เสถียรภาพของมิติพิเศษในจักรวาลวิทยาของสตริงแก๊สแบบ O(d,d) โคแวเรียนต์

นายปรีดา พัชรมณีปกรณ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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STABILITY OF EXTRA DIMENSIONS IN $O(d, d)$-COVARIANT STRING GAS COSMOLOGY

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A Thesis Submitted in Partial Fulfillment of the Requirements
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ทฤษฎีสตริงเป็นทฤษฎีแรงโน้มถ่วงควอนตัมที่น่าจะประสบความสำเร็จในอนาคตมากที่สุดทฤษฎีหนึ่ง ดังนั้นจึงเป็นที่คาดหมายว่าจักรวาลวิทยาของสตริงแก๊สจะสามารถอธิบายพลวัตรของเอกภพในช่วงต้นซึ่งมีพลังงานสูงได้ จักรวาลวิทยาของสตริงแก๊สสามารถแก้ปัญหาภาวะเอกฐานเริ่มต้นและปัญหามิติของจักรวาลวิทยามาตรฐานได้โดยใช้ภาวะคู่กัน-ที และภาวะสตริงไวน์ดิง ตามที่กลไก Brandenberger-Vafa ได้กล่าวไว้ว่า กระบวนการการประลัยระหว่างภาวะสตริงไวน์ดิงและภาวะสตริงปฏิไวน์ดิงสามารถเกิดขึ้นได้อย่างมีประสิทธิผลในกาลอวกาศ 4 มิติ จึงเป็นผลให้อวกาศ 3 มิติเกิดการขยายตัวไปพร้อมกับการวิวัฒน์ของเอกภพในขณะที่มิติพิเศษ 6 มิติที่เหลือถูกทำให้เสถียรที่จุดตรึงคู่กันเนื่องจากผลของภาวะสตริงโมเมนตัมและภาวะสตริงไวน์ดิงที่เหลืออยู่ตามทิศทางของมิติพิเศษ ถึงแม้ว่าจักรวาลวิทยาของสตริงแก๊สจะประสบความสำเร็จในการแก้ปัญหาบางปัญหาในจักรวาลมาตรฐานตามที่กล่าวมาข้างต้น แต่แบบ จำลองนี้กลับพบปัญหาการทำให้เกิดเสถียรภาพของโมดูลีโดยเกี่ยวข้องกับเสถียรภาพของโมดูลีซึ่งเป็นตัวแปรกระทบที่อธิบายขนาด รูปร่าง และฟลักซ์ของมิติพิเศษ จากการคำนวณเชิงตัวเลขซึ่งเป็นผลการศึกษาที่วางอยู่บนสมมติฐานการหายไปของฟลักซ์พบว่า ขนาดของมิติพิเศษแกว่งกวัดอย่างหน่วงรอบจุดตรึงคู่กันในขณะที่ไดเลตอนมีค่าเข้าสู่การคู่ควบอย่างอ่อน สำหรับจุดประสงค์ของวิทยานิพนธ์นี้คือการศึกษาจักรวาลวิทยาของสตริงแก๊สแบบ O(d,d)-โคแวเรียนต์ และนำมาใช้แก้ปัญหาการทำให้เกิดเสถียรภาพของโมดูลีเชิงคุณภาพ โดยในวิทยานิพนธ์นี้เราได้พิจารณาตัวก่อกำเนิดและจุดตรึงที่มีสมมาตรเพิ่มขึ้นของสมมาตร O(d,d;$\mathbb{Z}$) ซึ่งเป็นรูปแบบทั่วไปของภาวะคู่กัน-ที ของพื้นหลังทรงห่วงยาง d มิติที่มีฟลักซ์ โดยเราพบว่าในกรณีที่มิติพิเศษเป็นพื้นหลังทรงห่วงยาง 2 มิติ เราจะได้จุดตรึงที่มีสมมาตรเพิ่มขึ้นอีกหนึ่งจุดนอกจากจุดตรึงคู่กัน ซึ่งมาจากกรุ๊ปซิมพลีเลซ อันดับ 2 และ ไดเลตอนถูกทำให้เสถียร โดยพจน์ที่เกี่ยวข้องกับการขยายตัวของอวกาศ - มิติ ณ จุดตรึงทั้ง 2 จุดนั้น

ภาควิชาฟิสิกส์ ลายมือชื่อนิสิต ปรีดา พัชรมณีปกรณ์

สาขาวิชาฟิสิกส์ ลายมือชื่ออาจารย์ที่ปรึกษา อรรถกฤต ฉัตรภูติ

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String theory is one of the most promising candidates for theory of quantum gravity. String gas cosmology is thus expected to be a plausible cosmological model which provides an excellent explanation for dynamics of the early universe. By means of T-duality and string winding modes, string gas cosmology can solve the initial singularity problem and the dimensionality problem in standard cosmology. According to Brandenberger-Vafa mechanism, the annihilation process of winding and anti-winding modes can take place effectively in four-dimensional spacetime. As a consequence, the expansion of three spatial dimensions comes with the evolution of the early universe; on the other hand, six extra dimensions are stabilized at the self-dual point due to the existence of momentum and winding modes in these directions. Despite its success, string gas cosmology encounters the moduli stabilization problem concerning the stability of moduli describing size, shape and, flux of extra dimensions for the late-time universe. The numerical calculation from the previous work based on the assumption of the vanishing flux showed that the size of the extra dimensions dampedly oscillated around the self-dual point as the dilaton ran slowly to weak coupling. The aim of this thesis is to study string gas cosmology in the $O(d,d)$-covariant formalism and resolve the moduli stabilization problem qualitatively. In this thesis we considered the generators and the enhanced symmetric fixed points of $O(d,d;\mathbb{Z})$ symmetry, which is regarded as a generalization of T-duality for d-dimensional toroidal background with the non-vanishing flux. The result showed that in the case of two-dimensional toroidal extra dimensions, there were two enhanced symmetric fixed points corresponding to the simply laced groups, $SU(2)\times SU(2)$, and $SU(3)$, and the dilaton was stabilized by Hubble damping term at both fixed points.

Department Physics — Student's signature ....

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CHAPTER I

INTRODUCTION

One of the most crucial problems for physicists concerns the number of spacetime dimensions, and is sometime referred as to the dimensionality problem. In fundamental theories of modern physics, such as conventional quantum field theory and general relativity, the dimensionality of spacetime is set to four by hand. Therefore, any would-be theory of everything should provide the ways to derive the spacetime dimensionality. Certainly, string theory emerging as a candidate for such a theory cannot avoid this challenge. The purpose of this thesis is to review the basic concepts of string theory and application to cosmology. We emphasize on string gas cosmology in order to study the methods for solving the dimensionality problem and other important concepts.

1.1 Brief Overview

1.1.1 History of String Theory

At present string theory is regarded as one of the most promising candidates for theory of quantum gravity. However, it is the fact that a true original purpose of string theory is to describe an enormous proliferation of hadrons of strong interaction physics discovered in 1960s [1, 2, 3]. The hadronic spectrum contains particles with high spin. It turns out that the relation between mass squared of the lightest particles and spin J is approximately $m^2 = \frac{J}{\alpha'}$. The constant α' takes the value around 1 GeV^{-2} and is known as the Regge slope. This behavior was tested up to spin $\frac{11}{2}$ and it seemed to continue infinitely. It is obvious that in that period there was no quantum field theory that could provide the acceptable explanation for this proliferation since the consistent quantum field theory seemed to be limited for particles with low spin. The difficulty for constructing quantum field theory of high spin particles is that tree diagrams for the exchanged particles with high spin do not maintain the unitary bounds at high energy limit. More precisely, the high-energy behaviors of s- and t-channel amplitudes for high spin particles are divergent. Therefore the dual model proposed by Veneziano was used to overcome this problem in 1968. According to Veneziano's model, there exists the duality between s- and t- channel amplitudes; as a consequence, these two amplitudes are equivalent and total scattering amplitude may be convergent at high energy limit. A surprising result of Veneziano's model was that it inevitably included the massless spin-2 particles. At the first glance, this seemed to be incorrect since in Yang-Mills field theory there are only the massless fields of spin-1 particles. However, it is important to note that in general relativity the gauge fields whose quanta are called gravitons are massless spin-2 particles. This means that the Veneziano's dual model sheds some light on the unification between gravity and other interactions. In 1970, Yoichiro Nambu showed that the dual model could be explained by quantum theory of stringy particles and it was possible to describe all interactions by means of the same circumstance. This could be regarded as the origin of string theory for a candidate of the unified theory and string theory arose in this way was known as bosonic string theory. Although bosonic string theory provided many elegant explanations for interaction processes, it still encountered some crucial problems that were the lack of fermion states and the existence of negative mass-squared particles, namely tachyons. In the late 1970s the Ramond-Neveu-Schwarz string model, which possesses the worldsheet supersymmetry, became an active area of research. This model was regarded as the first superstring theory. In 1984, M. Green and J. H. Schwarz discovered the anomaly cancellation in superstring theories [4]. This was the first superstring revolution. At that time there were three possible superstring theories, i.e., type I superstring, type IIA superstring and type IIB superstring. In 1985, two other superstring theories, namely the $SO(32)$ and the $E_8 \times E_8$ heterotic string theories [5], which can provide the non-Abelian gauge group, were introduced by D. Gross and others [6]. Between 1994 and 1997 there was the second string revolution pioneered by E. Witten. The different string theories were connected to the eleven-dimensional theory, namely M-theory by a set of symmetries, such as S-duality, T-duality, and U-duality. New object called D-brane was discovered and became the center of interest. This object led to the relation between strongly coupled gauge theory in $d+1$-dimensional Anti de Sitter spacetime and string theory, in d-dimensional spacetime, known as AdS/CFT correspondence. This topic was considered as a realization of holographic principle introduced by 't Hooft and developed by L. Susskind. In 1990s the holographic principle became the active area of research and its applications, such as thermodynamics of black hole and brane world cosmology, were developed significantly. In 2000s the string landscape [34, 35, 36], which concerns a conjecture that there exist a large number of possible string vacua, is believed to agree with the anthropic principle and becomes the center of interest among string theorists.

1.1.2 A Cursory Look on String Theory

In string theory all fundamental particles arise from the same origin and all forces are elegantly unified in a significant way. According to this theory, each particle can be identified by a specific vibrational mode of a fundamental microscopic open or closed strings and different particles correspond to different vibrational modes of these strings. In analogy with music, different elementary particles represent different musical notes and also different frequencies. There are many differences between string theory and quantum field theory for point particles. The first issue we mention here is that the interaction process of stringy particles. For example, a decay process $\alpha \to \beta + \gamma$ in the view point of string theory can be thought as a process that a single string with a vibrational mode corresponding to particle α decays to two strings with different modes corresponding to particles β and γ. It arises from the fact that in string theory there is no well-defined point at which a single string is decomposed into two strings. In other words, the interacting string theory is not Lorentz invariant. The consequent advantage is that string theory is free from the ultraviolet divergence that occurs in quantum field theory for point particles due to the existence of the well-defined interaction point. Another advantage of string theory is that it is quite unique since it contains only one adjustable dimensional parameter, namely the string length l_s, that is roughly interpreted as the typical size of a string. We can see, for instance, the standard model contains nineteen adjustable dimensionless parameters that are required to be precisely adjusted from experiments and also includes ten parameters for taking into account the neutrino masses. The different values for adjustable parameters lead to different theories with different predictions. Therefore, there can be a large number of distinct standard models due the different values of these dimensionless parameters.

One of the most striking predictions of string theory is that the dimensionality of spacetime is ten (twenty-six) for superstring (bosonic string) theory. This seems to be a contradiction to human's sense since one realizes only the existence of four-dimensional spacetime and never observes any extra dimensions yet. However, in string theory it is possible to avoid this problem by using an assumption that extra dimensions are compactified in so small space that they cannot be detected. There are many types for compactifications, such as toroidal, orbifold, or $K3$ compactifications. It is convenient to consider string compactification in terms of Lie algebra lattice. This idea was introduced by Narain and others [25, 26]. It is also essential to note that string theory provides not only the different pictures for the elementary particles, interactions and spacetime but also provides different symmetries from those of point-particle field theory. The important symmetries in string theory concern some certain number of discrete symmetries, namely dualities. It appears that different string theory relates to one another via a duality. In string theory there are three important discrete symmetries; T-, S- and U-dualities. In this thesis we focus on T-duality, or Target-space duality that is a symmetry between two spaces compactified in large and small volume. An example for one-dimensional toroidal compactification is that physics in a circle with radius R is equivalent to physics in a circle with radius α'/R.

1.1.3 Application of String Theory to Cosmology

Standard cosmology has been developed and has been used to study the celestial phenomena and evolution of the universe for many decades. Although it can provide many elegant explanations for many problems, there are still some unsolvable difficulties. Most of these fatal problems concern the invalidity of quantum field theory in the region that the gravitational field is very strong, i.e. in the vicinity of black hole, or at the origin of the universe. This failure motivates the modification of standard cosmology by a theory of quantum gravity. An application of string theory to cosmology is known as string gas cosmology pioneered by Brandenberger and Vafa [16]. This model is constructed under the assumption that the early universe consisting of the hot gases of closed superstring is initially compactified into the $T^9 \times \mathbb{R}$ space where T^9 denotes the nine-dimensional torus with compactification radii of string scale (at the order of $\alpha' \approx 10^{-33}$cm in the conventional unit $\hbar = c = 1$). Based on T-duality in string theory, this model is remarkably successful in resolving the initial singularity problem that takes place at the very beginning of the universe. Moreover, it also provides the reasonable solution for the dimensionality problem. Tseytlin and Vafa [17] showed that as the late-time universe is expanding, only three spatial dimensions are growing larger whereas six extra spatial dimensions are still confined to the string scale due to the new degrees of string modes wrapping around these compact dimensions, namely the winding modes. The consequent problem is the moduli stabilization problem that concerns the stability of moduli describing the shape, size and flux of the extra dimensions [18, 22, 23]. This issue is the aim of this thesis and is studied in chapter 3 and 4. One of the most important consequences is that there is a great number of possible string vacua corresponding to different ways that moduli of extra dimensions are stabilized. It turns out that these enormous distinct string vacua possess very slightly different vacuum energies. In the energy space, string theory contains plenty of possible local minima of effective potential. This is known as the string landscape [34, 35, 36] and relates to the anthropic principle arguing that only the universe with some specific conditions sufficient to allow observers (human-beings) to exist are permitted. This subject is not in the scope of the thesis; however, we will mention it briefly in chapter 5.

1.2 Organization of Thesis

In chapter 2 the fundamental concepts of string theory are reviewed. All four types of string theories are introduced briefly. Theories consisting of closed strings, that are bosonic strings, Type II superstrings and heterotic strings, are emphasized. String low-energy effective theory is also studied.

For chapter 3, various cosmological models are studied. In §3.1, we review standard cosmology [8, 9]. The initial singularity, flatness and horizon problems are mentioned explicitly. In §3.2, inflationary models are introduced as successful cosmological models that can provide some elegant solutions to flatness, and horizon problems. In §3.3, a study of string gas cosmology is given. T-duality and winding modes are introduced in order to solve the initial singularity problem and explain the dimensionality of spacetime. The moduli stabilization problem arose from this model is solved numerically in this chapter.

Chapter 4 provides Narain's original idea to study string compactification on the Lie algebra lattice [25, 26]. In §4.1, the general discussion for d-dimensional toroidal compactification is given. The generalization of T-duality to $O(d,d;\mathbb{Z})$ symmetry is studied. In §4.2, we repeat a study of string gas cosmology in the $O(d,d;\mathbb{Z})$ formalism. In §4.3, we apply the methods we studied in §4.1 and §4.2 to solve moduli stabilization problem qualitatively.

For chapter 5, summary and discussion are provided. further developments, such as the string landscape and the anthropic principle, are also included in this chapter.

CHAPTER II

FUNDAMENTALS OF STRING THEORY

In string theory, all fundamental particles arise from the same origin and all forces are elegantly unified in a significant way. This can be easily understood by assuming that fundamental particles are one-dimensional objects as strings and each type of particles can be identified by a specific vibrational mode of these fundamental microscopic open or closed strings. However, the full description of string theory is more complicated since it is furnished with many mathematical aspects. Therefore, this whole chapter is devoted to an introduction to string theory. In the first part, we take the understanding of bosonic string theory [1, 2]. All types of superstring theory are then studied in the second part. We end up this chapter by studying dynamics of strings in the low-energy limit.

2.1 Bosonic String Theory

In this section we study a theory of free bosonic strings. First, we introduce bosonic string action and discuss its classical symmetries. The equations of motion and their solutions are also provided. Then, two equivalent canonical formalisms are introduced to quantize this theory. Thereafter, all consistency conditions for the quantum theory will be determined explicitly. We end up this section by studying in detail the spectrum of bosonic string states.

2.1.1 Classical Theory of Bosonic String

Strings can be thought of as one-dimensional objects moving in D-dimensional spacetime. Their spacetime coordinates are denoted by X^μ where $\mu = 0, 1, ..., D-1$. As a string propagates freely in the spacetime, it sweeps out a two-dimensional surface, namely a worldsheet. We can parametrize the worldsheet by using a timelike coordinate, τ, and a spacelike coordinate, σ. This means that $X^\mu(\tau, \sigma)$ can be thought as D scalar fields of τ and σ. The classical string action is proportional to the area of the worldsheet and is known as the **Nambu-Goto action**, S_{NG}. The Nambu-Goto action can be written in the form

$$S_{NG}[X] = -T \int d\tau d\sigma \sqrt{(\dot{X}^{\mu} X'_{\mu})^2 - (\dot{X}^{\mu}\dot{X}_{\mu})(X'^{\mu} X'_{\mu})}, \tag{2.1}$$

where we denote derivatives of $X^{\mu}(\tau, \sigma)$ with respect to τ and σ by $\dot{X}^{\mu}$ and X'^{μ}, respectively. T is the string tension, which can be related to the Regge slope parameter, α', and the intrinsic string length, l_s, by $T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2}$. Note that we use the conventional unit $\hbar = c = 1$ throughout this thesis.

Since the Nambu-Goto action is in a square-root form of $X^{\mu}(\tau, \sigma)$, it is quite difficult to quantize this action. Thus, we instead introduce another action which is physically equivalent to the Nambu-Goto action but obviously easier to be quantized. This action is known as the **Polyakov action**, S_P, and can be expressed in the form

$$S_P[X, h] \;=\; -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^{\mu} \partial_b X_{\mu}, \tag{2.2}$$

where h and h^{ab} are the determinant and the inverse of the worldsheet metric, respectively. We also label the worldsheet coordinates τ and σ by σ^a with Latin index $a = 0, 1$, where $\sigma^0 = \tau$ and $\sigma^1 = \sigma$.

Since there is no quadratic term of $\partial_0 h^{ab}$ appearing in the action (2.2), h^{ab} is a non-dynamical variable which acts like a Lagrange multiplier in the classical action of a constrained system. In general, the Polyakov action is not physically equivalent to the Nambu-Goto action. However, their equivalence can take place under certain conditions that will be discussed later.

The very important concept in the classical theory is the symmetry in the classical action and the corresponding conserved charge determined by **Noether's theorem**. There are two local gauge symmetries for the Polyakov action :

(1) Reparametrizations (or diffeomorphisms) of the worldsheet coordinates;

$$\delta\sigma^a \;=\; \epsilon^a(\tau, \sigma), \tag{2.3}$$

$$\delta h^{ab} \;=\; \epsilon^c \partial_c h^{ab} - \partial_c \epsilon^a h^{cb} - \partial_c \epsilon^b h^{ac}, \tag{2.4}$$

$$\delta X^{\mu} \;=\; \epsilon^a \partial_a X^{\mu}, \tag{2.5}$$

(2) Weyl rescaling of the worldsheet metric;

$$\delta h^{ab} \;=\; \omega(\tau, \sigma) h^{ab}(\tau, \sigma). \tag{2.6}$$

It is worth noting that the Polyakov action is also invariant under a (global) Poincare transformation of spacetime coordinates;

$$\delta X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu}. \tag{2.7}$$

We then determine the energy-momentum tensor, T_{ab}, by varying the Polyakov action with respect to h^{ab}

$$T_{ab} = -\frac{2}{T\sqrt{-h}}\frac{\delta S_P}{\delta h^{ab}} = \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^{\mu}\partial_d X_{\mu}, \tag{2.8}$$

where we use the relation $\delta h = -h_{ab}\delta h^{ab} = h^{ab}\delta h_{ab}$.

It is the fact that the reparametrization invariance implies the conservation of the energy-momentum tensor

$$\partial^a T_{ab} = 0, \tag{2.9}$$

and the Weyl invariance implies the tracelessness of energy-momentum tensor

$$T^a_a = 0. \tag{2.10}$$

We can show that if the worldsheet metric satisfies its equation of motion

$$0 = T_{ab} = \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^{\mu}\partial_d X_{\mu}, \tag{2.11}$$

the Polyakov action becomes the Nambu-Goto action.

According to gauge theory, any two different systems which relate to each other by gauge symmetries are physically equivalent. There are some difficulties in counting the quantum states of these systems. Thus, these gauge symmetries should be eliminated by fixing the appropriate gauge choice in the action.

As we mentioned above, the Polyakov action possesses three local gauge symmetries (two reparametrizations and one Weyl rescaling), which act on the worldsheet metric. Therefore, we can gauge away all three components of the worldsheet metric by using these local gauge symmetries. As a result, we can choose the worldsheet metric as a (Lorentzian) flat metric, $h_{ab} = \eta_{ab} = diag(-1,1)$. This gauge choice is known as a **conformal gauge**.

After gauge fixing, the Polyakov action then become

$$\begin{aligned} S_P[X] &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\eta}\eta^{ab}\partial_a X^{\mu}\partial_b X_{\mu} \\ &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma(-\dot{X}^{\mu}\dot{X}_{\mu} + X'^{\mu}X'_{\mu}). \end{aligned} \tag{2.12}$$

We can define the conjugate momenta corresponding to $\dot{X}_\mu$ and X'^μ by $P^\mu_\tau \equiv \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = T\dot{X}^\mu$ and $P^\mu_\sigma \equiv \frac{\partial \mathcal{L}_P}{\partial X'_\mu} = -TX^{\mu'}$, respectively. In accordance with classical mechanics, the variables X^μ and P^μ_τ must satisfy Poisson brackets (P.B.) evaluated at equal τ

$$\begin{aligned} \{X^\mu(\tau,\sigma), P^\nu_\tau(\tau,\sigma')\}_{P.B.} &= \eta^{\mu\nu}\delta(\sigma-\sigma'), & (2.13)\\ \{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{P.B.} &= \{P^\mu_\tau(\tau,\sigma), P^\nu_\tau(\tau,\sigma')\}_{P.B.} = 0. & (2.14) \end{aligned}$$

It is essential to note that there is still a residual symmetry left in the Polyakov action although we have already fixed all local gauge symmetries. In order to see this, we examine the first line in (2.12). It turns out that if there is the reparametrization which yields Weyl rescaling on the worldsheet flat metric

$$\begin{aligned} \sigma^a &\longrightarrow \sigma'^a = \sigma'^a(\sigma^0,\sigma^1), & (2.15)\\ \eta_{ab} &\longrightarrow \eta'_{ab}(\sigma'^0,\sigma'^1) = e^{\omega(\sigma'^0,\sigma'^1)}\eta_{ab}, & (2.16) \end{aligned}$$

then the gauge-fixed Polyakov action is invariant under this transformation.

The transformation mentioned above is known as a **conformal transformation**, which preserves the angle between any two vectors. This means that there exists the conformal symmetry in the gauge-fixed Polyakov action. The corresponding quantum field theory is called the **conformal field theory (CFT)**.

The variation of the gauged-fixed Polyakov action with respect to X_μ reads

$$\begin{aligned} 0 &= \delta_X S_P \\ &= \frac{1}{2\pi\alpha'}\int d\tau d\sigma(-\ddot{X}^\mu + X''^\mu)\delta X_\mu - \frac{1}{2\pi\alpha'}\int d\tau [X'^\mu \delta X_\mu]^\pi_0, \qquad (2.17) \end{aligned}$$

where the portion of a string is indicated by $\sigma \in [0,\pi]$.

The first term in the RHS of (2.17) provides us the equation of motion

$$-\ddot{X}^\mu + X''^\mu = 0. \qquad (2.18)$$

It is obvious that the equation of motion of the classical bosonics string is the wave equation.

The second term in the RHS of (2.17) corresponds to the surface term and its vanishing leads to the boundary conditions of the string

$$[X'^\mu \delta X_\mu]^\pi_0 = 0. \qquad (2.19)$$

There are three possible boundary conditions:

(1) Periodic boundary condition,

$$X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = \pi), \tag{2.20}$$

a string which satisfies this boundary condition is called a closed string.

(2) Neumann boundary condition,

$$X'^\mu(\tau, \sigma = 0) = X'^\mu(\tau, \sigma = \pi) = 0, \tag{2.21}$$

a string which satisfies this boundary condition is called an open string.

(3) Dirichlet boundary condition,

$$X^\mu(\tau, \sigma = 0) = a^\mu, X^\mu(\tau, \sigma = \pi) = b^\mu, \tag{2.22}$$

where a^μ and b^μ are constant vectors, a string which satisfies this boundary condition is an open string attached on a p-dimensional object, namely a D-brane or Dp-brane (D stands for Dirichlet and p is the integer $0, 1, 2, \ldots, D-2$).

In this thesis, we give an emphasis on the closed strings and open strings from the Neumann boundary condition as we neglect the open strings living on the D-brane.

From (2.18), a classical bosonic string propagates in the spacetime by obeying the wave equation. Therefore, $X^\mu(\tau, \sigma)$ can be decomposed into the left-moving sector, $X^\mu_L(\sigma^+)$, and right-moving sector, $X^\mu_R(\sigma^-)$, where we introduce the left-moving and right-moving worldsheet coordinates as $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$, respectively. Thus,

$$X^\mu(\tau, \sigma) \;\; = \;\; X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma). \tag{2.23}$$

The corresponding partial derivatives with respect to σ^+ and σ^- can be defined by $\partial_+ = \frac{1}{2}(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma})$ and $\partial_- = \frac{1}{2}(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma})$, respectively. At this point, we can determine explicitly the general solutions of $X^\mu(\tau, \sigma)$ of both closed and open strings .

For closed strings, equation of motion (2.18) can be reexpressed in the form

$$\partial_+ X^\mu_L(\tau + \sigma) - \partial_- X^\mu_R(\tau - \sigma) = \partial_+ X^\mu_L(\tau + \sigma + \pi) - \partial_- X^\mu_R(\tau - \sigma - \pi), \tag{2.24}$$

This means that $\partial_+ X^\mu_L(\tau + \sigma)$ and $\partial_- X^\mu_R(\tau - \sigma)$ are periodic with the period π, and they can be expanded by exponential Fourier series. After some calculation, X^μ_L and X^μ_R can be expressed as

$$X^\mu_L(\tau + \sigma) \;\; = \;\; \frac{1}{2}x^\mu_L + \sqrt{2\alpha'}\alpha^\mu_0(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0}\frac{\alpha^\mu_n}{n}e^{-2ni(\tau+\sigma)}, \tag{2.25}$$

$$X^\mu_R(\tau - \sigma) \;\; = \;\; \frac{1}{2}x^\mu_R + \sqrt{2\alpha'}\tilde{\alpha}^\mu_0(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2ni(\tau-\sigma)}. \tag{2.26}$$

Note that from (2.20), (2.25) and (2.26) we obtain the relation $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. Therefore, the general solution of a closed string $X^\mu(\tau,\sigma)$ can be expressed as

$$X^\mu(\tau,\sigma) \;=\; x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^\mu e^{-2ni(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-2ni(\tau-\sigma)}), \quad (2.27)$$

where we define $x_0^\mu = \frac{1}{2}(x_L^\mu + x_R^\mu)$ and $p^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^\mu$ as the center-of-mass coordinate at $\tau = 0$ and the center-of-mass momentum, respectively . The reality condition , $X^{\mu *}(\tau,\sigma) = X^\mu(\tau,\sigma)$, is required in order that $X^\mu(\tau,\sigma)$ is real. This condition yields $\alpha_{-n}^\mu = \alpha_n^{\mu *}$ and $\tilde{\alpha}_{-n}^\mu = \tilde{\alpha}_n^{\mu *}$. From the general solutions for both left-moving and right-moving modes, we can see that these modes are obviously independent to each other.

It is also important to determine the Poisson brackets in terms of α_n^μ and $\tilde{\alpha}_n^\mu$. From (2.13),(2.14) and (2.27) the equal-τ Poisson brackets are written as

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{P.B.} \;=\; in\delta_{-m,n}\eta^{\mu\nu}, \quad (2.28)$$

$$\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{P.B.} \;=\; in\delta_{-m,n}\eta^{\mu\nu}, \quad (2.29)$$

$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\}_{P.B.} \;=\; 0. \quad (2.30)$$

In addition, it is found that x_0^μ and p^μ also satisfy the Poisson bracket

$$\{x_o^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}. \quad (2.31)$$

For open strings, from equation of motion (2.18) we obtain

at $\sigma = 0$

$$\partial_+ X_L^\mu(\tau) = \partial_- X_R^\mu(\tau), \quad (2.32)$$

at $\sigma = \pi$

$$\partial_+ X_L^\mu(\tau+\pi) \;=\; \partial_- X_R^\mu(\tau-\pi). \quad (2.33)$$

We can impose the periodicity condition to $X^\mu(\tau,\sigma)$ by extending the domain of σ from $[0,\pi]$ to $[-\pi,\pi]$ and assuming $X^\mu(\tau,\sigma) = X^\mu(\tau,-\sigma)$. Combining this with (2.32) and (2.33), we can determine the general solutions of X_L^μ and X_R^μ as

$$X_L^\mu(\tau+\sigma) \;=\; \frac{1}{2}x_L^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-ni(\tau+\sigma)}, \quad (2.34)$$

$$X_R^\mu(\tau-\sigma) \;=\; \frac{1}{2}x_R^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-ni(\tau-\sigma)}. \quad (2.35)$$

Therefore, the general solution of open strings $X^\mu(\tau,\sigma)$ can be written as

$$X^\mu(\tau,\sigma) \;=\; x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-ni\tau}\cos n\sigma, \qquad (2.36)$$

where we define $x_0^\mu = \frac{1}{2}(x_L^\mu + x_R^\mu)$ and $p^\mu = \sqrt{\frac{1}{2\alpha'}}\alpha_0^\mu$ and the reality condition is also imposed on $X^\mu(\tau,\sigma)$. From the general solutions for X_L^μ and X_R^μ, we can interpret that these two modes reflect at the ends of open string and then form the standing wave.

Now, we will consider the equations of constraint for both open and closed strings. It is the fact that in the gauged-fixed Polyakov action we lose (2.11) as the equation of motion of the worldsheet because we use the conformal gauge ($h_{\mu\nu} = \eta_{\mu\nu}$) to fix all degrees of freedom of the worldsheet metric . Thus, we must impose (2.11) as an equation of constraint in order that the gauged-fixed polyakov action is physically equivalent to the Nambu-Goto action. For convenience, we will express this constraint in terms of σ^+ and σ^-. Thus, we introduce the Lorentzian metric of the left- and right-moving coordinates as $\eta = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$. As a consequence, the energy-momentum tensor (2.11) can be rewritten as

$$T_{++}(\sigma^+) \;=\; (\dot{X}^\mu + X'^\mu)(\dot{X}_\mu + X'_\mu) = \partial_+ X_L^\mu \partial_+ X_{L\mu} = 0 \qquad (2.37)$$

$$T_{--}(\sigma^-) \;=\; (\dot{X}^\mu - X'^\mu)(\dot{X}_\mu - X'_\mu) = \partial_- X_R^\mu \partial_- X_{R\mu} = 0 \qquad (2.38)$$

$$T_{+-} \;=\; T_{-+} = 0. \qquad (2.39)$$

Equations (2.37) and (2.38) are called **Virasoro constraints**. It is obvious that the periodicity and dependence of $\partial_+ X_L^\mu \partial_+ X_{L\mu}$ and $\partial_- X_R^\mu \partial_- X_{R\mu}$ are as same as those of $\partial_+ X_L^\mu$ and $\partial_- X_R^\mu$, respectively. Therefore, we can also expand the Virasoro constraints by exponential Fourier series (at $\tau = 0$, for convenience).

For closed strings, (2.37) and (2.38) can be expanded as

$$T_{++}(\sigma) \;=\; 4\alpha'\sum_{n\in\mathbb{Z}} L_n e^{-2ni\sigma}, \qquad (2.40)$$

$$T_{--}(\sigma) \;=\; 4\alpha'\sum_{n\in\mathbb{Z}} \tilde{L}_n e^{2ni\sigma}, \qquad (2.41)$$

with

$$L_n \;=\; \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{++} e^{2ni\sigma}, \qquad (2.42)$$

$$\tilde{L}_n \;=\; \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{--} e^{-2ni\sigma}, \qquad (2.43)$$

where $4\pi\alpha'$ is given to make L_n and $\tilde{L}_n$ dimensionless as well as for convenience. From (2.42) and (2.43) we define L_n and $\tilde{L}_n$ as the **Virasoro generators**, which generate the conformal transformations on σ^+ and σ^-, respectively. This corresponds to the conformal symmetry which is left in the gauge-fixed Polyakov action. Using (2.25) and (2.26), we can determine the Virasoro generators in terms of α_n^μ and $\tilde{\alpha}_n^\mu$

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m \cdot \alpha_{n-m} = 0, \tag{2.44}$$

$$\tilde{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0, \tag{2.45}$$

where $\alpha_m \cdot \alpha_l = \alpha_m^\mu \alpha_{l\mu}$ and $\tilde{\alpha}_m \cdot \tilde{\alpha}_l = \tilde{\alpha}_m^\mu \tilde{\alpha}_{l\mu}$. Then, we can determine straightforwardly the Poisson brackets of the Virasoro generators, known as the **Virasoro algebra**

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}, \tag{2.46}$$

$$\{\tilde{L}_m, \tilde{L}_n\}_{P.B.} = -i(m-n)\tilde{L}_{m+n}, \tag{2.47}$$

$$\{L_m, \tilde{L}_n\}_{P.B.} = 0. \tag{2.48}$$

According to classical mechanics, the Hamiltonian, H, can be interpreted as the generator of translation in τ and can be determined in the form

$$\begin{aligned} H &= \int_0^\pi d\sigma (T_{++} + T_{--}) \\ &= 2(L_0 + \tilde{L}_0) = 0. \end{aligned} \tag{2.49}$$

Similar to the Hamiltonian, the momentum, P, is interpreted as the generator of translation in σ and can be determined in the form

$$\begin{aligned} P &= \int_0^\pi d\sigma (T_{++} - T_{--}) \\ &= 2(L_0 - \tilde{L}_0) = 0. \end{aligned} \tag{2.50}$$

The equation (2.50) is one of the most important constraint , known as **level-matching condition**, when the classical theory of closed strings is quantized. From the definition of p^μ and H, we also express the mass squared of closed strings in the form

$$\begin{aligned} M^2 &= -p^\mu p_\mu \\ &= \frac{1}{\alpha'}\sum_{n\in\mathbb{Z}} (\alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}). \end{aligned} \tag{2.51}$$

For open strings, expanding (2.37) and (2.38) yields

$$T_{++}(\sigma) = \alpha' \sum_{n\in\mathbb{Z}} L_n e^{-ni\sigma}, \tag{2.52}$$

$$T_{--}(\sigma) = \alpha' \sum_{n\in\mathbb{Z}} L_n e^{ni\sigma}. \tag{2.53}$$

In analogy with closed strings, L_n can be interpreted as the Virasoro generators of the open strings and can be written in terms of α_n^μ

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m \cdot \alpha_{n-m} = 0. \tag{2.54}$$

Obviously, we can obtain the Virasoro algebra of L_n, which is in the same form as (2.46). The Hamiltonian, momentum and mass squared in open strings can be written as

$$H = L_0 = 0, \tag{2.55}$$

$$P = 0, \tag{2.56}$$

and

$$M^2 = \frac{1}{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n \cdot \alpha_{-n}. \tag{2.57}$$

Notice that from the Virasoro constraints all generators of translations in the worldsheet coordinates (the Virasoro generators of translations in σ^+ and σ^- or the Hamiltonian and momentum of translations in τ and σ, respectively) for the gauge-fixed Polyakov action must be equal to zero. This means that we implicitly set all conserved charges (generators) to vanish in order that there is no gauge transformation linking the physical systems.

