CHAPTER III

THE FUNCTIONAL EQUATION :

$f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$ ON GROUP

In this chapter we will discuss the functional equation :

(A) $$f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

where $f : G \times G \longrightarrow G'$, G and G' are ablian groups and $x, y, z,$ in G.

Our purpose is to give conditions under which a function $g : G \rightarrow G'$ such that

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all $x, y,$ in G exists.

We shall denote the identities of G and G' by e and e' respectively.

Let $f' : G \times G \longrightarrow G'$ be defined by $f'(x, y) = f(x, y)-f(e,e)$. It can be verified that f' satisfied (A) and $f'(e, e) = e'$. It follows that f satisfied (A) if and only if there exists $k \in G'$ and there exists $f' : G \times G \longrightarrow G'$, satisfying (A) and $f'(e,e) = e'$, such that $f = f' + k$. Hence there is no loss of generality to assume that

(*) $$f(e, e) = e' .$$

In (A), replace $y = z = e$. Then

$$f(x,e) + f(x, e) = f(e, e) + f(x, e).$$

Hence by cancellation we have

(A.1) $$f(x, e) = e'$$

for all x in G.

Similarly

(A.2) $$f(e, y) = e'$$

for all y in G.

Definition 3.1 A function f on $G \times G$ is said to be symmetric if

$$f(x, y) = f(y, x)$$

for all x, y, in G.

Definition 3.2 Let γ be an ordinal. By a γ - sequence in a group G, we mean a one - to - one function x on γ into $G - \{e\}$, where e is the identity of G. For each $\beta < \gamma$, we define a subgroup S_β as follows:

$$S_\beta = \langle \{x_\alpha / \alpha < \beta\} \rangle ,$$

$\{S_\beta\}(\beta < \gamma)$ will be called the γ - sequence of subgroups determined by the γ - sequence $\{x_\alpha\}(\alpha < \gamma)$.

Definition 3.3 Let S be a subgroup of a group G. For any $t \in G - S$, we shall denote the subgroup generated by S and t by $S[t]$, i.e.,

$$S[t] = \langle S \cup \{t\} \rangle .$$

Lemma 3.4 Let S be a proper subgroup of a group G.

For any element $t \in G - S$.

a) if $mt \in S$ for some nonzero integer m, then for every element $y \in S[t] - S$, there exists an element s of S and positive integer p such that $y = s + pt$,

b) if $mt \notin S$ for any nonzero integer m, then for every element $y \in S[t] - S$, there exists a unique element $s \in S$ and a unique integer n such that $y = s + nt$.

Proof. a) Assume that $mt \in S$ for some non-zero integer m.

Since $(-m)t = (-mt)$, hence $(-mt) \in S$.

Therefore it is sufficient to prove the lemma for the case that $mt \in S$ where m is positive.

Let $y \in S[t] - S$.

Since $y \in S[t]$, hence y can be expressed in the form $y = s + nt$, where $s \in S$ and n is an integer.

By Archimedean property, we can choose a positive integer q such that $qm > -n$.

Observe that

$$y = s - qmt + qmt + nt,$$

$$= (s-qmt) + (qm + n)t.$$

Since $s - qmt \in S$ and $qm + n > 0$, hence $y = s^* + pt$, where $s^* = (s - qmt) \in S$ and $p = qm+n$ is a positive integer.

b) Assume that $mt \notin S$ for any nonzero integer m.

Let $y \in S[t] - S$.

Since $y \in S[t]$, hence y can be expressed in the form $y = s + nt$ where $s \in S$ and n is an integer.

To show the uniqueness of s and n, let

$$y = s + nt \quad \text{and} \quad y = s' + n't$$

where $s, s' \in S$ and n, n' are integers.

Without loss of generality we may assume that $n > n'$.

Therefore $$(n - n')t = s' - s \in S.$$

By the assumption that there is no $m \neq 0$ such that $mt \in S$, we have $n - n' = 0$. Hence $s' - s = 0t = e$.

It follows that $n = n'$ and $s = s'$.

<u>Lemma</u> 3.5 Let $(G, +)$ and $(G', +)$ be abelian groups. Let $f : G \times G \longrightarrow G'$ be a symmetric function such that

(*) $$f(e,e) = e'$$

and

(A) $$f(x, y) + f(x+ y, z) = f(y, z) + f(x, y+ z)$$

for all x, y, z, in G. Let S be a subgroup of G such that there exist a function $g : S \longrightarrow G'$ satisfying

(B) $$f(x,y) = g(x) + g(y) - g(x+y)$$

for all $x,y,$ in S. Let $t \in G - S$. Then there exists an extension $\hat{g}$ of g such that f and $\hat{g}$ satisfies (B) on $S[t]$, i.e.

