CHAPTER I

PRELIMINARIES

Let X be a set and B_X the set of all binary relations on X. For $\rho, \sigma \in B_X$, define their composition $\rho\sigma$ by

$$\rho\sigma = \{(a,b) \in X \times X \mid (a,x) \in \rho \text{ and } (x,b) \in \sigma \text{ for some } x \in X\}.$$

Then B_X is a semigroup under composition of relations which is called the semigroup of binary relations on X. For a binary relation ρ on X, the domain and range of ρ are denoted by $\Delta\rho$ and $\nabla\rho$, respectively.

By a transformation of X we mean a map of X into itself. A partial transformation of X is a map from a subset of X into X. The empty transformation of X is the partial transformation of X with empty domain and it is denoted by 0. For $\alpha, \beta \in P_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \emptyset$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \emptyset$, let

$\alpha\beta = (\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}})(\beta|_{(\nabla\alpha\cap\Delta\beta)})$ (the composition of the maps $\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha\cap\Delta\beta}$) where $\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha\cap\Delta\beta}$ denote the restrictions of α and β to $(\nabla\alpha\cap\Delta\beta)\alpha^{-1}$ and $\nabla\alpha\cap\Delta\beta$, respectively. Then P_X is a semigroup having 0 and 1_X as its zero and identity, respectively where 1_X is the identity map on X and P_X is called the partial transformation semigroup on X. In fact, P_X is a subsemigroup of B_X.

Let T_X be the set of all transformations of X. Then T_X is a subsemigroup of P_X with identity 1_X and it is called the full transformation semigroup on X. Let G_X be the symmetric group on X. Then G_X is a subgroup of T_X containing 1_X as its identity and $G_X \subseteq T_X \subseteq P_X \subseteq B_X$.

Let V be a vector space. For $A \subseteq V$, let $\langle A \rangle$ denote the subspace of V generated by A. The following facts of vector spaces will be used in this research :

(1) The cardinal numbers of any two bases of V are equal.

(2) Let W and Z be subspaces of V such that $W \subseteq Z$. Then $\dim (V/W) \geq \dim (V/Z)$.

(3) If W is a subspace of V, then $\dim V = \dim W + \dim(V/W)$.

Let LT_V, LG_V, LM_V and LE_V be given as follows :

LT_V = the set of all linear transformations of V,

LG_V = the set of all 1 - 1 onto linear transformations of V,

LM_V = the set of all 1 - 1 linear transformations of V and

LE_V = the set of all onto linear transformations of V.

Then under composition of maps, LT_V, LG_V, LM_V and LE_V are subsemigroups of T_V, P_V and B_V containing 1_V as their identity and the semigroups LT_V, LG_V, LM_V and LE_V are refered respectively as the multiplicative semigroup of all linear transformations of V, the multiplicative group of all 1 - 1 onto linear transformations of V, the multiplicative semigroup of all 1 - 1 linear transformations of V and the multiplicative semigroup of all onto linear transformations of V. Then LG_V is a subgroup of LM_V, LE_V, LT_V, T_V, P_V and B_V, $LG_V \subseteq LM_V \subseteq LT_V \subseteq T_V \subseteq P_V \subseteq B_V$ and $LG_V \subseteq LE_V \subseteq LT_V \subseteq T_V \subseteq P_V \subseteq B_V$. From the fact that if $\dim V < \infty$ and $\alpha \in LT_V$, then α is 1 - 1 if and only if α is onto, it follows that if $\dim V < \infty$, then $LM_V = LG_V = LE_V$.

If F is a field and n is a positive integer, let $M_n(F)$ be the multiplicative semigroup of all $n \times n$ matrices over F and $G_n(F)$ the multiplicative group of all $n \times n$ nonsingular matrices over F, so we have $M_n(F) \cong LT_{F^n}$ and $G_n(F) \cong LG_{F^n}$ where F^n denotes the vector space $F \times \dots \times F$ (n times) over F. Moreover, if V is a vector space over a field F with $\dim V = n < \infty$, then $LT_V \cong M_n(F)$ and $LG_V \cong G_n(F)$.

A subset A of a semigroup S is said to be <u>dense</u>* in S if for any semigroup T and for any homomorphisms $\alpha, \beta : S \to T$, $\alpha|_A = \beta|_A$ implies $\alpha = \beta$.

A subsemigroup U of a semigroup S is said to be <u>closed</u>* in S if for any element $d \in S \setminus U$, there are a semigroup T and homomorphisms $\varphi, \psi : S \to T$ such that $\varphi|_U = \psi|_U$ and $d\varphi \neq d\psi$.

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* In Topology, it is known that for a metric space X and for $A \subseteq X$,

(1) A is dense in X if and only if for any metric space Y and for any continuous mappings $f, g : X \to Y$ $f|_A = g|_A$ implies $f = g$ and

(2) A is closed in X if and only if for any point $x \in X \setminus A$, there are a metric space Y and continuous mappings $f, g : X \to Y$ such that $f|_A = g|_A$ and $f(x) \neq g(x)$.

A semigroup S is said to be <u>absolutely closed</u> if S is closed in every extension of S, that is, S is closed in T for any semigroup T containing S as a subsemigroup.

Let S be a semigroup. For $A \subseteq S$, let $\langle A \rangle$ denote the subsemigroup of S generated by A. Let U be a subsemigroup of S. For any element d of S, d is said to be <u>dominated</u> by U or U <u>dominates</u> d if for any semigroup T and for any homomorphisms $\varphi, \psi : S \to T$, $\varphi|_U = \psi|_U$ implies $d\varphi = d\psi$. The set of all elements of S which are dominated by U is called the <u>dominion</u> of U in S and it is denoted by Dom(U, S).

