we need one lemma for each case.

Lemma 2.15. If V is a vector space of infinite dimension, then $LM_V \setminus LG_V$ is an ideal of LM_V.

Proof : Since $\dim V = \infty$, $LM_V \neq LG_V$, so $LM_V \setminus LG_V \neq \emptyset$. To show that $LM_V \setminus LG_V$ is an ideal of LM_V, let $\alpha \in LM_V$ and $\beta \in LM_V \setminus LG_V$. Then $\alpha\beta$, $\beta\alpha \in LM_V$. If $\alpha\beta \in LG_V$, then $\nabla\beta \supseteq \nabla\alpha\beta = V$ which implies that $\beta \in LG_V$, a contradiction. Then $\alpha\beta \in LM_V \setminus LG_V$. Suppose that $\beta\alpha \in LG_V$. Then $\nabla\alpha \supseteq \nabla\beta\alpha = V$, so $\alpha \in LG_V$. This implies that $\beta = (\beta\alpha)\alpha^{-1} \in LG_V$, a contradiction. Therefore $\beta\alpha \in LM_V \setminus LG_V$. This proves that $LM_V \setminus LG_V$ is an ideal of LM_V.

#

Theorem 2.16. For any vector space V over a field F, LM_V has a proper dense subsemigroup if and only if one of the following statements holds :

(1) $\dim V = \infty$.

(2) F has a nonzero element of infinite order under multiplication.

Proof : Assume that LM_V has a proper dense subsemigroup. To prove dim V = ∞ or F has a nonzero element of infinite order under multiplication, suppose that dim V < ∞. Then $LM_V = LG_V \cong G_n(F)$ where n = dim V. Then $G_n(F)$ has a proper dense subsemigroup. It follows from Corollary 2.13 that F has a nonzero element of infinite order under multiplication. This proves that dim V = ∞ or F has a nonzero element of infinite order under multiplication.

For the converse, assume that dim V = ∞ or F has a nonzero element of infinite order under multiplication. If dim V < ∞ , then $LM_V \cong G_n(F)$ and by Corollary 2.13, $G_n(F)$ has a proper dense subsemigroup which implies that LM_V has a proper dense subsemigroup.

Next, assume that dim V = ∞ . Then by Lemma 2.15, $LM_V \setminus LG_V$ is an ideal of LM_V. By Theorem 2.12, LG_V has a proper dense subsemigroup. It then follows from Lemma 2.9 that LM_V has a proper dense subsemigroup.

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Corollary 2.17. If V is a vector space over a field of characteristic 0, then LM_V has a proper dense subsemigroup.

Proof : It follows directly from Lemma 2.2 and Theorem 2.16.

#

Lemma 2.18. If V is a vector space of infinite dimension, then $LE_V \setminus LG_V$ is an ideal of LE_V.

Proof : Since $\dim V = \infty$, $LE_V \setminus LG_V \neq \emptyset$. To show that $LE_V \setminus LG_V$ is an ideal of LE_V, let $\alpha \varepsilon LE_V$ and $\beta \varepsilon LE_V \setminus LG_V$. Then $\alpha\beta$, $\beta\alpha \varepsilon LE_V$. If $\beta\alpha \varepsilon LG_V$, then β is 1-1, so $\beta \varepsilon LG_V$, a contradiction. Thus $\beta\alpha \varepsilon LE_V \setminus LG_V$. Suppose that $\alpha\beta \varepsilon LG_V$. Then α is 1-1, so $\alpha \varepsilon LG_V$. This implies that $\beta = \alpha^{-1}(\alpha\beta) \varepsilon LG_V$ which is a contradiction. Hence $\alpha\beta \varepsilon LE_V \setminus LG_V$. This proves that $LE_V \setminus LG_V$ is an ideal of LE_V, as desired.

#

Theorem 2.19. For any vector space V over a field F, LE_V has a proper dense subsemigroup if and only if one of the following statements holds :

(1) $\dim V = \infty$.

(2) F has a nonzero element of infinite order under multiplication.

Proof : Assume that LE_V has a proper dense subsemigroup. To prove that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication, suppose that $\dim V < \infty$. Then $LE_V = LG_V \cong G_n(F)$ where $n = \dim V$. Then $G_n(F)$ has a proper dense subsemigroup. It follows from Corollary 2.13 that F has a nonzero element of infinite order under multiplication.