(B′) $$f(x,y) = \hat{g}(x) + \hat{g}(y) - \hat{g}(x + y)$$

for all $x,y \in S[t]$. Furthermore, if a is any element in G', $\hat{g}$ can be chosen in such a way that $\hat{g}(t) = a$.

<u>Proof.</u> Let a be any element in G'.

<u>Case</u> I Assume that t is such that $mt \notin S$ for any nonzero integer m. Define $\hat{g}$ on $S[t]$ as follows : For any $s \in S$, let

(3.5.1) $$\hat{g}(0t + s) = g(s) ,$$

(3.5.2) $$\hat{g}(nt + s) = -a + \hat{g}((n+1)t+s)+f(t,nt+s) \quad \text{for } n \leq -1,$$

(3.5.3) $$\hat{g}(nt + s) = a + \hat{g}((n-1)t+s)-f(t,(n-1)t+s) \quad \text{for } n \geq 1 .$$

Since each $y \in S[t] - S$ has a unique representation in the form $y = nt + s$, hence $\hat{g}$ is well - defined. It follows from (3.5.1) that $\hat{g}$ is an extension of g. It remains to be shown that f and $\hat{g}$ satisfies (B′).

Let s, s' be any elements in S. For each integers n, m, let $B(n,m)$ be the statement

"$f(nt+s,\ mt + s') = \hat{g}(nt+s) + \hat{g}(mt+s') - \hat{g}((n+m)t+s+s')$."

For each nonnegative integer N, let P(N) be the proposition :

"B(n,m) holds for all integers n,m with $|n| \leq N$, $|m| \leq N$."

Since $s, s' \in S$, we have

$$f(s, s') = g(s) + g(s') - g(s + s').$$

By (3.5.1) we have

$$f(ot + s, ot+s') = \hat{g}(ot+s) + \hat{g}(ot+s') - \hat{g}(ot+s+s').$$

Hence P(0) holds.

Let k be any nonnegative integer. Assume that P(k) holds. We shall show that P(k+1) holds.

Let n, m be any integers such that $|n| \leq k+1$, $|m| \leq k+1$. By the assumption P(k), we have B(n,m) holding for all n,m such that $|n| < k+1$ and $|m| < k+1$. It remains to be verified that B(n,m) holds in the following cases :

Case 1. $n = k+1$, $|m| \leq k$.

Case 2. $m = k+1$, $|n| \leq k$.

Case 3. $n = k+1$, $m = -k-1$.

Case 4. $n = -k-1$, $m = k+1$.

Case 5. $n = -k-1$, $|m| \leq k$.

Case 6. $|n| \leq k$, $m = -k-1$.

Case 7. $n = k+1$, $m = k+1$.

Case 8. $n = -k-1$, $m = -k-1$.

Case 1. $n = k+1$, $|m| \leq k$.

Note that $k+1$ and $k+m+1$ are positive. Hence, by (3.5.3) we have

$$\hat{g}((k+1)t+s) = a+\hat{g}(kt+s) - f(t,kt+s),$$

$$\hat{g}((k+m+1)t+s+s') = a+\hat{g}((k+m)t+s+s') - f(t,(k+m)t+s+s').$$

These imply

$$\hat{g}((k+1)t+s) + \hat{g}(mt+s') - \hat{g}((k+m+1)t+s+s')$$

$$= a+\hat{g}(kt+s)-f(t,kt+s)+\hat{g}(mt+s')-a-\hat{g}((k+m)t+s+s')$$

$$+ f(t,(k+m)t+s+s'),$$

$$= \hat{g}(kt+s)-f(t,kt+s)+\hat{g}(mt+s')-\hat{g}((k+m)t+s+s')$$

$$+ f(t,(k+m)t+s+s'),$$

$$= \hat{g}(kt+s)+\hat{g}(mt+s')-\hat{g}((k+m)t+s+s') - f(t,kt+s)$$

$$+ f(t,(k+m)t+s+s'),$$

$$= f(kt+s,mt+s')-f(t,kt+s) + f(t,(k+m)t+s+s'),$$

where the last equality follows from the inductive hypothesis. Replacing x, y, z in (A) by t, $kt+s$, $mt+s'$ respectively, we have

$$f(t,kt+s) + f((k+1)t+s,mt+s') = f(kt+s,mt+s') + f(t,(k+m)t+s+s').$$

This implies

$$f(kt+s,mt+s') - f(t,kt+s) + f(t,(k+m)t+s+s') = f((k+1)t+s,\ mt+s').$$

Hence

$$\hat{g}((k+1)t+s)+\hat{g}(mt+s') - \hat{g}((k+m+1)t+s+s') = f((k+1)t+s,mt+s').$$

