CHAPTER IV

APPLICATION TO THE CASE OF IDENTICALLY DISTRIBUTED RANDOM VARIABLES

In this chapter, we specialize Theorem 3.11 to the case of identically distributed integral-valued random variables. Theorem 4.1 give a general result along this line. We also provide worked out examples for the cases where the random variables have common distributions.

<u>Theorem 4.1.</u> Let X_j, $j = 1,2,\ldots,n$, be independent identically distributed integral-valued random variables. Assume that each X_j has positive variance and has finite moment up to order $2p_o + 2$ where p_o is a positive integer. If

i) each $\theta_{X_j}(t)$ has $(2p_o+2)$- th derivative on $(-\mathcal{T},\mathcal{T})$ and there exists an a such that $\left|\theta_{X_j}^{(2p_o+2)}(t)\right| \leq a$ on $(-\mathcal{T},\mathcal{T})$ and

ii) $\theta_{S_n}^{(2p_o+1)}(t)$ is continuous on $[-\mathcal{T},\mathcal{T}]$

where
$$\mathcal{T} = \frac{1}{2\sqrt[4]{n}\sqrt{\sigma^2(X_1)+1}},$$

then there exist polynomial functions $K_p(T)$, $p = 1,2,\ldots,6p_o-5$ and positive constants A, B, C such that

$$R(T) = \frac{1}{\sqrt{2\pi}}\int_0^T e^{\frac{-t^2}{2}}\,dt + \frac{e^{\frac{-T^2}{2}}}{\sqrt{2\pi}}\sum_{p=1}^{6p_o-5}\frac{K_p(T)}{(\sqrt{n})^p} + \Delta,$$

where

$$|\Delta| < \frac{A}{n^{p_o}} + Be^{-C\sqrt{n}}$$

Proof : Note that Theorem 3.11 can be applied to our random variables $X_1, X_2, \ldots, X_n$. Hence

(4.1.1)...

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \frac{\sin(T\sqrt{B_n}t)}{t} A(t)dt$$

$$+ \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_1(t)+h_1(t)h_3(t)+h_3(t))dt$$

$$- \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \cos(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_4(t)+h_1(t)h_4(t))dt + \Delta$$

and

$$|\Delta| < \frac{e^{-\mathfrak{T}^2 c}}{\mathfrak{T}^2 c} + \frac{1}{2\pi} \int_0^\infty e^{-ct^2} B(t)dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} h_2(t)\left|\frac{1}{t} + \frac{A(t)}{t}\right| dt$$

$$+ \frac{1}{\pi} \left| \int_0^{\mathfrak{T}} e^{-\frac{1}{2}B_n t^2} (1+h_1(t))h_5(t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)dt \right| +$$

$$\frac{1}{\pi} \int_{\mathfrak{T}}^\infty \left| e^{-\frac{1}{2}B_n t^2} (1+h_1(t))(1+h_3(t)+h_4(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt$$

where $A(t), h_1(t), h_2(t), h_3(t), h_4(t), h_5(t)$ and $B(t)$ are as defined in Chapter III.

In Chapter III, we have

(4.1.2) .... $q_p(X_j) = \sum_{m=0}^{2p} \binom{2p}{m}(-1)^{2p-m} E(X_j^m)E(X_j^{2p-m})$, for $p = 1, 2, \ldots, p_0+1$,

$$(4.1.3)\ldots\ldots \quad \tilde{H}_{X_j,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{(-1)^{m-1}}{2(2m)!}\tilde{g}_m(X_j)t^{2m}\right]^P,$$

$$(4.1.4)\ldots\ldots \quad H_{X_j,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{1}{2(2m)!}g_m(X_j)t^{2m}\right]^P,$$

$$(4.1.5)\ldots\ldots \quad c_p(X_j) = -\frac{1}{2}\sum_{m=1}^{p_o} \frac{2^m}{m(2p)!}\tilde{H}^{(2p)}_{X_j,m}(0), \text{ for } p = 1,2,\ldots,p_o,$$

$$(4.1.6)\ldots\ldots \quad c_{p_0+1}(X_j) = \frac{1}{2}\sum_{p=1}^{p_o}\sum_{q=p_0+1}^{p(p_o+1)} \frac{2^p}{p(2q)!}\tilde{H}^{(2q)}_{X_j,p}(0) + \sigma^{2p_0+2}(X_j),$$

$$(4.1.7)\ldots\ldots \quad K_p = \sum_{j=1}^{n} c_p(X_j),$$

$$(4.1.8)\ldots\ldots \quad K = \sum_{p=1}^{p_o+1} |K_p|,$$

$$(4.1.9)\ldots\ldots \quad \tilde{G}(t) = \sum_{p=2}^{p_o+1} K_p t^{2p},$$

$$(4.1.10)\ldots\ldots \quad G(t) = \sum_{p=2}^{p_o+1} |K_p| t^{2p},$$

$$(4.1.11) \quad h_1(t) = \begin{cases} 0 & \text{if } p_o = 1, \\ \sum_{m=1}^{p_o} \frac{1}{m!}\sum_{q=2}^{m+p_o-1} \frac{1}{(2q)!}\tilde{G}^{(2q)}_{[m]}(0)t^{2q} & \text{if } p_o > 1, \end{cases}$$

$$(4.1.12) \quad h_2(t) = \begin{cases} K_2 e^{\frac{K}{16n}t^4} & \text{if } p_o = 1, \\ \sum_{m=1}^{p_o} \frac{1}{m!}\left[\sum_{q=m+p_o}^{m(2p_o+2)} \frac{1}{(2q)!} G^{(2q)}_{[m]}(0)\right] t^{2(m+p_o)} \\ \quad + \frac{1}{(p_o+1)} e^{\frac{K}{16n}} (G_{[1]}(1))^{p_o+1} t^{4p_o+2} & \text{if } p_o > 1, \end{cases}$$