It is also essential to determine the generators of the Poincare transformation of the spacetime as well as we do for the conformal transformation of the worldsheet. Using Noether's theorem, we can obtain the generator of translation, $\mathbf{P}^\mu$, and the Lorentz generator, $\mathbf{M}^{\mu\nu}$, as

$$\mathbf{P}^\mu = \int_0^\pi d\sigma P_\tau^\mu = p^\mu, \tag{2.58}$$

and

$$\begin{aligned} \mathbf{M}^{\mu\nu} &= T\int_0^\pi d\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) \\ &= l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu}, \end{aligned} \tag{2.59}$$

for closed strings and

$$\mathbf{M}^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}, \tag{2.60}$$

for open strings ,where

$$\begin{aligned} l^{\mu\nu} &= x_0^\mu p^\nu - x_0^\nu p^\mu, \\ E^{\mu\nu} &= -\frac{i}{2}\sum_{n\in\mathbb{Z}} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu), \\ \tilde{E}^{\mu\nu} &= -\frac{i}{2}\sum_{n\in\mathbb{Z}} \frac{1}{n}(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu). \end{aligned}$$

It is obvious that these Lorentz generators satisfy the **Lorentz algebra**

$$\{\mathbf{M}^{\mu\nu}, \mathbf{M}^{\rho\sigma}\}_{P.B.} = \eta^{\nu\rho}\mathbf{M}^{\mu\sigma} + \eta^{\mu\sigma}\mathbf{M}^{\nu\rho} - \eta^{\nu\sigma}\mathbf{M}^{\mu\rho} - \eta^{\mu\rho}\mathbf{M}^{\nu\sigma}. \tag{2.61}$$

It turns out that $\mathbf{P}_\tau^\mu$ and $\mathbf{M}^{\mu\nu}$ also satisfy the relations

$$\{\mathbf{P}^\mu, \mathbf{P}^\nu\}_{P.B.} = 0, \tag{2.62}$$

$$\{\mathbf{P}^\rho, \mathbf{M}^{\mu\nu}\}_{P.B.} = \eta^{\rho\nu}\mathbf{P}^\mu - \eta^{\rho\mu}\mathbf{P}^\nu, \tag{2.63}$$

Equations (2.61)-(2.63) are known as the **Poincare algebra**.

It is the fact that we are now able to quantize the classical string theory in order to obtain the quantum field theory which is manifestly Lorentz invariant. The corresponding procedure is called the **Lorentz covariance quantization**. This means that the forms of all equations for all variables are unchanged regardless of the Lorentz frames in which an observer is. However, we cannot use the Virasoro constraints as the operator equations since we cannot solve these constraints in terms of α^μ and $\tilde{\alpha}^\mu$, explicitly. Therefore, we must impose them on the Fock space in order to obtain the subspace of physical states. In general, it is quite difficult to study the quantum string theory by this approach. Therefore, we introduce an alternative method which is less difficult to quantize but compensates for the manifest Lorentz invariance. This method is known as the **light-cone gauge quantization**. In the last part of this subsection, the preliminaries for this method are provided before quantizing in next subsection.

The purpose of the light-cone gauge method is to use the appropriate extra gauge fixing to eliminate the conformal symmetry in the gauge-fixed Polyakov action. Thus, we start this by reparametrizing $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ and $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$ in such a way that

$$\begin{aligned} \tau' &= \frac{1}{2}(\sigma'^+(\tau+\sigma) + \sigma'^-(\tau-\sigma)), \\ \sigma' &= \frac{1}{2}(\sigma'^+(\tau+\sigma) - \sigma'^-(\tau-\sigma)). \end{aligned}$$

As a result, τ' and σ' can satisfy the wave equations as well as the spacetime coordinates X^μ. However, τ' and σ' relate to each other by the condition of conformal transformations. This means that we can choose a gauge choice by setting τ' or σ' to be proportional to one of X^μ. Next, we must choose the appropriate condition for this gauge choice.

The condition we choose is that a physical string may be a spacelike or occasionally lightlike object but cannot be a timelike object. In order to achieve this condition, we examine a relation

$$\tau' = \lambda n_\mu X^\mu, \tag{2.64}$$

where n^μ is an arbitrary vector and λ is a constant. At a fixed τ', all vectors X^μ satisfying this relation are assigned by the same τ' and so indicate the entire portion of a string. Furthermore, any vector joining between two of those X^μ lives on the hyperplane orthogonal to n^μ . This means that the entire portion of a string also lies on that hypersurface. It is the fact that if n^μ is a lightlike vector, any vector living on the hyperplane orthogonal to n^μ can be only a spacelike or lightlike vector but cannot be a timelike vector. As a consequence, if we choose a lightlike n^μ, a string can be a spacelike or lightlike object but not a timelike object as we want. In this case, we choose a unit lightlike vector $n_\mu = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \ldots & 0 \end{pmatrix}$ and introduce the spacetime coordinates corresponding to this choice as the light-cone coordinates. As a result, we define

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{D-1}), \tag{2.65}$$

$$X^- = \frac{1}{\sqrt{2}}(X^0 - X^{D-1}), \tag{2.66}$$

and X^i with $i = 1, \ldots, D-2$ as the light-cone timelike, spacelike and transverse coordinates, respectively. As a result, the flat spacetime metric, $g_{\mu\nu} = \eta_{\mu\nu}$, changes from $\eta_{\mu\nu} = diag(-1, 1, 1, ..., 1)$ to $\eta_{+-} = \eta_{-+} = -1$, $\eta_{++} = \eta_{--} = 0$ and $\eta_{ij} = \delta_{ij}$. Thus, the components of any vector V are

$$V^+ = \frac{1}{\sqrt{2}}(V^0 + V^{D-1}), \tag{2.67}$$

$$V^- = \frac{1}{\sqrt{2}}(V^0 - V^{D-1}), \tag{2.68}$$

and V^i with $i = 1, ..., D-2$. The inner product of two vectors V and W is written as

$$V \cdot W = -V^+W^- - V^-W^+ + \sum_{i=1}^{D-2} V^iW^i \tag{2.69}$$

$$= V^+W_+ + V^-W_- + V^iW_i, \tag{2.70}$$

where $W_+ = -W^-$, $W_- = -W^+$ and $W^i = W_i$.

Omitting the sign "prime" of the worldsheet coordinate for convenience, we can reexpress (2.64) in the form

$$\begin{aligned} \tau &= \frac{1}{2\alpha' p^+} X^+, \\ X^+ &= 2\alpha' p^+ \tau. \end{aligned} \tag{2.71}$$

The gauge choice (2.71) is called the **light-cone gauge**. The associated σ is written in the form

$$\sigma = \frac{1}{2\alpha' p^+} \int_0^\sigma d\xi \dot{X}^+(\tau).$$

The gauge-fixed Polyakov action with light-cone gauge fixing can be rewritten as

$$\begin{aligned} S_P &= -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-2X'^+ X'^- + 2\dot{X}^+ \dot{X}^- + X'^i X'_i - \dot{X}^i \dot{X}_i) \\ &= \int d^2\sigma (-\frac{p^+}{\pi} \dot{X}^- + \frac{1}{4\pi\alpha'} (\dot{X}^i \dot{X}_i - X'^i X'_i)), \end{aligned} \tag{2.72}$$

where we define respectively the conjugate momenta corresponding to $\dot{X}^-$, $\dot{X}^i$ and X'^i as

$$\begin{aligned} P^+ &= \frac{\partial \mathcal{L}_\mathcal{P}}{\partial \dot{X}_+} = -\frac{\partial \mathcal{L}_\mathcal{P}}{\partial \dot{X}^-} = \frac{1}{\pi} p^+, \\ P^i_\tau &= \frac{\partial \mathcal{L}_\mathcal{P}}{\partial \dot{X}_i} = \frac{1}{2\pi\alpha'} \dot{X}^i, \end{aligned}$$

and

$$P^i_\sigma = \frac{\partial \mathcal{L}_\mathcal{P}}{\partial X'_i} = -\frac{1}{2\pi\alpha'} X'^i.$$

From (2.72) there is no quadratic term of $\dot{X}^-$ appearing in the action, so $\dot{X}^-$ acts like a Lagrange multiplier . This means that only the transverse coordinates X^i can be interpreted as the physical coordinates and possesses the same properties of X^μ in the Lorentz covariance case such as the boundary conditions for closed and open strings and equations of motion.

For closed strings, the general solutions of X^i can be determined in the form

$$X^i(\tau, \sigma) = x^i_0 + 2\alpha' p^i \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} (\alpha^i_n e^{-2ni(\tau+\sigma)} + \tilde{\alpha}^i_n e^{-2ni(\tau-\sigma)}), \tag{2.73}$$

It is worth noting that once the light-cone gauge is fixed, we can solve the Virasoro constraints to determine the relation between the α's. This is one of the most important advantages of the light-cone gauge approach. As a consequence, we can express $X^-(\tau,\sigma)$ in the form

$$X^- \;=\; x_0^- + 2\alpha' p^-\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^- e^{-2ni(\tau+\sigma)} + \tilde{\alpha}_n^- e^{-2ni(\tau-\sigma)}), \quad (2.74)$$

where

$$\begin{aligned} p^- &= \sqrt{\frac{2}{\alpha'}}\alpha_0^- = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^-, \\ \alpha_n^- &= \frac{1}{p^+\sqrt{2\alpha'}}\sum_m \alpha_m^i \alpha_{i_{n-m}}, \\ \tilde{\alpha}_n^- &= \frac{1}{p^+\sqrt{2\alpha'}}\sum_m \tilde{\alpha}_m^i \tilde{\alpha}_{i_{n-m}}. \end{aligned}$$

The result is that we can express α_n^- in terms of α_m^i and $\tilde{\alpha}_m^i$.

It is the fact that X^i, P_τ^i, x^- and p^+ must satisfy the Poisson bracket as well as those in the covariance case. Thus, their Poisson brackets are written as

$$\{X^i(\tau,\sigma), P_\tau^j(\tau,\sigma')\}_{P.B.} \;=\; \eta^{ij}\delta(\sigma-\sigma') \quad (2.75)$$

$$\{x^-, p^+\}_{P.B.} \;=\; \eta^{-+} = -1. \quad (2.76)$$

As a result, the Poisson brackets of α^i and $\tilde{\alpha}^i$ are in the similar form

$$\{\alpha_m^i, \alpha_n^j\}_{P.B.} \;=\; in\delta_{-m,n}\eta^{ij}, \quad (2.77)$$

$$\{\tilde{\alpha}_m^i, \tilde{\alpha}_n^j\}_{P.B.} \;=\; in\delta_{-m,n}\eta^{ij}, \quad (2.78)$$

$$\{\alpha_m^i, \tilde{\alpha}_n^j\}_{P.B.} \;=\; 0. \quad (2.79)$$

We can define the transverse Virasoro generators , $L^\perp$ and $\tilde{L}^\perp$, in the same way as L and $\tilde{L}$,

$$L_n^\perp \;=\; \frac{1}{\pi}\int_0^\pi d\sigma e^{2in\sigma}\partial X_L^i \partial X_{iL} = p^+\sqrt{\frac{\alpha'}{2}}\alpha_n^-, \quad (2.80)$$

$$\tilde{L}_n^\perp \;=\; \frac{1}{\pi}\int_0^\pi d\sigma e^{-2in\sigma}\partial X_L^i R\partial X_{iR} = p^+\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_n^-, \quad (2.81)$$

These transverse Virasoro generators must satisfy the Virasoro algebras as well,

$$\{L_m^\perp, L_n^\perp\}_{P.B.} \;=\; -i(m-n)L_{m+n}^\perp, \quad (2.82)$$

$$\{\tilde{L}_m^\perp, \tilde{L}_n^\perp\}_{P.B.} \;=\; -i(m-n)\tilde{L}_{m+n}^\perp, \quad (2.83)$$

$$\{L_m^\perp, \tilde{L}_n^\perp\}_{P.B.} \;=\; 0. \quad (2.84)$$

The Hamiltonian, momentum and mass squared are also written as

$$H = 2(L_0^{\perp} + \tilde{L}_0^{\perp}) = \sum_n (\alpha_n^i \alpha_{i-n} + \tilde{\alpha}_n^i \tilde{\alpha}_{i-n}), \quad (2.85)$$

$$P = 2(L_0^{\perp} - \tilde{L}_0^{\perp}) = \sum_n (\alpha_n^i \alpha_{i-n} - \tilde{\alpha}_n^i \tilde{\alpha}_{i-n}), \quad (2.86)$$

$$M^2 = 2p^+p^- - p^i p_i = \frac{1}{\alpha'} \sum_n (\alpha_n^i \alpha_{i-n} + \tilde{\alpha}_n^i \tilde{\alpha}_{i-n}). \quad (2.87)$$

We now consider the situations of the Lorentz generators. From the light-cone gauge, we notice that the worldsheet timelike coordinate τ is fixed to be proportional to the light-cone timelike coordinates X^+. This means that X^+ cannot change usually according to the Lorentz transformations, in the other words, the classical string theory with this gauge choice is not manifestly Lorentz invariant. However, we can investigate whether this theory is really Lorentz invariant by considering the Lorentz algebra of corresponding generators. If the Lorentz generators satisfy the Lorentz algebra, this theory still maintains the Lorentz invariance. From the definition of the Lorentz generators, we can express the Lorentz generators in the light-cone gauge case as

$$\mathbf{M}^{+-} = -x_0^- p^+, \quad (2.88)$$

$$\mathbf{M}^{ij} = l^{ij} + E^{ij} + \tilde{E}^{ij}, \quad (2.89)$$

$$\mathbf{M}^{-i} = l^{-i} + E^{-i} + \tilde{E}^{-i}, \quad (2.90)$$

where l^{ij}, E^{ij}, $\tilde{E}^{ij}$, E^{-i} and $\tilde{E}^{-i}$ are in the similar forms as the Lorentz covariance case. However, we must modify the term l^{-i} due to the effect of the reparametrization compensating for maintaining the gauge choice on the Lorentz transformations of X^+. It appears that the term l^{-i} must be in the form

$$l^{-i} = x_0^- p^i - \frac{1}{2}(x_0^i p^- + p^- x_0^i). \quad (2.91)$$

The term $\frac{1}{2}(x_0^i p^- + p^- x_0^i)$ appears in l^{-i} to verify that the Poisson bracket between $\mathbf{M}^{-i}$ and $\mathbf{M}^{-j}$ must vanish. Therefore, all classical Lorentz generators in this approach obey the Lorentz algebra and the classical theory maintains the Lorentz invariance. However, we must check the Lorentz algebra for $\mathbf{M}^{-i}$ and $\mathbf{M}^{-j}$ carefully when this theory is quantized.

The situation for open strings is very similar to that of closed strings, so we will neglect all derivations for open strings. At this point, we have studied all preliminary concepts for light-cone gauge quantization. Therefore, we are ready to study the quantum theory of bosonic string in next subsection.

2.1.2 Quantum Theory of Bosonic String

In this subsection we study the quantum theory of free bosonic strings by using two basic canonical quantizaions. The first one is the Lorentz covariance formalism, which is manifestly Lorentz invariant but not manifestly ghost-free while the second one, known as the light-cone gauge formalism, is not manifestly Lorentz invariant but manifestly ghost-free. As a consequence of anomaly cancellation, we can obtain the consistency conditions for the quantum theory. Moreover, we can show the equivalence for both formalisms. Finally, we also discuss some low-level string states as examples for the representation theory.

We will use a standard shortcut to the understanding of quantum string theory by means that we consider a variable as an operator and replace Poisson bracket by commutator via substitution $\{\quad,\quad\}_{P.B.} \to -i[\quad,\quad]$. As a result, a nonzero commutation relation between $X^\mu(\tau,\sigma)$ and $P^\nu_\tau(\tau,\sigma')$ reads

$$[X^\mu(\tau,\sigma), P^\nu_\tau(\tau,\sigma')] \;=\; i\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{2.92}$$

We also obtain all commutation relations of α^μ_m and/or $\tilde{\alpha}^\mu_n$ in the forms

$$[\alpha^\mu_m, \alpha^\nu_n] \;=\; m\delta_{m,-n}\eta^{\mu\nu}, \tag{2.93}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] \;=\; m\delta_{m,-n}\eta^{\mu\nu}, \tag{2.94}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] \;=\; 0. \tag{2.95}$$

That of x^μ_0 and p^μ is also expressed as

$$[x^\mu_o, p^\nu] = i\eta^{\mu\nu}. \tag{2.96}$$

For $m>0$, the α^μ_m and $\tilde{\alpha}^\mu_m$ can be interpreted as the annihilation operators while $\alpha^\mu_{-m} = \alpha^{\mu\dagger}_m$ and $\tilde{\alpha}^\mu_{-m} = \tilde{\alpha}^{\mu\dagger}_m$ can be interpreted as the creation operators. It is conventional to use the normalized operators, a^μ_m and $\tilde{a}^\mu_m$, instead of α^μ_m and $\tilde{\alpha}^\mu_m$. The normalized operators can be expressed in the forms

$$a^\mu_m \;=\; \frac{1}{\sqrt{m}}\alpha^\mu_m, \qquad m>0, \tag{2.97}$$

$$\tilde{a}^\mu_m \;=\; \frac{1}{\sqrt{m}}\tilde{\alpha}^\mu_m, \qquad m>0. \tag{2.98}$$

As usual, we can define the ground state of a string with momentum k^μ, $|0;k\rangle$, in the Fock space by using the relation

$$a^\mu_m|0;k\rangle = \tilde{a}^\mu_m|0;k\rangle = 0, \qquad m>0. \tag{2.99}$$

However, it is the fact that we cannot solve the Virasoro constraints to determine the relation between the α's in this approach. Thus, the states created by a^{μ}_{-m} and $\tilde{a}^{\mu}_{-m}$ are not all independent. This will produce the inconsistent states in the theory. We can demonstrate this by examining a norm of the state $a^{0}_{-m}|0\rangle$ or $\tilde{a}^{0}_{-m}|0\rangle$

$$\langle 0|a^0_m a^0_{-m}|0\rangle = \langle 0|(a^0_{-m}a^0_m + m\eta^{00})|0\rangle = -1. \tag{2.100}$$

The states with negative norms are called "ghosts" and are not allowed to exist in any physical theories. According to the quantization of gauge fields, we can solve this problem by imposing the Virasoro constraints on the Fock space to choose only the independent states. However, we cannot impose the Virasoro constraints in the classical form immediately, since all operators in quantum string theory must be defined by means of the normal ordering, $:\quad:$. This means that we must check whether the quantum Virasoro generators are in the same forms of those in the classical case. As a consequence, we must reconsider the expressions for L_0 and $\tilde{L}_0$ because they contain the terms of $\alpha^{\mu}_m\alpha^{\nu}_n$ or $\tilde{\alpha}^{\mu}_m\tilde{\alpha}^{\nu}_n$, with $m = -n$. Thus, the normal ordered L_0 and $\tilde{L}_0$ can be written as

$$L_0 = \frac{1}{2}\sum_m :\alpha_m\cdot\alpha_{-m}: = \frac{1}{2}\alpha_0\cdot\alpha_0 + N, \tag{2.101}$$

$$\tilde{L}_0 = \frac{1}{2}\sum_m :\tilde{\alpha}_m\cdot\tilde{\alpha}_{-m}: = \frac{1}{2}\tilde{\alpha}_0\cdot\tilde{\alpha}_0 + \tilde{N}, \tag{2.102}$$

where

$$N = \sum_{m=1}^{\infty}\alpha^{\dagger}_m\cdot\alpha_m = \sum_{m=1}^{\infty} m a^{\dagger}_m\cdot a_m, \tag{2.103}$$

$$\tilde{N} = \sum_{m=1}^{\infty}\tilde{\alpha}^{\dagger}_m\cdot\tilde{\alpha}_m = \sum_{m=1}^{\infty} m\tilde{a}^{\dagger}_m\cdot\tilde{a}_m \tag{2.104}$$

are defined as the left-moving and right-moving number operators, respectively. Thus, the zeroth-mode Virasoro constraints from classical theory can be written in the forms

$$0 = \frac{1}{2}\sum_m \alpha_m\cdot\alpha_{-m} = L_0 + \eta^{\mu}_{\mu}\sum_{m=1}^{\infty} m, \tag{2.105}$$

$$0 = \frac{1}{2}\sum_m \tilde{\alpha}_m\cdot\tilde{\alpha}_{-m} = \tilde{L}_0 + \eta^{\mu}_{\mu}\sum_{m=1}^{\infty} m. \tag{2.106}$$

The constant $\eta^{\mu}_{\mu}\sum_{m=1}^{\infty} m$ can be determined by means of zeta-function regularization. This means that we can modify the zeroth-mode Virasoro constraints by adding some constant to the zeroth-mode Virasoro generators. It is reasonable to suppose that the zeroth-mode Virasoro constraints in quantum string theory are not necessary to be exactly the same as those in classical theory. Thus, the Virasoro constraints can be expressed as

$$L_m - a\delta_{m,0} = 0, \tag{2.107}$$

$$\tilde{L}_m - a\delta_{m,0} = 0, \tag{2.108}$$

where a is some constant and known as the **normal ordering constant**.

Furthermore, we must recalculate the Virasoro algebra. As a result, the modified Virasoro algebra can be written in the forms

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m,-n}, \tag{2.109}$$

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m,-n}. \tag{2.110}$$

We call (2.109) and (2.110) the **Virasoro algebra with central charge**.

It is conventional to introduce the notation for the quantum states of open and closed strings. In general, the n^{th} excited state of open string with the center-of-mass momentum k^μ is denoted by $|n;k\rangle$ where

$$N|n;k\rangle = n|n;k\rangle, \tag{2.111}$$

$$p^\mu|n;k\rangle = k^\mu|n;k\rangle. \tag{2.112}$$

The situation is quite similar for the closed string but the difference is that the quantum state of closed string is the tensor product of two copies of open-string states corresponding to the left- and right-moving sectors.

At this point, we can eliminate the ghost states by imposing the Virasoro constraints on the Fock space. This procedure corresponds to the Gupta-Bleuler formalism for treating the Lorentz gauge in quantum electrodynamics. Thus, the physical states, $|\phi\rangle$, can be defined as the quantum states satisfying the nonnegative-mode Virasoro constraints

$$(L_m - a\delta_{m,o})|\phi\rangle = 0, \qquad m \geq 0 \tag{2.113}$$

$$(\tilde{L}_m - a\delta_{m,o})|\phi\rangle = 0, \qquad m \geq 0. \tag{2.114}$$

This definition is equivalent to the vanishing of the expectation values of the energy-momentum tensors

$$\langle\phi|T_{ab}|\phi\rangle = 0. \tag{2.115}$$

It is worth noting that we can determine the Hamiltonian and the mass squared from the zeroth-mode Virasoro constraint

$$H = 2(L_0 + \tilde{L}_0), \tag{2.116}$$
$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2a), \tag{2.117}$$

where

$$N = \tilde{N}, \tag{2.118}$$

for closed strings and

$$H = L_0, \tag{2.119}$$
$$M^2 = \frac{1}{\alpha'}(N - a), \tag{2.120}$$

for open strings.

Equation (2.117) or (2.120) is called the mass-shell condition while (2.118) is called the level-matching condition. This means that imposing the zeroth-mode Virasoro constraints for the left- and right-moving sectors in the case of closed strings is equivalent to imposing the mass-shell condition and the level-matching condition while imposing the zeroth-mode Virasoro constraint in the case of open strings is equivalent to imposing only the mass-shell condition.

After imposing the constraints on the Fock space, we obtain the physical states from which we can decouple the timelike operators α_m^0 and $\tilde{\alpha}_m^0$. We can see this by considering that the number of operators with timelike component is equal to the number of constraints. Thus, it is possible to express the timelike-component operators in terms of the spacelike-component operators and avoid the ghosts contributed from these operators. However, we must examine such conditions for the normal ordering constant a in the Virasoro constraints and dimension of spacetime D in the Virasoro algebra. This leads to the consistency conditions for the quantum string theory.

We start an investigation by considering the mass-shell condition on the ground state of open string with momentum k^μ, $|0;k\rangle$. As a result, we obtain the squared mass $m^2 = -k^2 = -\frac{1}{\alpha'}a$. Then, we consider the first excited state $\zeta \cdot a_{-1}|0;k\rangle$ where ζ^μ is a polarization vector. The norm of this state is ζ^2 and the mass square is $m^2 = -k^2 = \frac{1}{\alpha'}(1-a)$. From the first-mode Virasoro constraint, we obtain $\zeta \cdot k = 0$. This means that only $(D-1)$ independent polarizations are allowed in this state. For convenience, we assume that the momentum k^μ lies in the $(0,1)$ direction. Thus, there are certainly $(D-2)$ spacelike polarizations which all contribute the positive norms regardless of the momentum. Our real task is to examine the conditions for the last polarization. There are three possible cases for the last polarization according to a .

(1) If $a > 1$, then k^μ is spacelike and the associated ζ^μ is timelike. This means that the last polarization contributes to the negative-norm state.

(2) If $a = 1$, then k^μ is lightlike and the associated ζ^μ is also lightlike. This means that the last polarization contributes to the zero-norm state.

(3) If $a < 1$, then k^μ is timelike and the associated ζ^μ is spacelike. This means that the last polarization contributes to the positive-norm state.

It is obvious that only the cases (2) and (3) are ghost-free. Thus, we obtain the consistency condition for the normal ordering constant a

$$a \leq 1. \tag{2.121}$$

This condition is also valid in the case of closed string. It is worth noting that at the boundary ($a = 1$) we obtain the state with zero norm as the extra state. This state is certainly not included in the S-matrix because of its zero norm. The appearance of the zero-norm state mentioned above is essential to find the consistency conditions in quantum string theory. Similarly, we can obtain the consistency condition for D as

$$D \leq 26. \tag{2.122}$$

Actually, the only consistency conditions for the interacting string theory are $a = 1$ and $D = 26$. However, it is difficult to achieve those conditions. Thus, we assume that the consistent quantum theory of bosonic string is that with $a = 1$ and $D = 26$. In addition, we can study string theory with $D < 26$ by choosing an appropriate a . Such string theory is interpreted as the theory constructed from the subspace of ground states of $D = 26$ string theory. It is worth noting that $D = 26$ is called the critical dimension for bosonic string since the theory with $D = 26$ contains the extra zero-norm states but no ghost states at all. Accordingly, any string theory with higher dimensions must contain the ghost states unavoidably.

Now, we study the light-cone gauge quantization. This approach offers the effective framework for the understandings of quantum string theory and consistency conditions. In analogy with the Lorentz covariance quantization, we consider all variable as operators and replace Poisson bracket by commutator via substitution $\{\quad,\quad\}_{P.B.} \rightarrow -i[\quad,\quad]$. Thus, commutators of X^i, P^i_τ, x^- and p^+ are written as

$$[X^i(\tau,\sigma), P^j_\tau(\tau,\sigma')] \;=\; i\eta^{ij}\delta(\sigma-\sigma'), \tag{2.123}$$

$$[x^-, p^+] \;=\; -i. \tag{2.124}$$

It is worth noting that in this theory the only independent operators are x_0^- , p^+ , α^i_m and $\tilde{\alpha}^i_m$. Then the commutators of α^i and $\tilde{\alpha}^i$ are written in the familiar forms

$$[\alpha^i_m, \alpha^j_n] \;=\; m\delta_{-m,n}\eta^{ij}, \tag{2.125}$$

$$[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] \;=\; m\delta_{-m,n}\eta^{ij}, \tag{2.126}$$

$$[\alpha^i_m, \tilde{\alpha}^j_n] \;=\; 0. \tag{2.127}$$

We also define the normalized operators a^i_m in the same way as a^μ in the covariance case. As a result, the ground state of a string with momentum k^+ and k^i, $|0;k^+,k^i\rangle$, can be defined as

$$a^i_m|0;k^+,k^i\rangle = \tilde{a}^i_m|0;k^+,k^i\rangle = 0, \qquad\qquad m>0. \tag{2.128}$$

As a consequence of normal ordering, we obtain the zeroth-mode Virasoro constraints as

$$0 \;=\; L^\perp_0 - a = \frac{1}{2}\alpha^i_0\alpha_{0i} + N^\perp - a, \tag{2.129}$$

$$0 \;=\; \tilde{L}^\perp_0 - a = \frac{1}{2}\tilde{\alpha}^i_0\tilde{\alpha}_{0i} + \tilde{N}^\perp - a, \tag{2.130}$$

where

$$N^\perp \;=\; \sum_{m=1}^{\infty}\alpha^{\dagger i}_m\alpha_{mi} = \sum_{m=1}^{\infty} m a^{\dagger i}_m a_{mi}, \tag{2.131}$$

$$\tilde{N}^\perp \;=\; \sum_{m=1}^{\infty}\tilde{\alpha}^{\dagger i}_m\tilde{\alpha}_{mi} = \sum_{m=1}^{\infty} m \tilde{a}^{\dagger i}_m \tilde{a}_{mi}, \tag{2.132}$$

and the normal ordering constant a can be determined by **ζ-function regularization**. Like the Lorentz covariance formalism, we obtain the Virasoro algebra with central charge

$$[L^\perp_m, L^\perp_n] \;=\; (m-n)L^\perp_{m+n} + \frac{D-2}{12}(m^3-m)\delta_{m,-n}, \tag{2.133}$$

$$[\tilde{L}_m^\perp, \tilde{L}_n^\perp] = (m-n)\tilde{L}_{m+n}^\perp + \frac{D-2}{12}(m^3-m)\delta_{m,-n}. \quad (2.134)$$

We can determine the Hamiltonian and the mass squared

$$H = 2(L_0^\perp + \tilde{L}_0^\perp), \quad (2.135)$$
$$M^2 = \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp - 2a), \quad (2.136)$$

where

$$N^\perp = \tilde{N}^\perp, \quad (2.137)$$

for closed strings and

$$H = L_0^\perp, \quad (2.138)$$
$$M^2 = \frac{1}{\alpha'}(N^\perp - a), \quad (2.139)$$

for open strings.

Since only a_m^i and $\tilde{a}_m^i$ are independent and appear in the Hamiltonian, we can use the negative modes of these operators to construct other states after defining the ground state. This means that the Fock space in the light-cone gauge case contains only the independent states. Hence this theory is certainly ghost-free. Nevertheless, the Lorentz invariance is still subtle for this theory. Similar to classical theory, we can examine whether the quantum string theory in the light-cone gauge case is still Lorentz invariant by means of the Lorentz algebra. In the classical case we concern $\{\mathbf{M}^{-i}, \mathbf{M}^{-j}\}_{P.B.}$ since $\mathbf{M}^{-i}$ is the generator of transformation on X^+. In the quantum case it turns out that the corresponding commutator is still ambiguous because of the normal ordering. After some calculations, the result for open string is

$$[\mathbf{M}^{-i}, \mathbf{M}^{-j}] = \frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} (\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i) \times \left(m(1 - \frac{D-2}{24}) + \frac{1}{m}(\frac{D-2}{24} - a)\right). \quad (2.140)$$

The quantum string theory maintains the Lorentz invariance provided that $[\mathbf{M}^{-i}, \mathbf{M}^{-j}] = 0$ as well as $\{\mathbf{M}^{-i}, \mathbf{M}^{-j}\}_{P.B.} = 0$ in the classical theory. It is obvious that the above commutator vanishes if and only if $D = 26$ and $a = 1$. The situation is similar for the closed strings. Thus, only quantum string theory with $D = 26$ and $a = 1$ is Lorentz invariant and is chosen as the consistent theory. We can conclude that in the final result the quantum theory of bosonic string in the light-cone gauge case is equivalent to that in the Lorentz covariance case.

Finally, we consider the string states of low-lying levels by using the light-cone gauge quantization. For open strings, the ground state, $|0;k^+,k^i\rangle$, satisfies $N^\perp|0;k^+,k^i\rangle = 0$ and then has the negative squared mass, $M^2 = \frac{1}{\alpha'}(-a) = -\frac{1}{\alpha'}$. Thus, this state represents a scalar field, known as a tachyon. For the first excited state, $N^\perp$ takes the first nonzero value, $N^\perp = 1$ and yields the massless state, $M^2 = \frac{1}{\alpha'}(1-a) = 0$. Since such a state is constructed by acting any a^i_{-1} to the ground state, there are $D-2 = 24$ first excited states. The general expression for this state is $\zeta_i a^i_{-1}|0;k^+,k^i\rangle$, where ζ^i is a polarization vector. From above expression and other properties, we can interpret that the first excited state represents the photon which is a massless spin-1 particle. This means that we obtain the photon states unexpectedly from the quantization of classical theory of free open string. For higher excited states, $N^\perp > 1$ and these states represent the massive particles.

For closed strings, the ground state $|0;k^+,k^i\rangle_L \bigotimes |0;k^+,k^i\rangle_R$ represents the tachyon, as well. The difference is that its squared mass, $M^2 = -\frac{4}{\alpha'}$ is four times larger than that of open string. For the first excited state, we obtain $N^\perp = \tilde{N}^\perp = 1$. This means that the first excited state of closed string is also massless. We can construct this state by applying a^i_{-1} and $\tilde{a}^j_{-1}$ to the ground state. Since i and j are chosen independently, the number of these states is $(D-2)^2$. The general expression of the first excited state is $R_{ij}a^i\tilde{a}^j|0;k^+,k^i\rangle$ where R^{ij} can be considered as an element of the $(D-2)\times(D-2)$ matrix. We can decompose it into three parts

$$R_{ij}a^i\tilde{a}^j|0;k^+,k^i\rangle = (S_{ij} + A_{ij} + R\eta_{ij})a^i\tilde{a}^j|0;k^+,k^i\rangle, \qquad (2.141)$$

where S^{ij} is an element of the symmetric traceless matrix, A^{ij} is an element of the antisymmetric matrix and $R = \frac{1}{D-2}\eta_{ij}R^{ij}$ is the average of the trace of R^{ij}. The first term $S_{ij}a^i\tilde{a}^j|0;k^+,k^i\rangle$ has $\frac{1}{2}(D-3)D$ degrees of freedom and represents the graviton, $G_{\mu\nu}$, which is the massless spin-2 particle. For the second term on the RHS of (2.141), it contains $\frac{1}{2}(D-3)(D-2)$ degrees of freedom and represents the Kalb-Ramond field, $B_{\mu\nu}$, which can be interpreted as the generalization of Maxwell gauge field in theory of electromagnetism. The last term of the RHS of (2.141) has only one degree of freedom and represents the dilaton field, ϕ, which is interpreted as the field of string coupling. Similar to open string, we can obtain the unexpected massless particles, such as gravitons, from quantum theory of free closed string.

2.2 Supersymmetric String Theory

The existence of tachyon in the spectrum and the lack of fermions indicate that the bosonic string theory is not an appropriate candidate for a theory of quantum gravity. In this section, we study a more feasible string theory called superstring theory. This theory possesses the fermionic fields as well as bosonic fields. An understanding of superstring theory can be carried out in the same way as the bosonic string theory. This means that we start with the classical superstring theory. All preliminaries for the Lorentz covariance and light-cone gauge quantizations are prepared. Then, the quantum superstring theory and its consistency conditions are developed, explicitly. Finally, we introduce the condition required for eliminating inappropriate states and study the low-lying states of superstring spectrum.

