CHAPTER IV

THEOREMS ON CONVERGENCE TO THE STANDARD NORMAL DISTRIBUTION FUNCTION.

In this chapter we apply our main result of Chapter III (Theorem 3.2.6) to the case in which the given limit distribution function is the standard normal distribution function. A condition for convergence is stated and proved in section 4.1. In section 4.2, we state theorem on convergence to the standard normal distribution function of Bethmann [1]. In section 4.3, we compare convergence theorems in section 4.1 and section 4.2.

4.1 <u>Theorem On Convergence The Standard Normal Distribution Function.</u>

The standard normal distribution function usually denoted by Φ and is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-t^2}{2}}\,dt.$$

Its mean and variance are 0 and 1 respectively. Its characteristic function is given by

$$\varphi(t) = e^{\frac{-t^2}{2}}.$$

We can represented its logarithm by

$$\mathrm{Log}\varphi(t) = \int_{-\infty}^{\infty} f(t,x)dK(x),$$

where

$$K(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Theorem 4.1.1 Let $(Z_n;X_{nj})$ be a random double sequence of random variables which are independent in each row and satisfy the condition $(\tilde{\alpha})$. Then there exists a double sequence (A_{nj}) of real numbers such that

(i-a) the distribution functions of random sums

$$S_{Z_n} = X_{n1}+X_{n2}+X_{n3}+.....+X_{nZ_n}-A_{nZ_n}$$

converge weakly to Φ, and

(i-b) $\hat{\varphi}_{l_n(q)}(t) \longrightarrow e^{\frac{-t^2}{2}}$ for each $q \in (0,1)$ and each real number t, where $\hat{\varphi}_{Z_n}$ are random accompanying characteristic functions associated with (Z_n),(φ_{nj}) and (A_{nj}), and

(ii) $\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1,$

if and only if

(i') $\sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1$

for every $\varepsilon > 0$, and

(ii') $\sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1.$

The constants A_{nk} may be chosen according to the formula

$$A_{nk} = \sum_{j=1}^{k} \mu_{nj}.$$

Proof. Since the constant μ, the function K in Kolmogorov formula of the characteristic function of Φ are given by

$$\mu = 0,$$

$$K(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0, \end{cases}$$

and the variance of Φ is 1, it follows that the conditions (i') and (ii') are equivalent to the following conditions.

(1) $K_{Z_n}(u) \xrightarrow{P} K(u)$, for every continuity point u of K, and

(2) $K_{Z_n}(+\infty) \xrightarrow{P} K(+\infty)$.

Hence the theorem follows.

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Remark 4.1.2. For a random double sequence $(Z_n;X_{nj})$ of random variables, the condition (ii) of Theorem 4.1.1 is equivalent to condition (ii') of Theorem 4.1.1, i.e.

$$\text{(ii)} \quad \sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1,$$

is equivalent to

$$\text{(ii')} \quad \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1.$$

Proposition 4.1.3 Let $(Z_n;X_{nj})$ be a random double sequence of random variables which satisfies the following conditions.

$$\text{(i')} \qquad \sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1, \text{and}$$

$$\text{(ii')} \qquad \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1.$$

Then $(Z_n;X_{nj})$ satisfies the condition $(\tilde{\alpha})$.

Proof. By (i') and (ii') we have

$$\sum_{j=1}^{Z_n} \int_{|x|\geq\varepsilon} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 0.$$

for every $\varepsilon > 0$. From this fact and the fact that

$$\sup_{1\leq j\leq Z_n} P(|X_{nj}-\mu_{nj}| \geq \varepsilon) = \sup_{1\leq j\leq Z_n} \int_{|x|\geq\varepsilon} dF_{nj}(x+\mu_{nj})$$

$$\leq \frac{1}{\varepsilon^2} \sum_{j=1}^{Z_n} \int_{|x|\geq\varepsilon} x^2 dF_{nj}(x+\mu_{nj}),$$

we have $(Z_n;X_{nj})$ satisfies the condition $(\tilde{\alpha})$.

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From the above proposition we have the following corollary.