(4.1.13)... $$\tilde{F}(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + na \frac{t^{2p_o+2}}{(2p_o+2)!},$$

(4.1.14)... $$F(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \left|\theta_{X_j}^{(2m+1)}(0)\right| \frac{t^{2m+1}}{(2m+1)!} + na \frac{t^{2p_o+2}}{(2p_o+2)!},$$

(4.1.15)... $$h_3(t) = \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q)!} \tilde{F}_{[2i]}^{(2q)}(0) t^{2q},$$

(4.1.16)... $$h_4(t) = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q-1)!} \tilde{F}_{(2i-1]}^{(2q-1)}(0) t^{2q-1},$$

(4.1.17)... $$h_5(t) = \sum_{i=1}^{p_o} \frac{1}{(2i)!} \left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} F_{[2i]}^{(q)}(0)\right] t^{4i+2p_o}$$

$$+ \sum_{i=1}^{p_o} \frac{1}{(2i-1)!} \left[\sum_{q=4i+2p_o-2}^{(2i-1)(2p_o+2)} \frac{1}{q!} F_{[2i-1]}^{(q)}(0)\right] t^{4i+2p_o-2}$$

$$+ \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1} t^{6p_o+3},$$

and for each $j = 1,2,\ldots,n$

$$\hat{c}_j = \frac{2}{\pi^2} \sum_{k=-\infty}^{\infty} p_{X_j}(k) p_{X_j}(k+1),$$

and

(4.1.18)... $$c = \sum_{j=1}^{n} \hat{c}_j$$

Since $X_1, X_2, \ldots, X_n$ are identically distributed. Therefore for a positive integer p such that $1 \le p \le p_o + 1$ and positive integers k, q, we have

$$\varphi_{X_1}(t) = \varphi_{X_2}(t) = \ldots\ldots\ldots = \varphi_{X_n}(t),$$

$$E(X_1^k) = E(X_2^k) = \ldots\ldots\ldots = E(X_n^k),$$

$$\sigma^2(X_1) = \sigma^2(X_2) = \cdots\cdots = \sigma^2(X_n),$$

$$\theta_{X_1}(t) = \theta_{X_2}(t) = \cdots\cdots = \theta_{X_n}(t),$$

$$g_p(X_1) = g_p(X_2) = \cdots\cdots = g_p(X_n),$$

$$c_p(X_1) = c_p(X_2) = \cdots\cdots = c_p(X_n),$$

$$\tilde{H}_{X_1,q}(t) = \tilde{H}_{X_2,q}(t) = \cdots\cdots = \tilde{H}_{X_n,q}(t),$$

$$H_{X_1,q}(t) = H_{X_2,q}(t) = \cdots\cdots = H_{X_n,q}(t),$$

and $$P_{X_1}(q) = P_{X_2}(q) = \cdots\cdots = P_{X_n}(q).$$

We shall denote $\varphi_{X_j}(t), E(X_j^k), \sigma^2(X_j), \theta_{X_j}(t), g_p(X_j), c_p(X_j), \tilde{H}_{X_{j,q}}(t), H_{X_{j,q}}(t)$ and $P_{X_j}(q)$ by $\varphi(t), \mu_k, \sigma^2, \theta(t), g_p, c_p, \tilde{H}_q(t), H_q(t)$ and $P(q)$ respectively

Hence for each $p = 1,2,\ldots,p_o+1$ and a positive integer j, we have the following.

From (4.1.2), (4.1.3) and (4.1.4)

$$\tilde{H}_p(t) = \left[\sum_{m=1}^{p_o+1} \frac{(-1)^{m-1}}{2(2m)!}\left[\sum_{q=0}^{2m} \binom{2m}{q}(-1)^{2m-q}\mu_q\mu_{2m-q}\right]t^{2m}\right]^p,$$

$$H_p(t) = \left[\sum_{m=1}^{p_o+1} \frac{1}{2(2m)!}\left[\sum_{q=0}^{2m} \binom{2m}{q}(-1)^{2m-q}\mu_q\mu_{2m-q}\right]t^{2m}\right]^p,$$

From (4.1.7), $K_p = nc_p$

so, from (4.1.8) $$K = n\sum_{p=1}^{p_o+1} |c_p|.$$

Since $K_p = nc_p$, from (4.1.9), (4.1.10),(4.1.11) and (4.1.12),

$$\tilde{G}(t) = n\tilde{g}(t) \quad \text{where } \tilde{g}(t) = \left(\sum_{p=2}^{p_o+1} c_p t^{2p}\right)$$

$$G(t) = ng(t) \quad \text{where} \quad g(t) = (\sum_{p=2}^{p_o+1} |c_p| t^{2p}),$$

$$(4.1.19)\ldots \quad h_1(t) = \begin{cases} 0 & \text{if } p_o = 1, \\ \sum_{m=1}^{p_o} \frac{n^m}{m!} \sum_{q=2}^{m+p_o-1} \frac{1}{(2q)!} \tilde{g}_{[m]}^{(2q)}(0) t^{2q} & \text{if } p_o > 1, \end{cases}$$

$$h_2(t) = \begin{cases} nc_2 e^{\frac{|c_1|+|c_2|}{16}} t^4 & \text{if } p_o = 1, \\ \sum_{m=1}^{p_o} \frac{n^m}{m!} \sum_{q=m+p_o}^{m(2p_o+2)} \frac{1}{(2q)!} g_{[m]}^{(2q)}(0)\, t^{2(m+p_o)} + \\ \frac{n^{p_o+1}}{(p_o+1)!} e^{\frac{1}{16} \sum_{p=1}^{p_o+1} |c_p|} (g_{[1]}(1))^{p_o+1} t^{4p_o+2} & \text{if } p > 1. \end{cases}$$