<u>Case</u> 2 $\qquad m = k+1, \quad |n| \leqslant k$.

The verification of $B(n,m)$ in this case is similar to the case 1.

<u>Case</u> 3 $\qquad n = k+1, \quad m = -k-1.$

Note that $k+1$ is positive and $-k-1$ is negative, hence by (3.5.3) and (3.5.2) respectively we have

$$\hat{g}((k+1)t+s) = a+\hat{g}(kt+s) - f(t,kt+s).$$

$$\hat{g}((-k-1)t+s') = -a + \hat{g}(-kt+s') + f(t,(-k-1)t+s') .$$

These imply

$$\hat{g}((k+1)t+s) + \hat{g}((-k-1)t+s') - \hat{g}(s+s')$$

$$= a+\hat{g}(kt+s)-f(t,kt+s)-a+\hat{g}(-kt+s') + f(t,(-k-1)t+s')-\hat{g}(s+s'),$$

$$= \hat{g}(kt+s)-f(t,kt+s)+\hat{g}(-kt+s')+f(t,(-k-1)t+s')-\hat{g}(s+s'),$$

$$= \hat{g}(kt+s)+\hat{g}(-kt+s')-\hat{g}(s+s')-f(t,kt+s)+f(t,(-k-1)t+s'),$$

$$= f(kt+s,\ -kt+s') -f(t,kt+s) + f(t,(-k-1)t+s'),$$

where the last equality follows from the inductive hypothesis. Replacing x,y,z in (A) by $kt+s$, t, $(-k-1)t+s'$ respectively, we have

$$f(kt+s,t) + f((k+1)t+s, (-k-1)t+s') = f(t,(-k-1)t+s') + f(kt+s, -kt+s').$$

This implies

$$f(kt+s,-kt+s') - f(t,kt+s)+f(t,(-k-1)t+s') = f((k+1)t+s,(-k-1)t+s').$$

Hence

$$\hat{g}((k+1)t+s)+\hat{g}((-k-1)t+s') - \hat{g}(s+s') = f((k+1)t+s,(-k-1)t+s').$$

Case 4 $\qquad n = -k-1 \;, \quad m = k+1 \;.$

The verification of $B(n,m)$ in this case is similar to the case 3.

Case 5 $\qquad n = -k-1 \;, \quad |m| \leq k \;.$

Note that $-k-1$ and $-k+m-1$ are negative. Hence by (3.5.2) we have

$$\hat{g}((-k-1)t+s) = -a+\hat{g}(-kt+s)+f(t,(-k-1)t+s),$$

$$\hat{g}((-k+m-1)t+s+s') = -a+\hat{g}((-k+m)t+s+s') +f(t,(-k+m-1)t+s+s').$$

These imply

$$\hat{g}((-k-1)t+s)+\hat{g}(mt+s') - \hat{g}((-k+m-1)t+s+s')$$

$$= -a+\hat{g}(-kt+s)+f(t,(-k-1)t+s)+\hat{g}(mt+s')$$

$$+a-\hat{g}((-k+m)t+s+s')- f(t,(-k+m-1)t+s+s'),$$

$$= \hat{g}(-kt+s)+f(t,(-k-1)t+s)+\hat{g}(mt+s')- \hat{g}((-k+m)t+s+s')$$

$$-f(t,(-k+m-1)t+s+s'),$$

$$= \hat{g}(-kt+s)+\hat{g}(mt+s')-\hat{g}((-k+m)t+s+s')+f(t,(-k-1)t+s)$$

$$-f(t,(-k+m-1)t+s+s'),$$

$$= f(-kt+s,mt+s') + f(t, (-k-1)t+s)-f(t,(-k+m-1)t+s+s'),$$

where the last equality follows from the inductive hypothesis.