<u>Corollary 4.1.4</u> Let $(Z_n;X_{nj})$ be a random double sequence of random variables which are independent in each row and

$$\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1.$$

Then there exists a double sequence (A_{nj}) of real numbers such that

(i-a) the distribution functions of random sums

$$S_{Z_n} = X_{n1}+X_{n2}+X_{n3}+\ldots+X_{nZ_n}-A_{nZ_n}$$

converge weakly to Φ, and

(i-b) $\hat{\varphi}_{l_n(q)}(t) \longrightarrow e^{\frac{-t^2}{2}}$ for each $q \in (0,1)$ and real number t, where $\hat{\varphi}_{Z_n}$ are random accompanying characteristic functions associated with (Z_n), (φ_{nj}) and (A_{nj}), and

(ii) $(Z_n;X_{nj})$ satisfies the condition $(\tilde{\alpha})$,
if only if

$$\sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x+\mu_{nj}) \xrightarrow{P} 1$$

for every $\varepsilon > 0$.
The constants A_{nj} may be chosen according to the formula

$$A_{nk} = \sum_{j=1}^{k} \mu_{nj}.$$

When we specialize Corollary 4.1.4 to the cases in which the X_{nj}'s have zero means, we can take all the A_{nj}'s to be zeros. In this special case, the condition (i') and (ii') will be simplified.
The Condition (i') becomes the following.

(RL) $$\sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x) \xrightarrow{P} 1$$

for every $\varepsilon > 0$. This condition is known as the random Lindeberg condition [1]. So, as a special case of Theorem 4.1.4 we have the following theorem.

Theorem 4.1.5 Let $(Z_n;X_{nj})$ be a random double sequence of random variables which are independent in each row and satisfy the following conditions.

(a) For each n and j, $\mu_{nj} = 0$.

(b) $\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1$.

Then

(i-a) the distribution functions of random sums

$$S_{Z_n} = X_{n1}+X_{n2}+X_{n3}+.....+X_{nZ_n}$$

converge weakly to Φ, and

(i-b) $\hat{\varphi}_{l_n(q)}(t) \longrightarrow e^{\frac{-t^2}{2}}$ for each $q \in (0,1)$ and real number t, where $\hat{\varphi}_{Z_n}$ are random accompanying characteristic functions associated with (Z_n), (φ_{nj}) and (A_{nj}), and

(ii) $(Z_n;X_{nj})$ satisfies the condition $(\tilde{\alpha})$,

if and only if $(Z_n;X_{nj})$ satisfies the random Lindeberg condition, i.e.

$$\sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x) \xrightarrow{P} 1$$

for every $\varepsilon > 0$.

4.2 Bethmann Theorem On Convergence To The Standard Normal Distribution Function.

In this section, we state Bethmann Theorem on convergence to the standard normal distribution function in our terminology.

Theorem 4.2.1 ([1]) Let (Z_n, X_{nj}) be a random double sequence of random variables which are independent in each row and satisfy the following conditions.

(a) For each n and j, $\mu_{nj} = 0$

(b) $E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] \to 1$.

(c) $Z_n \xrightarrow{P} \infty$.

Then

(i) the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \dots\dots + X_{nZ_n}$$

converge weakly to Φ, and

(ii) (X_{nj}) satisfies the condition $(\tilde{\alpha})$

if and only if $(Z_n;X_{nj})$ satisfies the random Lindeberg condition satisfied, i.e.

$$\sum_{j=1}^{Zn} \int_{|x|<\varepsilon} x^2 dF_{nj}(x) \xrightarrow{P} 1.$$

for every $\varepsilon > 0$.

4.3 A Comparision Between Bethmann Theorem (Theorem 4.2.1) And Theorem 4.1.5.

The differences between Bethmann's Theorem (Theorem 4.2.1) and our theorem (Theorem 4.1.5) is in the general assumption. There are cases in which our theorem can be applied, but Bethmann Theorem can not be applied. There are also cases in which Bethmann can be applied but our theorem can not be applied. These are illustrated in the following examples.