Hence there exist $k_1, k_2, \ldots, k_{p_o+1}$ such that

$$(4.1.20) \ldots\ldots \quad h_2(t) = \sum_{m=1}^{p_o+1} k_m n^m t^{2(m+p_o)}.$$

From (4.1.13)-(4.1.17) we have

$$\tilde{F}(t) = n\tilde{f}(t) \quad \text{where } \tilde{f}(t) = \sum_{m=1}^{p_o} \theta_X^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + \frac{at^{2p_o+2}}{(2p_o+2)!},$$

$$F(t) = nf(t) \quad \text{where } f(t) = \sum_{m=1}^{p_o} |\theta_X^{(2m+1)}(0)| \frac{t^{2m+1}}{(2m+1)!} + \frac{at^{2p_o+2}}{(2p_o+2)!},$$

$$(4.1.21)\ldots\quad h_3(t) = \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} n^{2i} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q)!} \tilde{f}^{(2q)}_{[2i]}(0) t^{2q},$$

$$(4.1.22)\ldots\quad h_4(t) = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} n^{2i-1} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q-1)!} \tilde{f}^{(2q-1)}_{[2i-1]}(0) t^{2q-1},$$

$$(4.1.23)\ldots\quad h_5(t) = \sum_{i=1}^{p_o} \frac{n^{2i}}{(2i)!} \left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} f^{(q)}_{[2i]}(0) \right] t^{4i+2p_o}$$

$$+ \sum_{i=1}^{p_o} \frac{n^{2i-1}}{(2i-1)!} \left[\sum_{q=4i+2p_o-2}^{(2i-1)(2p_o+2)} \frac{1}{q!} f^{(q)}_{[2i-1]}(0) \right] t^{4i+2p_o-2}$$

$$+ \frac{2n^{2p_o+1}}{(2p_o+1)!} f_{[2p_o+1]}(1) t^{6p_o+3}.$$

and from (4.1.18)

$$c = nd$$

where

$$d = \frac{2}{\pi^2} \sum_{k=-\infty}^{\infty} p_X(k) p_X(k+1).$$

Since $B_n = \sum_{j=1}^{n} \sigma^2(X_j)$, we have $B_n = n\sigma^2$.

By Lemma 3.10, we have

$$(4.1.24)\ \ldots\ldots\quad \frac{1}{t} + \frac{A(t)}{t} = \sum_{k=0}^{p_o-1} a_{2k} t^{2k-1}$$

and

$$(4.1.25)\ \ldots\ldots\quad B(t) = d_o t^{2p_o-1}$$

where

$$a_0 = 1,$$

$$a_{2m} = \frac{1}{(2m)!} \sum_{k=1}^{p_o-1} (-1)^k \tilde{D}_{[k]}^{(2m)}(0) \quad \text{for } m = 1,2,\dots,p_o-1,$$

$$|d_o| = \begin{cases} \frac{\pi}{24} & \text{if } p_o = 1, \\ \sum_{k=1}^{p_o-1} \sum_{m=p_o}^{kp_o} \frac{1}{(2m)!} D_{[k]}^{(2m)}(0) + \frac{\pi}{2}\left(\sum_{p=1}^{p_o} \frac{1}{(2p+1)!}\right) & \text{if } p_o > 1, \end{cases}$$

$$\tilde{D}(t) = \sum_{p=1}^{p_o} \frac{(-1)^p}{(2p+1)!} t^{2p},$$

and $$D(t) = \sum_{p=1}^{p_o} \frac{1}{(2p+1)!} t^{2p}.$$

Hence; if $X_1, X_2, \dots, X_n$ are identically distributed, from (4.1.1), (4.1.19)-(4.1.24), we can write R(T) as follow.

Case $p_o = 1$, we have

$$\text{(4.1.26)} \quad R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt - \frac{n\theta^{(3)}(0)}{6\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \cos(T\sqrt{B_n}t)t^2 dt + \Delta,$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt - \frac{\theta^{(3)}(0)}{6\sqrt{2\pi n}\sigma^3} e^{-\frac{T^2}{2}} [T^2+1] + \Delta ,$$

which can be rewritten in the form

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{K_1(T)}{\sqrt{n}} e^{-\frac{T^2}{2}} + \Delta ,$$

where $K_1(T)$ is a polynomial in T.