The following statements clearly hold :

(i) $U \subseteq \mathrm{Dom}(U,S)$.

(ii) Dom(U, S) is a subsemigroup of S.

(iii) U is dense in S if and only if Dom(U, S) = S.

(iv) U is closed in S if and only if Dom(U, S) = U.

(v) If U and V are subsemigroups of S such that $U \subseteq V$, then $\mathrm{Dom}(U,S) \subseteq \mathrm{Dom}(V,S)$ and $\mathrm{Dom}(U,V) \subseteq \mathrm{Dom}(U,S)$ and hence U is closed in S implies U is closed in V.

Let $\mathbb{N}$ denote the set of all positive integers.

Let U be a subsemigroup of a semigroup S. A <u>zigzag</u> of length $m \in \mathbb{N}$ in U over S with value $d \in S$ is a system of equalities :

$$d = u_o y_1, \quad u_o = x_1 u_1$$

$$(*) \qquad x_i u_{2i} = x_{i+1} u_{2i+1}, \quad u_{2i-1} y_i = u_{2i} y_{i+1}, \quad i = 1, 2, \ldots, m-1,$$

$$u_{2m-1} y_m = u_{2m},$$

where $u_o, u_1, \ldots, u_{2m} \in U$, $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m \in S$.

Remark. If (*) holds, then we have the following equalities :

$$d = u_o y_1$$

$$d = x_i u_{2i-1} y_i = x_i u_{2i} y_{i+1} = x_{i+1} u_{2(i+1)-1} y_{i+1}, \quad i = 1, 2, \ldots, m-1$$

and

$$d = x_m u_{2m},$$

that is,

$$\begin{aligned} d &= u_o y_1 \\ &= x_1 u_1 y_1, \quad u_o = x_1 u_1 \\ &= x_1 u_2 y_2, \quad u_1 y_1 = u_2 y_2 \\ &= x_2 u_3 y_2, \quad x_1 u_2 = x_2 u_3 \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= x_m u_{2m-1} y_m, \quad x_{m-1} u_{2m-2} = x_m u_{2m-1} \\ &= x_m u_{2m}, \quad u_{2m-1} y_m = u_{2m}. \end{aligned}$$

Proof : Since $x_1 u_1 = u_o$, we have $x_1 u_1 y_1 = u_o y_1 = d$. From $u_{2i-1} y_i = u_{2i} y_{i+1}$ and $x_i u_{2i} = x_{i+1} u_{2i+1}$ for all $i = 1, 2, \ldots, m-1$, we have that

$$x_i u_{2i-1} y_i = x_i u_{2i} y_{i+1}$$

and

$$x_i u_{2i} y_{i+1} = x_{i+1} u_{2i+1} y_{i+1}, \quad i = 1, 2, , \ldots, m-1.$$

Now we have

$$d = x_1 u_1 y_1$$

and

$$x_i u_{2i-1} y_i = x_i u_{2i} y_{i+1} = x_{i+1} u_{2i+1} y_{i+1} , \; i = 1,2, \ldots ,m-1 .$$

Then it follows inductively that

$$d = x_i u_{2i-1} y_i = x_i u_{2i} y_{i+1} = x_{i+1} u_{2(i+1)-1} y_{i+1} , \; i = 1,2, \ldots , m-1 .$$

In particular $d = x_m u_{2m-1} y_m$. But $u_{2m} = u_{2m-1} y_m$, so we get $x_m u_{2m} = x_m u_{2m-1} y_m = d$. Hence the remark is proved.

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The following quoted results will be used in this thesis :

Theorem 1.1 (Isbell's Zigzag Theorem,[4]). Let U be a subsemigroup of a semigroup S. Then $d \in \mathrm{Dom}(U,S)$ if and only if $d \in U$ or there is a zigzag in U over S with value d.

It is clearly seen from Theorem 1.1 that if A is an ideal of a semigroup S, then $\mathrm{Dom}(A,S) = A$.

Theorem 1.2 ([5]). Every inverse semigroup is absolutely closed. In particular, every group is absolutely closed.

Theorem 1.3 ([3]). If G is a group and U is a subsemigroup of G containing the identity of G, then $U^{-1} \subseteq \mathrm{Dom}(U,G)$ where $U^{-1} = \{x^{-1} \mid x \in U\}$.

Theorem 1.4 ([3]). Let G be a group and U a subsemigroup of G. Then U is dense in G if and only if $<U \cup U^{-1}> = G$ where $<U \cup U^{-1}>$ is the subsemigroup of G generated by $U \cup U^{-1}$.

Theorem 1.5 ([3]). Let X be an infinite countable set , A a subset of X such that $|A| = |X| = |X \setminus A|$ and B a subset of X. Then the following statements hold :

(i) If $A \cap B$ is infinite, then there exists $\alpha \in G_X$ such that $A \subseteq A\alpha$ and $A \subseteq B\alpha$

(ii) If $X \setminus (A \cup B)$ is infinite, then there exists $\eta \in G_X$ such that $A\eta \subseteq A$ and $B\eta \subseteq A$.

(iii) If B is infinite and $X \setminus (A \cup B)$ and $A \cap B$ are finite, then there exists $\lambda \in G_X$ such that $A\lambda \subseteq A$ and $A \cap B\lambda$ is infinite.

Theorem 1.6 ([3]). Let F be a field and n a positive integer. If F has a proper dense subsemigroup under multiplication, then the matrix semigroups $M_n(F)$ and $G_n(F)$ have proper dense subsemigroups.

Theorem 1.7 ([3]). Let F be a field and n a positive integer. Let S be a subsemigroup of $M_n(F)$ such that $G_n(F) \subseteq S$. If S contains a matrix in $M_n(F)$ of rank $k < n$, then S contains all matrices in $M_n(F)$ of rank $\leq k$.