2.2.1 Classical Theory of Supersymmetric String

Since the bosonic string theory does not possess any fermions, it is reasonable to construct a prototype of superstring theory by adding some fermionic term to the Polyakov action. To do this, we introduce a Majorana spinor, Ψ^μ, as a worldsheet fermion. This means that we require Ψ^μ to be a two-component real spinor and can be written in the form

$$\Psi^\mu = \begin{pmatrix} \Psi^\mu_R \\ \Psi^\mu_L \end{pmatrix}, \tag{2.142}$$

where Ψ^μ_R and Ψ^μ_L are defined as the right- and left-moving sectors, respectively. It is worth noting that Ψ^μ is considered as a spinor in the view point of worldsheet but it is considered as a vector in the view point of spacetime. However, we must introduce a zwiebein, e^α_a, which satisfies

$$h_{ab} = \eta_{\alpha\beta} e^\alpha_a e^\beta_b, \tag{2.143}$$

in order to incorporate the spinor in a curved worldsheet. Although e^α_a has one more component than h_{ab} for two-dimensional worldsheet, we obtain an additional local (worldsheet) Lorentz symmetry for the action. This means that we can use these new bases whereas we do not change the context of general relativity in the worldsheet. Therefore, we start with the action

$$S_0 \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}(h^{ab}\partial_a X^\mu \partial_b X_\mu - i\overline{\Psi}^\mu \rho^a \nabla_a \Psi_\mu), \tag{2.144}$$

where

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.145}$$

are two-dimensional Dirac matrices which satisfy the anti-commutation relation

$$\{\rho^a, \rho^b\} = -2\eta^{ab} I_{2\times 2}, \tag{2.146}$$

and $\nabla_a = \partial_a + \frac{1}{8}\omega_a^{\alpha\beta}[\rho_\alpha, \rho_\beta]$ is the covariant derivative operator for any spinor. Since the spin connection, $\omega_a^{\alpha\beta}$, does not contribute to the two-dimensional spinor, this implies that we can replace ∇_a by ∂_a in (2.144). The action S_0 certainly possesses a reparametrization invariance and a local (worldsheet) Lorentz symmetry. In additional, there is a global worldsheet supersymmetry in the action. The corresponding global worldsheet supersymmetry transformation is

$$\delta X^\mu = \bar{\xi}\Psi^\mu, \tag{2.147}$$

$$\delta\Psi^\mu = -i\rho^a\partial_a X^\mu \xi, \tag{2.148}$$

where ξ is a two-component Majorana spinor. In order to achieve the constraints in superstring theory, it is essential to obtain the action which is invariant under the local worldsheet supersymmetry transformation. This means that we now consider ξ as a function of τ and σ. However, it turns out that the action S_0 is no longer invariant under the local worldsheet supersymmetry transformation. We can see this by varying this action with respect to $\xi(\tau, \sigma)$; as a result, we obtain

$$\delta S_0 = -\frac{1}{\pi\alpha'}\int d^2\sigma(-h)^{\frac{1}{2}}\partial_a\xi J^a, \tag{2.149}$$

where

$$J^a \equiv \frac{1}{2}\rho^b\rho^a\Psi^\mu\partial_b X_\mu, \tag{2.150}$$

is defined as the worldsheet supercurrent. Cancelling this non-vanishing term, we obtain the locally supersymmetric action, which is known as the **Ramond-Neveu-Schwarz (RNS) action**, as

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}(h^{ab}\partial_a X^\mu\partial_b X_\mu - i\overline{\Psi}^\mu\rho^a\partial_a\Psi_\mu + 2\overline{\chi}_b\rho^a\rho^b\Psi^\mu\partial_a X_\mu + \frac{1}{2}\overline{\Psi}_\mu\Psi^\mu\overline{\chi}_a\rho^b\rho^a\chi_b), \tag{2.151}$$

where the two-component Majorana spinor, χ_a, is called the gravitino and is introduced in the same way as h_{ab}.

The RNS action not only possesses the local worldsheet supersymmetry

$$\delta X^\mu = \bar{\xi}\Psi^\mu, \tag{2.152}$$

$$\delta\Psi^\mu = -i\rho^a(\partial_a X^\mu - \overline{\Psi}^\mu\chi_a)\xi, \tag{2.153}$$

$$\delta\chi_a = \partial_a\xi, \tag{2.154}$$

$$\delta e_a^\alpha = -2i\bar{\xi}\rho^\alpha\chi_a, \tag{2.155}$$

but also possesses the local fermionic symmetry

$$\delta\chi_a = i\rho_a\eta(\tau,\sigma), \tag{2.156}$$

$$\delta e_a^\alpha = \delta X^\mu = \delta\Psi^\mu = 0, \tag{2.157}$$

and the extended Weyl symmetry

$$\delta e_a^\alpha = \omega(\tau,\sigma)e_a^\alpha, \tag{2.158}$$

$$\delta\Psi^\mu = -\frac{1}{2}\omega(\tau,\sigma)\Psi^\mu, \tag{2.159}$$

$$\delta\chi_a = \frac{1}{2}\omega(\tau,\sigma)\chi_a. \tag{2.160}$$

At this point we can fix all components of e_a^α and χ_a by using all local symmetries. We use four local bosonic symmetries corresponding to two reparametrizations, one Weyl rescaling and one Lorentz transformation to gauge away all four components of the zwiebein, so we can choose $e_a^\alpha = \delta_a^\alpha$. Similarly, we use the local worldsheet supersymmetry and the local fermionic symmetry to fix all four components of the Majorana spinors χ_a, so we can set $\chi_a = 0$. This gauge choice is called the **superconformal gauge**. After gauge fixings, the RNS action reduces to

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma(\partial^a X^\mu\partial_a X_\mu - i\overline{\Psi}^\mu\rho^a\partial_a\Psi_\mu). \tag{2.161}$$

Similar to the bosonic string, this gauge-fixed action is still invariant under the residual symmetry, namely the **superconformal symmetry**. The variation of this action leads to the equations of motion and boundary conditions for both bosonic fields X^μ and fermionic fields Ψ^μ. In the case of X^μ, their equations of motion and boundary conditions are as same as those in the previous subsection. In the case of Ψ^μ, we obtain

$$\partial_-\Psi_L^\mu = 0, \tag{2.162}$$

$$\partial_+\Psi_R^\mu = 0. \tag{2.163}$$

These equations show that it is reasonable to define Ψ_L^μ and Ψ_R^μ as the left- and right-moving sectors, respectively. We also obtain the boundary term

$$[i\overline{\Psi}^\mu\rho^1\delta\Psi_\mu]|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.164}$$

There are two fermionic boundary conditions. The first one is the periodic or **Ramond (R) boundary condition** $\Psi^\mu(\tau, \sigma = 0) = \Psi^\mu(\tau, \sigma = \pi)$. The second one is the anti-periodic or **Neveu-Schwarz (NS) boundary condition** $\Psi^\mu(\tau, \sigma = 0) = -\Psi^\mu(\tau, \sigma = \pi)$. In order to obtain the general solutions, we must express these conditions in terms of Ψ^μ_L and Ψ^μ_R for closed and open strings, separately.

In the case of closed string, we can choose arbitrarily the R- or NS-boundary conditions for the left-moving and right-moving sectors since the left- and right-moving sectors are independent to each other. The boundary condition for Ψ^μ_L and Ψ^μ_R can be expressed in the forms

$$\Psi^\mu_L(\tau + \sigma) = \pm\Psi^\mu_L(\tau + \sigma + \pi), \tag{2.165}$$

and

$$\Psi^\mu_R(\tau - \sigma) = \pm\Psi^\mu_R(\tau - \sigma - \pi), \tag{2.166}$$

where the sign $+$ is for R-boundary condition and the sign $-$ is for NS-boundary condition. In analogy with the bosonic string theory, we can determine the general solutions by using the exponential Fourier series.

For R-boundary condition, the general solutions of Ψ^μ_L and Ψ^μ_R can be expressed as

$$\Psi^\mu_L(\tau + \sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau+\sigma)}, \tag{2.167}$$

$$\Psi^\mu_R(\tau - \sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau-\sigma)}. \tag{2.168}$$

For NS-boundary condition, we obtain

$$\Psi^\mu_L(\tau + \sigma) = \sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir(\tau+\sigma)}, \tag{2.169}$$

$$\Psi^\mu_R(\tau - \sigma) = \sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau-\sigma)}. \tag{2.170}$$

By pairing the left- and right-moving sectors, we obtain four distinct closed-string sectors denoted by R-R, R-NS, NS-R and NS-NS.

In the case of open string, there are two possible sectors according to the R- or NS-boundary conditions. It comes from the fact that the left- and right-moving sectors for each mode must have the same amplitude and form a standing wave together.

The boundary condition then can be written as

$$\Psi^\mu_L(\tau) = \Psi^\mu_R(\tau), \quad (2.171)$$

$$\Psi^\mu_L(\tau+\pi) = \pm\Psi^\mu_R(\tau-\pi), \quad (2.172)$$

where + is for R-boundary condition and − is for NS-boundary condition as well. Then, we can express the general solutions of Ψ^μ_L and Ψ^μ_R as

$$\Psi^\mu_L(\tau+\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau+\sigma)}, \quad (2.173)$$

$$\Psi^\mu_R(\tau-\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau-\sigma)}, \quad (2.174)$$

for R-boundary condition and

$$\Psi^\mu_L(\tau+\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-ir(\tau+\sigma)}, \quad (2.175)$$

$$\Psi^\mu_R(\tau-\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-ir(\tau-\sigma)}, \quad (2.176)$$

for NS-boundary condition.

Since assuming that Ψ^μ is the two-component Majorana spinor is equivalent to imposing the reality condition to the spinor, we obtain $d^\mu_{-n} = d^{\mu *}_n$, $b^\mu_{-r} = b^{\mu *}_r$, $\tilde{d}^\mu_{-n} = \tilde{d}^{\mu *}_n$ and $\tilde{b}^\mu_{-r} = \tilde{b}^{\mu *}_r$ for positive n and r. It is also the fact that the spinor Ψ^μ must satisfy the anti-commutation relation

$$\{\Psi^\mu_A(\tau,\sigma), \Psi^\nu_B(\tau,\sigma')\} = 2\pi\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma')\delta_{AB}. \quad (2.177)$$

As a result, the anti-commutation relation for b^μ_r and $\tilde{b}^\mu_r$ is

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}, \quad (2.178)$$

$$\{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}, \quad (2.179)$$

$$\{b^\mu_r, \tilde{b}^\nu_s\} = 0. \quad (2.180)$$

The anti-commutation relation for d^μ_m and $\tilde{d}^\mu_m$ can be expressed in the similar forms.

We also obtain the constraints in superstring theory. Obviously, the bosonic constraint is the energy-momentum tensor

$$T_{++} = \partial_+ X^\mu_L \partial_+ X_{\mu L} - \frac{i}{2}\Psi^\mu_L \partial_+ \Psi_{\mu L} = 0, \quad (2.181)$$

$$T_{--} = \partial_- X^\mu_R \partial_- X_{\mu R} - \frac{i}{2}\Psi^\mu_R \partial_- \Psi_{\mu R} = 0, \quad (2.182)$$

which can be determined in the same way as that in the case of bosonic string. In addition, we also obtain the worldsheet supercurrent as the fermionic constraint

$$J_+ = \Psi^\mu_L \partial_+ X_{\mu L} = 0, \tag{2.183}$$

$$J_- = \Psi^\mu_R \partial_- X_{\mu R} = 0. \tag{2.184}$$

Equations (2.181)-(2.184) are known as the **super-Virasoro constraints**.

Then, the super-Virasoro generators can be determined from these constraints. Now, we will discuss only the case of open string for conciseness. The super-Virasoro generators can be expressed as

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m \cdot \alpha_{n-m} + \frac{1}{2}\sum_{m\in\mathbb{Z}} (\frac{n}{2} - m) d_m \cdot d_{n-m} = 0, \tag{2.185}$$

$$F_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} d_m \cdot \alpha_{n-m} = 0, \tag{2.186}$$

for the R-boundary condition and

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m \cdot \alpha_{n-m} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (\frac{n}{2} - r) b_r \cdot b_{n-r} = 0, \tag{2.187}$$

$$G_r = \frac{1}{2}\sum_{m\in\mathbb{Z}} b_m \cdot \alpha_{r-m} = 0, \tag{2.188}$$

for NS-boundary condition. Obviously, these super-Virasoro generators altogether form the algebra, namely the **super-Virasoro algebra**.

For R-sector, the super-Virasoro algebra for left-moving generators is

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}, \tag{2.189}$$

$$\{L_m, F_n\}_{P.B.} = -i(\frac{1}{2}m - n)F_{m+n}, \tag{2.190}$$

$$\{F_m, F_n\} = 2L_{m+n}. \tag{2.191}$$

Then, the Hamiltonian and mass squared for the R-sector can be written as

$$H = \sum_{n\in\mathbb{Z}} (\alpha_n \cdot \alpha_{-n} - n d_n \cdot d_{-n}) = 0, \tag{2.192}$$

$$M^2 = \frac{1}{2\alpha'}\sum_{n\in\mathbb{Z}} (\alpha_n \cdot \alpha_{-n} - n d_n \cdot d_{-n}). \tag{2.193}$$

For the NS-sector, we can obtain these quantities by replacing d_n with b_r.

It is the fact that the RNS action still maintains the global Poincare symmetry of spacetime as well as the Polyakov action does. Therefore, it is important to determine the Poincare generators for this action. In the case of closed string, we obtain the momentum, $\mathbf{P}^\mu$, and the Lorentz generator, $\mathbf{J}^{\mu\nu}$, as

$$\mathbf{P}^\mu = p^\mu, \quad (2.194)$$

$$\mathbf{J}^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu}, \quad (2.195)$$

where the terms $l^{\mu\nu}$ and $E^{\mu\nu}$ are in the similar forms as those in bosonic string but $K^{\mu\nu}$ is

$$K^{\mu\nu} = -\frac{i}{2}\sum_{n=\mathbb{Z}}(d^\mu_{-n}d^\nu_n - d^\nu_{-n}d^\mu_n),$$

for R-boundary condition and

$$K^{\mu\nu} = -\frac{i}{2}\sum_{r=\mathbb{Z}+\frac{1}{2}}(b^\mu_{-r}b^\nu_r - b^\nu_{-r}b^\mu_r),$$

for NS-boundary condition. These Poincare generators certainly obey the Poincare algebra.

Next, we briefly mention the light-cone gauge formalism in classical superstring theory. In order to fix the light-cone gauge in the action, we must introduce the fermionic light-cone coordinates Ψ^+, Ψ^-, and Ψ^i as well as the bosonic light-cone coordinates X^+, X^-, and X^i. From the superconformal invariance, we can fix X^+, and Ψ^+ into the forms

$$X^+ = 2\alpha' p^+\tau, \quad (2.196)$$

$$\Psi^+ = 0. \quad (2.197)$$

The choices (2.196) and (2.197) are called the light-cone gauge for superstring theory. Then, the action can be written in the form

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma(4\alpha' p^+\partial_\tau X^- + \partial^a X^i\partial_a X_i - i\overline{\Psi}^i\rho^a\partial_a X_i). \quad (2.198)$$

It is obvious that only X^i and Ψ^i are considered as the physical variables and their equations of motion are

$$\partial^a\partial_a X^i = 0, \quad (2.199)$$

$$\rho^a\partial_a\Psi^i = 0, \quad (2.200)$$

respectively. Fixing the light-cone gauge in the action, we can solve the super-Virasoro constraints. As a consequence, we can determine all light-cone spacelike modes (the minus modes) in terms of transverse modes. We can obtain all expressions for super-Virasoro generators, the squared mass, and other quantities by using the transverse oscillators instead. For the Poincare symmetry, it turns out that we can easily obtain the Lorentz generators $\mathbf{J}^{ij}$ and $\mathbf{J}^{i-}$ by adding the ferminic modes, K^{ij} and K^{i-} to the Lorentz generator in the bosonic case. It is also obvious that all Poincare generators obey the Poincare algebra in any dimension of spacetime.

2.2.2 Quantum Theory of Supersymmetric String

In this subsection we will use only the light-cone gauge quantization to study the quantum superstring theory. In analogy with bosonic string, we start by considering all variables as operators and replacing the Poisson bracket $\{\quad\}_{P.B.}$ with $-i[\quad]$. Then, we introduce the annihilation and creation operators as the positive- and negative-mode oscillators, respectively. The normalized positive-mode bosonic oscillators and fermionic oscillators are used to define the ground state. For brevity, we consider only the case of open string.

All commutation relations and expressions for the bosonic transverse oscillators are as same as those in the previous subsection. For fermionic transverse oscillators, we can express all operators and commutation relations in the similar forms of those in the previous subsection via the normal ordering. The difference is that we must replace the general oscillators by transverse oscillators. Therefore, we can obtain the mass-squared operators as

$$M^2 \;=\; \frac{1}{\alpha'}(N^{\perp} - a_R), \tag{2.201}$$

where

$$N^{\perp} \;=\; \sum_{m=1}^{\infty} m(a_m^{\dagger i} a_{mi} + d_m^{\dagger i} d_{mi}), \tag{2.202}$$

for R sector and

$$M^2 \;=\; \frac{1}{\alpha'}(N^{\perp} - a_{NS}), \tag{2.203}$$

where

$$N^{\perp} \;=\; \sum_{m=1}^{\infty} m a_m^{\dagger i} a_{mi} + \sum_{r=\frac{1}{2}}^{\infty} r b_r^{\dagger i} b_{ri}, \tag{2.204}$$

for NS sector.

Next, we determine the consistency conditions for superstring theory. As we have mentioned in the previous section, demanding the Lorentz invariance, we obtain the consistency condition for string theory in light-cone gauge approach. Similar to the bosonic string theory, we concern whether the generator $\mathbf{J}^{-i}$ obeys the Lorentz algebra. If the commutator $[\mathbf{J}^{-i}, \mathbf{J}^{-j}]$ vanishes, then Lorentz algebra still maintains in this theory. So this theory is Lorentz invariant. After some calculation, we can determine this commutator in the case of NS sector as

$$\begin{aligned}[\mathbf{J}^{-i}, \mathbf{J}^{-j}] &= \frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} (\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) \\ &\times \ (m(\frac{D-10}{8}) - \frac{1}{m}(\frac{D-2}{8} - 2a_{NS})). \end{aligned} \tag{2.205}$$

Therefore, the consistency condition for NS sector is

$$a_{NS} = \frac{1}{2}, \tag{2.206}$$

$$D = 10. \tag{2.207}$$

With the same method, we obtain $a_R = 0$ and $D = 10$ as the consistency condition for R sector. This means that the critical dimension for superstring is ten no matter which boundary condtion we consider.

Analogous to bosonic string, the ground state of a superstring with momentum k^+ and k^i, $|0; k^+, k^i\rangle$ can be defined by using the condition

$$a^i_m|0; k^+, k^i\rangle = d^i_n|0; k^+, k^i\rangle = 0, \qquad m, n > 0, \tag{2.208}$$

for R-boundary condition and

$$a^i_m|0; k^+, k^i\rangle = b^i_r|0; k^+, k^i\rangle = 0, \qquad m, r > 0, \tag{2.209}$$

for NS-boundary condition. Obviously, the Fock space in this approach is manifestly ghost-free. It is worth noting that the ground state of R sector is degenerated. In order to see this, we examine the anti-commutation relation of d^i_0

$$\{d^i_0, d^j_n\} = \eta^{ij}\delta_{0,n}. \tag{2.210}$$

This also implies that d^μ_0 commutes with $N^\perp$ and M^2. It turns out that if $|0\rangle$ is applied by a combination of d^i_0, then this new state also obeys the condition of the ground state as well as the state $|0\rangle$. In the case that both oscillators in the algebra are of zero mode, we obtain

$$\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}. \tag{2.211}$$

This is similar to the Dirac algebra $\{\Gamma^i, \Gamma^j\} = -2\eta^{ij}$ where Γ^i are defined as the $D-2$ -dimensional Dirac matrices. This means that these degenerate ground states of R sector are the irreducible representation of the Clifford algebra. In other words, we can express these degenerate ground states as the spinor representation of $SO(D-2)$. Therefore, the ground state of R sector can be written as

$$|\mathbf{n}_\alpha\rangle = \prod_{\alpha=1}^{4}(e_\alpha^\dagger)^{n_\alpha}|0\rangle \qquad ; n_\alpha = 0, 1 \quad , \tag{2.212}$$

where

$$e_\alpha = d_0^{2\alpha-1} + id_0^{2\alpha} \qquad ; \alpha = 1, \ldots, 4 \quad , \tag{2.213}$$

and we denote $|0; k^+, k^i\rangle$ by $|0\rangle$. For convenience, we introduce $n = (\sum n_\alpha)\mathrm{mod}2$. So $n = 0$ is for even number of creation operators that we apply to $|0\rangle$ and $n = 1$ is for odd number. Choosing whether each creation operator $e_\alpha^\dagger$ is or is not applied on $|0\rangle_L$, we can obtain $2^4 = 16$ degenerate ground states in R sector. Therefore, the R sector of superstring is called the fermionic sector; on the other hand, the NS sector is called the bosonic sector.

Next, we study the condition which must be imposed on superstring spectrum in order to eliminate an inappropriate ground state. We consider the string spectrum of closed string as an example. It is obvious that the ground state of R sector is the massless state whereas that of NS sector is the state with negative mass squared or tachyon. It is the fact that the tachyonic ground state implies the instability of the theory. We can solve this problem by discarding the tachyonic ground state, all states that have the same chirality as the tachyonic ground state in NS sector and all states with the certain chirality in R sector. Practically, this truncation can be performed by projecting onto states with definite chirality for left- and right-moving modes in both NS and R sectors, independently. The corresponding projection is called the GSO projection that was proposed by Gliozzi, Scherk and Olive in 1976.

In the case of R sector, the projection operator, P, on the left-moving sector can be written as

$$P = \frac{1 + \eta(-1)^{F+1}}{2}, \tag{2.214}$$

where the number operator, F, is defined as

$$F = \sum_{m=1}^{\infty} d_m^{\dagger i} d_{mi}, \tag{2.215}$$

and η is either $+1$ or -1 independently for left- and right-moving sectors. For right-moving sector, we obtain the similar expressions for projection and number operators by replacing d_m^i with $\tilde{d}_m^i$.

In the case of NS sector, the projection operator, P can be expressed in the form

$$P = \frac{1 + (-1)^{F+1}}{2}, \tag{2.216}$$

where

$$F = \sum_{r=\frac{1}{2}}^{\infty} b_r^{\dagger i} b_{ri}. \tag{2.217}$$

Obviously, the tachyonic ground state, which corresponds to $F = 0$ in NS sector, is certainly eliminated by this projection. It is worth noting that after this projection, the bosonic (NS) ground state and the fermionic (R) ground state of left-moving sector are both massless. Besides, they also have the same number of degrees of freedom. We can see this by considering that sixteen ground states of R sector can be separated into eight states with $n = 0$ and eight states with $n = 1$. From GSO projection, we keep only former (or latter) eight states if we set $\eta = 1$ (or -1). Thus, the number of selected ground state is as same as $b^i_{-\frac{1}{2}}|0\rangle_L$ of NS sector. The situation is similar for the states in right-moving sector. The matching of degrees of freedom is an important evidence for the spacetime supersymmetry. Therefore, we can infer that the two-dimensional (worldsheet) superstring theory with GSO projection is equivalent to the ten-dimensional (spacetime) superstring theory. We then end up this section by examining the lowest states of open and closed strings.

In the case of open string, we obtain eight massless bosonic states $b^i_{-\frac{1}{2}}|0\rangle$, which represent the photons for NS sector. For R sector, we can choose either eight massless ground states with $n = 0$ or those with $n = 1$ since we finally obtain the other eight states in the string spectrum. This is also a result of the GSO projection. We can easily see this by assuming that the eight ground states with $n = 0$ are chosen. We can construct first-excited states by applying one of d^i_{-1} and one of other $e^\dagger_\alpha$'s to the selected ground states in order to preserve the number of fermionic oscillators. Equivalently, we obtain the states constructed from the ground states with $n = 1$ in this level. For this sector, we can interpret that these eight massless fermionic states represent the superpartners of photons from NS sector. This open superstring and an unoriented closed superstring, which is invariant under a parity of worldsheet spatial coordinate ($\sigma \to \pi - \sigma$), together form the **type I** superstring theory.

In the case of (oriented) closed string, the ground states of NS-NS sector can be written as $b^i_{-\frac{1}{2}}\tilde{b}^j_{-\frac{1}{2}}|0\rangle_L \otimes |0\rangle_R$. Similar to bosonic string theory, we obtain the massless fields which represent the graviton $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton ϕ. Number of these bosonic states is 64. The ground states of NS-R and R-NS sectors can be written as $b^i_{-\frac{1}{2}}|0\rangle_L \otimes |\mathbf{n}_\alpha\rangle_R$ and $\tilde{b}^i_{-\frac{1}{2}}|\mathbf{n}_\alpha\rangle_L \otimes |0\rangle_R$, respectively. From these sectors, we can obtain $(8 \times 8) + (8 \times 8) = 128$ massless fermionic states. If we choose the different values of η for left- and right-moving sectors, the fermions in both sector have the opposite chirality. This theory is called **type IIA** superstring theory. On the other hand, we choose the same value of η , this theory is called **type IIB** superstring theory and contains the fermions with the same chirality in both sectors. For R-R sector, there are 64 massless bosonic ground states, $|\mathbf{n}_\alpha\rangle_L \otimes |\mathbf{n}_\beta\rangle_R$. In the type-IIA theory, we obtain the Maxwell field, A^μ, and the antisymmetric tensor, $A^{\rho\mu\nu}$, with three indices. In the type-IIB theory, we obtain the scalar field, A, the Kalb-Ramond field, $A^{\mu\nu}$, and the totally antisymmetric tensor, $A^{\rho\sigma\mu\nu}$, with four indices for this sector.

2.3 Heterotic String Theory

Although the superstring theory that we have studied in the previous section possesses both bosonic and fermionic fields, it does not contain any non-Abelian gauge fields which are required to describe the strong interaction. In this section we study one of superstring theories which include non-Abelian gauge symmetries [6]. This theory comes from the fact that the left- and right-moving modes of closed strings are independent. This implies that we can choose one of these modes to possess the worldsheet supersymmetry and the other mode to possess non-Abelian gauge symmetries. A theory consisting of such closed strings is called a heterotic string theory. Similar to the previous section, we begin a study of heterotic string theory with a classical theory. Then, we use a shortcut to take the understanding of a quantum theory. After that, we end up this theory with an analysis of string spectrum.

2.3.1 Classical Theory of Heterotic String

In this context, we choose the right mover to possess the worldsheet supersymmetry and the left mover to possess the non-Abelian gauge symmetries. This means that we obtain the bosonic fields X^μ_R and fermionic fields Ψ^μ_R with $\mu = 0, 1, \ldots, D-1$ for right-moving. For left-moving mode, we still obtain the bosonic fields X^μ_L but replace Ψ^μ_L by Majorana fermions, λ^A_L with $A = 1, \ldots, n$. The n

fermions λ_L^A are Lorentz singlets which possess internal degrees of freedom. In order to achieve the dynamics of heterotic string as soon as possible, we start our study with the gauge-fixed action which can be written in the form

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma(\partial^a X^\mu \partial_a X_\mu - 2i\Psi_R^\mu \partial_+ \Psi_{R\mu} - 2i\sum_{A=1}^{n}\lambda_L^A \partial_- \lambda_L^A). \qquad (2.218)$$

For right-moving sector, all results of X_R^μ and Ψ_R^μ are as same as those in Section 2.2. For left-moving sector, X_L^μ still have the same formulae as those in Section 2.1 and λ_L^A can satisfy the boundary conditions like Ψ_L^μ do. The difference is that there is no supersymmetry between X_L^μ and λ_L^A. In fact, we can proceed the work on λ_L^A in order to obtain the non-Abelian gauge fields. However, it is quite difficult for this approach to demonstrate the non-Abelian gauge symmetries. Therefore, we choose another framework to study the heterotic string theory. In this framework we replace the fermionic fields λ_L^A by bosonic fields, X_L^I with $I = 1, \ldots, d$. Since two Majorana fermions have the same number of degrees of freedom as one boson, we obtain the relation $d = \frac{n}{2}$. In this approach we can consider a heterotic string as a closed string consisting of the right mover from superstring and the left mover from bosonic string.

It is the fact that the bosonic fields X_L^I can carry non-Abelian gauge degrees of freedom if they are compactified into a d-dimensional torus with correct radii. Now we briefly mention the toroidal compactification. In this context we generalize our case by assuming that there are still the corresponding right-moving modes, X_R^I and then eliminate these modes for heterotic string later. First, we introduce a lattice, Γ, with basis vectors (vielbeins) e_i^I; $i = 1, \ldots, d$. If a d-dimensional space is compactified into a d-dimensional torus, the spatial coordinate x^I must satisfy the equivalent relation

$$x^I \sim x^I + 2\pi \sum_{i=1}^{d} R^{(i)} n^i e_i^I \qquad ; n^i \in \mathbb{Z}, \qquad (2.219)$$

where $R^{(i)}$ are compactification radii and n^i are called winding numbers. This means that we compactify the d-dimensional space by winding it along each direction i with n^i numbers of round as a circle with radius $R^{(i)}$. It is conventional to choose the vielbeins such that they are orthonormal

$$g_{IJ} e_i^I e_j^J = \delta_{ij}. \qquad (2.220)$$

As a consequence, the coordinates of closed string $X^I(\tau, \sigma)$ must satisfy the boundary condition

$$X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + 2\pi \sum_{i=1}^{d} R^{(i)} n^i e_i^I = X^I(\tau, \sigma) + 2\pi\alpha' w^I. \qquad (2.221)$$

Here, we define the string winding mode, w^I, as

$$w^I \equiv \sum_{i=1}^{d} \frac{R^{(i)} n^i}{\alpha'} e_i^I. \tag{2.222}$$

Then, the mode expansions of X_L^I and X_R^I can be expressed as

$$X_L^I(\tau+\sigma) = \frac{1}{2}x_L^I + \alpha' p_L^I(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^I}{n}e^{-2ni(\tau+\sigma)}, \tag{2.223}$$

$$X_R^I(\tau-\sigma) = \frac{1}{2}x_R^I + \alpha' p_R^I(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^I}{n}e^{-2ni(\tau-\sigma)}, \tag{2.224}$$

where $p_L^I = \sqrt{\frac{2}{\alpha'}}\alpha_0^I$, $p_R^I = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^I$. From the boundary condition we can obtain the relation between p_L^I and p_R^I as

$$p_L^I - p_R^I = 2w^I. \tag{2.225}$$

Therefore, the general solution of $X^I(\tau,\sigma)$ can be written in the form

$$\begin{aligned} X^I(\tau,\sigma) &= x^I + 2\alpha' p^I \tau + 2\alpha' w^I \sigma \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^I e^{-2ni(\tau+\sigma)} + \tilde{\alpha}_n^I e^{-2ni(\tau-\sigma)}), \end{aligned} \tag{2.226}$$

where

$$x^I = \frac{1}{2}(x_L^I + x_R^I), \tag{2.227}$$

$$p^I = \frac{1}{2}(p_L^I + p_R^I). \tag{2.228}$$

We then eliminate the right-moving mode X_R^I by imposing a (second-class) constraint

$$(\partial_\tau - \partial_\sigma)X^I = 0. \tag{2.229}$$

In order to obtain the context of heterotic string. Because of the second-class constraint, we must use the Dirac bracket, $\{\ \ ,\ \ \}^*$, instead of the Poisson bracket. In this case, the equal-τ Dirac bracket of $X^I(\tau,\sigma)$ and $P^J(\tau,\sigma')$ can be written in the form

$$\{X^I, P^J\}^* = \frac{1}{2}\delta^{IJ}\delta(\sigma-\sigma'). \tag{2.230}$$

After we imposed the constraint, the center-of-mass coordinates and all oscillators of right-moving mode vanish

$$x_R^I = 0, \qquad \tilde{\alpha}_n^I = 0 \quad ; n \in \mathbb{Z}.$$

As a result, the x_L^I must satisfy the condition

$$\frac{1}{2}x_L^I = \frac{1}{2}x_L^I + 2\pi\alpha' w^I, \tag{2.231}$$

and the Dirac bracket of x_L^I and p_L^I can be written in the form

$$\{x_L^I, p_{LJ}\}^* = 2\delta_J^I. \tag{2.232}$$

At this point we can combine X_L^μ and X_L^I as bosonic coordinates X_L^α where $\alpha = 0, 1, \ldots, D+d-1$. Therefore, we can treat X_L^α in the same way as the left-moving mode in Section 2.1. The difference is that the squared mass is defined from the momentum of non-compactified dimensions only. Now we turn to the quantum theory of heterotic string.

2.3.2 Quantum Theory of Heterotic String

Similar to the previous section, we make a shortcut to the understanding of quantum heterotic string theory by considering all variables as operators and replacing the Poisson bracket and Dirac bracket with commutator $-i[\quad,\quad]$. For conciseness, we study only the Lorentz covariance formalism in this subsection.

In the case of right-moving mode, we can refer all results from those of superstring in Section 2.2. This means that the critical dimension of heterotic string theory is ten ($D = 10$). From the zeroth bosonic mode of super-Virasoro constraint the right-moving mass-squared operator, M_R^2, and number operator $\tilde{N}$ can be expressed as

$$M_R^2 = \frac{4}{\alpha'}\tilde{N}, \tag{2.233}$$

$$\tilde{N} = \sum_{n=1}^{\infty} n(\tilde{a}_n^\dagger \cdot \tilde{a}_n + \tilde{d}_n^\dagger \cdot \tilde{d}_n), \tag{2.234}$$

for R sector and

$$M_R^2 = \frac{4}{\alpha'}(\tilde{N} - \frac{1}{2}), \tag{2.235}$$

$$\tilde{N} = \sum_{n=1}^{\infty} n\tilde{a}_n^\dagger \cdot \tilde{a}_n + \sum_{r=\frac{1}{2}}^{\infty} r\tilde{b}_r^\dagger \cdot \tilde{b}_r, \tag{2.236}$$

for NS sector. It is worth noting that we also use GSO projection on the Fock space of right-moving sector to eliminate the tachyonic ground state and obtain spacetime supersymmetry.

In the case of left-moving mode, we can obtain the same results as those in Section 2.1. Since the bosonic string theory is ghost-free and Lorentz invariant if the number of dimension is twenty-six ($D + d = 26$), the number of compactified dimension is sixteen ($d = 16$). Then, we consider commutator of x^I_L and p^J_L

$$[x^I_L, p_{LJ}] = 2i\delta^I_J. \tag{2.237}$$

From this commutator we can infer that $\frac{1}{2}p^I_L$ is a generator of translation in x^I_L . Therefore, the wave function of x^I_L is proportional to

$$\exp(i\sum_{I=1}^{d}\frac{1}{2}p_{LI}x^I_L) \tag{2.238}$$

and is single-valued under the condition (2.238). As a consequence, the p^I_L can be written in the form

$$p_{LI} = \sum_{i=1}^{d}\frac{m_i}{R^{(i)}}e^{*i}_I \qquad ; m_i \in \mathbb{Z}, \tag{2.239}$$

where m_i are called momentum numbers and e^{*i}_I are basis vectors of a dual lattice, Γ^* , and satisfy

$$\sum_{I=1}^{d} e^I_i e^{*j}_I = \delta^j_i. \tag{2.240}$$

It is essential to note that in order to avoid the gauge and gravitational anomalies, string theory must possess the modular invariance, which is a reparametrization invariance on a one-loop string scattering amplitude. As a consequence, the lattice Γ must be an even self-dual lattice, which satisfies

$$\Gamma = \Gamma^*, \tag{2.241}$$

and contains the vectors whose squared length is even number.