Example 4.3.1. For each n, let Z_n be such that

$$P(Z_n = n) = 1 - \frac{1}{n^2} \text{ and } P(Z_n = n+1) = \frac{1}{n^2}.$$

For each n and j, we define X_{nj} as follow.

If $j \neq n+1$, then X_{nj} is defined by

$$P(X_{nj} = \frac{1}{\sqrt{n}}) = P(X_{nj} = -\frac{1}{\sqrt{n}}) = \frac{1}{2}.$$

In case $j = n+1$, let X_{nj} be defined by

$$P(X_{nj} = 2^n) = P(X_{nj} = -2^n) = \frac{1}{2}$$

It can be seen that

$\mu_{nj} = 0$ for every n and j, and

$$\sigma_{nj}^2 = \begin{cases} \frac{1}{n} & \text{if } j \neq n+1 \\ 2^{2n} & \text{if } j = n+1. \end{cases}$$

Hence for every $\eta > 0$, we have

$$P(|\sum_{j=1}^{Z_n} \sigma_{nj}^2 - 1| \geq \eta) \leq P(Z_n = n+1)$$

$$= \frac{1}{n^2}$$

which converge to 0. So

$$\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1.$$

Hence all the assumptions of Theorem 4.1.5 are satisfied.

Next, we show that $(Z_n;X_{nj})$ defined above does not satisfy the general assumption of Bethmann Theorem.

Observe that

$$E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] = P(Z_n = n) \sum_{j=1}^{Z_n} \sigma_{nj}^2 + P(Z_n = n+1) \sum_{j=1}^{Z_n} \sigma_{nj}^2$$

$$= \frac{1}{2} + \frac{1}{2}(1 + 2^{2n}),$$

which converge to ∞. Hence the condition

$$\lim_{n \to \infty} E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] = 1,$$

does not hold. So the general assumption of Bethmann Theorem is not satisfied.

In applying our theorem to the double sequence $(Z_n;X_{nj})$ of random variables in example 4.3.1, we find that, for any positive real numbers ε and η we have

$$P(|\sum_{j=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nj}(x) - 1| \geq \eta) \leq P(Z_n = n+1)$$

$$= \frac{1}{n^2}$$

for all $n > \frac{1}{\varepsilon}$. It follows that $(Z_n;X_{nj})$ satisfies the random Lindeberg condition. Hence the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \dots\dots\dots + X_{nZ_n}$$

converge weakly to Φ and $(Z_n;X_{nj})$ satisfies the condition $(\tilde{\alpha})$.

Example 4.3.2. For each n, let Z_n be such that

$$P(Z_n = n) = \frac{1}{2} \quad \text{and} \quad P(Z_n = n+1) = \frac{1}{2}.$$

For each n and j, we define X_{nj} as follow.

If $j \neq n+1$, then let

$$X_{nj} = 0.$$

If $j = n+1$, let X_{nj} defined by

$$P(X_{nj} = \sqrt{2}) = P(X_{nj} = -\sqrt{2}) = \frac{1}{2}.$$

It can be seen that

$\mu_{nj} = 0$ for every n and j, and

$$\sigma_{nj}^2 = \begin{cases} 0 & \text{if } j \neq n+1 \\ 2 & \text{if } j = n+1. \end{cases}$$

Hence

$$E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] = 1,$$

for every n. It follows that

$$\lim_{n\to\infty} E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] = 1.$$

Let η be any positive real number. Choose n_o to be a positive integer such that $n_o \geq \eta$. Then we have

$$P(Z_n \geq \eta) = 1$$

for all $n \geq n_o$. It follows that

$$\lim_{n\to\infty} P(Z_n \geq \eta) = 1.$$

Therefore

$$Z_n \xrightarrow{p} \infty.$$

Hence all the assumptions of Bethmann Theorem are satisfied.

Next, we show that $(Z_n;X_{nj})$ does not satisfy the general assumption of Theorem 4.1.5.

Observe that

$$P(\left|\sum_{j=1}^{Z_n} \sigma_{nj}^2 - 1\right| \geq \frac{1}{2}) = 1$$

for all n. Hence the condition

$$\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{p} 1$$

does not hold. Therefore the general assumption of Theorem 4.1.5 is not satisfied.