Case $p_o > 1$, we have

(4.1.27) ... $R(T) =$

$$\frac{1}{\sqrt{2\pi}} \int_0^T e^{\frac{-t^2}{2}} dt + \frac{1}{\pi} \sum_{m=1}^{p_o-1} a_{2m} \int_0^\infty e^{\frac{-1}{2} n \sigma^2 t^2} \sin(T\sqrt{n}\sigma t) t^{2m-1} dt$$

$$+ \frac{1}{\pi} \sum_{m=1}^{p_o} \sum_{k=0}^{p_o-1} a_{2k} \int_0^\infty e^{\frac{-1}{2} n \sigma^2 t^2} \sin(T\sqrt{n}\sigma t) \left[\frac{n^m}{m!} \sum_{q=2}^{m+p_o-1} \frac{1}{(2q)!} \tilde{g}^{(2q)}_{[m]}(0) t^{2k+2q-1} \right.$$

$$+ \sum_{i=1}^{p_o} \frac{(-1)^i n^{m+2i}}{(2i)!m!} \sum_{j=2}^{m+p_o-1} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2j)!(2q)!} \tilde{g}^{(2j)}_{[m]}(0) \tilde{f}^{(2q)}_{[2i]}(0) t^{2j+2k+2q-1}$$

$$\left. + \frac{(-1)^m}{(2m)!} n^{2m} \sum_{q=2m}^{2m+p_o-1} \frac{1}{(2q)!} \tilde{f}^{(2q)}_{[2m]}(0) t^{2k+2q-1} \right] dt - \frac{1}{\pi} \sum_{m=1}^{p_o} \sum_{k=0}^{p_o-1} a_{2k} \int_0^\infty e^{\frac{-1}{2} n \sigma^2 t^2}$$

$$\cos(T\sqrt{n}\sigma t) \left[\frac{(-1)^m}{(2m-1)!} n^{2m-1} \sum_{q=2m}^{2m+p_o-1} \frac{1}{(2q-1)!} \tilde{f}^{(2q-1)}_{[2m-1]}(0)\, t^{2k+2q-2} \right.$$

$$+ \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i-1)!m!} n^{m+2i-1} \sum_{q=2}^{m+p_o-1} \sum_{j=2i}^{2i+p_o-1} \frac{1}{(2q)!(2j-1)!} \tilde{g}^{(2q)}_{[m]}(0) \tilde{f}^{(2j-1)}_{[2i-1]}(0)$$

$$\left. t^{2q+2k+2j-2} \right] dt + \Delta .$$

By applying integration by parts to each of the integrals on the right hand side of (4.1.27) and rearrange terms we have

$$(4.1.28) \qquad R(T) = \frac{1}{\sqrt{2\pi}}\int_0^T e^{-\frac{t^2}{2}}dt + \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\left[\frac{K_1(T)}{n^{1/2}} + \frac{K_2(T)}{n} + \dots + \frac{K_{6p_o-5}(T)}{n^{3p_o-5/2}}\right] + \Delta$$

where $K_1(T),\dots,K_{6p_o-5}(T)$ are polynomials in T.

Next, we consider Δ, such that

$$(4.1.29) \quad |\Delta| < \frac{e^{-n\delta\tau^2}}{n\delta\tau^2} + \frac{1}{\pi}\int_0^\infty e^{-ndt^2}B(t)dt + \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}n\delta^2t^2} h_2(t)\left|\frac{1}{t}+\frac{A(t)}{t}\right|dt$$

$$+ \frac{1}{\pi}\int_0^\tau e^{-\frac{1}{2}n\delta^2t^2}\left|(1+h_1(t))h_5(t)\left(\frac{1}{t}+\frac{A(t)}{t}\right)\right|dt +$$

$$+ \frac{1}{\pi}\int_\tau^\infty e^{-\frac{1}{2}n\delta^2t^2}\left|(1+h_1(t))(1+h_3(t)+h_4(t))\left(\frac{1}{t}+\frac{A(t)}{t}\right)\right|dt.$$

The first integral on the right hand side of (4.1.29) can be obtained by using (4.1.25). We have

$$(4.1.30) \quad \dots \quad \int_0^\infty e^{-ndt^2}B(t)dt = d_o\int_0^\infty e^{-ndt^2}t^{2p_o-1}dt,$$

$$= \frac{d_2}{n^{p_o}}$$

for some constant d_2.

The second integral on the right hand side of (4.1.29) can be treated by using (4.1.20). We have

$$\int_0^\infty e^{-\frac{1}{2}n\sigma^2t^2} h_2(t)\left|\frac{1}{t} + \frac{A(t)}{t}\right| dt$$

$$= \sum_{m=1}^{p_o+1} \frac{n^m}{m!} c_m \int_0^\infty e^{-\frac{1}{2}n\sigma^2t^2} t^{2(m+p_o)} \sum_{k=0}^{p_o-1} |a_{2k}| t^{2k-1} dt$$

$$\leq \sum_{m=1}^{p_o+1} \sum_{k=0}^{p_o-1} \frac{n^m}{m!} |a_{2k}| c_m \int_0^\infty e^{-\frac{1}{2}n\sigma^2t^2} t^{2(m+p_o+k)-1} dt ,$$

$$= \sum_{m=1}^{p_o+1} \sum_{k=0}^{p_o} n^m c_m |a_{2k}| \frac{(m+p_o+k-1)!2^{(m+p_o+k-1)}}{(n\sigma^2)^{m+p_o+k}}$$

$$= \sum_{m=1}^{p_o+1} \sum_{k=0}^{p_o-1} \frac{c_m|a_{2k}|(m+p_o+k-1)!2^{(m+p_o+k-1)}}{\sigma^{2(m+p_o+k)} . n^{p_o+k}}$$

$$\leq \sum_{m=1}^{p_o+1} \sum_{k=0}^{p_o-1} \frac{c_m|a_{2k}|(m+p_o+k-1)!2^{(m+p_o+k-1)}}{\sigma^{2(m+p_o+k)}} . \frac{1}{n^{p_o}} .$$