From the zeroth-mode Virasoro constraints the left-moving mass-squared operator, M^2_L, and number operator N can be expressed as

$$M^2_L = \sum_{I=1}^{d}(p^I_L)^2 + \frac{4}{\alpha'}(N-1), \tag{2.242}$$

$$N = \sum_{n=1}^{\infty} n(a^\dagger_n \cdot a_n + \sum_{I=1}^{d} a^{\dagger I}_n a^I_n). \tag{2.243}$$

Combining the results from both left- and right-moving modes, we can obtain the mass-shell condition and the level matching condition as

$$M^2 = \frac{1}{2}(M_L^2 + M_R^2) = \frac{2}{\alpha'}(N + \tilde{N} - 1) + \frac{1}{2}\sum_{I=1}^{d}(p_L^I)^2, \quad (2.244)$$

$$0 = N - \tilde{N} + \frac{\alpha'}{4}\sum_{I=1}^{d}(p_L^I)^2 - 1, \quad (2.245)$$

for R sector and

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - \frac{3}{2}) + \frac{1}{2}\sum_{I=1}^{d}(p_L^I)^2, \quad (2.246)$$

$$0 = N - \tilde{N} + \frac{\alpha'}{4}\sum_{I=1}^{d}(p_L^I)^2 - \frac{1}{2}, \quad (2.247)$$

for NS sector. It is obvious that the level-matching condition does not allow the (bosonic) string states in the left-moving sector to possess the negative mass-squared. In the other words, there are no tachyons in the heterotic string theory as well as in any superstring theory. We can see that it is more convenient if the ordering constant ($a_R = 0$ or $a_{NS} = 1/2$) is included in the right-moving number operator $\tilde{N}$ for each sector in order that the mass-squared operator and the level-matching condition for both sectors take the same forms (2.244) and (2.245), respectively.

Now we turn to a study of ground states of heterotic string. We will see the emergence of non-Abelian gauge groups in this theory. Before we consider the string states, it is important to note that p_L^I is the internal degree of freedom of string state since it is the momentum of the compactified dimensions. Therefore, we introduce $|0; p_L^I\rangle_L \otimes |0\rangle_R$ for NS sector or $|0; p_L^I\rangle_L \otimes |\mathbf{n}_\alpha\rangle_R$ for R sector as a ground state which has the momentum p_L^I in the compactified dimensions. In this context, we use the Light-cone gauge formalism to consider string states for convenience.

We start by considering the ground states which have no momentum in compactified dimensions ($p_L^I = 0$). For NS sector the massless bosonic states which are in the form $a_{-1}^k \tilde{b}_{-\frac{1}{2}}^l |0; 0\rangle_L \otimes |0\rangle_R$ can be decomposed into ten-dimensional graviton, Kalb-Ramond field, and dilaton. Their superpartners are from R sector and are expressed in the form $a_{-1}^k |0; 0\rangle_L \otimes |\mathbf{n}_\alpha\rangle_R$. We can see that heterotic string theory also possesses the string states which are similar to those in the previous section. Next, we will consider additional massless states of heterotic string which are constructed from oscillators in compactified dimensions and contribute an

Abelian gauge group. We will consider only the bosonic states from NS sector for conciseness. These additional bosonic states can be expressed as $a^I_{-1}\tilde{b}^k_{-\frac{1}{2}}|0;0\rangle_L \otimes |0\rangle_R$. According to Kaluza-Klein theory, these states provide an Abelian gauge group $U^{16}(1)$. Next, we will see that this gauge group can be enhanced to a non-Abelian gauge group for the particular lattice Γ and radii R_i .

The momentum p^I_L of ground state must have some specific value in order to obtain extra massless gauge fields. From the mass-shell condition the ground state can be massless if

$$\sum_{I=1}^{16}(\sqrt{\frac{\alpha'}{2}}p^I_L)^2 = 2. \tag{2.248}$$

These extra massless states can be written in the form $\tilde{b}^k_{-\frac{1}{2}}|0;p^I_L\rangle_L \otimes |0\rangle_R$. Due to the modular invariance, there are only two choices for the even self-dual lattice Γ and only one choice for the radii $R^{(i)}$ which can satisfy the condition (2.248) in the 16-dimensional space. For the compactification radii, the choice must be

$$R^{(i)} = \sqrt{\alpha'} \qquad ; \quad i = 1, \ldots, 16. \tag{2.249}$$

For a lattice Γ, the first choice is Γ_{16}, which contains the root lattice of $SO(32)$ as a sublattice. With this lattice the $U^{16}(1)$ gauge group is enlarged to the $SO(32)$ gauge group. The second choice is $\Gamma_8 \times \Gamma_8$, which leads to the $E_8 \times E_8$ gauge group. Therefore, there are two types for heterotic string theory, $SO(32)$ or $E_8 \times E_8$, depending on a non-Abelian gauge group that the theory possesses.

2.4 Low-Energy Effective String Theory

Until now, we have studied only the dynamics of free strings with no background fields. Therefore, we devote the last section to a brief discussion about the dynamics of strings with the presence of the background fields at low energy, which is lower than the energy scale of string $\alpha'^{-\frac{1}{2}}$. In this context only massless bosonic fields of closed string $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, and $\phi(X)$ are considered as a background since the massless fields are relevant at such a limit and we are interested in theories of closed strings; such as a bosonic string, a type-II superstring, and a heterotic string.

We start out by modifying the string action (2.2) with all massless background fields of closed string

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}(h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \; + \; \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \; + \; \alpha' R^{(2)}\phi(X)), \tag{2.250}$$

where ϵ^{ab} is an antisymmtric tensor and $R^{(2)}$ is a Ricci scalar in the two-dimensional worldsheet. Similar to the free theory of bosonic string, the action (2.251) possesses both reparametrization invariance and Weyl invariance. In the conformal gauge, this action reduces to

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma(\partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} - \alpha' R^{(2)}\phi). \qquad (2.251)$$

Obviously, the action (2.251) is conformally invariant. This is a non-trivial quantum field theory, which is known as a non-linear sigma model. We can consider the scattering amplitude as an expansion in powers of α'. Since the low-energy limit is heuristically equivalent to an approximation $\alpha' \to 0$, in such a limit we consider the expansion in the lowest order of α'. In quantum field theory the UV divergence causes the infiniteness of the scattering amplitude; therefore, the scale invariance is required for renormalization. The existence of nonzero β functions implies the breakdown of scale invariance. In string theory it turns out that the conformal invariance of the action (2.251) ensures the vanishing of all lowest-order β functions,

$$\beta^G_{\mu\nu} \approx \alpha'(R_{\mu\nu} - \frac{1}{4}H^{\rho\sigma}_\mu H_{\nu\rho\sigma} + 2\nabla_\mu\nabla_\nu\phi) = 0, \qquad (2.252)$$

$$\beta^B_{\nu\rho} \approx \alpha'(\nabla_\mu H^\mu_{\nu\rho} - 2(\nabla_\mu\phi)H^\mu_{\nu\rho}) = 0, \qquad (2.253)$$

$$\beta^\phi \approx \alpha'(4\nabla_\mu\phi\nabla^\mu\phi - R - 4\nabla_\mu\nabla^\mu\phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}) = 0, \qquad (2.254)$$

where ∇_μ and $R_{\mu\nu}$ are the D-dimensional covariant derivative and Ricci tensor, respectively and

$$H = \mathtt{d}B, \qquad (2.255)$$

is the field strength of Kalb-Ramond field.

Equivalently, we can study the dynamics of these background fields in the view point of spacetime by introducing the D-dimensional string action that yields equations (2.252)-(2.254) as equations of motion. This action is called the **low-energy effective action**, S_{eff}, and is expressed as

$$S_{eff} = -\frac{1}{2k_D^2}\int d^D x\sqrt{-G}e^{-2\phi}(R + 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}), \qquad (2.256)$$

where k_D relates to the D-dimensional gravitational coupling constant, G_D, by $k_D^2 = 8\pi G_D$. It is conventional to define the string coupling, g_s, as

$$g_s \equiv e^{2\phi}. \qquad (2.257)$$

We will see in the next chapter that under some assumptions we can use this action to study the dynamics of string gas in the early universe .

In summary, we begin this chapter with bosonic string theory. From anomaly cancellation, the quantum theory of bosonic string is consistent only in 26 dimensions. Adding the fermion terms to the bosonic string action, we obtain the consistent superstring theory in 10 dimensions. Combining the left-moving mode of closed string from bosonic string and the other mode from superstring, we obtain the heterotic string theory, which contains non-Abelian gauge fields. In the presence of background fields the conformal invariance implies that the background fields must satisfy the field equations of string theory themselves.

CHAPTER III

ASPECTS OF STRING GAS COSMOLOGY

The main purpose of this chapter is to study the aspects of string gas cosmology, which is a combination of the basic concepts of standard cosmology and the fundamentals of string theory. In the first part of this chapter we review the standard model of cosmology. The dynamics of the universe is discussed briefly. Then we point out some crucial problems in this model. Thereafter, we introduce one of the most successful cosmological models, namely the inflationary model. We will study the mechanism of the inflation and demonstrate the ways this model solves the problems. The final part of this chapter is devoted to the understandings of string gas cosmology. We give emphasis on the mechanism of hot gases of closed strings in the early universe. It will be shown that the new symmetry and new degrees of freedom emerging from this model play the crucial role in solving one of the most important problems in standard cosmology and explaining the evolution of the early universe. The further development and the striking problems in this model are also discussed.

3.1 Standard Model of Cosmology

For almost a century the standard model of cosmology has been developed and has been used to study the evolution of the universe. The perspective concept of this model is that the universe consists of enormous numbers of celestial objects which interact with each other under the effect of gravity. In the other words, the standard model of cosmology can be considered as an application of general relativity (GR). The dimension of spacetime in this model is set to be four corresponding to the number of dimensions we can observe. Therefore, we begin this section with the classical action of gravitation, namely the Einstein-Hilbert action, S_{EH}, in the 4-dimensional spacetime

$$S_{EH} = -\frac{1}{8\pi G}\int d^4x\sqrt{-g}(R-2\Lambda), \qquad (3.1)$$

where R, G and Λ are respectively the Ricci scalar, the gravitational constant and the cosmological constant, which contributes to the vacuum energy. Due to the presence of matters, we add the action, S_M, which provides the energy-momentum tensor

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{3.2}$$

to the Einstein-Hilbert action. Varying the total action, $S = S_{EH} + S_M$, with respect to $g^{\mu\nu}$, we obtain the Einstein's field equation with the presence of the cosmological constant and the matters

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.3}$$

It is essential to consider the conservation of energy-momentum, which can be expressed in the form

$$\nabla^\mu T_{\mu\nu} = 0. \tag{3.4}$$

It is very difficult to solve the Einstein's field equation and the equation of conservation in general case. However, under some suitable assumptions we can simplify these equations and use them to study the evolution of universe.

3.1.1 Cosmological Principle and Robertson-Walker Universe

It is believed that the universe viewed on the sufficiently large scale looks very similar in every directions and at every points. This assumption is known as the **cosmological principle**, which states that the universe is isotropic and homogeneous [8, 9]. The isotropy means that there is no preferred directions in the universe and implies the rotational symmetry. On the other hand, the homogeneity means that there is no preferred place in the universe and implies the translational symmetry. The standard choice of the metric corresponding to this principle is the **Friedmann-Robertson-Walker (FRW) metric**

$$ds^2 = -dt^2 + a(t)(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)), \tag{3.5}$$

where k can be $+1, 0, -1$ for the closed, flat or open universe respectively and $a(t)$ is known as the scale factor. Alternatively, we can express this metric in another form

$$ds^2 = a(\eta)(-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d^2\phi)), \tag{3.6}$$

where $\eta(t) \equiv \int^t \frac{dt'}{a(t')}$ is the conformal time. It is very useful to define the Hubble parameter, $H(t)$, as

$$H \equiv \frac{\dot{a}(t)}{a(t)}. \tag{3.7}$$

Before we study the dynamics of the universe, it is essential to consider some properties of matter in the universe. It is believed that the early universe is occupied by the hot gases of matter. This means that the matter in the universe can be considered as the perfect fluid with the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{3.8}$$

where ρ and p are the energy density and the pressure in the rest frame of fluid respectively and U^μ is the normalized fluid four-velocity. For simplicity, we use $U_\mu = (1, 0, 0, 0)$ and then obtain

$$T^\nu_\mu = \begin{pmatrix} -\rho & \\ & p\delta_{ij} \end{pmatrix}, \tag{3.9}$$

We can see that the pressures in all spatial dimensions are assumed to be equal due to the cosmological principle. Since the matter is assumed to be the perfect fluid, it is essential to introduce the equation of state for perfect fluid as

$$p = w\rho. \tag{3.10}$$

From this equation we can consider the matter in the cases of the radiation ($w = \frac{1}{3}$) and the pressureless dust ($w = 0$). At this stage, we are ready to solve the Einstein's field equation and study the dynamics of the universe.

3.1.2 Evolution of The Universe

Inserting the expressions for the energy-momentum tensor (3.9) and the FRW metric (3.5) into the Einstein's field equation (3.3), we find that the non-zero components of this equation are the 00-component, which is called the **Friedmann's equation** and is expressed as

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\rho_v - \frac{k}{a^2} \tag{3.11}$$

where $\rho_v \equiv \frac{\Lambda}{8\pi G}$ is the vacuum energy density contributed by the cosmological constant, and the ij- components, which can be written as

$$\frac{\ddot{a}}{a} + \frac{1}{2}H^2 = -4\pi G p - \frac{k}{2a^2} + \frac{\Lambda}{2}. \tag{3.12}$$

Combining these two equations, we obtain the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{3.13}$$

In the case that the cosmological constant is absent ($\Lambda = 0$), we can define the critical energy density, ρ_c, from the Friedmann's equation as

$$\rho_c \equiv \frac{3H^2}{8\pi G}. \tag{3.14}$$

It is obvious that the universe is flat ($k = 0$) if the energy density of all particles in the universe is equal to the critical density. It is also useful to define the density parameter, Ω, as

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G}{3H^2}\rho. \tag{3.15}$$

From the observable data on Cosmic Microwave Background Radiation (CMBR), it is shown that the total density parameter at the present time, Ω_0, is

$$\Omega_0 = 1.02 \pm 0.02. \tag{3.16}$$

This means that our present universe is very flat. Obviously, it is not the unexpected result because it agrees with our intuition that we are living in the flat space.

The equation of the energy-momentum conservation (3.4) is also simplified into the form

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{3.17}$$

Using the equation of state, we can obtain the solution of the energy density

$$\rho(a) \propto a^{-3(1+w)}. \tag{3.18}$$

Inserting the expression of ρ into the Friedmann's equation with the assumptions $k = 0$ and $\Lambda = 0$, the solution of scale factor $a(t)$ can be determined as

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{3.19}$$

From the above equation it is obvious that the scale factor increases continuously with time for the radiation ($w = \frac{1}{3}$) and the pressureless dust ($w = 0$). Substituting this solution back into equation (3.17), we can obtain the energy density as a function of time

$$\rho(t) \propto t^{-2}. \tag{3.20}$$

From equation (3.17) we can see that the energy density of radiation falls down as a^{-4} while the energy density of matter (pressureless dust) falls down as a^{-3}. This means that in the past there might be some period for which the universe was dominated by the radiation and then the universe came into the era for which the matter became dominant instead of the radiation. The former epoch is called the **radiation-dominated period** and the latter epoch is called the **matter-dominated period**. We can see that it is reasonable for both radiation- and matter-dominated periods to assume $k = 0$ and $\Lambda = 0$ in Friedmann's equation (3.11), because the term $8\pi\rho/3$ becomes more relevant than the other two in the RHS during such periods. However, from the observations of Supernovae Type Ia (SNe Ia) and CMBR, we can estimate the density parameters of matter and cosmological constant at present as

$$\Omega_M \sim 0.3 \quad , \quad \Omega_\Lambda \sim 0.7. \tag{3.21}$$

It means that at the moment the universe is dominated by the cosmological constant instead of matter (and also radiation). As a consequence, we can determine the vacuum energy density ρ_v from the cosmological constant in the Planck unit

$$\frac{\rho_v}{M_P^4} \approx 0.8 \times 10^{-120}, \tag{3.22}$$

where $M_P = \frac{1}{\sqrt{8\pi}} m_P$ is the reduced Planck mass, and $m_P = \frac{1}{\sqrt{G}}$ is the Planck mass.

Next, we briefly study thermodynamics of the early universe and indicate some remarkable results before we undergo some difficulties emerging in this model.

3.1.3 Thermodynamics of The Early Universe

It is reasonable to consider the expansion of the universe as an adiabatic process since there is no heat transfer within the homogeneous universe. In the early universe the entropy density, s, the energy density and the pressure of particles which are in thermal equilibrium (their reaction rates are higher than the expansion rate of the universe) can be expressed in terms of temperature, T, as

$$s = \frac{2\pi^2}{45} N_* T^3, \tag{3.23}$$

$$\rho = \frac{\pi^2}{30} N_* T^4, \tag{3.24}$$

$$p = \frac{\pi^2}{90} N_* T^4, \tag{3.25}$$

where N_* is the number of total degrees of freedom from bosons and fermions. Since the expansion of the universe is adiabatic, the entropy, S, which can be written as

$$S = sa^3, \tag{3.26}$$

must be constant. From equations (3.23) and (3.26) we obtain the important relation between the scale factor and the temperature

$$aT = \text{constant}. \tag{3.27}$$

As a consequence, the temperature can be expressed in terms of time t as

$$T \propto t^{-\frac{1}{2}} \tag{3.28}$$

for the radiation-dominated period and

$$T \propto t^{-\frac{2}{3}} \tag{3.29}$$

for the matter-dominated period.

We end up the review of standard cosmology at this point. In the next subsection we will discuss some crucial difficulties in this model.

3.1.4 Problems in Standard Model of Cosmology

It is the fact that the standard model of cosmology provides the basic understandings of the universe and agrees with some observable data. However, there are still some crucial problems in this model. In this subsection we mention only three major problems.

The first problem is the **initial singularity problem**. This problem occurs when we study the dynamics of the universe at the origin or the Big Bang. This means that we take the limit $t \to 0$ for all expressions of physical quantities. It is worth noting that we assume the radiation-dominated period as the first era in the history of the universe. As a result, from equation (3.17) the scale factor approaches zero, $a(t) \to 0$. In other words, the universe was shrunk to a point at the time of Big Bang. At this stage we can see from equations (3.20) and (3.28) that the energy density and the temperature became infinite, $\rho(t), \quad T(t) \to \infty$. This result is unacceptable; therefore, we require the way to solve this fatal problem or a new model which can avoid it. In the last section we will see that string gas cosmology can overcome this difficulty gracefully.

From the CMBR data, it turns out that the background temperature of the universe at the present is extremely homogeneous no matter how far the distance is. This is so weird because it implies that the contact between two causally disconnected regions that even light signal cannot reach can occur. This inexplicable result is known as the **horizon problem**.

The last problem is called the **flatness problem**. This difficulty also emerges from the observation on CMBR. As we mentioned in §3.1.2, the CMBR data leads to the expected result that the universe at present is very flat. However, it is quite surprising when we determine the density parameter in the early universe. The result shows that the early universe is extremely flat

$$|\Omega - 1| \approx 1.66 \times 10^{-61}. \tag{3.30}$$

This number is unusually very small; therefore, the remarkable fine tuning on the density parameter in the early universe is essentially required.

In the next section we present a critical review on the inflationary model and its remarkable success in solving the horizon and flatness problems from the standard cosmology.

3.2 Inflationary Model

The inflation is a very rapid expansion of the universe which makes the scale factor grow larger with an incredible rate. The model of this cosmological scenario was first proposed by Guth [31] and is known as the inflationary model. This model not only solves both horizon and flatness problems mentioned in the previous section, but also provides the solution to the unwanted relics problem that we will discuss later. It is essential to note that there are, in fact, many inflationary models; however, in this thesis we mainly study the model of the slow-roll inflation and then discuss other models of inflation briefly.

3.2.1 Slow-roll Inflation model

In this model we can describe the inflation by using a scalar field called an inflaton, ϕ. It appears that the inflaton must roll very slowly at the beginning of the inflation and also rolls slowly along the nearly flat effective potential, $V(\phi)$, long enough to yield the sufficient inflation. We start our study with the action of inflaton

$$S_{inf} = \int d^4x\sqrt{-g}(\partial^\mu\phi\partial_\mu\phi - V(\phi)). \tag{3.31}$$

We can obtain the equation of motion

$$\begin{aligned} 0 &= \nabla_\mu\partial^\mu\phi + V'(\phi) \\ &= \ddot{\phi} + 3H\dot{\phi} + V'(\phi), \end{aligned} \tag{3.32}$$

where $V'(\phi)$ is the derivative of $V(\phi)$ with respect to ϕ. Since ϕ rolls slowly, we can estimate $|\frac{\ddot{\phi}}{3H\dot{\phi}}| \ll 1$ and we then obtain the expressions for $\dot{\phi}$ and $\ddot{\phi}$ as

$$\dot{\phi} = -\frac{V'(\phi)}{3H}, \tag{3.33}$$

$$\ddot{\phi} = -\frac{V''\dot{\phi}}{3H} + \frac{V'}{3H^2}\dot{H}. \tag{3.34}$$

It is the fact that the energy density of inflaton contributes to the vacuum energy ρ_v as well as the cosmological constant does. The energy-momentum tensor of inflaton can be expressed as

$$T_{00} = \rho_v = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.35}$$

$$T_{ij} = p\delta_{ij} = (\frac{1}{2}\dot{\phi}^2 - V(\phi))\delta_{ij}. \tag{3.36}$$

It is reasonable to assume that in the inflation period the vacuum energy is dominated by the potential $V(\phi)$; in other words, we assume that $\frac{\frac{1}{2}\dot{\phi}^2}{V(\phi)} \ll 1$. As a result, the Friedmann's equation (3.17) becomes

$$H^2 = \frac{8\pi G\rho_v}{3} \approx \frac{8\pi GV(\phi)}{3} = \frac{8\pi V(\phi)}{3m_p^2}. \tag{3.37}$$

We then obtain the slow-roll conditions

$$\frac{m_p^2}{24\pi}\frac{|V''(\phi)|}{V(\phi)} \ll 1, \tag{3.38}$$

$$\frac{m_p^2}{48\pi}(\frac{V'(\phi)}{V(\phi)})^2 \ll 1. \tag{3.39}$$

It is useful to introduce the e-fold number, N_e, which is the measurement of expansion rate of the inflation, as

$$N_e \equiv \ln(\frac{a(t_f)}{a(t_i)}) = \int_{t_i}^{t_f} H(t)dt, \tag{3.40}$$

where t_i and t_f are the time at the beginning and at the end of inflation, respectively. From Friedmann's equation and the fact that the inflaton rolls very slowly at the beginning of the inflation, we can express the e-fold number in terms of the inflaton field as

$$N_e \approx \frac{8\pi V(\phi)}{m_p^2|V''(\phi)|} \gg 1. \tag{3.41}$$

After the inflation ends, the inflatons must undergo a damped oscillation about the absolute minimum of potential and decay into radiation and massive particles in order that the radiation- and matter-dominated periods can occur. This scenario is known as the reheating process. We can study the reheating process by modifying equation (3.32) with a damping term $\Gamma_\phi\dot{\phi}$

$$0 \;=\; \ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} + V'(\phi), \tag{3.42}$$

where Γ_ϕ is the decay rate of inflatons. In general, we assume that the decay rate of inflatons is very rapid ($\Gamma_\phi \leq H$) and the inflatons decay into only fermions. As a result, equation (3.33) can be expressed in terms of the average of vacuum energy, $\langle\rho_v\rangle$,

$$\dot{\langle\rho_v\rangle} + 3H\langle\rho_v\rangle = 0. \tag{3.43}$$

However, if the inflatons decay into a large number of pairs of bosons before the reheating process takes place, the rapid decay can occur by means of the parametric resonance. This process is known as the preheating. As a consequence of the rapid decay, all regions throughout the universe can be rapidly equilibrated and have the same temperature known as the reheating temperature, T_{RH}. Thereafter, the universe undergoes the radiation- and matter-dominated eras respectively as well as the standard model. Up to this point, we can see obviously that the model of slow-roll inflation can solve the horizon problem because it offers the explanations for both tremendous expansion of the universe and isotropic background temperature. We also see that this model can solve the flatness problem by considering the equation (3.37) and equation (3.43). It turns out that the e-fold number must be at least sixty-four ($N_e \geq 64$) in order to satisfy expression (3.41). This is certainly possible for slow-roll inflation. As we mentioned above, this model can solve the unwanted relics problem. The inflation can dilute many particles which are produced during the phase transition period before the time of inflation, such as magnetic monopoles, and make them have so extremely low densities that we hardly observe them nowadays. Moreover, it appears that we can also explain the formation of celestial structures, such as clusters and galaxies, in the context of inflationary model by considering the quantum fluctuation of the inflaton fields, $\hat{\phi}$, which yields the density perturbation, $\frac{\delta\rho}{\rho}$. The spectrum of these density perturbations calculated by this model is in an excellent agreement with the recent data on the fluctuations of CMBR [32, 33]. However, we will neglect the details of the density perturbations for brevity. We can see in this subsection that the model of slow-roll inflation is very successful in describing the evolution of the universe and solving horizon and flatness problems in the standard model of cosmology. In the next subsection we will study other interesting models of inflation.

3.2.2 Other Inflationary Models

It is the fact that there are many models of inflation which can provide graceful explanations for the evolution of the universe, besides the model of slow-roll inflation. For conciseness, we briefly discuss only two more models in this subsection. One is the chaotic inflation model and the other is the hybrid inflation model.

In the slow-roll inflation model, we might require some fine tuning for the initial condition in order that the inflaton can start out in the flat region of the potential. We can avoid the fine tuning by assuming that the early universe starts in the time interval less than the Planck time, $\Delta t \leq t_p$, where we define the Plank time as $t_p \equiv m_p^{-1}$. From the uncertainty principle, we can expect that the inflatons ϕ should start with different values about the absolute minimum of the potential in different regions of the universe at the beginning of the inflation. As a consequence, various parts of the universe undergo various amounts of expansion during the inflation. The chaotic inflation model is named after this remarkable result.

For the hybrid model, there are two scalar fields, ϕ and ψ, involving in the inflation. As the field ϕ plays the important role in the slow roll, the other field ψ is the main character in the vacuum density. When the field ϕ rolls down below the critical value, ϕ_c, the field ψ is also unstable and rolls down from one vacuum with positive energy to the vacuum with zero energy.

In this section we can see that the inflationary model is very successful in both explaining the evolution of the universe and solving two major problems from the standard cosmology. However, a significant difficulty for this model is that there are still no physical concepts governing the mechanism of the inflation. In the next section we study one of the most promising cosmological model, namely string gas cosmology.

3.3 String Gas Cosmology

Since string theory emerged as one of the most impressive candidates for quantum gravity, this theory must explain any physical scenario which takes place at the Planck scale (at energy scale $\sim 10^{19}$ GeV) and can be verified by the experiments held at energy scale with the same order. Unfortunately, even the Large Hadron Collider (LHC) experiment, which will be in operation at CERN in November 2007, can provide the scenario of particle physics at energy scale approximately 10^4 GeV that is still very far from the Planck scale. However, there is still some hope to achieve this verification by applying string theory into the context of cosmology, especially in the early universe whose size is comparable to that of Planck scale. In other words, the cosmologists always encounter some unavoidable failure when they study any process taking place at very small scale that the quantum effect become relevant. This failure motivates the modification of standard cosmology with a theory of quantum gravity. String gas cosmology (SGC) pioneered by Brandenberger and Vafa [16] is an attempt in combining the ideas of standard cosmology with the essentials of string theory. The remarkable success of this model is that it can resolve the initial singularity problem which takes place at the beginning of the universe. Moreover, it also provides the reasonable explanation for the dimensionality of spacetime while the number of spacetime dimensions is set to four by hand in the standard cosmology .

3.3.1 Assumptions in String Gas Cosmology

In standard cosmology we assume the cosmological principle prior to the study of the evolution of the universe in order to reflect the remarkable properties of the universe and simplify the equations of motion. Similarly, in string gas cosmology three conditions are initially assumed before undergoing the evolution of the string gas universe. In this subsection we will state and discuss each assumption in detail.

The first assumption is that the string coupling is so weak that we do not take into account the back reaction of strings on the background. This simplifies thermodynamics of early universe in the case that we can ignore the effect of gravity on string gases.

The second assumption is that all nine spatial dimensions of the early universe were initially compactified in a shape of torus, T^9, with the size of Planck scale. We will see that from this view point we can provide the reasonable explanation for both expansion of the universe and the dimensionality of spacetime.

The third assumption is that the evolution of the universe is adiabatic. We will see in the next subsection that this assumption is acceptable for all types of superstring except the heterotic string at a certain compactification radius. Combining this assumption with the other assumptions, we can solve the initial singularity problem.

3.3.2 Toroidal Compactification and T-duality in SGC

It is essential to introduce a new symmetry and a new degree of freedom of string theory emerging from compactification of spatial dimensions. First, we will discuss a basic concept of compatification and indicate some important results from this mechanism. Eventually there are many kinds of compactifications, such as toroidal, $K3$, and orbifold; however, in this thesis we give emphasis on the simplest but quite plausible model, namely a toroidal compactification [25]. In this model we assume that the D-dimensional spacetime is separated into the (d_n+1)-dimensional non-compact spacetime $\mathbb{M}^{1+d_n}$ and the d-dimensional toroidal compact space T^d. Throughout the thesis, the components in D-dimensional spacetime are labelled by the indices $\mu, \nu, = 0, 1, \ldots, D-1$, the components of non-compact spacetime are labelled by the indices $\alpha, \beta = 0, 1, \ldots, d_n$ and the spatial components of non-compact spactime are labelled by $a, b = 1, \ldots, d_n$ (do not confuse to the indices a, b in the string frame). For the compact space, the indices of coordinates in an ordinary frame are denoted by $I, J = 1, \ldots, d$ and the indices of coordinates in an orthonormal frame are denoted by $i, j = 1, \ldots, d$.

As we mentioned in §2.3.2, the coordinate of toroidal compact space satisfies an equivalent relation

$$x^I \sim x^I + 2\pi \sum_{i=1}^{d} R^{(i)} n^i e_i^I, \quad ; i = 1, \ldots, d \quad ; n^i \in Z, \tag{3.44}$$

where $R^{(i)}$, n^i, and e_i^I are the compactfication radius, the integer that is called the winding number, and the lattice basis (vielbein) of the i^{th} direction of a torus, respectively. Next, we provide the basic concept of toroidal compactificaton in the context of string theory. For simplicity, in this subsection we choose the vielbein as $e_i^I = \delta_i^I$ and use the second assumption $(d_n = 0, d = D-1)$. As a result, the mass squared formula for type II superstring (and heterotic string) can be respectively expressed as

$$
\begin{aligned}
M^2 &= \sum_{I=1}^{d}(p_I)^2 + \sum_{I=1}^{d}(w^I)^2 + \frac{2}{\alpha'}(N+\tilde{N}-a), \\
&= \sum_{I=1}^{d}(\frac{m_I}{R^{(I)}})^2 + \sum_{I=1}^{d}(\frac{n^I R^{(I)}}{\alpha'})^2 + \frac{2}{\alpha'}(N+\tilde{N}-a)
\end{aligned} \tag{3.45}
$$

where $p_I = \frac{m_I}{R^{(I)}}$, m_I, and $w^I = \frac{n^I R^{(I)}}{\alpha'}$ are the momentum mode, the integer that is called the momentum number, and the winding mode of I^{th} compactified spatial dimension, respectively and the normal ordering constant a takes value 2 (1) for type II superstring (heterotic string). From the formula above, there are three types of degrees of freedom in the string spectrum, i.e., the oscillatory modes (N and $\tilde{N}$), the momentum mode (p_I), and the winding modes (w^I). We can see that the winding mode is a new degree of freedom arising from the compactification. It is obvious that the oscillatory modes, which can govern the types of particles according to the values of the number operators, do not depend on how large the radius is; therefore, these modes are quite irrelevant in this scenario. The momentum modes p_I, which take a responsibility for strings moving in the I^{th} compact direction, vary as $1/R^{(I)}$. On the other hand, the winding modes w^I, which represent strings wrapping around a torus in I^{th} direction with n^I round(s), are proportional to $R^{(I)}$. This means that the larger the radius, the lighter the momentum modes but the heavier the winding modes, and vice versa. We will see in the next subsection that these outstanding properties of momentum modes and winding modes play crucial roles in the expansion of the early universe.

Next, we will discuss a new symmetry which is believed to be one of many essential symmeties for both perturbative and non-perturbative string theories; it is T-duality (T denotes for Target space). Literally, from T-duality it appears that physics in small scale compared to the string scale (or Planck scale) is as same as physics in large scale with the interchange of definitions between the momentum modes and winding modes, and vice versa. Mathematically, the expressions of all physical quantities, such as the string mass spectrum, the scattering amplitude, must be invariant under the duality transformations

$$
\frac{R^{(I)}}{\sqrt{\alpha'}} \longrightarrow \frac{\sqrt{\alpha'}}{R^{(I)}}, \tag{3.46}
$$

or

$$
\ln(\frac{R^{(I)}}{\sqrt{\alpha'}}) \longrightarrow -\ln(\frac{R^{(I)}}{\sqrt{\alpha'}}), \quad n^I \longleftrightarrow m_I. \tag{3.47}
$$

We can see for example that the expressions of string mass spectrum for type II superstring or heterotic string (3.45) is invariant under the transformations above . It is very important to note that at $R^{(I)} = \sqrt{\alpha'}$, the duality transformations map a target space into itself; as a consequence, this compactification radius ($R^{(I)} = \sqrt{\alpha'}$) is called the self-dual radius. In the next subsection we will see that these new degrees of freedom and symmetry in string theory together play important roles in the evolution of the early universe and in solving the initial singularity problem emerging from the standard cosmology.

3.3.3 String Thermodynamics of The Early Universe

It is the fact that the main purpose of the first assumption is to simplify a study of thermodynamics in theory of gravity. From this assumption we can ignore the gravitational effect contributed by the non-zero expectation value for energy density due to the strings on the background. In other words, from the first assumption we choose the string background to be classical background instead of the quantum background in this model for simplicity. In this subsection we will discuss thermodynamics of the early universe by using the microcanonical ensemble. It will be shown that this ensemble is appropriate and reasonable for explaining the mechanism of string gas in the early universe in the time that a quantum description for this scenario is still under development. We will also see the way in which this approach solves the initial singularity problem and offers the explanation for dimensionality of spacetime.