Replacing x,y,z, in (A) by $t,(-k-1)t+s$ and $mt+s'$ respectively, we have

$$f(t,(-k-1)t+s)+f(-kt+s, mt+s') = f((-k-1)t+s, mt+s')+f(t,(-k+m-1)t+s+s')$$

This implies

$$f(-kt+s,mt+s')+f(t,(-k-1)t+s)-f(t,(-k+m-1)t+s+s') = f((-k-1)t+s,mt+s').$$

Hence

$$\hat{g}((-k-1)t+s)+\hat{g}(mt+s')-\hat{g}((-k+m-1)t+s+s') = f((-k-1)t+s,mt+s').$$

<u>Case</u> 6 $\quad |n| \leq k \;,\; m = -k-1 \,.$

The verification of $B(n,m)$ in this case is similar to the case 5.

Case 7 $\quad n = k+1,\ m = k+1.$

Since $k+1$, $2k+1$, $2k+2$ are positive, hence by (3.5.3), we have

$$\hat{g}((k+1)t+s) = a+\hat{g}(kt+s)-f(t,kt+s),$$

$$\hat{g}((k+1)t+s') = a+\hat{g}(kt+s')-f(t,kt+s'),$$

$$\begin{aligned}\hat{g}((2k+2)t+s+s') &= a+\hat{g}((2k+1)t+s+s')-f(t,(2k+1)t+s+s'),\\ &= a+(a+\hat{g}(2kt+s+s') - f(t,2kt+s+s'))\\ &\quad -f(t,(2k+1)t+s+s'),\\ &= 2a+\hat{g}(2kt+s+s') - f(t,2kt+s+s')\\ &\quad -f(t,(2k+1)t+s+s').\end{aligned}$$

These imply

$$\begin{aligned}&\hat{g}((k+1)t+s)+\hat{g}((k+1)t+s')- \hat{g}((2k+2)t+s+s')\\ &= a+\hat{g}(kt+s)-f(t,kt+s)+a+\hat{g}(kt+s') - f(t,kt+s')\\ &\quad - 2a-\hat{g}(2kt+s+s')+ f(t, 2kt+s+s') + f(t,(2k+1)t+s+s'),\\ &= \hat{g}(kt+s)+\hat{g}(kt+s')-\hat{g}(2kt+s+s') -f(t,kt+s)\\ &\quad -f(t,kt+s')+f(t,2kt+s+s') +f(t,(2k+1)t+s+s'),\\ &= f(kt+s,kt+s')-f(t,kt+s)-f(t,kt+s')\\ &\quad + f(t,2kt+s+s') +f(t,(2k+1)t+s+s'),\end{aligned}$$

where the last equality follows from the inductive hypothesis.

Replacing x, y, z in (A) by t, $kt+s'$ and $kt+s$ respectively, we have

$$f(t,kt+s') + f((k+1)t+s',kt+s) = f(kt+s', kt+s) + f(t,2kt + s + s')$$

This implies

$$f(kt+s,kt+s') - f(t,kt+s') + f(t,2kt+s+s') = f((k+1)t+s', kt+s).$$

Adding $-f(t,kt+s)+ f(t,(2k+1)t+s+s')$ to the both sides, we have

$$f(kt+s,kt+s')-f(t,kt+s')-f(t,kt+s)+ f(t,2kt+s+s') + f(t,(2k+1)t+s+s')$$

$$= f((k+1)t+s', kt+s)-f(t,kt+s)+f(t,(2k+1)t+s+s').$$

Replacing x,y,z in (A) by t, $kt+s$ and $(k+1)t+s'$ respectively, we have

$$f(t,kt+s)+f((k+1)t+s, (k+1)t+s') = f(kt+s, (k+1)t+s')$$

$$+ f(t,(2k+1)t+s+s') .$$

This implies

$$f((k+1)t+s', kt+s) - f(t,kt+s)+f(t,(2k+1)t+s+s')$$

$$= f((k+1)t+s, (k+1)t+s').$$

Hence

$$\hat{g}((k+1)t+s)+\hat{g}((k+1)t+s') - \hat{g}((2k+2)t+s+s') = f((k+1)t+s,(k+1)t+s').$$