So we have

$$\int_0^\infty e^{-\frac{1}{2}n\sigma^2t^2} h_2(t)\left|\frac{1}{t} + \frac{A(t)}{t}\right| dt \leq \frac{d_3}{n^{p_o}} .$$

where $d_3 = \sum_{m=1}^{p_o+1} \sum_{k=0}^{p_o-1} \frac{c_m|a_{2k}|(m+p_o+k-1)!2^{(m+p_o+k-1)}}{\sigma^{2(m+p_o+k)}}$

So the second integral is bounded by $\frac{d_3}{n^{p_o}}$, ie. we have

(4.1.31) ... $$\int_0^\infty e^{-\frac{1}{2}n\sigma^2t^2} h_2(t)\left|\frac{1}{t} + \frac{A(t)}{t}\right| dt \leq \frac{d_3}{n^{p_o}} .$$

A bound on the third integral on the right hand side of (4.1.29) can be obtained similarly. In this case we have

$$(4.1.32)\quad \int_0^{\tau} e^{-\frac{1}{2}n\sigma^2 t^2} \left|(1+h_1(t))h_5(t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt \le \frac{d_4}{n^{p_o}}$$

for some constant d_4.

Finally we shall obtain a bound on the last integral on the right hand side of (4.1.29). Observe that

$$\left(\frac{1}{t} + \frac{A(t)}{t}\right)(1+h_1(t))(1+h_3(t)+h_4(t))$$

can be written in the form

$$\left(\frac{1}{t} + \frac{A(t)}{t}\right)(1+h_1(t))(1+h_3(t)+h_4(t)) = \frac{1}{t} + b_o + b_1 t + \dots + b_q t^q,$$

where q is a positive integer and $b_o, b_1, \dots, b_q$ are constants.

so

$$\int_{\varsigma}^{\infty} e^{-\frac{1}{2}n\sigma^2 t^2} \left|\left(\frac{1}{t} + \frac{A(t)}{t}\right)(1+h_1(t))(1+h_3(t)+h_4(t))\right| dt$$

$$= \int_{\varsigma}^{\infty} e^{-\frac{1}{2}n\sigma^2 t^2} \left|\frac{1}{t} + b_o + b_1 t + \dots + b_q t^q\right| dt$$

$$\le \int_{\varsigma}^{\infty} \frac{e^{-\frac{1}{2}n\sigma^2 t^2}}{t}\, dt + \sum_{k=1}^{q} |b_k| \int_{\varsigma}^{\infty} e^{-\frac{1}{2}n\sigma^2 t^2} t^k dt$$

$$= \int_{\frac{\sqrt{n}\sigma\varsigma}{\sqrt{2}}}^{\infty} \frac{e^{-u^2}}{u}\, du + \sum_{k=1}^{q} \left(\frac{\sqrt{2}}{\sqrt{n}\sigma}\right)^{k+1} |b_k| \int_{\frac{\sqrt{n}\sigma\varsigma}{\sqrt{2}}}^{\infty} u^k e^{-u^2} du.$$

$$\le \frac{e^{-\frac{n\sigma^2\varsigma^2}{2}}}{n\sigma^2\varsigma^2} + \sum_{k=1}^{q} \left(\frac{\sqrt{2}}{\sqrt{n}\sigma}\right)^{k+1} |b_k| \int_{\frac{\sqrt{n}\sigma\varsigma}{\sqrt{2}}}^{\infty} u^k e^{-u^2} du.$$

where the last inequality follow from the fact that $\int_x^\infty \frac{e^{-t^2}}{t}\,dt \le \frac{e^{-x^2}}{2x^2}$,

for $x \ge 1$.

For large n, we have

$$\int_{\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}}^\infty u^k e^{-u^2} du \;\le\; \int_{\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}}^\infty u^{2m+1} e^{-u^2} du,$$

where 2m+1 be the smallest integer such that $k \le 2m+1$. Since

$$\int_x^\infty e^{-t^2} t^{2m+1} dt = \frac{e^{-x^2}}{2} (x^{2m} + mx^{2m-2} + m(m-1)x^{2m-4} + \ldots + m!),$$

we have

$$\int_{\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}}^\infty u^k e^{-u^2} du \;\le\; \frac{e^{-\frac{n\sigma^2\mathfrak{T}^2}{}}}{2} \left[\left(\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}\right)^{2m} + m\left(\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}\right)^{2m-2} + \ldots + m! \right]$$

Since $\mathfrak{T} = \frac{1}{2^4\sqrt{n}(\sigma^2+1)}$, there is constant η_k such that

$$\int_{\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}}^\infty u^k e^{-u^2} du \;\le\; \eta_k n^{\frac{m}{2}} e^{-\frac{\sigma^2\sqrt{n}}{8(\sigma^2+1)}}$$

Hence

$$\left(\frac{\sqrt{2}}{\sqrt{n}\sigma}\right)^{k+1} |b_k| \int_{\frac{\sqrt{n}\sigma\mathfrak{T}}{\sqrt{2}}}^\infty u^k e^{-u^2} du \;\le\; \eta_k' e^{-\frac{\sigma^2\sqrt{n}}{8(\sigma^2+1)}}$$

for some constant η_k'.

So,

(4.1.33)...

$$\int^{\infty} e^{-\frac{1}{2}n\sigma^2 t^2} \left|\left(\frac{1}{t}+\frac{A(t)}{t}\right)(1+h_1(t))(1+h_3(t)+h_4(t))\right| dt \leq d_5 e^{-\frac{\sigma^2\sqrt{n}}{8(\sigma^2+1)}} ,$$

for some constant d_5.