We begin this subsection by demonstrating that the microcanonical ensemble is more suitable than canonical ensemble for describing thermodynamics of strings in the early universe. We can see this by investigating the number of partitions at the asymptotic limit (large number of oscillation $N \longrightarrow \infty$) in the canonical ensemble. For simplicity we consider the case that there are only bosonic strings in the uncompactified spacetime with critical dimension; as a result, the number of partitions, $p(E)$, can be determined by the Hardy-Ramanujan method and can be expressed as

$$p(E) = \frac{1}{\sqrt{2}}\alpha'^{-\frac{27}{4}}E^{-\frac{27}{2}}\exp(4\pi\sqrt{\alpha'}E). \tag{3.48}$$

Then we can determine the entropy and temperature of bosonic strings as

$$S(E) = k\ln p(E) = k(4\pi\sqrt{\alpha'}E - \frac{27}{2}\ln(E) + \ln(\frac{1}{\sqrt{2}}\alpha'^{-\frac{27}{4}})), \tag{3.49}$$

$$\frac{1}{T(E)} = \frac{\partial S(E)}{\partial E} = 4\pi k\sqrt{\alpha'} - \frac{27k}{2E}. \tag{3.50}$$

It is very important to note that while the energy of string increases as much as possible, it turns out that the temperature tends to be constant and we define such a temperature as the Hagedorn temperature, T_H,

$$T_H = (\frac{\partial S(E)}{\partial E}|_{E\longrightarrow\infty})^{-1} = \frac{1}{4\pi k\sqrt{\alpha'}}. \tag{3.51}$$

In other words, as we increase the energy of string, the temperature of string increases corresponding to an increase of energy in such a way that the temperature approaches to the Hagedorn temperature T_H, the maximum allowed temperature in context of string theory. However, it turns out that the partition function, $Z(V,T)_{str}$, which can be written as

$$Z_{str} = \sum_i e^{-\beta E_i} \approx Z_0 + \frac{2^{11}}{\pi} V(kTkT_H)^{\frac{25}{2}}(\frac{T}{T_H - T})e^{-4\pi\sqrt{\alpha'}E_0(\frac{T_H}{T}-1)}, \tag{3.52}$$

where V is a volume of transverse dimensions, Z_0 and E_0 are partition function and the energy under which are suitable for the Hardy-Ramanujan approximation, diverges at $T = T_H$. This does not mean that physics is invalid at the Hagedorn temperature. We can see that all physical quantities, i.e. the energy density and specific heat, are still finite at this temperature. The situation changes drastically when we turn to thermodynamics of strings in compactified spacetime. It turns out that if number of uncompactified dimensions is less than or equal to three, the energy density diverges at the Hagedorn temperature. This comes from the fact that the aspect of winding modes, which are new degrees of freedom in the context of string compactification, is not appropriate in the infinite volume. The larger the finite volume of universe is, the more energy the winding mode acquires but not infinite. This implies that the universe should start with the finite volume. Moreover, we see that we cannot trust the canonical ensemble when the temperature reaches the Hagedorn temperature because of large energy fluctuation. The situation is very similar for the superstring thermodynamics.

This implies that it is more appropriate to use the microcanonical ensemble, which is more fundamental than the canonical approach, to study string thermodynamics of compactified spacetime in the vicinity of the Hagedorn temperature. However, it appears that if some of spatial dimensions are uncompactified, the specific heat then becomes negative and leads to the unacceptable description for strings thermodynamics. The situation is quite different when all spatial dimensions are all compactified. In this approach the number of partition for both type II superstring and heterotic string at high energy limit and the entropy can be expressed as

$$dp(E) = \frac{\exp(\beta_H E)}{E} dE, \quad (3.53)$$

$$S = \beta_H E - \frac{c_1}{E} + c_2, \quad (3.54)$$

where $\beta_H = kT_H$, c_1 and c_2 are some constants. It turns out that the specific heat determined by this approach is positive. This means that the microcanonical ensemble is appropriate for describing string thermodynamics in compact spacetime.

Now we undergo a study of string thermodynamics in the early universe from the view point of SGC. We will see that this model offers the qualitative way to solve the initial singularity problem. From the previous subsection we introduce T-duality in string theory. As a consequence of this duality, the temperature must be symmetric around the self-dual radius ($R^{(I)} = \sqrt{\alpha'}$). We will explain this scenario qualitatively. For large $R^{(I)}$, it turns out that T decreases as $1/R^{(I)}$ similar to the standard cosmology since the winding mode becomes irrelevant. As the radius approaches the self-dual radius ($R^{(I)} \longrightarrow \sqrt{\alpha'}$), the temperature does not increase in the same way as in the standard model but it tends to the limiting temperature, Hagedorn temperature T_H. The curve in the vicinity of the Hagedorn temperature is so flat that this region can be called a plateau. It turns out that the temperature of the plateau is $T_{plateau} = T_H - \frac{c}{S^2}$, where c is a constant. Moreover, it is found that the width of the plateau, $\Delta R^{(I)}_{plateau}$, is also proportional to the entropy S as $\Delta R^{(I)}_{plateau} \propto \frac{S^{\frac{1}{9}}}{T_H}$.

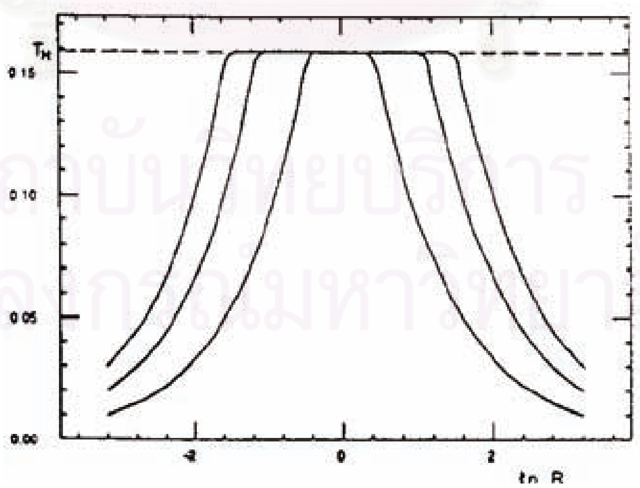


Figure 3.1: The temperature of Plateau approaches to the Hagedorn temperature. The higher entropy, the larger the plateau. The temperature is symmetric around the self-dual radius, $\ln R = 0$ ($\alpha' = 1$). From [16].

Thereafter, when $R^{(I)}$ becomes much smaller than the self-dual radius, it is very striking that the temperature again decreases as $1/R^{(I)}$. This occurs because the winding modes which stay dominantly in thermal equilibrium become lighter and lighter. As a result, the adiabaticity dictates the temperature to decrease as well as the winding modes. This scenario is illustrated in the Figure 3.1.

Before we continue our study, it is worth noting that the figure above is normally valid for type II superstring theories but it requires little modification for heterotic string theory. We can see from [16] that at the self-dual radius the heterotic strings acquire the enhanced gauge symmetry in the same way as they possess the non-Abelian gauges for which we discussed in §2.3. In this case we assume that all nine spatial dimensions are compactified as a nine-dimensional torus; therefore, the enhanced symmetry for the heterotic strings is $SU(2)^9$. As a consequence, at the self-dual radius ($R^{(I)} = \sqrt{\alpha'}$), there exist in the string spectrum the extra massless states contributed by the compactified heterotic strings which have $|p| = |w| = \frac{1}{\alpha'}$ (or equivalently $|m_I| = |n^I| = 1$ for only one index I and $m_J = n^J = 0$ for other indices J) and these massless states carry so little energy and temperature. Therefore, the adiabacity implies that the temperature must decrease to zero at this point and this feature is quite unacceptable. A reasonable solution to this subtlety is that we will relax the assumption for adiabaticity and we instead assume that the energy and the temperature of heterotic strings are constant when the radius R passes the self-dual radius. This modification makes the curves for heterotic strings are similar to those for type II superstrings. However, it also yields the differences in the number of degenerate states and in the Hagedorn temperature between the heterotic strings and type II superstrings.

Next, we will investigate the interpretation of the scenario we mentioned above. It is the fact that in the compact space the momentum modes and the winding modes can interchange to each other when we cross from large $R^{(I)}$ to small $R^{(I)}$ or vice versa due to T-duality. We can see that when $R^{(I)}$ is larger than the self-dual radius ($R^{(I)} > \sqrt{\alpha'}$), we use the momentum mode ($p_I = \frac{m_I}{R^{(I)}}$) which are light in this region to create the light particle in order to define the position. On the other hand, when $R^{(I)}$ become smaller than the string scale ($R^{(I)} < \sqrt{\alpha'}$), the momentum modes become heavier while the winding modes ($w^I = \frac{n^I R^{(I)}}{\alpha'}$) become lighter. Therefore, it is easier to create light by using the winding modes instead of the momentum modes in this region. However, we can interpret the winding modes in this scenario as the momentum modes

$$\tilde{p}_I = w^I \longrightarrow \frac{\tilde{m}_I}{\tilde{R}^{(I)}} = \frac{n^I R^{(I)}}{\alpha'}. \tag{3.55}$$

As a result, we realize that we measure the position by using photon with the momentum number $\tilde{m}_I = n^I$ and thus we are living in the universe with the radius $\tilde{R^{(I)}} = \frac{\alpha'}{R^{(I)}}$ which also appears larger than the self-dual radius. In other words, all observers who use some certain way to measure the position realize that the self-dual radius ($\sqrt{\alpha'}$) is the smallest effective length that they can observe and they cannot distinguish whether they are living in the universe with radius R or in the universe with radius $\tilde{R} = \frac{\alpha'}{R}$. This means that we can never encounter the state for which the universe become a point ($R^{(I)} = 0$). As a result, we can conclude that SGC provides a satisfactory exit for the initial singularity problem because of the existence of T-duality and winding modes.

Next, we will explore the way that SGC can explain why we live in four-dimensional spacetime. In other words, we will show that the universe with nine compact spatial dimensions and one time direction ($\mathbb{R}^1 \times \mathrm{T}^9$) from the starting point evolves in such a way that one time direction combines with three spatial dimensions which grow much larger than the Planck scale but the other six dimensions are still fixed at Planck scale ($\mathbb{M}^4 \times \mathrm{T}^6$). We start by considering one of the most important properties of winding modes. It is the fact that the winding modes contribute to the negative pressure; as a result, these modes prevent the universe from expansion. On the other hand, the momentum modes contribute to the positive pressure and allow the universe to expand. We can see this obviously by examining the mass spectrum of superstring or heterotic string. The larger the universe expands (the larger the radius is), the more energy the winding modes require. Therefore, the expansion can occur if and only if the number of winding modes decreases. This means that the universe continues expanding as long as the winding modes stay in thermal equilibrium and thus annihilate to their anti-particles, namely anti-winding modes. In this context we give emphasis on the number of spacetime dimensions in which the winding modes can maintain thermal equilibrium. Up to this point it is clear that the winding modes do not maintain thermal equilibrium certainly if all nine compact dimensions expand at the early universe. Next, we investigate the condition for which the winding modes can stay in thermal equilibrium as the universe expands. It is the fact that two strings with equal winding number but opposite signs (winding mode and anti-winding mode) can interact to each other if their worldsheets come to intersect. Since a worldsheet of one string has two dimensions, this means that the largest number of expanding spacetime in which two worldsheet can intersect certainly is $2+2 = 4$. As a result, the most possible way for which the winding and anti-winding modes can interact to each other is that the winding modes in three spatial compact directions must be annihilated and only three spatial dimensions are expanding along the expansion of the universe. This is a reasonable explanation from SGC for the dimensionality of spacetime. However, for this scenario there is still a major problem which concerns how we know that the size of other six dimensions are really fixed at the Planck length in later time in order to avoid from the observation at the present time. This difficulty is known as the **moduli stabilization problem** for the late-time universe. We will discuss the numerical analysis on this problem which was done by Watson and Brandenberger [18] in §3.3.5.

3.3.4 Evolution of The Late-time Universe

In this subsection we study of the evolution of the late-time universe without the flux ($B_{\mu\nu} = 0$) [17, 18]. Since the energy of strings in the late-time universe is very small compared to that in the early universe, it is reasonable to use the low-energy effective action (2.256) in ten-dimensional spacetime ($D = 10$) without flux

$$S_0 = -\frac{1}{2k_D^2} \int d^D x \sqrt{-G} e^{-2\phi} (R + 4\nabla_\mu \phi \nabla^\mu \phi). \tag{3.56}$$

It is important to note that it is difficult to consider T-duality in this form. However, we will see in the next chapter that we can express this action in the particular form which makes T-duality more transparent.

Because of the presence of string gas, we introduce the string action for matter, S_M, as

$$S_M = \int d^D x \sqrt{-G} \rho, \tag{3.57}$$

where ρ is the total energy density of string gas. Varying the total action, $S = S_0 + S_M$, with respect to the inverse metric $G^{\mu\nu}$, and the dilaton ϕ, respectively, we obtain the equations of motion

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi = k_D^2 e^{2\phi} T_{\mu\nu}, \tag{3.58}$$

$$R - 4\nabla_\mu \phi \nabla^\mu \phi + 4\nabla_\mu \nabla^\mu \phi = 0. \tag{3.59}$$

From the second assumption ($d_n = 0, d = 9$), we introduce the metric ansatz as

$$ds^2 = -dt^2 + a^2(t) \sum_{I=1}^{3} dx^{I2} + b^2(t) \sum_{J=1}^{6} dy^{J2}, \tag{3.60}$$

where $a(t) \equiv \frac{R^{(I)}(t)}{\sqrt{\alpha'}} = e^{\lambda(t)}$ ($b(t) \equiv \frac{R^{(J)}(t)}{\sqrt{\alpha'}} = e^{\nu(t)}$) and dx^I (dy^J) are the scale factor and the dual basis of non-compact (compact) space, respectively. We separate the scale factors into two parts in order that we will use the first part $a(t)$ to represent the scale factor of three expanding spatial dimensions and the second part $b(t)$ to represent that of the six extra dimensions. As a consequence of the second assumption, not only the scale factor become a function of time, but also do the dilaton $\phi = \phi(t)$ and the string coupling $g_s = e^{2\phi(t)}$.

Since we assume that the string gas behaves like the perfect fluid, it can be simplified into the familiar form

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & & \\ & p_\lambda & \\ & & p_\nu \end{pmatrix}, \tag{3.61}$$

where

$$\begin{aligned} \rho &= \frac{E_{total}}{V}, && (3.62)\\ p_\lambda &= -\frac{1}{V}\frac{\partial E_{total}}{\partial \lambda}, && (3.63)\\ p_\nu &= -\frac{1}{V}\frac{\partial E_{total}}{\partial \nu}. && (3.64) \end{aligned}$$

Here, we define E_{total}, p_λ and p_ν as the total energy contributed by all numbers of string from all modes, the pressure densities in non-compact and compact directions, respectively. Substituting the metric ansatz into (3.58) and (3.59), we obtain the equations of motion

$$\begin{aligned} -3\ddot{\lambda} - 3\dot{\lambda}^2 - 6\ddot{\nu} - 6\dot{\nu}^2 + 2\ddot{\phi} &= k_{10}^2 e^{2\phi}\rho, && (3.65)\\ \ddot{\lambda} + 3\dot{\lambda}^2 + 6\dot{\lambda}\dot{\nu} - 2\dot{\lambda}\dot{\phi} &= k_{10}^2 e^{2\phi} p_\lambda, && (3.66)\\ \ddot{\nu} + 6\dot{\nu}^2 + 3\dot{\lambda}\dot{\nu} - 2\dot{\nu}\dot{\phi} &= k_{10}^2 e^{2\phi} p_\nu, && (3.67)\\ 3\ddot{\lambda} + 6\dot{\lambda}^2 + 6\ddot{\nu} + 21\dot{\nu}^2 + 18\dot{\lambda}\dot{\nu} + 2\dot{\phi}^2 - 2\ddot{\phi} - 6\dot{\lambda}\dot{\phi} - 12\dot{\nu}\dot{\phi} &= 0. && (3.68) \end{aligned}$$

It is interesting to examine these equations in the case that the dilaton is constant and the extra dimensions are ignored ($\nu = 0$). As a result, we obtain the equations analogous to (3.11) and (3.13)

$$H^2 \equiv \dot{\lambda}^2 = \frac{8\pi G_{10}}{3}\rho, \tag{3.69}$$

$$\ddot{\lambda} + \dot{\lambda}^2 = -\frac{4\pi G_{10}}{3}(\rho + 3p_\lambda), \tag{3.70}$$

with a constraint equation

$$R^{(4)} = \ddot{\lambda} + 2\dot{\lambda}^2 = 0, \tag{3.71}$$

where $R^{(4)}$ is the Ricci tensor in four-dimensional spacetime and the gravitational coupling is restored by using the relation $k_{10}^2 = 8\pi G_{10} = \frac{8\pi}{m_{P(10)}^2} = \frac{1}{M_{P(10)}^2}$. It is important to note that there are very slight differences between the expressions above and those in standard cosmology that are the extra term $\dot{\lambda}^2$ in (3.70) and the ten-dimensional gravitational coupling G_{10} instead of the four-dimensional coupling G. However, we can obtain the relation $\rho - 3p = 0$ which implies that the application of general relativity in cosmology can be considered as the string low-energy effective theory and then the evolution of the early universe starts in the radiation-dominated period. For the constraint equation (3.71), it is obvious that this equation verifies the conformal invariance in the context of string theory ($T^{\mu}_{\mu} = 0$).

Since all nine spatial dimensions are compactified, the string mass squared is equal to the energy squared. Then we can express respectively the energy squared of type II superstring or the heterotic strings from equation (3.45) as

$$M^2 = E^2 = E_{osc}^2 + \sum_I (E_p^{(I)})^2 + \sum_I (E_w^{(I)})^2, \tag{3.72}$$

where

$$E_{osc} = \sqrt{\frac{2}{\alpha'}(N + \tilde{N} - a)}, \tag{3.73}$$

and

$$E_p^{(I)} = p_I = \frac{m_I}{R^{(I)}} = \frac{1}{\sqrt{\alpha'}} m_I \left(\frac{R^{(I)}}{\sqrt{\alpha'}}\right)^{-1}, \tag{3.74}$$

$$E_w^{(I)} = w^I = \frac{n^I R^{(I)}}{\alpha'} = \frac{1}{\sqrt{\alpha'}} n^I \left(\frac{R^{(I)}}{\sqrt{\alpha'}}\right). \tag{3.75}$$

In the expression above E_{osc} is the energy contributed by the oscillatory mode while $E_p^{(I)}$ and $E_w^{(I)}$ are contributed by the momentum mode, and the winding mode for the I^{th} spatial direction, respectively. It is obvious that the energy contributed by the oscillatory mode does not depend on the scale factor $\frac{R^{(I)}}{\sqrt{\alpha'}}$; therefore, we can ignore the energy from this mode when we consider the evolution of the universe. Thereafter, we define $N_{(|n^I|)}$ as the total number of winding strings which have the winding number $|n^I|$ in the I^{th} direction. In other words, $N_{(|n^I|)}$ is the sum of the winding and anti-winding strings which wrap a torus along the I^{th} direction with $|n^I|$ number of round ($+$ for winding mode and $-$ for anti-winding modes). In the similar way we define the total number of strings with momentum modes as $M_{(|m_I|)}$. Consequently, we can express the total energy, E_{total}, contributed by momentum and winding modes in all directions as

$$
\begin{aligned}
E_{total} &= \sum_I \sum_{|m_I|} M_{(|m_I|)} |E_p^{(I)}| + \sum_I \sum_{|n^I|} N_{(|n^I|)} |E_w^{(I)}| && (3.76)\\
&= \frac{1}{\sqrt{\alpha'}} \sum_I (M^{(I)} (\frac{R^{(I)}}{\sqrt{\alpha'}})^{-1} + N^{(I)} \frac{R^{(I)}}{\sqrt{\alpha'}}), && (3.77)
\end{aligned}
$$

where $M^{(I)} \equiv \sum_{|m_I|} M_{(|m_I|)} |m_I|$ and $N^{(I)} \equiv \sum_{|n^I|} N_{(|n^I|)} |n^I|$. It is important to note that we can assume that the three spatial dimensions are isotropic and so do the other six dimensions. As a consequence, we can set $M^{(I)} = M^{(3)}$ and $N^{(I)} = N^{(3)}$; $I = 1, 2, 3$ for three spatial dimensions with the scale factor $a(t)$. Similarly, we choose $M^{(J)} = M^{(6)}$ and $N^{(J)} = N^{(6)}$; $J = 1, \ldots, 6$ for the other six spatial dimensions with the scale factor $b(t)$. If the total volume can be expressed as $V \equiv va^3(t)b^6(t) = ve^{3\lambda+6\nu}$ where v is the comoving spatial volume with dimension $[L]^{D-1}$, the energy density ρ and the pressure p_λ and p_ν can be calculated by using equations (3.62)-(3.64) and can be written as

$$
\begin{aligned}
\rho &= \frac{1}{v\sqrt{\alpha'}} e^{-3\lambda-6\nu} (3M^{(3)} e^{-\lambda} + 3N^{(3)} e^{\lambda} + 6M^{(6)} e^{-\nu} + 6N^{(6)} e^{\nu}), && (3.78)\\
p_\lambda &= \frac{1}{v\sqrt{\alpha'}} e^{-3\lambda-6\nu} (M^{(3)} e^{-\lambda} - N^{(3)} e^{\lambda}), && (3.79)\\
p_\nu &= \frac{1}{v\sqrt{\alpha'}} e^{-3\lambda-6\nu} (M^{(6)} e^{-\nu} - N^{(6)} e^{\nu}). && (3.80)
\end{aligned}
$$

As a result, the equations of motion (3.65)-(3.68) become

$$
\begin{aligned}
-3\ddot{\lambda} - 6\ddot{\nu} - 3\dot{\lambda}^2 - 6\dot{\nu}^2 + 2\ddot{\phi} = \frac{e^{2\phi-3\lambda-6\nu}}{v\sqrt{\alpha'}} (3M^{(3)} e^{-\lambda} + 3N^{(3)} e^{\lambda} &\\
+6M^{(6)} e^{(-\nu)} + 6N^{(6)} e^{\nu}), && (3.81)\\
\ddot{\lambda} + 3\dot{\lambda}^2 + 6\dot{\lambda}\dot{\nu} - 2\dot{\lambda}\dot{\phi} = \frac{e^{2\phi-3\lambda-6\nu}}{v\sqrt{\alpha'}} (M^{(3)} e^{-\lambda} - N^{(3)} e^{\lambda}), && (3.82)\\
\ddot{\nu} + 6\dot{\nu}^2 + 3\dot{\lambda}\dot{\nu} - 2\dot{\nu}\dot{\phi} = \frac{e^{2\phi-3\lambda-6\nu}}{v\sqrt{\alpha'}} (M^{(6)} e^{-\nu} - N^{(6)} e^{\nu}), && (3.83)\\
4\ddot{\phi} - 6\ddot{\lambda} - 12\ddot{\nu} - 4\dot{\phi}^2 + 4\dot{\phi}(3\dot{\lambda} + 6\dot{\nu}) - (3\dot{\lambda} + 6\dot{\nu})^2 - 3\dot{\lambda}^2 - 6\dot{\nu}^2 = 0. && (3.84)
\end{aligned}
$$

Here, we set $k_{10} = 1$ and work in the unit of the reduced Planck mass. It is very important to note that the equations of motion (3.81) - (3.84) are obviously invariant under T-duality transformation

$$
\lambda(t) \longrightarrow -\lambda(t), \quad \nu(t) \longrightarrow -\nu(t), \quad \phi(t) \longrightarrow \phi(t) - 3\lambda(t) - 6\nu(t), \quad M^{(I)} \leftrightarrow N^{(I)}. \quad (3.85)
$$

These equations are too complicated to be solved in the form of general solutions; however, we can investigate how the scale factors $a(t)$, $b(t)$ and the string coupling g_s evolve with time numerically. In the next subsection we will use the results from §3.3.3 and §3.3.4 to solve the moduli stabilization problem for the late-time universe in a quantitative way. We also determine the conditions for which the scale factor $a(t)$ of three dimensions grows large continuously as the scale factor $b(t)$ are still stabilized at the self-dual radius.

3.3.5 Moduli Stabilization Problem

As we mentioned in §3.3.3, it is essential for SGC to demonstrate that six extra dimensions must be stabilized at the self-dual radius during the expansion of the late-time universe in order that we cannot detect these dimensions nowadays. We can see that the winding modes contribute the negative pressure from (3.79) and (3.80) and lead to the negative effective potential in the equations of motion for λ in (3.82) and for ν in (3.83). This means that as the universe expands, the winding modes become more massive and then halt the expansion of the universe. However, the expansion of the universe can occur if and only if the string winding modes in three dimensions are all annihilated according to the Brandenberger and Vafa mechanism mentioned in §3.3.3. On the other hand, the momentum modes, which are dual to the winding modes, contribute the positive pressure and become heavier as the universe collapses, and prevent the universe from collapsing to the singularity.

In our simulation we first set $N^{(3)} = 0$ or equivalently $N_{(|n^I|)} = 0 \quad ; I = 1, 2, 3$. From the first assumption we set the initial value of string coupling g_s to become very small and to change very slowly. From the second assumption the scale factor of both extended and extra dimensions starts at self-dual radius ($a(0) = b(0) = 1$); as a result, we obtain $\lambda(0) = \nu(0) = 0$. In addition, we assume that the expansion rate of the scale factor in six extra dimensions is initially equal to zero ($\dot{b}(0) = 0$) and this yields the result $\dot{\nu}(0) = \ddot{\nu}(0) = 0$. From the assumptions above, we obtain the important condition from equation (3.87) as

$$N^{(6)} = M^{(6)}, \tag{3.86}$$

which is always true at the self-dual radius. Thereafter, we use the equations of motion (3.81) - (3.84) to simulate the evolution of scale factors $a(t) = e^{\lambda(t)}$, $b(t) = e^{\nu(t)}$ and the string coupling $g_s = e^{2\phi}$ and we obtain the results that are shown in the Figures 3.2. We can see that the scale factor of three large dimensions grows continuously as the scale factor of the extra dimensions is still fixed at the self-dual radius and the string coupling is still very weak. However, we cannot investigate whether the momentum modes and winding modes of the extra dimensions really drive the scale factor $b(t)$ toward the self-dual radius. Therefore, we assume that the scale factors $a(t)$ and $b(t)$ starts at the point slightly away from the self-dual point. However, it is not so far that the relation (3.86) is still valid. The evolutions of the scale factors $a(t)$, $b(t)$, and the string coupling g_s are shown in Figure 3.3. From Figure3.3 we can see that the scale factor $b(t)$ oscillates around the self-dual point due to the effect of momentum modes and the winding modes. The string coupling and the scale factor $a(t)$ together play the role in damping the oscillation as we can see from (3.83). In other words, the large scale factor $a(t)$ and the very weak string coupling damp the oscillation of the scale factor $b(t)$.

From this analysis, we can conclude that the expansion of the universe can really occur when all winding modes in three dimensions are all annihilated and the existence of both winding modes and momentum modes stabilize the scale factor of the extra dimensions to the self-dual point. This scenario takes place in the regime of which the string coupling is very weak during the consideration. We can see that SGC is successful in the explanations for the dimensionality of spacetime and for the moduli stabilization for the late-time universe. However, it is more appreciate if there is a systematic way to examine this problem qualitatively.

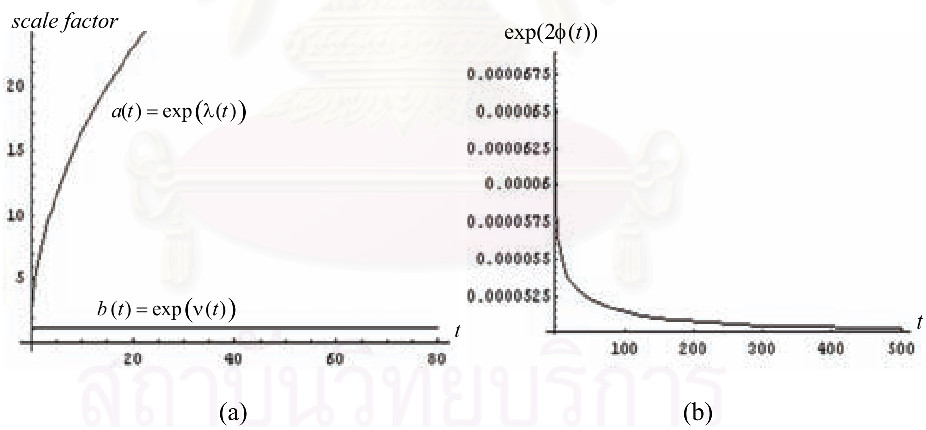


Figure 3.2: Figure 3.2(a) shows that the scale factor of non-compact dimensions $a(t)$ grows larger while the scale factor of compact dimensions $b(t)$ is stabilized at the self-dual radius. Figure 3.2(b) shows that the string coupling is very weak.

3.3.6 Further Development and Striking Problems

In the last subsection we will discuss some outstanding problems and the further development in SGC briefly.

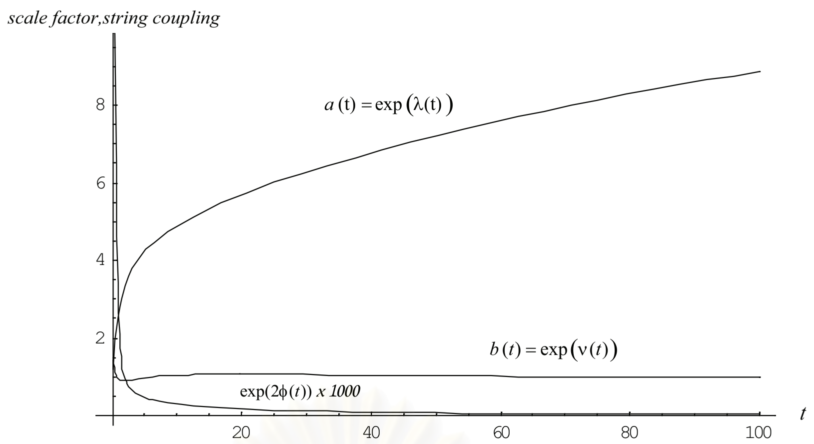


Figure 3.3: Figure 3.3 shows that the scale factor of compact dimensions $b(t)$ oscillates around the self-dual with the decreasing amplitude as the scale factor of non-compact dimensions grows larger and the string coupling runs slowly to weak coupling.

The first issue emerges from the fact that it is very important for SGC to provide the explanation of the evolution of the universe for matter-dominated period. This means that the solution for the moduli stabilzation problem must be provided in this era. If it can be possible, the phase transition of the universe and supersymmetry breaking should be explained in the context of SGC.

Second, it is essential for the string gas cosmologists to obtain the mechanism for explaining the inflation or the mechanism which is equivalent to the inflation in the context of SGC. As a consequence of this issue, we can solve the other two crucial problems, i.e. the horizon and the flatness problems.

The last issue is that we require the qualitative analysis for the moduli stabilization for the late-time universe and also for the the universe consisting of gas of heterotic strings.

CHAPTER IV

MODULI STABILIZATION IN $O(d,d)$-COVARIANT FORMALISM

In the previous chapter we mentioned the moduli stabilization problem that concerned the stability of size and shape of the compact space T^6 for the late-time universe and solved it numerically. In this chapter we will resolve the moduli stabilization problem by using another approach, known as the $O(d,d)$-covariant formalism. This approach arose from the discovery of the $O(d,d)$ symmetry in the space of the string background. This symmetry reveals obviously when one studies the toroidal compactification by using the concept of compactification lattice proposed by Narain [25, 26]. Therefore, in the first part of this chapter we will follow the Narain's idea of the compactification lattice for the d-dimensional toroidal background. The space of string background and its symmetry are then studied explicitly. Thereafter, it will be shown that the low-energy effective action for the late-time string gas universe can be expressed in the $O(d,d)$-covariant form. We then end up this section by analyzing the moduli stabilization problem qualitatively in the context of $O(d,d)$-covariant string cosmology.

4.1 $O(d,d)$ Symmetry of The Moduli Space of d-dimensional Toroidal Compactification

In perturbative string theory, each background (solution) is provided by one conformal field theory (CFT) and a deformation of the background is described by a set of operators in the spectrum, known as the truly marginal operators. The space of all backgrounds connected by the deformation is called the moduli space. In general, if a CFT possesses a Lagrangian density $\mathcal{L}$ that is exactly solvable, there can be a neighbor CFT with a Lagrangian density $\mathcal{L}'$ expanded as

$$\mathcal{L}' = \mathcal{L} + \sum_{i}^{d} g_i f_i(\tau, \sigma), \tag{4.1}$$

where g_i are suitable couplings and $f_i(\tau, \sigma)$ are the truly marginal operators. This fact implies that one can use the couplings g_i as basis vectors of the local neighborhood of the background corresponding to the CFT with the Lagrangian density $\mathcal{L}$ and the number, d, of marginal operators $f_i(\tau, \sigma)$ is locally the number of dimension of the moduli space. It is important to note that in some cases the local neighborhood of the background of the CFT with $\mathcal{L}$ can be generated by a certain continuous group $\mathcal{G}$ of $\mathcal{L}$. Furthermore, there exists a subgroup $\mathcal{G}_d$ of $\mathcal{G}$ which leaves the spectrum invariant and is considered as a symmetry of the physical theories. This means that one can obtain another CFT by applying an element $g \in \mathcal{G}$ to one CFT and these two CFTs are physically equivalent when $g \in \mathcal{G}_d$.

In our case the couplings g_i correspond to the allowed target space backgrounds which are d-dimensional torus and are generally collected into the background metric G_{IJ}, the Kalb-Ramond field B_{IJ}, and the dilaton ϕ. The continuous group $\mathcal{G}$ which generates the moduli space is $O(d, d; \mathbb{R})$ while its subgroup $O(d, d; \mathbb{Z})$ plays a role of a symmetry for physically equivalent theories. We then study the T-duality which is the element of the $O(d, d; \mathbb{Z})$ group explicitly.

4.1.1 Compactification Lattice on d-dimensional Torus

We start by supposing that we live in the spacetime with critical dimension D which can be separated into the $(1 + d_n)$-dimensional non-compact spacetime and the d-dimensional compact space. Remember that the components in D-dimensional spacetime are labelled by the indices $\mu, \nu, = 0, 1, \ldots, D-1$, the components of non-compact spacetime are labelled by the indices $\alpha, \beta = 0, 1, \ldots, d_n$ and the spatial components of non-compact spactime are labelled by $a, b = 1, \ldots, d_n$ (not confuse to the indices a, b of coordinates τ, σ in the string frame). For the compact space, the indices of coordinates in an ordinary frame are denoted by $I, J = 1, \ldots, d$ and the indices of coordinates in an orthonormal frame are denoted by $i, j = 1, \ldots, d$. Similar to the previous chapter, the compact space is assumed to be compactified as T^d. Thus, we can use the worldsheet action (2.251) as the worldsheet action, S_{BG}, for bosonic part of type-II superstrings or heterotic strings in the d-dimensional toroidal background

$$S_{BG} = -\frac{1}{4\pi\alpha'} \int d^2\sigma (G_{IJ}\eta^{ab}\partial_a X^I \partial_b X^J + B_{IJ}\epsilon^{ab}\partial_a X^I \partial_b X^J - \frac{1}{2}\phi R), \qquad (4.2)$$

where X^I is the compactified coordinate which satisfies the periodic condition

$$X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + 2\pi\alpha' w^I \quad ; I = 1, \ldots, d. \qquad (4.3)$$

Here, the winding mode w^I, which represents the string wrapping around the compact coordinate X^I, is defined by

$$w^I \equiv \sum_{i=1}^{d} \frac{n^i R^{(i)}}{\alpha'} e_i^I, \tag{4.4}$$

where the vielbeins e_i^I are the basis vectors of the compactification lattice Λ^d, $R^{(i)}$ and n^i are the radius and the number of wound for the compactification in the i-direction, respectively. It is worth noting that the total action is, in fact, the sum of the worldsheet action for the non-compact spacetime and the worldsheet action for the compact space.