Conversely, assume that $\dim V = \infty$ or F has a nonzero element of infinite order under multiplication. If $\dim V < \infty$, then $LE_V = LG_V \cong G_n(F)$ where $n = \dim V$ and by Corollary 2.13, $G_n(F)$ has a proper dense subsemigroup which implies that LE_V has a proper dense subsemigroup.

Next, assume that $\dim V = \infty$. Then by Lemma 2.18, $LE_V \setminus LG_V$ is an ideal of LE_V and by Theorem 2.12, LG_V has a proper dense subsemigroup which implies by Lemma 2.9 that LE_V has a proper dense subsemigroup.

#

Corollary 2.20. If V is a vector space over a field of characteristic 0, then LE_V has a proper dense subsemigroup.

Proof : It follows directly from Lemma 2.2 and Theorem 2.19.

#

Remark. Let V be a vector space over a field F. Consider the following two statements :

(1) $\dim V = \infty$.

(2) F has a nonzero element of infinite order under multiplication.

By Lemma 2.2, if char(F) = 0, then (2) holds. We can see that if (2) holds, then F must be infinite. The following two examples of fields show that there exists a field which satisfies (2) and whose characteristic is not 0 and there exists an infinite field which does not satisfy (2). It is easy to see that for any field K, there exist a vector space over K of finite dimension and a vector space over K of infinite dimension. Hence the statement (2) of Theorem 2.6, Theorem 2.12, Theorem 2.16 and Theorem 2.19 cannot be replaced by any one of the statements : " char(F) = 0 " and " F is an infinite field ".

<u>Example 1</u>. Let p be a prime. Then the field $\mathbb{Z}_p(x)$ (the quotient field of the ring $\mathbb{Z}_p[x]$) has characteristic $p \neq 0$ and x is a nonzero element of infinite order under multiplication in the field $\mathbb{Z}_p(x)$.

<u>Example 2</u>. Let p be a prime. Set $F_o = \mathbb{Z}_p$. Then there exists an irreducible polynomial $f_1(x_1)$ in the polynomial ring $F_o[x_1]$ of degree 2 (see [11], page 304). Let $F_1 = F_o[x_1] / <f_1(x)>$. Then F_1 is a field and

$$F_1 = \{a + bx_1 + <f_1(x_1)> / a, b \in F_o\}$$

Since $|F_o| = p$ and $\text{char}(F_o) = p$, it follows that F_1 is a finite field of order p^2 with characteristic p. Then every nonzero element of F_1 has finite order under multiplication. Also, F_o can be considered as a subfield of F_1 by the map $a \to a + <f_1(x_1)>$. Assume that $k \in \mathbb{N} \cup \{0\}$ and $F_o, F_1, \ldots, F_k$ are constructed such that $F_o \subseteq F_1 \subseteq \ldots \subseteq F_k$, $|F_i| = p^{2^i}$ and $\text{char}(F_i) = p$ for $i = 0,1,2,\ldots,k$. Let $f_{k+1}(x_{k+1})$ be an irreducible polynomial of degree 2 in the polynomial ring $F_k[x_{k+1}]$. Set $F_{k+1} = F_k[x_{k+1}] / <f_{k+1}(x_{k+1})>$. Then F_{k+1} is a field and

$$F_{k+1} = \{ a + bx_{k+1} + <f_{k+1}(x_{k+1})> \mid a, b \in F_k \}.$$

Since $|F_k| = p^{2^k}$ and $\text{char}(F_k) = p$, we have that F_{k+1} is a finite field of order $p^{2^{k+1}}$ with characteristic p. Then every nonzero element of F_{k+1} has finite order under multiplication. Also, F_k can be considered as a subfield of F_{k+1} by the map $a \to a + <f_{k+1}(x_{k+1})>$. By this induction process, we have a sequence of fields $(F_n)_{n=o}^{\infty}$ such that $F_o \subseteq F_1 \subseteq F_2 \subseteq \ldots$, $|F_n| = p^{2^n}$

and char(F_n) = p for n = 0,1,2, ... Set $F = \bigcup_{n=o}^{\infty} F_n$. Define the addition $\oplus$ and the multiplication $\odot$ on F as follows : For a,b ε F, if a,b ε F_n, let a $\oplus$ b and a $\odot$ b be the addition of a, b in F_n and the multiplication of a, b in F_n, respectively. Then the operations $\oplus$ and $\odot$ are well-defined and under these operations, F is a field containing F_n as a subfield for every $n \varepsilon \mathbb{N} \cup \{0\}$. Hence F is an infinite field with characteristic p and every nonzero element has finite order under multiplication.