From (4.1.29)-(4.1.33) we have

$$|\Delta| < \frac{4(\sigma^2+1)}{\sqrt{n}d_1} e^{-\frac{d_1\sqrt{n}}{4(\sigma^2+1)}} + \frac{d_2}{n^{p_o}} + \frac{d_2}{n^{p_o}} + \frac{d_4}{n^{p_o}} + d_5 e^{-\frac{\sigma^2\sqrt{n}}{8(\sigma^2+1)}}$$

$$< \frac{(d_2+d_3+d_4)}{n^{p_o}} + \left(\frac{4(\sigma^2+1)}{\sqrt{n}d_1} + d_5\right)e^{-\min\left\{\frac{d_1}{4(\sigma^2+1)}, \frac{\sigma^2}{8(\sigma^2+1)}\right\}\sqrt{n}}$$

So there exist positive constants A, B and C such that

$$|\Delta| < \frac{A}{n^{p_o}} + Be^{-C\sqrt{n}} . \qquad \#$$

The following examples illustrate how Theorem 4.1 can be applied.

Example 1. Let X_j, $j = 1, 2, \ldots , n$, be independent identically distributed random variables such that for each $k = q+1, q+2, \ldots, q+m$,

(E1 - 1)... $$P(X_j = k) = \frac{1}{m} ,$$

$j = 1, 2, \ldots , n$. We shall apply Theorem 4.1 with $p_o = 1$ to these random variables. From (4.1.26), we have

$$(E1-2)\ldots \quad R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{\frac{-t^2}{2}} dt - \frac{\theta^{(3)}(0)\, e^{\frac{-T^2}{2}}}{6\sqrt{2\pi n}\, 6^3} (T^2+1) + \Delta ,$$

where Δ is the error term, which, according to Theorem 4.1, satisfies

$$|\Delta| \leq \frac{A}{n} + B e^{-C\sqrt{n}} ,$$

for some positive constants A, B and C.

From (E1-1) we have

$$\varphi(t) = \sum_{j=k}^{m} \frac{1}{m} e^{i(q+k)t} ,$$

$$= \frac{1}{m} e^{i(q+1)t} \left(\frac{1-e^{imt}}{1-e^{it}}\right) ,$$

$$= \frac{1}{m} e^{i(q+1)t} \frac{e^{\frac{imt}{2}}}{e^{\frac{it}{2}}} \left[\frac{e^{-\frac{imt}{2}} - e^{\frac{imt}{2}}}{e^{-\frac{it}{2}} - e^{\frac{it}{2}}}\right] ,$$

$$= \frac{1}{m} \frac{\sin(\frac{mt}{2})}{\sin \frac{t}{2}} e^{i(q+\frac{m}{2}+\frac{1}{2})t} .$$

Hence

$$\theta(t) = (q+\frac{m}{2}+\frac{1}{2})t .$$

So

$$\theta^{(3)}(0) = 0.$$

Hence, from (E1-2), we have

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \Delta ,$$

where

$$|\Delta| = \frac{A}{n} + B\, e^{-C\sqrt{n}} ,$$

for some constants A, B and C

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Example 2. Let X_j, $j = 1, 2, \ldots, n$, be independent identically distributed random variables such that

$$\text{(E2-1)}\ldots\ldots \quad \begin{cases} P(X_j = 0) = q , \\ P(X_j = 1) = p , \end{cases}$$

$j = 1, 2, \ldots, n$, where $0 < p, q < 1$ and $p + q = 1$. We shall apply Theorem 4.1 with $p_o = 2$ to these random variables. From (4.1.27) we have

$$\text{(E2-2)}\ldots \quad R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{1}{6\pi} \int_0^\infty e^{-\frac{1}{2}n\sigma^2 t^2} \sin(T\sqrt{n}\,\sigma\, t)t\, dt$$

$$+ \frac{1}{\pi} \sum_{m=1}^{2} \sum_{k=o}^{1} a_{2k} \int_0^\infty e^{-\frac{1}{2}n\sigma^2 t^2} \sin(T\sqrt{n}\,\sigma\, t)$$

$$\left[\frac{n^m}{m!} \sum_{q=2}^{m+1} \frac{1}{(2q)!} \tilde{g}^{(2q)}_{[m]}(0)\, t^{2k+2q-1} \right.$$

$$+ \sum_{i=1}^{2} \frac{(-1)^i}{(2i)!} \frac{n^{m+2i}}{m!} \sum_{j=2}^{m+1} \sum_{q=2i}^{2i+1} \frac{1}{(2j)!(2q)!} \tilde{g}^{(2j)}_{[m]}(0)\, \tilde{f}^{(2q)}_{[2i]}(0)\, t^{2j+2k+2q-1}$$

$$\left. + \frac{(-1)^m}{(2m)!} n^{2m} \sum_{q=2m}^{2m+1} \frac{1}{(2q)!} \tilde{f}^{(2q)}_{[2m]}(0)\, t^{2k+2q-1} \right] dt$$

$$- \frac{1}{\pi} \sum_{m=1}^{2} \sum_{k=o}^{1} a_{2k} \int_0^\infty e^{-\frac{1}{2} n\sigma^2 t^2} \cos(T\sqrt{n}\,\sigma\, t) \left[\frac{(-1)^m}{(2m-1)!} n^{2m-1} \sum_{q=2m}^{2m+1} \frac{1}{(2q-1)!} \right.$$

$$\tilde{f}^{(2q-1)}_{[2m-1]}(0)\, t^{2k+2q-2} + \sum_{i=1}^{2} \frac{(-1)^i}{(2i-1)!m!} n^{m+2i-1} \sum_{q=2}^{m+1} \sum_{j=2i}^{2i+1} \frac{1}{(2q)!(2j-1)!}$$

$$\left. \tilde{g}^{(2q)}_{[m]}(0)\, \tilde{f}^{(2j-1)}_{[2i-1]}(0)\, t^{2q+2k+2j-2} \right] dt + \Delta$$

where Δ is the eror term, which according to Theorem 4.1, satisfies

$$|\Delta| \leq \frac{A}{n^2} + B\, e^{-C\sqrt{n}} ,$$

for some positive constants A, B, C.