It is the fact that the number of truly marginal operators for a d-dimensional toroidal background is d^2. The d^2 truly marginal operators can be separated into $d(d+1)/2$ operators from

$$G_{IJ}\eta^{ab}\partial_a X^I \partial_b X^J, \tag{4.5}$$

and $d(d-1)/2$ operators from

$$B_{IJ}\epsilon^{ab}\partial_a X^I \partial_b X^J. \tag{4.6}$$

The dilaton term $\frac{1}{2}\phi R$ does not play any role in describing the string background, so we neglect this term for convenience. At this point we will introduce the field that contains those d^2 truly marginal operators, namely the background matrix, E,

$$E_{IJ} \;=\; G_{IJ} + B_{IJ}. \tag{4.7}$$

It is easy to show that the equations of motion take the same forms as those of uncompactified coordinates

$$(\partial_\tau^2 - \partial_\sigma^2)X^I = 0. \tag{4.8}$$

From the periodic condition (4.3) and the equations of motion (4.8), it is obvious that X^I can be separated into the left-moving and right-moving coordinates expressed as

$$X_L^I(\tau+\sigma) \;=\; \frac{1}{2}x_L^I + \sqrt{2\alpha'}\alpha_0^I(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^I}{n}e^{-2ni(\tau+\sigma)}, \tag{4.9}$$

$$X_R^I(\tau-\sigma) \;=\; \frac{1}{2}x_R^I + \sqrt{2\alpha'}\tilde{\alpha}_0^I(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^I}{n}e^{-2ni(\tau-\sigma)}. \tag{4.10}$$

Thus, the general solution of $X^I(\tau,\sigma)$ reads

$$X^I(\tau,\sigma) = x^I + \sqrt{2\alpha'}(\alpha_0^I + \tilde{\alpha}_0^I)\tau + \sqrt{2\alpha'}(\alpha_0^I - \tilde{\alpha}_0^I)\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}(\frac{\alpha_n^I}{n}e^{-2ni(\tau+\sigma)} + \frac{\tilde{\alpha}_n^I}{n}e^{-2ni(\tau-\sigma)}), \quad (4.11)$$

where $x^I \equiv \frac{1}{2}(x_L^I + x_R^I)$ is the string center-of-mass coordinate. The momenta conjugate along τ-and σ-directions are denoted by $P_{I\tau}$ and $P_{I\sigma}$ and are expressed as

$$\begin{aligned} P_{\tau I} &= \frac{1}{2\pi\alpha'}(G_{IJ}\partial_\tau X^J + B_{IJ}\partial_\sigma X^J) \\ &= \frac{1}{2\pi\alpha'}\{\sqrt{2\alpha'}E_{IJ}\alpha_0^J + \sqrt{\frac{\alpha'}{2}}E_{IJ}\sum_{n\neq 0}\alpha_n^J e^{-2ni(\tau+\sigma)} \\ &\quad +\sqrt{2\alpha'}E_{IJ}^T\tilde{\alpha}_0^J + \sqrt{\frac{\alpha'}{2}}E_{IJ}^T\sum_{n\neq 0}\tilde{\alpha}_n^J e^{-2ni(\tau-\sigma)}\}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} P_{\sigma I} &= -\frac{1}{2\pi\alpha'}(G_{IJ}\partial_\sigma X^J + B_{IJ}\partial_\tau X^J) \\ &= \frac{1}{2\pi\alpha'}\{\sqrt{2\alpha'}E_{IJ}^T\alpha_0^J + \sqrt{\frac{\alpha'}{2}}E_{IJ}^T\sum_{n\neq 0}\alpha_n^J e^{-2ni(\tau+\sigma)} \\ &\quad +\sqrt{2\alpha'}E_{IJ}\tilde{\alpha}_0^J + \sqrt{\frac{\alpha'}{2}}E_{IJ}\sum_{n\neq 0}\tilde{\alpha}_n^J e^{-2ni(\tau-\sigma)}\}. \end{aligned} \quad (4.13)$$

The center-of-mass momenta p_I can be determined by

$$\begin{aligned} p_I &= \int_0^\pi d\sigma P_{\tau I} \\ &= \frac{1}{\sqrt{2\alpha'}}(E_{IJ}\alpha_0^J + E_{IJ}^T\tilde{\alpha}_0^J). \end{aligned} \quad (4.14)$$

Inserting the general solution (4.11) into the periodic condition (4.3), we obtain the relation between the string winding mode w^I and the zero-mode oscillators α_0^I, $\tilde{\alpha}_0^I$

$$w^I = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^I - \tilde{\alpha}_0^I). \quad (4.15)$$

As we mentioned in §2.3.1, if the basis vectors of the compactification lattice are laid along the directions of compactified coordinates X^I ($e_i^I = \delta_i^I$) and we work in quantum theory, the momentum mode p_I and the winding mode w^I take the forms

$$p_I = \frac{m_I}{R^{(I)}}, \quad (4.16)$$

$$w^I = \frac{n^I R^{(I)}}{\alpha'}, \quad (4.17)$$

where m_I and n^I are integers that represent the momentum number and winding number, respectively. As a consequence, the left-and right-moving momenta $p_L^I = \sqrt{\frac{2}{\alpha'}}\alpha_0^I$ and $p_R^I = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^I$ can be written in terms of momentum and winding modes as

$$p_L^I = G^{IJ}\frac{m_J}{R^{(J)}} + G^{IJ}(G_{JK} - B_{JK})\frac{n^K R^{(K)}}{\alpha'}, \tag{4.18}$$

$$p_R^I = G^{IJ}\frac{m_J}{R^{(J)}} - G^{IJ}(G_{JK} + B_{JK})\frac{n^K R^{(K)}}{\alpha'} \tag{4.19}$$

Analogous to bosonic string, we can determined the commutation relations for the center-of-mass coordinates and momenta and for the oscillators by using the equal-time commutation relation $[X^I(\tau,\sigma), P_{\tau J}(\tau,\sigma')] = i\delta^I_J\delta(\sigma-\sigma')$. The non-vanishing commutation relations are

$$[x^I, p_J] = i\delta^I_J, \quad [\alpha^I_m, \alpha^J_n] = [\tilde{\alpha}^I_m, \tilde{\alpha}^J_n] = mG^{IJ}\delta_{m,-n}. \tag{4.20}$$

We can see that the commutation relations of oscillators are background dependent. At this point we can determine the (bosonic) Virasoro operators and number operators of closed strings by using the procedure in Chapter 2 and then obtain

$$L_0 = \frac{1}{2}G_{IJ}\alpha_0^I\alpha_0^J + N, \quad N = \sum_{n=1}^{\infty}(\alpha^\mu_{-n}\alpha_{\mu n} + G_{IJ}\alpha^I_{-n}\alpha^J_n), \tag{4.21}$$

$$\tilde{L}_0 = \frac{1}{2}G_{IJ}\tilde{\alpha}_0^I\tilde{\alpha}_0^J + \tilde{N}, \quad \tilde{N} = \sum_{n=1}^{\infty}(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{\mu n} + G_{IJ}\tilde{\alpha}^I_{-n}\tilde{\alpha}^J_n). \tag{4.22}$$

Here, α^μ_n and $\tilde{\alpha}^\mu_n$ are the oscillators in the non-compact spacetime.

It is obvious that the toroidal compactification we mentioned above is quite trivial since we set $e^I_i = \delta^I_i$. In general, we choose the vielbiens e^I_i to be the orthonormal basis vectors which satisfy

$$G_{IJ}e^I_i e^J_j = \delta_{ij}. \tag{4.23}$$

We also introduce the dual basis vectors e^{*i}_I of the dual lattice Λ^{*d} which satisfy

$$\sum_{i=1}^{d} e^I_i e^{*i}_J = \delta^I_J. \tag{4.24}$$

Therefore, in these bases the metric G_{IJ} and its inverse G^{IJ} can be expressed as

$$G_{IJ} = \sum_{i=1}^{d} e^{*i}_I e^{*i}_J, \quad G^{IJ} = \sum_{i=1}^{d} e^I_i e^J_j. \tag{4.25}$$

We continue our study by considering the left-and right-moving momenta in the orthonormal frame

$$p_{Li} \;=\; p_{LI}e_i^I = (\frac{m_I}{R^{(I)}} + E_{IJ}^T\frac{n^J R^{(J)}}{\alpha'})e_i^I, \tag{4.26}$$

$$p_{Ri} \;=\; p_{RI}e_i^I = (\frac{m_I}{R^{(I)}} - E_{IJ}\frac{n^J R^{(J)}}{\alpha'})e_i^I. \tag{4.27}$$

Using the zeroth-mode Virasoro constraints, we can determine the mass-squared operator M^2 and the level-matching condition for type-II superstring as

$$M^2 \;=\; \frac{1}{2}(p_L^2 + p_R^2) + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{4.28}$$

$$\tilde{N} - N \;=\; m^I n_I. \tag{4.29}$$

It is important to note that the vectors $(\sqrt{\frac{\alpha'}{2}}p_{Li}, \sqrt{\frac{\alpha'}{2}}p_{Ri})^T$ form the even self-dual (d,d) Lorentzian lattice $\Gamma^{(d,d)}$ for type-II superstrings since the squared length of any vector in this lattice is even

$$(\sqrt{\frac{\alpha'}{2}}p_{Li})^2 - (\sqrt{\frac{\alpha'}{2}}p_{Ri})^2 = 2m_I n^I \in 2\mathbb{Z}, \tag{4.30}$$

and the lattice is self-dual

$$\Gamma^{(d,d)} = \Gamma^{*(d,d)}. \tag{4.31}$$

As we mentioned in §2.3.2, this implies that string theory possesses the modular invariance and avoids the gauge and gravitational anomalies. Furthermore, we will see in the next subsection that the continuous group which generates the moduli space is the group of transformation that preserves the properties of $\Gamma^{(d,d)}$.

Next, we explore the string spectrum by considering the massless states. If $p_{Li} = p_{Ri} = 0$ (or equivalently, $m_I = n^I = 0$), we can see from (4.28) that the massless states, which are $\alpha_{-1}^\mu \tilde{\alpha}_{-1}^I |0;0\rangle_L \bigotimes |0;0\rangle_R$ and $\tilde{\alpha}_{-1}^\mu \alpha_{-1}^I |0;0\rangle_L \bigotimes |0;0\rangle_R$, generate the local $U(1)^d \times U(1)^d$ gauge symmetry. Moreover, there can be extra massless states if $\sqrt{\frac{\alpha'}{2}}p_{Li}$ and $\sqrt{\frac{\alpha'}{2}}p_{Ri}$ are roots of the simply laced groups G_L and G_R of rank d with root length $\sqrt{2}$, respectively. As a consequence, the $U(1)_L^d \times U(1)_R^d$ gauge symmetry is enlarged to $G_L \times G_R$. However, it appears that the toroidal compactification of type-II superstring theories do not give rise to any non-Abelian gauge symmetry. This result arises from the fact that the gauge symmetry G_L is as same as G_R and consequently, theses symmetries act on both left-and right-moving fermions in the similar way.

In the case of heterotic string we must add the extra components $\sqrt{\frac{\alpha'}{2}}p_{Lj}; j = 1, \ldots, 16$ that provide the non-Abelian gauge group $SO(32)$ or $E_8 \times E_8$ for ten-dimensional heterotic string theory we mentioned in §2.3.2 to the vectors; therefore, the even self-dual Lorentzian lattice in this case is $\Gamma^{(16+d,d)}$. However, we can ignore these extra degrees of freedom since our aim is to study the stability of the d-dimensional toroidal space. This means that we can focus on the $d \times d$-dimensional sublattice of $\Gamma^{(16+d,d)}$. Thus, the mass-squared operator and the level-matching condition in our consideration are

$$M^2 = \frac{1}{2}(p_L^2 + p_R^2) + \frac{2}{\alpha'}(N + \tilde{N} - 1), \tag{4.32}$$

$$\tilde{N} - N = m^I n_I - 1, \tag{4.33}$$

where N and $\tilde{N}$ are the (bosonic) left-and right-moving number operators of heterotic string. Similar to type-II superstring theory, if $p_{Li} = p_{Ri} = 0; i = 1, \ldots, d$ and $p_{Lj} = 0; j = 1, \ldots, 16$, the massless states of heterotic string generate the local $U(1)_L^{16+d} \times U(1)_R^d$ gauge symmetry. However, the difference for heterotic string is that the Abelian gauge group can be enhanced to the non-Abelian gauge group for some certain specific compactification lattices as we mentioned in §2.3.2. It turns out that the enhancement of gauge group can take place if the vectors with $(\sqrt{\frac{\alpha'}{2}} p_{Li})^2 = 2$ and $\sqrt{\frac{\alpha'}{2}} p_{Ri} = 0$ belong to the lattice $\Gamma^{(16+d,d)}$, and the largest non-Abelian gauge group we can obtain is $G_L = SO(32 + 2d)$.

In this subsection we provided the cursory review of the toroidal compactification of type-II superstring and heterotic string in the standard form. However, it is quite difficult for this approach to reveal the symmetries of the moduli space. In the next subsection we will take a study of the toroidal compactification in the particular form which makes the symmetries of the moduli space more transparent.

4.1.2 Generating Group $O(d, d; \mathbb{R})$ and Symmetry Group $O(d, d; \mathbb{Z})$ of The Moduli Space

This subsection is devoted to a study of the continuous group that generates the moduli space and its discrete subgroup that leaves the spectrum invariant and is regarded as a symmetry group of theory. We can determine the generating group of the moduli space from the fact that an $O(d, d; \mathbb{R})$ transformation preserves both even and self-dual conditions of the even self-dual Lorentzian (d, d) lattice $\Gamma^{(d,d)}$. Moreover, each $\Gamma^{(d,d)}$ is also unique up to such a transformation. This means that the continuous group $O(d, d; \mathbb{R})$ generates the whole set of the even self-dual (d, d) Lorentzian lattices that possibly give rise to the backgrounds of the d-dimensional toroidal compactification. Therefore, the $O(d, d; \mathbb{R})$ group is considered as the generating group $\mathcal{G}$ of the moduli space. However, not all the $O(d, d; \mathbb{R})$ rotations provide different string spectra. It is found that as $(\sqrt{\frac{\alpha'}{2}} p_{Li}, \sqrt{\frac{\alpha'}{2}} p_{Ri})^T$ transforms as a vector under the action of $O(d,d;\mathbb{R})$, its squared length is invariant under the continuous subgroup $O(d;\mathbb{R})\times O(d;\mathbb{R})$ that corresponds to the d-dimensional rotations of $\sqrt{\frac{\alpha'}{2}}p_{Li}$ and $\sqrt{\frac{\alpha'}{2}}p_{Ri}$, separately. Therefore, at this point we can state that the moduli space of d-dimensional compactification is isomorphic to the quotient space $O(d,d;\mathbb{R})/(O(d;\mathbb{R})\times O(d;\mathbb{R}))$.

In order to reveal the $O(d,d)$ symmetry of string spectrum obviously, we introduce a $2d$-dimensional vector, Z, which represents the (d,d) Lorentzian vector $(\sqrt{\frac{\alpha'}{2}}p_{Li},\sqrt{\frac{\alpha'}{2}}p_{Ri})$ in terms of the momentum and winding numbers

$$Z \;=\; (m_1,\ldots,m_d,n^1,\ldots,n^d)^T. \tag{4.34}$$

Let Ω be an element of the $O(d,d;\mathbb{R})$ group and take the form

$$\Omega=\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}, \tag{4.35}$$

where $\Omega_{11},\Omega_{12},\Omega_{21},\Omega_{22}$ are $d\times d$ real matrices. It is obvious that Ω must preserve the form of a $2d\times 2d$, matrix, $\eta=\begin{pmatrix}0 & I_{d\times d}\\ I_{d\times d} & 0\end{pmatrix}$ in such a way that

$$\Omega^T\eta\Omega=\eta. \tag{4.36}$$

We then introduce a $2d\times 2d$ matrix, $M(E)$, that contains the information of the background matrix E as

$$M(E)=\begin{pmatrix}G^{-1} & -G^{-1}B\\ BG^{-1} & G-BG^{-1}B\end{pmatrix}. \tag{4.37}$$

Here, we rescale the metric G_{IJ} and the Kalb-Ramond field B_{IJ} by

$$G_{IJ}\to G'_{IJ}=\frac{R^{(I)}R^{(J)}}{\alpha'}G_{IJ},\quad B_{IJ}\to B'_{IJ}=\frac{R^{(I)}R^{(J)}}{\alpha'}B_{IJ}, \tag{4.38}$$

and drop the sign $'$ for convenience. We can see that $p_L^2+p_R^2$ can be expressed in terms of $M(E)$ and Z as (see **(B.5) - (B.6)**)

$$\begin{aligned}p_L^2+p_R^2 &= \frac{2}{\alpha'}(G^{IJ}m_Im_J+G_{IJ}n^In^J+G^{IJ}B_{IK}B_{JL}n^Kn^L\\ &\quad -2G^{IJ}B_{JK}m_In^K) \qquad (4.39)\\ &= \frac{2}{\alpha'}Z^TMZ. \qquad (4.40)\end{aligned}$$

Thus, the mass squared operator M^2 can be written as

$$M^2 \;=\; \frac{1}{\alpha'}Z^TMZ+\frac{2}{\alpha'}(N+\tilde{N}-a), \tag{4.41}$$

where a is the normal ordering constant that takes the value 2 (1) for Type-II superstring (heterotic string).

Under an $O(d,d;\mathbb{R})$ transformation Ω, a matrix $M(E)$ and a vector Z change by

$$M \quad \rightarrow \quad M' = \Omega M \Omega^T, \tag{4.42}$$

$$Z \quad \rightarrow \quad Z' = \Omega Z, \tag{4.43}$$

respectively. The string mass spectrum (4.41) is obviously invariant under this transformation. At this point it is essential to consider the fact that in order to determine the string spectrum, one must know p_{LI} and p_{Ri} or equivalently E, m_I, and n^I besides the number operators. However, it is easy to see that the number operators do not change under the $O(d,d;\mathbb{R})$ transformation but the others do. This means that any point in the moduli space can be labelled by E, m_I, and n^I.

Next, we will find a transformation corresponding to the background matrix E. In order to do this, we consider the particular element Ω_E of $O(d,d;\mathbb{R})$ that is expressed as

$$\Omega_E = \begin{pmatrix} (e^{*T})^{-1} & 0 \\ B(e^{*T})^{-1} & e^* \end{pmatrix}, \tag{4.44}$$

where e^* is a $d \times d$ matrix representation of the vielbeins e^{*i}_I. With this element, the matrix $M(E)$ takes the form

$$M(E) = \Omega_E \Omega_E^T. \tag{4.45}$$

We then see that the transformed matrix M' can be expressed in the form

$$M' = M(E') = \Omega M(E) \Omega^T = \Omega_{E'} \Omega_{E'}^T \quad ; \Omega_{E'} = \Omega \Omega_E. \tag{4.46}$$

Now we introduce a $d \times d$ matrix, ω_Ω, which transforms any $d \times d$ matrix under the action of the element Ω. The action of ω_Ω on a $d \times d$ matrix F is defined by

$$\omega_\Omega(F) \equiv (\Omega_{11} F + \Omega_{12})(\Omega_{21} F + \Omega_{22})^{-1}. \tag{4.47}$$

It means that we can obtain the background matrix by applying ω_{Ω_E} to the identity matrix

$$\omega_{\Omega_E}(I) = E = G + B. \tag{4.48}$$

Thus, the expression of the transformed background matrix E' under the action of Ω is

$$E' = \omega_{\Omega_{E'}}(I) = \omega_\Omega(E) = (\Omega_{11} E + \Omega_{12})(\Omega_{21} E + \Omega_{22})^{-1}. \tag{4.49}$$

We can determine the transformed metric G' and the transformed anti-symmetric matrix B' by using the relation

$$G' = (E^T\Omega_{21}^T + \Omega_{22}^T)^{-1}G(\Omega_{21}E + \Omega_{22})^{-1} \quad (4.50)$$

to obtain G' and then substituting it back into (4.49) to obtain B'.

Next, we will study the discrete subgroup $\mathcal{G}_d$ of $\mathcal{G}$, which provides the discrete symmetries of the moduli space. The property of the subgroup we require is that it does not change the lattice $\Gamma^{(d,d)}$ to a different one but it indeed changes the lattice basis in terms of which the lattice is defined [25]. It turns out that such a subgroup is the $O(d,d;\mathbb{Z})$ group [29]. There are three transformation groups whose elements generate the $O(d,d;\mathbb{Z})$ group:

1. Theta-parameter shift group $\Theta(\mathbb{Z})$

This transformation arises from the fact that the spectrum is still unchanged if an antisymmetric integral-valued matrix, $\Theta \in \Theta(\mathbb{Z})$, is added to the antisymmetric matrix B. As a consequence, an element of $O(d,d;\mathbb{Z})$ group that is generated by Θ can be expressed in the form

$$\Omega_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}. \quad (4.51)$$

2. Basis-change group $GL(d;\mathbb{Z})$

This transformation is the basis change of the compactification lattice Λ^d. The action of $A \in GL(d;\mathbb{Z})$ on the background matrix E is $E' = AEA^T$. Then, an element of $O(d,d;\mathbb{Z})$ corresponding to A can be written as

$$\Omega_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}. \quad (4.52)$$

3. Factorized-duality group

An $O(d,d;\mathbb{Z})$ transformation corresponding to an element of this group is

$$\Omega_{D_I} = \begin{pmatrix} I - \Delta_{(I)} & \Delta_{(I)} \\ \Delta_{(I)} & I - \Delta_{(I)} \end{pmatrix}, \quad (4.53)$$

where $\Delta_{(I)}$ is a $d \times d$ matrix whose components are $[\Delta_{(I)}]_{JK} = \delta_{IJ}\delta_{IK}$. It is the fact that the factorized duality D_I is a transformation that changes the compactification radius in the direction X^I from R_I to $\frac{\alpha'}{R_I}$ but leaves the compactification radii in other directions unchanged. In other words, the factorized duality D_I is a generalization of $R_I \to \alpha'/R_I$ duality and thus concerns the target-space duality we mentioned in the previous chapter.

Actually, the $O(d,d;\mathbb{Z})$ group is not the complete discrete symmetry group $\mathcal{G}_d$. The complete discrete symmetry group includes the worldsheet-parity symmetry $\sigma \to -\sigma$ which acts on the antisymmetric matrix B as $B \to -B$ to the $O(d,d;\mathbb{Z})$ group. However, we ignore this discrete symmetry since it does not affect our consideration on the target space directly. Thus, the moduli space of d-dimensional toroidal background is the quotient space

$$O(d,d;\mathbb{Z})\backslash O(d,d;\mathbb{R})/(O(d;\mathbb{R}) \times O(d;\mathbb{R})), \tag{4.54}$$

where the quotient of the continuous groups is applied from the right and that of the discrete group is applied from the left.

4.1.3 T-duality of d-dimensional Toroidal Background

In this subsection we will study the particular elements of $O(d,d;\mathbb{Z})$ that generate T-duality or equivalently, the inversion of the background matrix ($E \to E^{-1}$). We then see that the generators of T-duality concern the symmetric tensor G and the antisymmetric tensor B. This arises from the fact that the generators of the discrete group $O(d,d;\mathbb{Z})$ are eventually the moduli fields. We start our study by considering a transformation that changes the background matrix to its inverse or the inversion duality

$$E \to E' = G' + B' = E^{-1}, \tag{4.55}$$

where

$$\begin{aligned} G' &= (G - BG^{-1}B)^{-1}, & (4.56)\\ B' &= (B - GB^{-1}G)^{-1}. & (4.57) \end{aligned}$$

It is important to note that this transformation preserves the symmetric (antisymmetric) property of G (B) and if it combines with the interchange between winding modes and momentum modes ($n \leftrightarrow m$), the consequent transformation is called the T-duality transformation which leaves the lattice $\Gamma^{(d,d)}$ and the mass spectrum invariant.

First, we consider the operator of the inversion duality that corresponds to the element of the factorized-duality transformation only

$$\Omega_D = \prod_I \Omega_{D_I} = \begin{pmatrix} I - \sum \Delta_{(I)} & \sum \Delta_{(I)} \\ \sum \Delta_{(I)} & I - \sum \Delta_{(I)} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{4.58}$$

We can see from (4.43) and (4.49) that the transformation generated by Ω_D not only interchanges the winding numbers with momentum numbers ($n^I \leftrightarrow m_I$), but also inverses the background matrix E to E^{-1}.

It turns out that there exist in the moduli space the fixed points that do not change under the inversion duality. It means that at the fixed point the background matrix is as same as its inverse

$$E = E^{-1}. \tag{4.59}$$

In this case, the fixed point corresponding to

$$E = G = I, B = 0. \tag{4.60}$$

is only the solution. We can see that if we express the equation (4.59) as $I = EE^{-1} = E^2 = (G+B)^2$, the unique solution of this relation is (4.60). Under the background inversion generated by (4.58) the metric at the fixed point changes as

$$G \rightarrow G' = G^{-1} = G. \tag{4.61}$$

If we restore the compactification radii by using the equation (4.38) and set their length to be equal ($R^{(I)} = R$), we obtain the relation between the transformed radius and the original radius at the fixed point

$$\frac{R^2}{\alpha'} \rightarrow \frac{R'^2}{\alpha'} = \frac{\alpha'}{R^2\sqrt{\mathrm{detG}}} = \frac{R^2}{\alpha'}. \tag{4.62}$$

It is conventional to define G_{IJ} as the metric of toroidal background with unit volume ($\sqrt{\mathrm{detG}} = 1$). As a consequence, at the fixed point the compactification radius is the self-dual radius

$$R = \sqrt{\alpha'}. \tag{4.63}$$

From the equation above, the fixed point in this case is usually called the self-dual point. This is the toroidal compactification we mentioned in Chapter 3.

However, we must include other $O(d,d;\mathbb{Z})$ transformations generated by Theta-parameter shift and basis change in order to obtain the complete background inversion. To do this, we consider the fixed point of the inversion duality $E \rightarrow E^{-1}$ modulo the basis change $A \in SL(d;\mathbb{Z})$ and the Theta-parameter shift $\Theta \in \Theta(\mathbb{Z})$

$$E \rightarrow E' = E^{-1} = A^T(E+\Theta)A. \tag{4.64}$$

It is the fact that the background satisfying the equation above possesses the maximally enhanced symmetry (see [27]). In this case the background matrix E at the fixed point must obey

$$E_{IJ} = C_{IJ}; \quad I > J, \quad E_{II} = \frac{1}{2}C_{II}, \quad E_{IJ} = 0; \quad I < J, \tag{4.65}$$

where C_{IJ} is the Cartan matrix. Equivalently,

$$G_{IJ} = \sum_i e_I^{i*} e_J^{i*} = \frac{1}{2}C_{IJ}, \quad B_{IJ} = \{ \begin{array}{cc} G_{IJ}, & I < J \\ 0, & I = J \\ -G_{IJ}, & I > J \end{array} . \tag{4.66}$$

Here, we the vielbein is defined by $e_I^{i*} \equiv \frac{1}{\sqrt{2}}\alpha_I^i$ where α_I spans the **root lattice** Λ_{root} of simply laced group G of rank d with root length $\sqrt{2}$. It appears that the solutions of equation 4.64 are

$$E \in SL(d;\mathbb{Z}), \tag{4.67}$$

$$A = E^{-1}, \tag{4.68}$$

$$\Theta = E^T - E. \tag{4.69}$$

This fact implies that a fixed point corresponding to the maximally enhanced symmetry G is a symmetric point under the non-trivial $O(d,d;\mathbb{Z})$ transformation generated by Theta-parameter shift, basis change, and factorized duality. However, it is difficult to determine the fixed points and their enhanced gauge symmetries for arbitrary d-dimensional compactification. Therefore, we provide one of the simplest but non-trivial example for toroidal compactification that is the identification of the fixed points and the associated enhanced gauge symmetries for two-dimensional compactification ($d = 2$).

As we mentioned in the previous subsection, the generating group and the symmetry group of two-dimensional toroidal background are $O(2,2;\mathbb{R})$ and $O(2,2;\mathbb{Z})$, respectively. Although $O(2,2;\mathbb{R})$ can be decomposed into $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$, its discrete subgroup $O(2,2;\mathbb{Z})$ cannot be separated into the analogous product $SL(2;\mathbb{Z}) \times SL(2;\mathbb{Z})$. In fact, $O(2,2;\mathbb{Z})$ is isomorphic to $SL(2;\mathbb{Z}) \times SL(2;\mathbb{Z}) \otimes \mathbb{Z}_2 \times \mathbb{Z}_2$, and the moduli space is isomorphic to $SL(2,\mathbb{R})/U(1) \times SL(2,\mathbb{R})/U(1)$.

Next, we examine the generators of $O(2,2;\mathbb{Z})$. For the Theta-parameter shift, there is one transformation, Ω_Θ, generated by Θ, where

$$\Omega_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}; \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.70}$$

In the case of the basis change group $GL(2;\mathbb{Z})$, there are three transformations:

1. The permutation between two directions of the torus generated by P_{12}

$$\Omega_{P_{12}} = \begin{pmatrix} P_{12} & 0 \\ 0 & P_{12} \end{pmatrix}; \quad P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.71}$$

2. The reflection on the I direction, generated by $R_I; I = 1, 2$

$$\Omega_{R_I} = \begin{pmatrix} R_I & 0 \\ 0 & R_I \end{pmatrix}; \quad R_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.72}$$

3. The transformation generated by T_{12}

$$\Omega_{T_{12}} = \begin{pmatrix} T_{12} & \Theta \\ 0 & (T_{12}^{-1})^T \end{pmatrix}; \quad T_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.73}$$

It is worth noting that the generators of $GL(2;\mathbb{R})$ actually are P_{12}, R_1 and T_{12} since we can generate R_2 by $R_2 = P_{12}R_1P_{12}$. Furthermore, it appears that the $SL(2;\mathbb{Z})$ subgroup of $GL(2;\mathbb{R})$ can be generated by $T \equiv T_{12}$ and $S \equiv R_1P_{12}$.

For the factorized duality, there are two generators, D_1 and D_2. The corresponding transformations are

$$\Omega_{D_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Omega_{D_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.74}$$

We end this section by investigating the fixed points with maximally enhanced symmetry. Since the dimension of compact space in this case is two, the maximally enhanced symmetry groups must be the **simply laced groups of rank** 2, which are $SU(2) \times SU(2)$ and $SU(3)$. The first fixed point corresponding to the enhanced symmetry group $SU(2)$ is described by

$$E = G = I, \quad B = 0. \tag{4.75}$$

Therefore, the basis vectors e_I^{*i} of the compactification lattice and the transformed vectors are

$$e_1^{*i} = e_i^{'1} = (1, 0), \quad e_2^{*i} = e_i^{'2} = (0, 1). \tag{4.76}$$

The $SU(2) \times SU(2)$ root lattice is illustrated in the Figure 4.1.

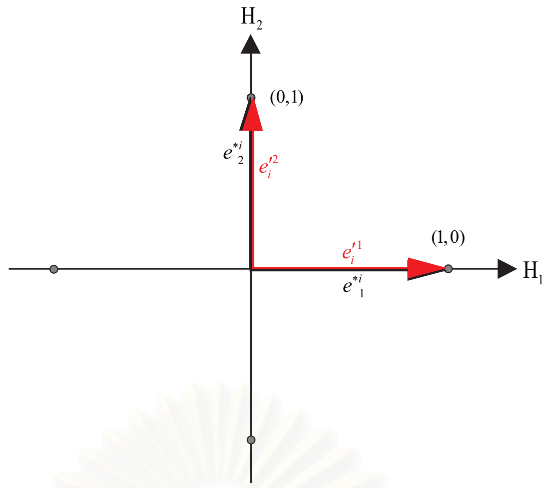


Figure 4.1: The original basis e_I^{*i} and the transformed basis $e_i^{\prime I}$ are the same in the $SU(2)\times SU(2)$ root lattice.

The second fixed point corresponding to the maximally enhanced symmetry $SU(3)$ is described by

$$E=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix},\quad G=\frac{1}{2}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix},\quad B=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{4.77}$$

The basis vectors e_I^{*i} of the compactification lattice are

$$e_1^{*i}=(\frac{1}{2},\frac{\sqrt{3}}{2}),\quad e_2^{*i}=(\frac{1}{2},-\frac{\sqrt{3}}{2}). \tag{4.78}$$

Under the background inversion, the transformed background, symmetric and anti-symmetric matrices are

$$E'=E^{-1}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad G'=\frac{1}{2}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\quad B'=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{4.79}$$

The transformed basis vectors are

$$e_i^{\prime 1}=(\frac{1}{2},\frac{\sqrt{3}}{2}),\quad e_i^{\prime 2}=(1,0). \tag{4.80}$$

It is worth noting that the original basis vectors (4.78) and the transformed ones (4.80) span the same $SU(3)$ lattice as shown in Figure 4.2.

Obviously, this arises from the fact that the basis change is one of the generators of the background inversion.

Up to this point all preliminaries for the $O(d,d)$-covariant formalism are prepared. In the next section we will review string gas cosmology in the context of the $O(d,d)$-covariant approach and use it to study the evolution of the late-time string gas.

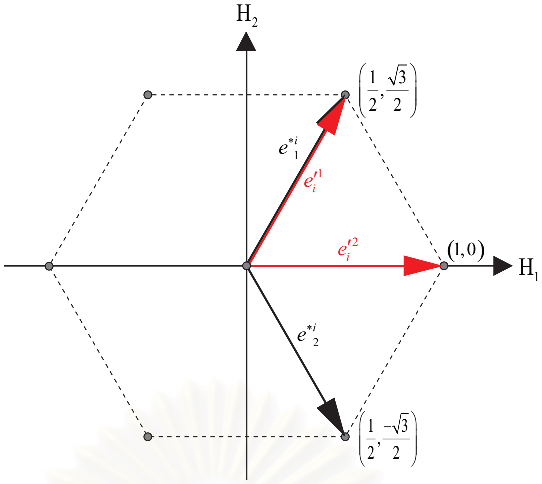


Figure 4.2: The original basis e^{*i}_I and the transformed basis e'^I_i are different in the $SU(3)$ root lattice.

4.2 $O(d,d)$-Covariant String Cosmology

In the final section of Chapter 3 we used the low-energy effective action with a specific condition (the vanishing of the Kalb-Ramond field, $B_{\mu\nu} = 0$) to study the evolution of the late-time universe and determined the solution for the moduli stabilization problem numerically. In this section we study the same circumstance in the $O(d,d)$-covariant formalism (see Refs[24, 30]).

Analogous to §3.3, we start with some prior assumptions before we continue our consideration. First, according to the Brandenberger-Vafa mechanism [16], the six spatial dimesions are toroidally compactified but the other four-dimensional spacetime is non-compact. Second, it is supposed that the metrics in both compact and non-compact space depend only on the non-compact coordinates; therefore, the metric ansatz is

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta + ds^2_{\text{torus}} = g_{\alpha\beta}(x)dx^\alpha dx^\beta + G_{IJ}(x)dy^I dy^J, \qquad (4.81)$$

where the metric and coordinates in non-compact and compact space are denoted by $g_{\alpha\beta}$ and G_{IJ}, respectively. It means that the compact space is flat with respect to the compact coordinate y^I. The last assumption is that the non-vanishing antisymmetric tensor exists only in the compact space ($B_{IJ} \neq 0$). As a consequence, the low-energy effective action (2.256) can be expressed in the $O(d,d;\mathbb{R})$-covariant form (see **(B.7)**)

$$S_o = -\frac{V_6}{2k_{10}^2}\int d^4x\sqrt{-g^{(4)}}e^{-\Phi}(R^{(4)} + \partial_\alpha\Phi\partial^\alpha\Phi + \frac{1}{8}\text{Tr}(\partial_\alpha M(E)\eta\partial^\alpha M(E)\eta), (4.82)$$

where V_6 is the volume of compact space, $M(E)$ is in the form of (4.37) and Φ is shifted dilaton expressed as

$$\Phi = 2\phi - \ln\sqrt{G^{(6)}}. \tag{4.83}$$

It is obvious that this action is invariant under the $O(d,d;\mathbb{R})$ transformation

$$\begin{aligned} M &\rightarrow M' = \Omega M \Omega^T, \\ \Phi &\rightarrow \Phi' = \Phi. \end{aligned} \tag{4.84}$$

The action of string gas source can be written in the form

$$S_{\rm M} = \int d^4x\sqrt{-g^{(4)}}\rho, \tag{4.85}$$

where g is the determinant of four-dimensional metric and ρ is the energy density of string gas expressed in the form

$$\rho = \frac{\mu_4}{\sqrt{g^{(s)}}}\sqrt{g^{ab}p_ap_b + M^2}. \tag{4.86}$$

Here, μ_4 is the comoving number density of string gas and g_{ab}, $g^{(s)}$, and p^a are metric, determinant of metric, and momentum in spatial part of four-dimensional spacetime, respectively. Note that M^2 in the above equation is obtained from the mass squared formula (4.41). It is obvious that the action of string gas source (4.85) is also invariant under the $O(d,d,\mathbb{R})$ transformation. This means that the total action $S = S_o + S_{\rm M}$ expressed in this form is $O(d,d)$-invariant. In other words, string gas cosmology possesses the $O(d,d)$ symmetry even with the presence of classical string source. It turns out that it is convenient to consider T-duality of toroidal background by expressing the action in the Einstein frame. If we choose the Einstein-frame metric $\overline{g}_{\alpha\beta} = e^{-\Phi}g_{\alpha\beta}$, then the total action $\overline{S}$ in the Einstein frame becomes

$$\begin{aligned} \overline{S} &= \overline{S}_o + \overline{S}_M \\ &= -\frac{V_6}{2k_{10}^2}\int d^4x\sqrt{-\overline{g}^{(4)}}(\overline{R}^{(4)} - \frac{1}{2}\partial_\alpha\Phi\partial^\alpha\Phi + \frac{1}{8}{\rm Tr}(\partial_\alpha M\eta\partial^\alpha M\eta)) \\ &\quad + \int d^4x\sqrt{-\overline{g}^{(4)}}V_{\rm eff}(\overline{g},\Phi,E), \end{aligned} \tag{4.87}$$

where the effective potential $V_{\rm eff}(\overline{g},\Phi,E)$ is defined by

$$V_{\rm eff}(\overline{g},\Phi,E) \equiv \frac{\mu_4}{\sqrt{\overline{g}^{(s)}}}\sqrt{\overline{g}^{ab}p_ap_b + e^{\Phi}M^2(E)}, \tag{4.88}$$

where $\overline{g}^{(s)} = \det\overline{g}_{ab}$ is the determinant of the spatial part of the four-dimensional metric in the Einstein frame.