In applying Bethman Theorem to the double sequence $(Z_n;X_{nj})$ of random variables in example 4.3.2, we see that

$$P\left(\left|\sum_{j=1}^{Z_n} \int_{|x_j|<\varepsilon} x^2 dF_{nj}(x) - 1\right| \geq \frac{1}{2}\right) \geq \frac{1}{2}$$

for every $\varepsilon>0$. It follow that $(Z_n;X_{nj})$ dose not satisfy the random Lindeberg condition. Hence the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \dots\dots\dots + X_{nZ_n}$$

do not converge to Φ, or

$(Z_n;X_{nj})$ does not satisfy the condition $(\hat{\alpha})$.

It is natural to ask whether there exists an example of random double sequence $(Z_n;X_{nj})$ that satisfies the general assumption of Bethmann's Theorem as well as the random Lindeberg condition but does not satisfy the general assumption of our theorem. The answer to this question is in the negative. The reason is in the following proposition.

Proposition 4.3.3. Let (Z_n,X_{nj}) be a random double sequence of random variables which are independent in each row and satisfy the following conditions.

(a) For each n and j, $\mu_{nj} = 0$

(b) $E[\sum_{j=1}^{Z_n} \sigma_{nj}^2] \to 1.$

(c) $Z_n \xrightarrow{P} \infty$

If $(Z_n;X_{nj})$ satisfies the random Lindeberg condition, then

$$\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} 1.$$

Proof. Let ($\sum_{j=1}^{Z_{n_k}} \sigma^2_{n_k j}$) be arbitrary subsequence of ($\sum_{j=1}^{Z_n} \sigma^2_{nj}$). By Lemma 3.1.4

$$E\left[\sum_{j=1}^{Z_{n_k}} \sigma^2_{n_k j}\right] = \int_0^1 \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq$$

Hence it follows from the assumption that

$$\lim_{k\to\infty} \int_0^1 \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq = 1.$$

By Fautou's Lemma we have that

$$\int_0^1 \liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq \le \liminf_{k\to\infty} \int_0^1 \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq$$

$$= \lim_{k\to\infty} \int_0^1 \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq$$

$$= 1,$$

i.e.

(1) $$\int_0^1 \liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}^{(q)}} \sigma^2_{n_k j}\, dq \le 1 .$$

Let $\varepsilon > 0$. Let

$$a_{nj} = \sum_{r=1}^{j} \int_{|x|<\varepsilon} x^2 dF_{nr}(x).$$

Since $(Z_n;X_{nj})$ satisfies the random Lindeberg condition, by Lemma3.1.5 we have that

(2) $$\lim_{n\to\infty} \sum_{j=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 dF_{nj}(x) = 1,$$

for every $q \in (0,1)$. From this fact and the fact that

(3) $$\sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j} \geq \sum_{j=1}^{l_{n_k}(q)} \int_{|x|<\varepsilon} x^2 dF_{n_k j}(x)$$

we have

(4) $$\int_0^1 \liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j}\, dq \geq 1.$$

From (1) and (4), we have

(5) $$\int_0^1 \liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j}\, dq = 1.$$

From (2) and (3) we have

(6) $$\liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j} \geq 1.$$

From (5) and (6) we have

$$\liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j} = 1 \text{ a.e.}$$

Since $\sum_{j=1}^{l_n(q)} \sigma^2_{nj}$ is non-decreasing on q, we have

$$\liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j} = 1$$

for every $q \in (0,1)$. Hence for arbitrary subsequence $(\sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j})$ of $(\sum_{j=1}^{l_n(q)} \sigma^2_{nj})$ we have

$$\liminf_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma^2_{n_k j} = 1.$$

From this fact, it can be seen that

$$\lim_{n\to\infty} \sum_{j=1}^{l_n(q)} \sigma^2_{nj} = 1$$

for every q in (0,1).

By Lemma3.1.5, we have that

$$\sum_{j=1}^{Z_n} \sigma^2_{nj} \xrightarrow{P} 1.$$

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