From(E2-1) we have

$$\varphi(t) = q + p e^{it}$$

$$\mu_k = p \quad \text{for } k = 1, 2, 3, \ldots$$

$$\sigma^2 = pq$$

Since $\varphi(t) = q + pe^{it}$, we have

$$\theta(t) = \arctan\left(\frac{p \sin t}{q+p\cos t}\right).$$

It can be shown that

$$\theta^{(3)}(0) = pq(p-q) ,$$

$$\theta^5(0) = pq(p-q)(pq-1)$$

and

$$|\theta^6(t)| \leq \frac{458}{\left(p^2 + q^2 + 2pq\cos\frac{1}{\sqrt{\sigma^2+1}}\right)^{16}}$$

for all t in $(-\zeta, \zeta)$. So, we can take a in (i) of Theorem 4.1 to be

$$\frac{458}{\left(p^2 + q^2 + 2pq\cos\frac{1}{\sqrt{\sigma^2+1}}\right)^{16}}$$

Hence $\tilde{f}$ in our proof of Theorem 4.1 is given by

(E2-3)... $$\tilde{f}(t) = pq(p-q)\frac{t^3}{3!} + pq(p-q)(12pq-1)\frac{t^5}{5!} + \frac{458}{\left(p^2+q^2+2pq\cos\frac{1}{\sqrt{\sigma^2+1}}\right)^{16}}\frac{t^6}{6!}$$

From (4.1.24), we have

(E2-4)... $$a_o = 1,$$

and

(E2-5)... $$a_1 = \frac{1}{6}.$$

Since $\mu_k = p$ for all positive integer p, it follows that

$$g_1 = g_2 = g_3 = 2pq ,$$

and

$$\tilde{H}_r(t) = \left[\sum_{m=1}^{3} \frac{(-1)^{m-1}}{(2m)!}(pq)t^{2m}\right]^r$$

for r = 1, 2.

Hence

$$C_2 = \frac{-pq}{2}(6pq-1)$$

and

$$C_3 = (pq - \frac{22061}{720} p^2q^2) + p^3q^3$$

So, $\tilde{g}$ in our proof of Theorem 4.1 is given by

$$\text{(E2-6)...} \quad \tilde{g}(t) = \frac{-pq}{2}(6pq-1)t^4 + \frac{1}{6!}((pq - \frac{22061}{720}p^2q^2) + p^3q^3)t^6$$

By substituting results from (E2-3)-(E2-6) into (E2-2) and work out the integrals involved, we have

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{e^{-T^2}}{\sqrt{2\pi}} \left[\frac{(p-q)(1-T^2)}{6(pq)^{\frac{1}{2}}} \cdot \frac{1}{n^{\frac{1}{2}}} \right.$$

$$+ \left[\frac{T}{6pq} - \frac{(6pq-1)}{2pq}(3-T^2)T^2 + \frac{(15-9T^3+T^5)\ T}{72p^2q^2} \right] \frac{1}{n}$$

$$+ \left[\frac{(1-6pq)}{12(pq)^{\frac{3}{2}}}(15-44T^2+14T^4-T^6) - \frac{(3-5T^2+T^4)}{36(pq)^{\frac{3}{2}}} \right.$$

$$\left. + \frac{(p-q)^2}{126(pq)^{\frac{3}{2}}}(105-413T^2+196T^4-27T^6+T^8) \right] \frac{1}{n^{\frac{3}{2}}}$$

$$+ \left[\frac{(6pq-1)}{36p^2q^2} - \frac{(p-q)}{1296p^3q^3}(105T-98T^3+20T^5-T^7) + \right.$$

$$\left. \frac{(6pq-1)(p-q)(945T-1197T^3+356T^5-35T^7+T^9}{432p^3q^3} \right] \frac{1}{n^2}$$

$$+ (p-q) \left[\frac{(6pq-1)}{42(pq)^{\frac{5}{2}}}(105-413T^2+196T^4-27T^6+T^8) \right.$$

$$+ \frac{(p-q)^2}{7776(pq)^{\frac{5}{2}}} (945-4662T^2+2961T^4+44T^8-T^{10})$$

$$+ \frac{(6pq-1)}{p^2q^2} (10395-61677T^2+46035T^4-13110T^6+1433T^8-65T^{10}+T^{12}) \Big] \frac{1}{n^{\frac{5}{2}}}$$

$$+ \left[\frac{(6pq-1)(p-q)(105-413T^2+196T^4-27T^6+T^8)}{72(pq)^{\frac{5}{2}}} \right] \frac{1}{n^3}$$

$$+ \frac{(6pq-1)}{2.6^5(pq)^{\frac{7}{2}}} (p-q)^3(135135-942696T^2+859716T^4-288957T^6+30017T^8-2926T^{10}$$

$$+90T^{12}-T^{14}) \frac{1}{n^{\frac{7}{2}}} \Bigg] + \Delta ,$$

where

$$|\Delta| \leq \frac{A}{n^2} + Be^{-C\sqrt{n}} ,$$

for some positve constants A, B, C

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Example 3. Let X_j, j = 1, 2, ... , n,be independent identically distributed random variables such that