As we mentioned in §4.1.3, it is very difficult to consider the $O(d,d;\mathbb{Z})$ group which generates T-duality in arbitrary dimensions. In Chapter 3, it was assumed that $T^6 = (T^1)^6$, so the one-dimensional torus T^1 was used to be a representative of the compact space. For this chapter, the $d = 2$ example was studied in the previous section. Obviously, it is appropriate for our consideration to assume that $T^6 = T^2 \times T^2 \times T^2$ in order to use T^2 as a representative of the compact space.

Before we continue our consideration, we require some adjustments for convenience. First, the metric and Kalb-Ramond tensors are described by four parameters ξ, η, β, and b^2 as follows

$$\xi = \frac{G'_{12}}{G'_{11}}, \quad \eta = \frac{\sqrt{\mathrm{detG'}}}{G'_{11}}, \quad \beta = B'_{12}, \quad b^2 = \sqrt{\mathrm{detG'}}, \tag{4.89}$$

or equivalently,

$$G' = \frac{b^2}{\eta}\begin{pmatrix} 1 & \xi \\ \xi & \xi^2+\eta^2 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}. \tag{4.90}$$

Note that the sign $'$ means the rescaled tensors in (4.38). Since we can consider each two-dimensional torus separately, the line element of the compact background ds^2_{torus} reduces to

$$ds^2_{\mathrm{torus}} = \frac{b^2}{\eta}((dy^1+\xi dy^2)^2 + \eta^2(dy^2)^2). \tag{4.91}$$

Second, we restore the compactification radii from (4.38) and suppose that two compactification radii of T^2 are equal ($R^{(1)} = R^{(2)} = R$). Third, the original metric G_{IJ} in (4.38) is assumed to have the unit volume ($\sqrt{\mathrm{detG}} = 1$). As a consequence, we obtain

$$\xi = \frac{G_{12}}{G_{11}}, \quad \eta = \frac{1}{G_{11}}, \quad \beta = \frac{R^2}{\alpha'}B_{12}, \quad b^2 = \frac{R^2}{\alpha'}. \tag{4.92}$$

Certainly, any point in the moduli space is now described by four parameters, ξ, η, β, and b^2. The shape of T^2 is characterized by two parameters ξ, η and illustrated in Figure 4.3.

The parameter β corresponds to the Kalb-Ramond field and is called the flux moduli. The last parameter b^2 obviously represents the scale factor of the compact space as we can see from (4.92).

In the final section we will resolve the moduli stabilization problem by using the $O(d,d)$-covariant formalism.

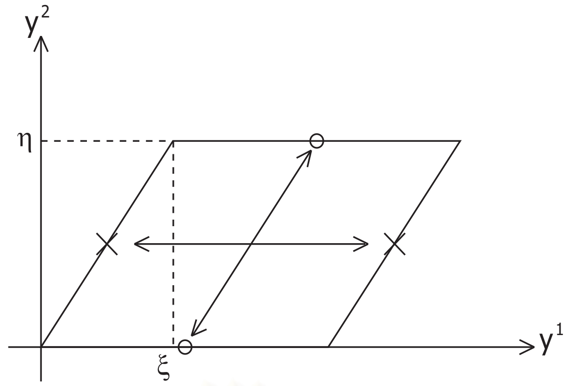


Figure 4.3: The shape of T^2 is described by two parameters ξ and η.

4.3 Moduli Stabilization in $O(d,d)$-Covariant Formalism

The moduli stabilization problem for the toroidal compactification of superstring was solved qualitatively by Kanno and Soda [22]. The result showed that at there was one stable fixed point However, the enhanced non-Abelian gauge symmetry did not appear since the fundamental strings used in that model was type-II superstrings. Moreover, T-duality in that model was generated by the factorized duality alone. Therefore, one could not obtain the maximally enhanced gauge symmetry. Later, Chatrabhuti [23] suggested that there could be more than one fixed point corresponding to the maximally enhanced gauge symmetries under T-duality if the full inversion of background matrix was taken into account. Furthermore, those enhanced gauge symmetries should be non-Abelian when the heterotic strings were used as fundamental strings. In this section we will follow the latter idea to solve the moduli stabilization problem. Our aim is to examine the stability of those fixed points.

4.3.1 $SU(2) \times SU(2)$ Fixed Point

As we mentioned in §4.1.3, there are two fixed points corresponding to the maximally enhanced gauge symmetries. The first fixed point occurs at

$$b^2 = \eta = 1, \quad \beta = \xi = 0. \tag{4.93}$$

From equation (4.92), this fixed point is the self-dual point ($R = \sqrt{\alpha'}$). The corresponding enhanced gauge symmetry for heterotic string is $G_L = SU(2) \times$

$SU(2)$. In order to examine the stability of the fixed point, we must determine all extra massless states in terms of momentum and winding numbers (m_1, m_2, n^1, n^2). Substituting (4.93) into (4.41), it turns out that at this fixed point there are four extra massless modes which are $(\pm 1, 0, \pm 1, 0)$ and $(0, \pm 1, 0, \pm 1)$. Next, we will determine the flat directions for each mode.

Mode $(1, 0, 1, 0)$

The mass squared in this mode can be expressed in terms of b, η, ξ, β as

$$M^2_{1010} = \frac{1}{b^2\eta}(1+\beta\xi)^2 + \frac{\beta^2\eta}{b^2} + \frac{b^2\xi^2}{\eta} + \eta b^2 - 2, \tag{4.94}$$

with the flat directions $\beta^2 = \frac{b^4\xi^2}{\eta^2}$ and $1 + \beta\xi = \eta b^2$. This mode corresponds to the closed string wrapping along the y^1 direction.

Mode $(0, 1, 0, 1)$

The mass squared in this mode can be expressed as

$$M^2_{0101} = \frac{1}{b^2\eta}(\xi - \beta)^2 + \frac{b^2}{\eta} + \frac{\eta}{b^2} - 2, \tag{4.95}$$

with flat directions $\beta = \xi$ and $b^2 = \eta$. This mode corresponds to the closed string wrapping along the y^2 direction.

It appears that two flat directions from two modes intersect at the self-dual point $b^2 = \eta = 1, \quad \beta = \xi = 0$; as a consequence, this self-dual point is stable in both y^1 and y^2 directions. We can obtain the same result if we examine other modes, $(-1, 0, -1, 0)$ and $(0, -1, 0, -1)$. Since these modes are massless, they stay at the minimum of the effective potential $V_{\rm eff}$ (4.88). The effective potential expanded around the minimum point can be expressed as

$$V_{\rm eff} = \mu_4 \sqrt{\frac{\overline{g}^{ab} p_a p_b}{\overline{g}^{(s)}}} + \frac{\mu_4 e^{\Phi}}{2\sqrt{\overline{g}^{(s)} \overline{g}^{ab} p_a p_b}} M^2(\beta, b, \eta, \xi). \tag{4.96}$$

We can examine the stability of the fixed point by perturbing the background around the minimum. If the flat directions at the minimum disappear, then the fixed point is stable. It turns out that the flat directions cancel each other in the total mass squared

$$\begin{aligned} \delta M^2_{1010} &= (\delta\xi - \delta\beta)^2 + (\delta\eta - 2\delta b)^2, && (4.97) \\ \delta M^2_{0101} &= (\delta\xi + \delta\beta)^2 + (\delta\eta + 2\delta b)^2, && (4.98) \\ \delta M^2_{\rm SU(2)} &= \delta M^2_{1010} + \delta M^2_{0101} = 2\delta\xi^2 + 2\beta^2 + 2\eta^2 + 8b^2. && (4.99) \end{aligned}$$

It implies that the $SU(2) \times SU(2)$ fixed point is stable.

4.3.2 $SU(3)$ Fixed Point

The second fixed point corresponding to the enhanced gauge symmetry $G_L = SU(3)$ appears at

$$b^2 = \eta = \frac{\sqrt{3}}{2}, \quad \beta = \xi = -\frac{1}{2}. \tag{4.100}$$

From (4.92) the compactification radius of this fixed point is $R = (\frac{\sqrt{3}}{2})^{\frac{1}{2}}\sqrt{\alpha'}$ and equivalent to $R = (\frac{2}{\sqrt{3}})^{\frac{1}{2}}\sqrt{\alpha'}$ by means of T-duality. It turns out that at this fixed point there are six massless modes corresponding to $(\pm 1, 0, \pm 1, 0)$, $(0, \pm 1, \pm 1, \pm 1)$ and $(\mp 1, \pm 1, 0, \pm 1)$. Analogous to the $SU(2) \times SU(2)$ fixed point, we will examine only three massless modes in order to determine the stable point.

Mode $(1, 0, 1, 0)$

The mass squared that contributes to the effective potential in this mode is

$$M^2_{1010} = \frac{1}{b^2\eta}(\xi - \beta)^2 + \frac{b^2}{\eta} + \frac{\eta}{b^2} - 2, \tag{4.101}$$

with the flat directions $\beta = \xi$ and $b^2 = \eta$. This mode corresponds to the closed string wrapping along the y^1 direction as well as that in the $SU(2) \times SU(2)$ fixed point.

Mode $(0, 1, 1, 1)$

The mass squared in this mode is

$$M^2_{0111} = \frac{1}{b^2\eta}(1 + \beta\xi)^2 + \frac{\eta\beta^2}{b^2} + \frac{b^2}{\eta}(\xi + 1)^2 + \eta b^2 + \frac{\beta}{b^2\eta}(\beta + 2\beta\xi + 2) - 2, \tag{4.102}$$

with flat directions $\xi + 1 = -\frac{\beta}{\beta^2 + b^4}$ and $(\xi + 1)^2 = \frac{\eta^2\beta^2}{b^4}$. This mode corresponds to the closed string wrapping along both y^1 and y^2 directions.

Mode $(-1, 1, 0, 1)$

$$\begin{aligned} M^2_{-1101} &= \frac{1}{b^2\eta}(1 + \beta\xi)^2 + \frac{\eta}{b^2}(1 + \beta)^2 + \frac{b^2}{\eta}(\eta^2 + \xi^2) \\ &\quad + \frac{\xi}{b^2\eta}(\xi + 2\beta\xi + 2) - 2, \end{aligned} \tag{4.103}$$

with flat directions $\beta + 1 = -\frac{\xi}{\xi^2 + \eta^2}$ and $(\beta + 1)^2 = \frac{b^4\xi^2}{\eta^2}$

We can see that these three flat directions from three modes intersect at the self-dual point of $SU(3)$: $b^2 = \eta = \frac{\sqrt{3}}{2}, \quad \beta = \xi = -\frac{1}{2}$. We can see that this fixed point is stable by examining the points around this fixed point. For simplicity we redefine the parameters b, ξ, η, β to

$$b^2 = \frac{\sqrt{3}}{2}\bar{b}^2, \quad \xi = -\frac{1}{2}\bar{\xi}, \quad \eta = \frac{\sqrt{3}}{2}\bar{\eta}, \quad \beta = -\frac{1}{2}\bar{\beta}. \tag{4.104}$$

From these new parameters, the $SU(3)$ fixed point is

$$\bar{b} = \bar{\xi} = \bar{\eta} = \bar{\beta} = 1. \tag{4.105}$$

The moduli parameters expanded around the fixed point are in the form

$$\bar{b} = 1 + \delta\bar{b}, \quad \bar{\xi} = 1 + \delta\bar{\xi}, \quad \bar{\eta} = 1 + \delta\bar{\eta}, \quad \bar{\beta} = 1 + \delta\bar{\beta}. \tag{4.106}$$

Then we can obtain the perturbed mass squared

$$\delta M^2_{1010} = \frac{1}{3}(\delta\bar{\xi} - \delta\bar{\beta})^2 + (\delta\bar{\eta} - 2\delta\bar{b})^2, \tag{4.107}$$

$$\begin{aligned} \delta M^2_{0111} &= \frac{1}{3}(\delta\bar{\xi}^2 + \delta\bar{\xi}\delta\bar{\beta} + \delta\bar{\beta}^2) \\ &\quad + (\delta\bar{\eta}^2 + 2\delta\bar{\eta}\delta\bar{b} + 4\delta\bar{b}^2 + \delta\bar{\eta}\delta\bar{\beta} - 2\delta\bar{\xi}\delta\bar{b}) \end{aligned} \tag{4.108}$$

$$\begin{aligned} \delta M^2_{-1101} &= \frac{1}{3}(\delta\bar{\xi}^2 + \delta\bar{\xi}\delta\bar{\beta} + \delta\bar{\beta}^2) \\ &\quad + (\delta\bar{\eta}^2 + 2\delta\bar{\eta}\delta\bar{b} + 4\delta\bar{b}^2 - \delta\bar{\eta}\delta\bar{\beta} + 2\delta\bar{\xi}\delta\bar{b}). \end{aligned} \tag{4.109}$$

It appears that the flat directions from all modes vanish in the total mass squared

$$\delta M^2_{\mathrm{SU(3)}} = \delta M^2_{1010} + \delta M^2_{0111} + \delta M^2_{-1101} = \delta\bar{\beta}^2 + \delta\bar{\xi}^2 + 3\delta\bar{\eta}^2 + 3\delta\bar{b}^2. \tag{4.110}$$

It means that the $SU(3)$ fixed point is the stable fixed point.

From these results one can conclude that the moduli of the compact space are stable at both fixed points. If the moduli of compact space start near the $SU(2)\times SU(2)$ fixed point, four string states, $(\pm 1, 0, \pm 1, 0)$ and $(0, \pm 1, 0, \pm 1)$, that are extra massless modes at this fixed point will become massive and thus stabilize the moduli at the $SU(2) \times SU(2)$ fixed point. Similarly, if the the moduli of compact space start around the $SU(3)$ fixed point, six string modes, $(\pm 1, 0, \pm 1, 0)$, $(0, \pm 1, \pm 1, \pm 1)$ and $(\mp 1, \pm 1, 0, \pm 1)$, which are extra massless modes at this fixed point, become massive and play the role in stabilizing the moduli.

Next, we investigate the dilaton stabilization by following the method in ref [22, 23]. The equations of motion are required to consider this issue. Since the parameter b^2 plays the role of scale factor, it can be considered as $b^2(t) \propto e^{2\nu(t)}$, where $\nu(t)$ is a function of the comoving time $x^0 = t$. The shape and flux moduli are then constrained as

$$
\begin{aligned}
\xi^2 + \eta^2 &= 1, & (4.111)\\
\beta &= \xi\frac{b^2}{\eta} = \frac{z}{2}e^{2\nu}, & (4.112)\\
\frac{b^2}{\eta} &= e^{2\nu}, & (4.113)
\end{aligned}
$$

where $z = 0$ for the $SU(2) \times SU(2)$ fixed point and $z = -1$ for the $SU(3)$ fixed point. As a consequence, the metric ansatz becomes

$$
ds^2 = -dt^2 + e^{2\lambda(t)}\sum_{a=1}^{3} dx^a dx^a + e^{2\nu(t)}((dy^1)^2 + zdy^1dy^2 + (dy^2)^2), \quad (4.114)
$$

and the background antisymmetric tensor takes the form

$$
B(t) = \frac{e^{2\nu(t)}}{2}\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}. \quad (4.115)
$$

Therefore, the equations of motion are (see **B.8 – B.10**)

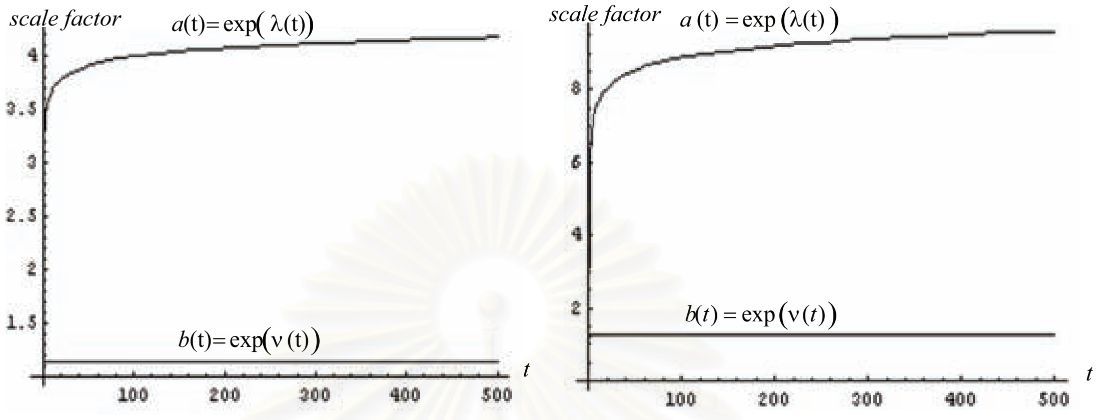
$$
\begin{aligned}
\ddot{\lambda} - 2\dot{\lambda}\dot{\phi} + 3\dot{\lambda}^2 + 2\dot{\lambda}\dot{\nu} &= k_{10}^2 p_\lambda e^{2\phi}, & (4.116)\\
\ddot{\nu} - 2\dot{\nu}\dot{\phi} + 3\dot{\lambda}\dot{\nu} + \frac{8}{4-z^2}\dot{\lambda}^2 &= k_{10}^2 p_\nu e^{2\phi}, & (4.117)\\
\ddot{\phi} - 2\dot{\phi}^2 + 3\dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\nu} + \frac{2z^2}{4-z^2}\dot{\lambda}^2 &= \frac{k_{10}^2}{2}Te^{2\phi}, & (4.118)
\end{aligned}
$$

where $T = -\rho + 3p_\lambda + 2p_\nu$ is the trace of the energy-momentum tensor, p_λ and p_ν are the pressure of the non-compact and compact directions, respectively. We can rewrite the last equation in the form

$$
\frac{d^2}{dt^2}(e^{-2\phi}) = -3\dot{\lambda}\frac{d}{dt}(e^{-2\phi}) - 2\dot{\nu}\frac{d}{dt}(e^{-2\phi}) - \frac{4z^2\dot{\nu}^2}{4-z^2}(e^{-2\phi}) - k_{10}^2 T. \quad (4.119)
$$

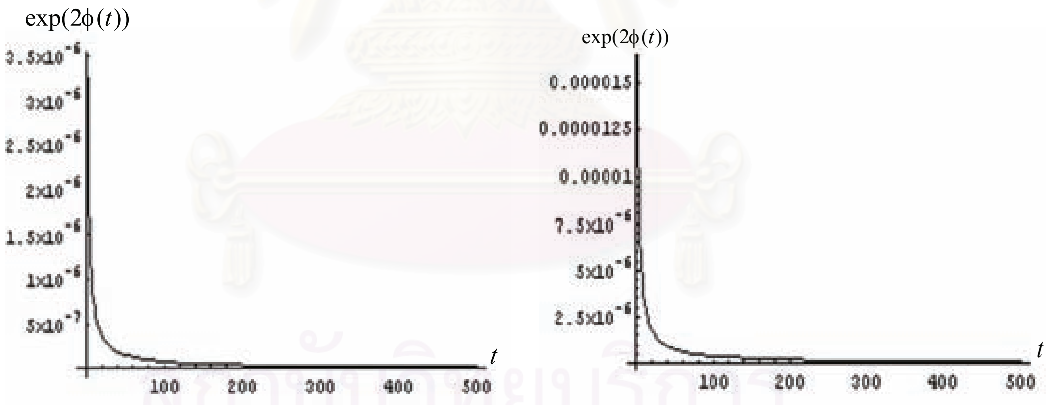
From the equation above it implies that at the fixed point $\nu = 0$ and $z = 0$ for the $SU(2) \times SU(2)$ fixed point ($z = -1$ for the $SU(3)$ fixed point), the energy-momentum tensor is traceless $T = 0$ and $\dot{\nu}$ is very close to zero. Therefore, the last three terms in the RHS of (4.119) vanish and the dilaton is stabilized by the damping term $-3\dot{\lambda}\frac{d}{dt}(e^{-2\phi})$ contributed from the expansion of the non-compact space. The situation is quite different for shifted dilaton Φ. It appears that as the mass squared $M^2(\beta, b, \eta, \xi)$ of heterotic string vanishes at the fixed points, the effective potential is independent to the shifted dilaton and the hubble expansion inhibits the shifted dilaton from moving along the flat direction. Therefore, the shifted dilaton is marginally stable at the fixed points.

To confirm the results obtained from the qualitative analysis, the evolutions of the scale factors of non-compact and compact space at $SU(2) \times SU(2)$ and $SU(3)$ fixed points are shown in the Figure 4.4 and the evolutions of the string coupling (dilaton) for both fixed points are shown in the Figure 4.5



(a) (b)

Figure 4.4: (a) The evolutions of scale factors at $SU(2) \times SU(2)$ fixed point (b) The evolutions of scale factors at $SU(3)$ fixed point



(a) (b)

Figure 4.5: (a) The evolution of string coupling at $SU(2) \times SU(2)$ fixed point (b) The evolution of string coupling at $SU(3)$ fixed point

CHAPTER V

SUMMARY AND DISCUSSION

In this chapter we review all articles throughout this thesis. Then the striking results in each part will be shown explicitly. We start from the basics of string theory and end at the moduli stabilization problem in the $O(d,d)$-covariant string gas cosmology. Thereafter, the ideas of further development are given in the final part of this chapter.

5.1 Concluding Remarks

In this thesis, various kinds of string theories are reviewed. Bosonic string theory describes the dynamics of one-dimensional fundamental particles, namely strings. There are three types of possible boundary conditions for the spacetime coordinates of bosonic strings which are periodic, Neumann, and Dirichlet conditions. Only strings satisfying periodic and Neumann boundary conditions or open and closed strings are emphasized. The constraint operators satisfy the Lie algebra, known as the Virasoro algebras. Because of the conformal anomaly, the only consistent quantum theory of bosonic strings is allowed for the 26-dimensional spacetime. The lowest states of closed and open strings represent tachyons whose masses squared are negative. The massless states of closed strings consist of the gravitons, Kalb-Ramond fields, and dilatons; on the other hand, those of open strings represent the photons. It appears that bosonic string theory cannot be a good candidate for unified theory due to the existence of the tachyonic states and the lack of fermionic states. In order to solve these problems the worldsheet supersymmetry and the two-dimensional Majorana spinors (the worldsheet fermions) are merged to basic descriptions of bosonic string theory. As a consequence, we obtain superstring theory in the worldsheet view point. Any fermionic field must satisfy either the periodic (R) or anti-periodic (NS) boundary condition. The constraint operators in this theory satisfy the graded Lie algebra, namely the super-Virasoro algebra. Similar to bosonic string, the anomaly cancellation allows the consistent quantum theory of superstring to exist in 10-dimensional spacetime. Truncating string states with the definite chirality, we can eliminate the tachyonic ground states and obtain the spacetime supersymmetry. This method is known as the GSO projection. Type I superstring theory consists of open strings and unoriented closed string. In this theory the lowest states of string spectrum are massless states representing the photons are their superpartners. For oriented closed strings, there are two types, type IIA and type IIB superstring theories corresponding to the chirality of left- and right-moving sectors. Both type IIA and type IIB theories contain gravitons and dilatons in the lowest states. The difference is that the massless states of type IIA also include the odd-form fields while those of type IIB include the even-form fields. Heterotic string theory consists of closed strings whose left-moving sectors are bosonic string and right-moving sectors are superstrings. There are two types for heterotic string theories corresponding to non-Abelian gauge groups, that are $SO(32)$ and $E_8 \times E_8$.

String gas cosmology (SGC) regarded as an application of string theory in cosmology is constructed under the assumption that the early universe of which all nine spatial dimensions are toroidally compactified with the size comparable to string scale consists of hot string gases with very weak coupling. T-duality implies that the smallest effective size of the universe is the string length; therefore, the universe is non-singular and never encounters the initial singularity problem. As a consequence, there exists the maximum temperature, known as the Hargedorn temperature. The annihilation of winding and anti-winding modes of strings allows the universe to expand and it can take place in the spacetime of which the number of dimensions does not exceed four. As a result, only three spatial dimensions can grow larger whereas the other six dimensions are still confined. The stability of extra dimensions becomes the essential problem for the late-time universe and this is called the moduli stabilization problem. In the first attempt, this problem is analyzed numerically under the assumption that the antisymmetric tensors vanish even in compact space. The result shows that the size moduli of extra dimensions are stabilized at the self-dual point and the string coupling is still very small. Studying the toroidal compactification of extra dimensions in the context of lattice, we can solve this problem qualitatively. Furthermore, this method allows us to deal with an $O(d, d; \mathbb{Z})$ symmetry, a generalization of T-duality, of the low energy effective action. The result shows that for the compact space $T^2 \times T^2 \times T^2$, there are two fixed points for heterotic strings, $SU(2) \times SU(2)$ and $SU(3)$. The former symmetry point is the fixed point for T-duality while the latter symmetry point is the fixed point for T-duality modulo the basis change and the shift on the antisymmetric tensor B with the integral-valued anti-symmetric matrix. The string coupling is still confined to the very small value in this approach. Furthermore, the effective potential of heterotic strings becomes the local minima at these fixed points.

5.2 Further Developments

5.2.1 Other Models for Compact Space

It is obvious that in this thesis the six-dimensional compact space is assumed to be $T^2 \times T^2 \times T^2$. In this toy model there are two stable fixed points for heterotic strings and the corresponding enhanced gauge symmetry groups, $SU(2) \times SU(2)$ and $SU(3)$. We can see that total number of simple roots for each symmetry group equal two, that is the number of dimensions of the torus T^2, and a maximally enhanced symmetry group must be a simply-laced group of rank 2 with the root length $\sqrt{2}$. This arises from the fact that for heterotic strings compactified in T^d the $U(1)^d$ gauge group at self-dual fixed point is enhanced to the $SU(2)^d$ gauge group and there can be other fixed points corresponding to the maximally enhanced symmetry groups which are the simply laced groups of rank d with the root length $\sqrt{2}$. As a consequence, if the compact space is assumed to be a product of other manifolds, we will obtain different fixed points with different enhanced gauge symmetry groups. However, the results always fall into the same conclusion.

As we mentioned in §3.3.2, there are many types of compactifications for string background. For the toroidal background, the six-dimensional compact metric G does not depend on the compact coordinates; as a consequence, the toroidal manifold is flat space and string equations of motion are in the same forms as those in the uncompactified theory. Despite the simplicity and the success of toroidal compactification, this model provides the four-dimensional $\mathcal{N} = 4$ supersymmetric theory and may not be an appropriate theory for our real world. This arises from the fact that a promising theory must be non-chiral or $\mathcal{N} = 1$ supersymmetric theory. We can solve this problem by replacing the toroidal compactification with more complicated and more plausible ones that provide the four-dimensional $\mathcal{N} = 1$ supersymmetric theories, for example, orbifold or Calabi-Yau compactificaions. An orbifold compactification can be simply considered as a six-dimensional toroidal compactification modified by identifying points on the torus that are mapped to each other by certain discrete symmetries of the lattice of torus. The resulting space is called the orbifold, that has some singular points corresponding to the fixed points of the discrete symmetries. The orbifold is flat all over, except at those singular points. At a singular point there exists a deficit angle corresponding to a rotational angle of the vector parallel transported along the closed path around this singular point. If the symmetry group, or equivalently the holonomy group in this case, is chosen to be an $SU(3)$ subgroup of $SO(6)$ group, one can obtain the $\mathcal{N} = 1$ supersymmetric theory in four-dimensional spacetime. A Calabi-Yau compactification can be regarded as a generalization of the orbifold compactification with an $SU(n)$ holonomy. More details on Calabi-Yau compactifications can be found in Refs [1, 2, 5, 10, 25, 27].

5.2.2 String Landscape and Anthropic Principle

From the result obtained in Chapter 4, it implies that there can be a vast number of string vacua depending on the possible configurations of the compact space. In other words, there is only one theory but many solutions that are characterized by the massless moduli describing the shape, volume and flux of compact space. The number of stable vacua calculated by Sen and others is of order between 10^{100} and 10^{500} [34]. If the energy of each possible vacuum is plotted as a function of the moduli, the graph looks similar to the landscape consisting of many peeks and valleys. This is known as the string landscape. From this paradigm each valley represents the stable vacuum corresponding to each fixed point of the moduli space. This means that each of these stable vacua, which correspond to the local minima of effective potential, possesses the exact spectrum of particles and the certain vacuum energy. The idea of string landscape sheds some light to the anthropic principle that concerns why the universe evolves in such a way that it allows the existence of carbon-based life forms, i.e. human beings. In cosmology the anthropic principle concerns the fine-tuning of the vacuum energy density ρ_v contributed from the cosmological constant. In spite of the value of vacuum energy density is very small, it cannot be zero ($\approx 0.8 \times 10^{-120}\mathrm{M}_\mathrm{P}^4$). Since the slight difference of the vacuum energy density leads to the different universe with obviously different rate of expansion, it is essential to determine the mechanism that provides such a vacuum energy density with very high precision or at least some possible solutions that contain such a vacuum energy density. We will see how string landscape solves this problem. It is the fact that the values of the vacuum energy density for possible string vacua range from $-\mathrm{M}_\mathrm{P}^4$ to $+\mathrm{M}_\mathrm{P}^4$ and the number of those possible vacua is about 100^{500}. Therefore, it is possible that there are some string vacua with the exact vacuum energy density we require. However, this also implies that there can be plenty of disconnected sub-universes with different expansion rates corresponding to their vacuum energy densities. We can demonstrate this scenario by supposing that the universe starts at some stable vacuum with the vacuum energy density around $+\mathrm{M}_{\mathrm{P}}^4$. Then there occurs the quantum tunnelling from this stable vacuum to other lower stable vacua at different points in the universe. Obviously, these different regions with different vacuum energy densities and different expansion rates are called sub-universes. More sub-universes can occur in a similar way. It is believed that the expansion rate of each sub-universe is greater than the speed of light; therefore, an event in one sub-universe cannot affect an event in another sub-universe, certainly.

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APPENDICES

APPENDIX A
LIE ALGEBRA LATTICE

A.1 Lie Algebras

A **Lie group** is a group of which all elements can be described by a function of continuous parameters. Let G be a Lie group and $g(\alpha)$ be a group element which depends on a set of N real continuous parameters, α_a; $a = 1, \ldots, n$. Since every group contains the identity element, e, it is conventional to parametrize elements in such a way that the identity element corresponds to $\alpha = 0$

$$g(\alpha)|_{\alpha=0} = e. \tag{A.1}$$

A representation of group G is a mapping, D of elements of group G onto a set of linear operators which obey

$$(i) \quad D(e) = \mathbb{I}, \qquad\qquad (ii) \quad D(g_1)D(g_2) = D(g_1 g_2). \tag{A.2}$$

Here, $\mathbb{I}$ is the identity operator. If a linear operator $D(g)$ acts on an N-dimensional vector space V, the **dimension** of the representation D is N. A representation D is **equivalent** to a representation D' if there exists a **similarity transformation** such that

$$D(g) \to D'(g) = S^{-1} D(g) S. \tag{A.3}$$

If any vector of a subspace of the linear vector space still leaves in the subspace under the action of $D(g)$, that subspace is called an **invariant subspace**. A representation D is said to be **reducible**, if it possesses invariant subspaces. A representation D is **irreducible** if it is not reducible. A representation D is **completely reducible** if it is equivalent to a representation in the **block diagonal form**

$$D(g) = \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{A.4}$$

where $D_i(g)$ is irreducible for $\forall i$. In other words, a representation $D(g)$ is completely reducible if it can be written as a **direct sum** of the irreducible representations $D_i(g)$

$$D(g) = D_1(g) \oplus D_2(g) \oplus \ldots . \tag{A.5}$$

It is obvious that a representation $D(g)$ of Lie group must be parametrized in the same way as the element $g(\alpha)$ of the Lie group; therefore, it is convenient to use $D(\alpha)$ in stead of $D(g(\alpha))$. From (1), one obtains

$$D(\alpha)|_{\alpha=0} = \mathbb{I}. \tag{A.6}$$

Then in the neighborhood of the identity element e, the linear operator $D(\alpha)$ can be expanded as

$$D(\alpha) = \mathbb{I} + i\alpha_a X_a + \mathcal{O}(\alpha^2), \tag{A.7}$$

where X_a are the generators of the group representation. From the expression above the generator T_a can be determined by

$$X_a = -i\frac{\partial D(\alpha)}{\partial \alpha_a}|_{\alpha=0}. \tag{A.8}$$

This parametrization suggests that the representation of the group element $g(\alpha)$ for finite α takes the exponential form

$$D(\alpha) = e^{i\alpha_a X_a}. \tag{A.9}$$

If D is the unitary representation ($D^{-1} = D^\dagger$), the generator X_a is hermitian ($X_a = X_a^\dagger$) and its eigenvalue is real number. The closure of the group G implies that the multiplication law of $D(\alpha) = e^{i\alpha_a X_a}$ must be

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\gamma_c X_c}. \tag{A.10}$$

The expression above is true if the generator X_a obeys the commutation relation

$$[X_a, X_b] = i f_{abc} X_c, \tag{A.11}$$

where f_{abc} are called the **structure constants** and satisfy

$$f_{abc} = -f_{bac}. \tag{A.12}$$

The **Lie algebra**, $\mathcal{G}$, is the vector space spanned by X_a with the commutation relation (A.12). In other words, the generators X_a of the representation the Lie group G form the Lie algebra $\mathcal{G}$. It is important to note that only the structure constants are sufficient for constructing the algebra and they are the same for all representations. It is the fact that the associativity of the group leads to another important relation, known as the **Jacobi identity**

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \tag{A.13}$$

A representation of the Lie algebra generated by structure constants are called the **adjoint representation**. The generators in this representation are the $n \times n$ matrices, T_a, defined by

$$[T_a]_{bc} = -if_{abc}. \tag{A.14}$$

Obviously, the dimension of the adjoint representation is the number of the independent generators, which is the number of the continuous parameters α_a. It is convenient to define the **Killing form** (T_a, T_b) of the generators as

$$(T_a, T_b) = \mathrm{Tr}(T_a T_b). \tag{A.15}$$

A **Lie subalgebra** $\mathcal{G}'$ of the Lie algebra $\mathcal{G}$ is a subspace of $\mathcal{G}$ whose elements themselves form the Lie algebra with the commutation relation (12). A subalgebra $\mathcal{G}'$ is said to be **invariant** if $[X_a, Y_b] \in \mathcal{G}'$ for $\forall X_a \in \mathcal{G}'$ and $\forall Y_b \in \mathcal{G}$. The whole algebra $\mathcal{G}$ and the empty set $\emptyset$ are also the invariant subalgebras, called the **trivial invariant subalgebras**. An invariant subalgebra $\mathcal{G}'$ is said to be **Abelian** if there exist elements of $\mathcal{G}'$ that commute with all elements of $\mathcal{G}$. A **simple Lie algebra** is a Lie algebra that is not Abelian and possesses only the trivial invariant subalgebras. It turns out that an adjoint representation of a simple Lie algebra is irreducible. A **semi-simple Lie algebra** is a Lie algebra that does not possess the Abelian invariant subalgebras. It is the fact that the semi-simple Lie algebra is a simple Lie algebra or a direct sum of the Lie subalgebras, i.e. the Lie algebra of $G_1 \times G_2$ is isomorphic to the direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$. Therefore, the structure of simple Lie algebra can be regarded as a basis of the semi-simple Lie algebras that play many important roles in physics.