$$\text{(E3-1)...} \qquad P_{X_j}(n) = \frac{\lambda^n e^{-\lambda}}{n!} \qquad (n = 0, 1, 2, \dots)$$

for some positive constant λ. We shall apply Theorem 4.1 with $p_o = 2$ to these random variables. Since $p_o = 2$, the formula for R(T) is the same as that given in (E2-2). From (E3-1) we have

$$\varphi(t) = e^{\lambda(e^{it}-1)},$$

$$\theta(t) = \lambda \sin t ,$$

$$\mu_k = \lambda^k,$$

and $$\sigma^2 = \lambda .$$

Since $\theta(t) = \lambda \sin t$, we have

$$\theta^3(0) = -\lambda ,$$

$$\theta^5(0) = \lambda ,$$

$$|\theta^6(t)| = \lambda ,$$

for all t in $(-\pi, \pi)$. So, we can take a in (i) of Theorem 4.1 to be λ.

Hence $\tilde{f}$ in our proof of Theorem 4.1 is given by

(E3-2) $$\tilde{f}(t) = -\frac{\lambda t^3}{3!} + \frac{\lambda t^5}{5!} + \frac{\lambda t^6}{6!} .$$

From (4.1.24), we have

(E3-3)... $$a_o = 1$$

and

(E3-4)... $$a_1 = \frac{1}{6} .$$

Since $\mu_k = \lambda^k$, for all positive integer k, it follows that

$$g_p = 0$$

for all positive p.

Hence $\tilde{H}_p = 0$ for all positive p.

So we have

$$c_2 = 0$$

and

$$c_3 = \lambda^3,$$

hence $\tilde{g}$ in our proof of Theorem 7.1 is given by

(E3-5)... $$\tilde{g}(t) = \lambda^3 t^6.$$

By substituting results from (E3-2)-(E3-5) into R(T) which is given by (E2-2), and work out the integrals involued, we have

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{e^{-\frac{T^2}{2}}}{\sqrt{2\pi}} \left[\frac{(1-T^2)}{6\lambda^{\frac{1}{2}} n^{\frac{1}{2}}} \right.$$

$$+ \left[\frac{T}{6} + \frac{(15-9T^3+T^5)}{72} T \right] \frac{1}{\lambda n}$$

$$- \left[213 - 593T^2 + 232T^4 - 27T^6 + T^8 \right] \frac{1}{\lambda^{\frac{3}{2}} n^{\frac{3}{2}}}$$

$$- \left[105T - 98T^3 + 20T^5 - T^7 \right] \frac{1}{432\lambda^2 n^2}$$

$$\left. + \left[945 - 4662T^2 + 2961T^4 - 599T^6 + 44T^8 - T^{10} \right] \frac{1}{6^5 \lambda^{\frac{5}{2}} n^{\frac{5}{2}}} \right] + \Delta ,$$

where

$$|\Delta| \leq \frac{A}{n^2} + B e^{-C\sqrt{n}},$$

for some positive constants A, B and C. #

APPENDIX

Let f be a real-valued function defined on [-a, a]. Assume that

(1) f has derivative of order $n+1$ everywhere on (-a, a) and

(2) $f^{(n)}(t)$ is continuous on [-a, a].

So that f has a Taylor's formula of the form

$$f(t) = a_o + a_1 t + a_2 t^2 + \ldots + a_n t^n + R_{n+1}(t) t^{n+1}.$$

In this appendix, we show that if f is odd on (-a, a) and $R_{n+1}(t)$ is bounded on a neighbourhood of 0, then

$$a_{2k} = 0$$

for all nonnegative integer k such that $2k \leq n$.

A proof is as follows :

Suppose that $a_{2k} \neq 0$ for some k such that $0 \leq 2k \leq n$. Let k_o be the samllest such k. Let g be defined on (-a, a) by

$$g(t) = \begin{cases} a_{2k_o} & \text{if } t = 0, \\ \dfrac{f(t) - \sum\limits_{2m-1 < 2k_o} a_{2m-1} t^{2m-1}}{t^{2k_o}} & \text{if } t \neq 0. \end{cases}$$

By definition of k_o, we see that

$$g(0) = a_{2k_o}$$

and

$$g(t) = a_{2k_o} + a_{2k_o+1}\, t + \dots + a_n t^{n-2k_o} + R_{n+1}(t)t^{n+1-2k_o}$$

for $t \neq 0$.

Since $R_{n+1}(t)$ is bounded on a neighbourhood of 0 and $n+1-2k_o > 0$, hence

$$\lim_{t \to 0} R_{n+1}(t)\, t^{n+1-2k_o} = 0.$$

It follows that g is continuous at 0. Observe that for $t \neq 0$

$$g(-t) = \frac{f(-t) + \sum\limits_{2m-1 < 2k_o} a_{2m-1} t^{2m-1}}{t^{2k_o}}$$

$$= \frac{-f(t) + \sum\limits_{2m-1 < 2k_o} a_{2m-1} t^{2m-1}}{t^{2k_o}}$$

$$= -g(t).$$

Hence g is odd on (-a, a).

Suppose that $|a_{2k_o}| > 0$. Since g is continuous at 0, there is a $\delta > 0$ such that $|g(x) - g(0)| < \frac{|a_{2k_o}|}{2}$ whenever $x \in (-a, a)$ and $|x - 0| < \delta$.

It follows that g(x) has the same sign as a_{2k_o} for all such x.

This is contary to the fact that g is odd. Hence $a_{2k} = 0$ for all k such that $0 < 2k \leq n$.

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