A.2 Structure of Simple Lie Algebras

Cartan Subalgebras, Weights, and Roots

A systematic way to study the representation of simple Lie algebras is the root system. First, we must determine the largest subspace of commuting hermitian generators, known as the Cartan subalgebra. In a representation D, the commuting generators corresponding to elements of the Cartan subalgebra are called the Cartan generators and denoted by H_i; $i = 1, \ldots, m$. Here, m is the number of commuting hermitian generators and is called the **rank** of the Cartan subalgebra. From definition, we obtain

$$[H_i, H_j] = 0. \tag{A.16}$$

Since any Cartan generator commutes to one another, it is possible to simultaneously diagonalize all Cartan generators and to use their eigenvalues to label the states. After the Cartan generators are simultaneously diagonalized, the states of representation D, denoted by $|\mu_i; D\rangle$, are the eigenvectors of the Cartan generators

$$H_i|\mu_i; D\rangle = \mu_i|\mu_i; D\rangle, \tag{A.17}$$

where the eigenvalue μ_i are called the **weights**. The m-dimensional vector consisting of μ_i as components is called the **weight vector**. We can choose a basis $\{H\}$ in such a way that the Killing form can be written as

$$(H_i, H_j) = \mathrm{Tr}(H_i H_j) = \lambda\delta_{ij} \tag{A.18}$$

where λ is a positive constant. The weight is **positive** if the first non-zero component is positive and the weight is **negative** if the first non-zero component is negative. Then, it is conventional to define the ordering of the weight is defined by

$$\mu > \nu \quad \text{if} \quad \mu - \nu \quad \text{is} \quad \text{positive}. \tag{A.19}$$

If a representation D is the adjoint representation, the dimension of representation is equal to the number of generators. It implies that the states of representation correspond to the generators themselves and can be labelled by $|X_a\rangle$. In this representation, the weights are called **roots** and the weight vector is called the **root vector**. Since every Cartan generator commutes to one another, all states, $|H_i\rangle$, corresponding to the Cartan generators have zero weight (root) vectors in the adjoint representation. The other states, denoted by $|E_\alpha\rangle$, do not correspond to the Cartan generators and have non-zero root vectors, α, with components α_i determined by

$$H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle. \tag{A.20}$$

This means that the commutator between H_i and E_α ($E_{-\alpha}$) is

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [H_i, E_{-\alpha}] = -\alpha_i E_{-\alpha}. \tag{A.21}$$

Then, we can define the raising and lowering operators $E_{\pm\alpha}$ and use them to construct other states. However, one must consider the ordering of the root vectors in order to specify which one is raising or lowering operator. It turns out that the positive (negative) root vector corresponds to the raising (lowering) operator $E_{+\alpha}$ ($E_{-\alpha}$). Since we use an entire root vector, not its components, to study the structure of the algebra, from now on we will call a root vector as a root for short. If $|\mu; D\rangle$ is a state with the weight μ of representation D and there are integer numbers $p^{(\mu)}$ and $q^{(\mu)}$ for each root α in such a way that

$$(E_{+\alpha})^{1+p^{(\mu)}}|\mu; D\rangle = 0, \quad (E_{-\alpha})^{1+q^{(\mu)}}|\mu; D\rangle = 0, \tag{A.22}$$

the weight μ satisfies the relation, namely the **master formula**

$$\frac{2\alpha \cdot \mu}{\alpha^2} = q^{(\mu)} - p^{(\mu)}. \tag{A.23}$$

At this point, it is essential to note that some positive roots can be considered as a linear combination of other roots. A **simple root** is defined as a positive root that cannot be written as a sum of other positive roots. This means that the simple roots are not only linearly independent, but also complete. As a consequence, the number of the simple roots is the rank of the Cartan subalgebra, and any positive root can be written as a linear combination of simple roots with non-negative integer coefficients. From now on, the I^{th} simple root is denoted by α_I where $I = 1, \ldots, m$. Therefore, the angle between two simple roots α_I and α_J can be determined by

$$\cos^2\theta = \frac{(\alpha_I \cdot \alpha_J)^2}{\alpha_I^2 \alpha_J^2} = \frac{(q_I^{(\alpha_J)} - p_I^{(\alpha_J)})(q_J^{(\alpha_I)} - p_J^{(\alpha_I)})}{4}. \tag{A.24}$$

where $p_I^{(\alpha_J)}$ and $q_I^{(\alpha_J)}$ satisfy the equation (22) for the state with the weight $\mu = \alpha_J$ acted by the operators $E_{\pm\alpha_I}$. It is important to note that there are only four possibilities for the angle between two simple roots, that are $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, and $\frac{5\pi}{6}$.

The advantage of the root system is that the whole algebra can be constructed from the simple roots. Any state annihilated by the generators of all simple roots is the state with the **highest weight**. We can construct the whole algebra by applying the appropriate lowering generators to the highest-weight state and this procedure is called the **highest weight construction**. If μ_J; $J = 1, \ldots, m$ is the weight vector satisfying

$$\frac{2\alpha_I \cdot \mu_J}{\alpha_I^2} = \delta^{IJ}, \tag{A.25}$$

every highest weight μ can be written as

$$\mu = \sum_{J=1}^{m} q_J^{(\mu)} \mu_J. \tag{A.26}$$

The vectors μ_J are called the **fundamental weights** and the m irreducible representations of which highest weights are components of the associated μ_J are called the **fundamental representations**,denoted by $D^{(J)}$.

From the simple roots, the **Cartan matrix**, C_{IJ}, is defined by

$$C_{IJ} = 2\frac{\alpha_I \cdot \alpha_J}{\alpha_J^2}. \tag{A.27}$$

The property of Cartan matrix is that all diagonal elements of Cartan matrix are equal to 2 and any off-diagonal element can be only 0,-1,-2, or -3. It is obvious that the Cartan matrix contains the information about the angle between two simple roots and the relative length of these roots.

Dynkin Diagram and Classification of Simple Lie Algebras

Next, we introduce the **Dynkin Diagram**, which is a short-notation for describing the simple roots of the algebra. It appears that the Dynkin diagram carries the information about angle and relative lengths of simple roots as well as the Cartan matrix, but in different form. In the Dynkin diagram, every simple root is indicated by an open dot. If a filled dot appears in the diagram, it means that the length of the corresponding simple root is shorter than that of open one. The line between two dots represents the angle between the pair of simple roots, as follows:

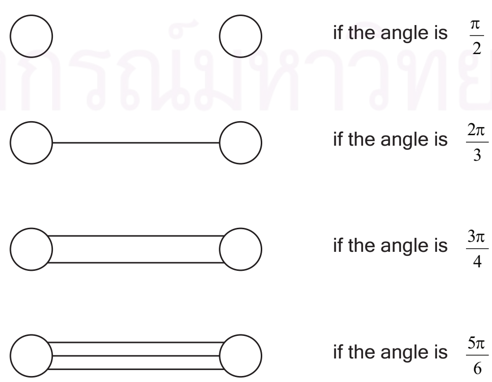

Figure A.1: Four possible angles between two simple roots

At this point, the classification of simple Lie algebras is given in terms of the Dynkin diagram. There are 7 types of simple Lie algebras, labelled by A to G, such as A_n where the subscript n is the rank of the algebra, which is the number of the simple roots. It turns out that these 7 types of simple Lie algebras can be divided into two groups, that are 4 infinite families, known as the classical Lie algebras and 3 finite families, known as the exceptional Lie algebras. The Dynkin diagrams of simple Lie algebras are shown in the figure below

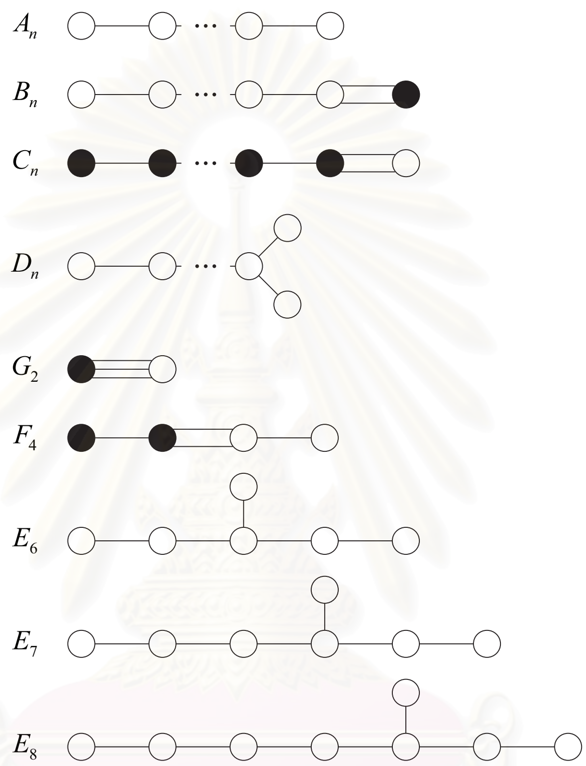

Figure A.2: Dynkin Diagrams of simple Lie algebras

For classical Lie algebras, the infinite series of root systems A_n, B_n; $n \geq 3$, C_n; $n \geq 2$, and D_n; $n \geq 4$ correspond to $SU(n+1)$, $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$ Lie algebras, respectively. We can see that the Lie groups generated by the classical Lie algebras can be considered as the groups of rotation in some vector spaces. For exceptional Lie group, there are only 5 allowed root systems, which are E_6, E_7, E_8, F_4, and G_2; besides, there is no geometrical picture for these algebras.

A **simply laced algebra** is the semi-simple Lie algebra whose all roots have the same lengths. From the Dynkin diagrams of simple Lie algebras above, we can see that a simply laced algebra can be a simple Lie algebra A_n, D_n, E_6, E_7, E_8, or a direct sum of these simple Lie algebras consisting of simple roots with equal lengths. The examples for simply laced lattices of rank one and two are provided as follows. For the simply laced lattice of rank one, it turns out that there is only one distinct lattice corresponding to A_1 or the $SU(2)$ Lie algebra. For the simply laced lattice of rank two, there are two possible lattices corresponding to $A_1 \times A_1$ or $SU(2) \times SU(2)$, and A_2 or $SU(3)$ Lie algebras.

A.3 Simply Laced Lattices as Self-Dual Lattice

Let V be an N-dimensional vector space, a **lattice** Λ is the space of all linear combinations of basis vectors of the vector space V with integer coefficients, and takes the form

$$\Lambda = \{\sum_{a=1}^{N} n^a \mathbf{e}_a | n^a \in, \mathbb{Z}\}, \tag{A.28}$$

where $\mathbf{e}_a$; $a = 1, \ldots, N$ is a basis vector of the vector space. The vector space we consider is either $\mathbb{R}^N$ with a **Euclidean** inner product, i.e., $v \cdot w = \sum_{i=1}^{N} v^i w^i$, where v^i and w^i are components of $v, w \in V$, or $\mathbb{R}^{(p,q)}$ $(p + q = N)$ with a **Lorentzian** inner product, i.e., $v \cdot w = \sum_{i=1}^{p} v^i w^i - \sum_{j=1}^{q} v^j w^j$. Then, it is conventional to define the metric of the lattice, denoted by g_{ab}, as the inner product of the basis vectors

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b. \tag{A.29}$$

The **unit cell** of a lattice is a set of points $\{x = \sum_a x_a \mathbf{e}_a \,|0 \leqslant x < 1, x \in \mathbb{R}\}$. Obviously, the unit cell contains only one lattice point, that is $x = 0$. The volume, vol(Λ), of the unit cell is determined by

$$\text{vol}(\Lambda) = \sqrt{|\text{det} g_{ab}|}. \tag{A.30}$$

A **dual lattice** Λ^* of Λ is a space of vectors of V whose inner products with elements of Λ are integers

$$\Lambda^* = \{w \in V | v \cdot w \in \mathrm{Z}, \forall v \in \Lambda\}. \tag{A.31}$$

If the basis vectors of Λ^* are given by $\mathbf{e}^{*a}$ with $\mathbf{e}^{*a} \cdot \mathbf{e}_b = \delta^a_b$, the dual lattice takes the form

$$\Lambda^* = \{\sum_{a=1}^{N} m_a \mathbf{e}^{*a} | m_a \in \mathbb{Z}\}. \tag{A.32}$$

The metric of the dual lattice Λ^* is denoted by g^{ab} and is determined by $g^{ab} = \mathbf{e}^{*a} \cdot \mathbf{e}^{*b}$. It is obvious that g^{ab} of Λ^* is the inverse of g_{ab} of Λ; therefore, we obtain $\text{vol}(\Lambda) = 1/\text{vol}(\Lambda^*)$.

A lattice Λ is said to be
- **unimodular** if $\text{vol}(\Lambda) = 1$,
- **integral** if $v \cdot w \in \mathbb{Z}$, $\forall v, w \in \Lambda$,
- **even** if Λ is integral, and $v^2 \in 2\mathbb{Z}$, $\forall v \in \Lambda$,
- **odd** if Λ is integral, but not even,
- **self-dual** if $\Lambda = \Lambda^*$.

It is the fact that Λ is integral if and only if $\Lambda \subset \Lambda^*$. As a consequence, Λ is self-dual if and only if Λ is integral and unimodular.

If the basis vectors of the vector space V are the simple roots of the semi-simple Lie algebra $\mathcal{G}$, a lattice Λ is called a **root lattice** Λ_R of $\mathcal{G}$. This means that the dimension of the vector space V is equal to the number of the simple roots as well as the rank of the algebra. A set of all linear combinations of the fundamental weights with integer coefficients is also a lattice, called the **weight lattice** Λ_W of $\mathcal{G}$. At this point, we can see from the master formula (A.23) that if all root are normalized in such a way that their squared lengths are equal to 2, the inner product of two roots is always an integer and the squared length of any vector in Λ_R or in Λ_W is also an integer. This means that

$$\Lambda_R \subset \Lambda_W, \quad \Lambda_W = \Lambda_R^*. \tag{A.33}$$

As a consequence, a self-dual lattice can be considered as a sublattice of the weight lattice of $\mathcal{G}$. The elements of the quotient space Λ_W/Λ_R are called the conjugacy classes, which form a group isomorphic to the center of the Lie algebra, i.e. the set of which element commutes with all elements. We can obtain the **even Lorentzian self-dual** lattice $\Gamma^{(p,p)}$ in the following way. First, we choose the semi-simple Lie algebra $\mathcal{G}$ of rank p. Then, we collect all vectors in $\mathbb{R}^{(p,p)}$ of the form $(\mathbf{v}_1, \mathbf{v}_2)$, such that $\mathbf{v}_1, \mathbf{v}_2$ belong to the same conjugacy class. As a result, the lattice we choose is unimodular, integral and even-self dual.

APPENDIX B

CALCULATIONS

Equations (3.67)-(3.70)

We give the indices $I = 1, \ldots, d_\lambda$ and the index $J = 1, \ldots, d_\nu$ label the components of expanding spatial dimensions ($d_\lambda = 3$) and the components of extra dimensions ($d_\nu = 6$). From the metric ansatz (3.60) and $B_{\mu\nu} = 0$, we can compute the connection coefficients and the Ricci tensors. The non-vanishing connection coefficients and Ricci tensors are

$$\begin{aligned}
\Gamma^I_{0I} &= \dot{\lambda}, \quad \Gamma^J_{0J} = \dot{\nu}, \quad \Gamma^0_{II} = \dot{\lambda} e^{2\lambda}, \quad \Gamma^0_{JJ} = \dot{\nu} e^{2\nu}, \\
R_{00} &= -(d_\lambda \ddot{\lambda} + d_\nu \ddot{\nu} + d_\lambda \dot{\lambda}^2 + d_\nu \dot{\nu}^2), \quad R_{II} = e^{2\lambda}(\ddot{\lambda} + d_\lambda \dot{\lambda}^2 + d_\nu \dot{\lambda}\dot{\nu}) \\
R_{JJ} &= e^{2\nu}(\ddot{\nu} + d_\nu \dot{\nu}^2 + d_\lambda \dot{\lambda}\dot{\nu}).
\end{aligned}$$

Then we express equations (3.60)-(3.61) in the forms

$$\begin{aligned}
g^{\mu\rho} R_{\rho\nu} + 2g^{\mu\rho}(\partial_\rho \partial_\nu \phi - \Gamma^\sigma_{\rho\nu} \partial_\sigma \phi) - \frac{1}{4} g^{\mu\kappa} H_{\kappa\rho\sigma} H^{\rho\sigma}_\nu &= k_D^2 e^{2\phi} T^\mu_\nu, \\
4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4g^{\mu\nu}(\partial_\mu \partial_\nu \phi - \Gamma^\kappa_{\mu\nu} \partial_\kappa \phi) &= 0.
\end{aligned}$$

The equations of motion then become

$$-d_\lambda \ddot{\lambda} - d_\nu \ddot{\nu} - d_\lambda \dot{\lambda}^2 - d_\nu \dot{\nu}^2 + 2\ddot{\phi} = k_D^2 e^{2\phi} \rho, \tag{B.1}$$

$$\ddot{\lambda} + d_\lambda \dot{\lambda}^2 + d_\nu \dot{\lambda}\dot{\nu} - 2\dot{\lambda}\dot{\phi} = k_D^2 e^{2\phi} p_\lambda, \tag{B.2}$$

$$\ddot{\nu} + d_\nu \dot{\nu}^2 + d_\lambda \dot{\lambda}\dot{\nu} - 2\dot{\nu}\dot{\phi} = k_D^2 e^{2\phi} p_\nu, \tag{B.3}$$

$$4\ddot{\phi} - 2d_\lambda \ddot{\lambda} - 2d_\nu \ddot{\nu} - 4\dot{\phi}^2 + 4\dot{\phi}(d_\lambda \dot{\lambda} + d_\nu \dot{\nu}) - (d_\lambda \dot{\lambda} + d_\nu \dot{\nu})^2 - d_\lambda \dot{\lambda}^2 - d_\nu \dot{\nu}^2 = 0. \tag{B.4}$$

Equations (4.39)-(4.40)

From equations (4.18) and (4.19), p_L^2 and p_R^2 can be expressed as

$$\begin{aligned}
p_L^2 &= G_{IJ} G^{IL} G^{JM} \left(\frac{m_L}{R^{(L)}} + (G_{LK} - B_{LK}) \frac{n^K R^{(K)}}{\alpha'}\right)\left(\frac{m_M}{R^{(M)}} + (G_{MN} - B_{MN} \frac{n^N R^{(N)}}{\alpha'}\right) \\
&= G^{LM} \Big(2G_{MN} \frac{m_L}{R^{(L)}} \frac{n^N R^{(N)}}{\alpha'} + G_{LK} G_{MN} \frac{n^K R^{(K)}}{\alpha'} \frac{n^N R^{(N)}}{\alpha'} - 2B_{MN} \frac{m_L}{R^{(L)}} \frac{n^N R^{(N)}}{\alpha'} \\
&\quad + \frac{m_L m_M}{R^{(L)} R^{(M)}} - 2G_{LK} B_{MN} \frac{n^K R^{(K)}}{\alpha'} \frac{n^N R^{(N)}}{\alpha'} + B_{LK} B_{MN} \frac{n^K R^{(K)}}{\alpha'} \frac{n^N R^{(N)}}{\alpha'}\Big),
\end{aligned}$$

$$
\begin{aligned}
p_R^2 &= G_{IJ}G^{IL}G^{JM}(\frac{m_L}{R^{(L)}} - (G_{LK}+B_{LK})\frac{n^K R^{(K)}}{\alpha'})(\frac{m_M}{R^{(M)}} - (G_{MN}+B_{MN}\frac{n^N R^{(N)}}{\alpha'}) \\
&= G^{LM}(-2G_{MN}\frac{m_L}{R^{(L)}}\frac{n^N R^{(N)}}{\alpha'} + G_{LK}G_{MN}\frac{n^K R^{(K)}}{\alpha'}\frac{n^N R^{(N)}}{\alpha'} - 2B_{MN}\frac{m_L}{R^{(L)}}\frac{n^N R^{(N)}}{\alpha'} \\
&\quad + \frac{m_L m_M}{R^{(L)}R^{(M)}} + 2G_{LK}B_{MN}\frac{n^K R^{(K)}}{\alpha'}\frac{n^N R^{(N)}}{\alpha'} + B_{LK}B_{MN}\frac{n^K R^{(K)}}{\alpha'}\frac{n^N R^{(N)}}{\alpha'}).
\end{aligned}
$$

After rescaling G_{IJ} and B_{IJ} by using (4.38), we combine (C.1) and (C.2) to obtain equation (4.39)

$$
p_L^2 + p_R^2 = \frac{2}{\alpha'}(G^{LM}m_L m_M + G_{LM}n^L n^M + G^{LM}B_{LK}B_{MN}n^K n^N - 2G^{LM}B_{MK}m_L n^K). \quad \text{(B.5)}
$$

It turns out that $p_L^2 + p_R^2$ can be expressed in terms of the background matrix M and the vector $Z = \begin{pmatrix} m \\ n \end{pmatrix}$

$$
\begin{aligned}
\frac{2}{\alpha'}Z^T M Z &= \frac{2}{\alpha'}\begin{pmatrix} m^T & n^T \end{pmatrix}\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} \\
&= \frac{2}{\alpha'}(m^T G^{-1} m + n^T G n - n^T B G^{-1} B n + n^T B G^{-1} m - m^T G^{-1} B n) \\
&= \frac{2}{\alpha'}(G^{LM}m_L m_M + G_{LM}n^L n^M \\
&\quad + G^{LM}B_{LK}B_{MN}n^K n^N - 2G^{LM}B_{MK}m_L n^K), \qquad \text{(B.6)}
\end{aligned}
$$

where $m^T = \begin{pmatrix} m_1, & \ldots, & m_d \end{pmatrix}$ and $n^T = \begin{pmatrix} n^1, & \ldots, & n^d \end{pmatrix}$.

Equation (4.82)

The metric and the Kalb-Ramond tensors are assumed to be in the forms

$$
(G_{\mu\nu}) = \begin{pmatrix} (g_{\alpha\beta}(x^\alpha)) & 0 \\ 0 & (G_{IJ}(x^\alpha)) \end{pmatrix}, \quad (B_{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & (B_{IJ}(x^\alpha)) \end{pmatrix}.
$$

The non-vanishing Christoffel coefficients are

$$
\begin{aligned}
\Gamma^I_{J\alpha} &= \frac{1}{2}G^{IK}\partial_\alpha G_{JK}, \quad \Gamma^\alpha_{IJ} = -\frac{1}{2}g^{\alpha\beta}\partial_\beta G_{IJ} \\
\Gamma^\alpha_{\beta\gamma} &= \frac{1}{2}g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}).
\end{aligned}
$$

In order to obtain the Ricci scalar, we determine the Ricci tensors $R_{\alpha\beta}$ and R_{IJ}

$$
\begin{aligned}
R_{\alpha\beta} &= (\partial_\gamma\Gamma^\gamma_{\alpha\beta} - \partial_\beta\Gamma^\gamma_{\gamma\alpha} + \Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\delta\gamma} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma}) - \partial_\beta\Gamma^I_{I\alpha} + \Gamma^\gamma_{\alpha\beta}\Gamma^I_{J\gamma} - \Gamma^J_{\alpha J}\Gamma^I_{\beta J} \\
&= R^{(d_n+1)}_{\alpha\beta} + \frac{1}{4}g^{\gamma\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta})G^{IJ}\partial_\gamma G_{IJ} - \frac{1}{2}\partial_\beta G^{IJ}\partial_\alpha G_{IJ} \\
&\quad -\frac{1}{2}G^{IJ}\partial_\alpha\partial_\beta G_{IJ} - \frac{1}{4}G^{JK}G^{IL}\partial_\alpha G_{IK}\partial_\beta G_{JL} \\
R_{IJ} &= \partial_\alpha\Gamma^\alpha_{IJ} + \Gamma^\alpha_{IJ}\Gamma^\beta_{\beta\alpha} + \Gamma^\alpha_{IJ}\Gamma^K_{K\alpha} - \Gamma^K_{I\alpha}\Gamma^\alpha_{JK} - \Gamma^\alpha_{IK}\Gamma^K_{L\alpha} \\
&= -\frac{1}{2}\partial_\alpha g^{\alpha\beta}\partial_\beta G_{IJ} - \frac{1}{2}g^{\alpha\delta}\partial_\alpha\partial_\delta G_{IJ} - \frac{1}{4}g^{\alpha\nu}\partial_\nu G_{IJ}g^{\beta\delta}(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\beta\alpha}) \\
&\quad -\frac{1}{4}g^{\alpha\beta}\partial_\beta G_{IJ}G^{KL}\partial_\alpha G_{KL} + \frac{1}{2}G^{KL}g^{\alpha\beta}\partial_\alpha G_{IL}\partial_\beta G_{JK}.
\end{aligned}
$$

Using the identity $\partial_\mu G^{\rho\sigma} = -G^{\rho\kappa}G^{\lambda\sigma}\partial_\mu G_{\kappa\lambda}$, we obtain the Ricci scalar

$$
\begin{aligned}
R &= g^{\alpha\beta}R_{\alpha\beta} + G^{IJ}R_{IJ} \\
&= g^{\alpha\beta}R^{(d_n+1)}_{\alpha\beta} - g^{\alpha\beta}G^{IJ}\partial_\alpha\partial_\beta G_{IJ} - \frac{3}{4}g^{\alpha\beta}\partial_\beta G^{IJ}\partial_\alpha G_{IJ} - \frac{1}{4}g^{\alpha\beta}G^{IJ}\partial_\alpha G_{IJ}G^{KL}\partial_\alpha G_{KL} \\
&\quad -G^{IJ}\partial_\alpha G_{IJ}\partial_\beta g^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}G^{IJ}\partial_\alpha G_{IJ}g^{\gamma\delta}\partial_\beta g_{\gamma\delta} \\
&= R^{(d_n+1)} + \frac{1}{4}g^{\alpha\beta}\partial_\beta G^{IJ}\partial_\beta G_{IJ} + g^{\alpha\beta}\partial_\alpha \ln\sqrt{G^{(d_c)}}\partial_\beta \ln\sqrt{G^{(d_c)}} - 2\partial_\beta g^{\alpha\beta}\partial_\alpha \ln\sqrt{G^{(d_c)}} \\
&\quad -2g^{\alpha\beta}\partial_\alpha \ln\sqrt{-g^{(1+d_n)}}\partial_\beta \ln\sqrt{G^{(d_c)}} - \frac{2g^{\alpha\beta}}{\sqrt{G^{(d_c)}}}\partial_\alpha\partial_\beta\sqrt{G^{(d_c)}},
\end{aligned}
$$

where $|\det g_{\alpha\beta}| = -g^{(1+d_n)}$ and $|\det G_{IJ}| = G^{(d_c)}$. Here, in the last line we used the identity $GG^{\mu\nu}\partial_\rho G_{\mu\nu} = \partial_\rho G$ and the following consequence $(1/\sqrt{G})\Box\sqrt{G} = \frac{1}{2}G^{\mu\nu}\Box G_{\mu\nu} + \frac{1}{2}\partial_\rho G^{\mu\nu}\partial^\rho G_{\mu nu} + \frac{1}{4}G^{\mu\nu}\partial_\rho G_{\mu\nu}G^{\kappa\lambda}\partial^\rho G_{\kappa\lambda}$.

The components of field strength $H = \mathtt{d}B$ in (2.255) can be determined by

$$H_{\mu\nu\rho} = 3_{[}\partial_\mu B_{\nu\rho]}.$$

Therefore, the non-vanishing components of H_{ABC} are

$$H_{\alpha IJ} = -H_{I\alpha J} = H_{IJ\alpha} = \partial_\alpha B_{IJ},$$

and the square of field strength H^2 is in the form

$$
\begin{aligned}
H^2 &= H^{\alpha IJ}H_{\alpha IJ} + H^{I\alpha J}H_{I\alpha J} + H^{IJ\alpha}H_{IJ\alpha} \\
&= 3G^{\alpha\beta}G^{IK}G^{JL}\partial_\alpha B_{IJ}\partial_\beta B_{KL}.
\end{aligned}
$$

After eliminating the divergence term, the low-energy effective action (2.256) takes the form

$$
\begin{aligned}
S_0 &= -\frac{V_{d_c}}{2k_D^2}\int d^{d_n+1}x\sqrt{-g^{(1+d_n)}}e^{-\Phi}(R^{(d_n)} + \partial_\alpha\Phi\partial^\alpha\Phi + \frac{1}{4}\partial^\alpha G^{IJ}\partial_\alpha G_{IJ} \\
&\quad -\frac{1}{4}G^{IK}G^{JL}\partial^\alpha B_{IJ}\partial_\alpha B_{KL}),
\end{aligned}
$$

where $\Phi \equiv 2\phi - \ln\sqrt{G^{(d_c)}}$ is the shifted dilaton and $V_{d_c} \equiv \int d^{d_c}x$ is the volume of compact space.

Next, we will see that the equation above can be written in terms of the background matrix $M(E)$. It is not difficult to show that $\mathrm{Tr}(\partial^\mu M\eta\partial_\mu M\eta)$ can be expressed as

$$\begin{aligned}
\mathrm{Tr}(\partial^\alpha M\eta\partial_\alpha M\eta) &= \partial^\alpha(B_{IJ}G^{JL})\partial_\alpha(B_{LK}G^{KI}) + \partial^\alpha(G_{IL} - B_{IJ}G^{JK}B_{KL})\partial_\alpha G^{LI} \\
&\quad + \partial^\alpha G^{IL}\partial_\alpha(G_{LI} - B_{LJ}G^{JK}B_{KI}) + \partial^\alpha(G^{IJ}B_{JL})\partial_\alpha(G^{LK}B_{KI}) \\
&= 2\partial^\mu G^{ad}\partial_\mu G_{ad} + 2G^{bd}G^{ca}\partial^\mu B_{ab}\partial_\mu B_{dc} \\
&= 2\partial^\alpha G^{IJ}\partial_\alpha G_{IJ} - 2G^{IK}G^{JL}\partial^\alpha B_{IJ}\partial_\alpha B_{KL}.
\end{aligned}$$

As a result, the low-energy effective action can be expressed in the $O(d,d)$-covariant form as

$$S_0 = -\frac{V_{d_c}}{2k_D^2}\int d^{d_n}x\sqrt{-g^{(1+d_n)}}e^{-\Phi}(R^{(d_n)} + \partial_\alpha\Phi\partial^\alpha\Phi + \frac{1}{8}\mathrm{Tr}(\partial^\alpha M\eta\partial_\alpha M\eta)). \quad \text{(B.7)}$$

Equations (4.16) - (4.18)

Since in our consideration we give $d_n = 2$, it is convenient to use $\mathbf{1}, \mathbf{2}$ as the indices of the compact components. From the metric ansatz (4.114), we can compute the connection coefficients and the Ricci tensors. The non-vanishing connection coefficients and Ricci tensors are

$$\begin{aligned}
\Gamma^a_{0a} &= \dot\lambda, \quad \Gamma^{\mathbf{1}}_{0\mathbf{1}} = \Gamma^{\mathbf{2}}_{0\mathbf{2}} = \dot\nu, \quad \Gamma^0_{aa} = \dot\lambda e^{2\lambda}, \quad \Gamma^0_{\mathbf{11}} = \Gamma^0_{\mathbf{22}} = \dot\nu e^{2\nu}, \quad \Gamma^0_{\mathbf{12}} = \frac{z}{2}\dot\nu e^{2\nu} \\
R_{00} &= -(d_n\ddot\lambda + d_c\ddot\nu + d_n\dot\lambda^2 + d_c\dot\nu^2), \quad R_{aa} = e^{2\lambda}(\ddot\lambda + d_n\dot\lambda^2 + d_c\dot\lambda\dot\nu) \\
R_{\mathbf{11}} &= R_{\mathbf{22}} = e^{2\nu}(\ddot\nu + d_c\dot\nu^2 + d_n\dot\lambda\dot\nu), \quad R_{\mathbf{12}} = \frac{1}{2}e^{2\nu}(\ddot\nu + d_c\dot\nu^2 + d_n\dot\lambda\dot\nu).
\end{aligned}$$

From the Kalb-Ramond tensor (4.115), the non-vanishing field strength tensors and the square of the field strength are

$$H_{0\mathbf{12}} = -H_{0\mathbf{21}} = z\dot\nu e^{2\nu}, \quad H^2 = -\frac{12d_c}{4-z^2}z^2\dot\nu^2.$$

Inserting these terms into the equations of motion in general case

$$\begin{aligned}
&R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\rho\sigma}H^{\rho\sigma}_\nu = k_D^2 e^{2\phi}T_{\mu\nu}, \\
&R - 4\nabla_\mu\phi\nabla^\mu\phi + 4\nabla_\mu\nabla^\mu\phi - \frac{1}{12}H^2 = 0, \\
&2\nabla_\mu\phi\nabla^\mu\phi - \nabla_\mu\nabla^\mu\phi - \frac{1}{12}H^2 = \frac{k_D^2 e^{2\phi}T}{2}.
\end{aligned}$$

we obtain

$$-d_n\ddot{\lambda} - d_c\ddot{\nu} - d_n\dot{\lambda}^2 - d_c\dot{\nu}^2 + 2\ddot{\phi} - \frac{4d_n}{4-z^2}\dot{\nu}^2 = k_D^2 e^{2\phi}\rho,$$

$$\ddot{\lambda} + d_n\dot{\lambda}^2 + d_c\dot{\lambda}\dot{\nu} - 2\dot{\lambda}\dot{\phi} = k_D^2 e^{2\phi} p_\lambda, \tag{B.8}$$

$$\ddot{\nu} + \frac{4d_c + (2-d_c)z^2}{4-z^2}\dot{\nu}^2 + d_n\dot{\lambda}\dot{\nu} - 2\dot{\nu}\dot{\phi} = k_D^2 e^{2\phi} p_\nu, \tag{B.9}$$

$$\ddot{\phi} - 2\dot{\phi}^2 + d_n\dot{\lambda}\dot{\phi} + d_c\dot{\nu}\dot{\phi} + \frac{k_D^2 e^{2\phi} T}{2}. \tag{B.10}$$

VITAE

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Presentations

1. Aspects of String Gas Cosmology: The Siam Physics Congress 2006, Chonburi, Thailand (March 25, 2006).
2. Stabilization of Extra dimensions in String Gas Cosmology: XII Vietnam School of Physics, Hanoi, Vietnam (January 7, 2006).

International Schools

1. Short Course on Cosmology, Chulalongkorn University, Bangkok, 16 - 27 January 2006.
2. XII Vietnam School of Physics, Hanoi, Vietnam, 26 December 2005 - 7 January 2006.
3. ThaiPhysUniverse III, Naresuan University, Thailand, 13 - 16 August 2005.
4. Short Course on Conformal Field Theory, Chulalongkorn University, Bangkok, 13 - 22 September 2004.