Case 8 $\qquad n = -k-1, \quad m = -k-1 .$

Since $-k-1$, $-2k-1$, $-2k-2$ are negative, hence by (3.5.2) we have

$$\hat{g}((-k-1)t+s) = -a+\hat{g}(-kt+s)+f(t,(-k-1)t+s) ,$$

$$\hat{g}((-k-1)t+s') = -a+\hat{g}(-kt+s') + f(t,(-k-1)t+s'),$$

$$\hat{g}((-2k-2)t+s+s') = -a+\hat{g}((-2k-1)t+s+s')+ f(t,(-2k-2)t+s+s'),$$

$$= -a+(-a+\hat{g}(-2kt+s+s')+f(t,(-2k-1)t+s+s'))$$

$$+ f(t,(-2k-2)t,+s+s'),$$

$$= -2a+\hat{g}(-2kt+s+s')+f(t,(-2k-1)t+s+s')$$

$$+f(t,(-2k-2)t+s+s').$$

These imply

$$\hat{g}((-k-1)t+s)+ \hat{g}((-k-1)t+s') - \hat{g}((-2k-2)t+s+s')$$

$$= -a+\hat{g}(-kt+s)+f(t,(-k-1)t+s) -a+\hat{g}(-kt+s')$$

$$+f(t,(-k-1)t+s')+2a-\hat{g}(-2kt+s+s')-f(t,(-2k-1)t+s+s')$$

$$-f(t,(-2k-2)t+s+s'),$$

$$= \hat{g}(-kt+s)+\hat{g}(-kt+s')-\hat{g}(-2kt+s+s')+f(t,(-k-1)t+s)$$

$$+ f(t,(-k-1)t+s')-f(t,(-2k-1)t+s+s')- f(t,(-2k-2)t+s+s').$$

$$= f(-kt+s,-kt+s')+ f(t,(-k-1)t+s)+f(t,(-k-1)t+s')$$

$$-f(t,(-2k-1)t+s+s')-f(t,(-2k-2)t+s+s'),$$

where the last equality follows from the inductive hypothesis.

Replacing x,y, z in (A) by t, $(-k-1)t+s'$ and $(-k-1)t+s$ respectively, we have

$$f(t,(-k-1)t+s')+f(-kt+s', (-k-1)t+s) = f((-k-1)t+s', (-k-1)t+s)$$
$$+f(t,(-2k-2)t+s+s'),$$

This implies

$$f(t,(-k-1)t+s')-f(t,(-2k-2)t+s+s')$$
$$= f((-k-1)t+s', (-k-1)t+s)-f(-kt+s',(-k-1)t+s).$$

Adding $f(-kt+s, -kt+s')+ f(t,(-k-1)t+s)-f(t,(-2k-1)t+s+s')$ to the both sides, we have

$$f(-kt+s,-kt+s')+f(t,(-k-1)t+s)+f(t,(-k-1)t+s') - \sum_{j=-2k-1}^{-2k-2} f(t,jt+s+s')$$

$$= f(-kt+s,-kt+s')+f(t,(-k-1)t+s)+f((-k-1)t+s', (-k-1)t+s)$$
$$-f(-kt+s',(-k-1)t+s)-f(t,(-2k-1)t+s+s').$$

Replacing x, y, z in (A) by t,$(-k-1)t+s$, $-kt+s'$ respectively, we have

$$f(t,(-k-1)t+s)+f(-kt+s, -kt+s') = f((-k-1)t+s,-kt+s')+f(t,(-2k-1)t+s+s').$$

This implies

$$f(-kt+s, -kt+s') + f(t,(-k-1)t+s)-f(-kt+s', (-k-1)t+s)-f(t,(-2k-1)t+s+s')$$
$$= \theta' .$$

Hence

$$\hat{g}((k-1)t+s) + \hat{g}((-k-1)t+s') - \hat{g}((-2k-2)t + s + s')$$

$$= f((-k-1)\ t+s' , (-k-1)t+s)$$

Therefore $P(k+1)$ holds.

<u>Case</u> II Assume that t is such that $mt \in S$ for some nonzero integer m. Let m_o be the smallest positive integer such that $m_o t \in S$. By lemma 3.4(a), for any $y \in S[t] - S$, there exists an element $s \in S$ and a positive integer n such that $y = s + nt$.

Define $\hat{g}$ on $S[t]$ as follows : For any $s \in S$, let

(3.5.4) $$\hat{g}(nt + s) = g(nt + s) \qquad \text{if } nt + s \in S$$

(3.5.5) $$\hat{g}(nt + s) = a + \hat{g}((n-1)t+s) - f(t,(n-1)t+s)$$

$$\text{for } n \geqslant 1 \text{ and } nt + s \notin S .$$

It follows from (3.5.4) that $\hat{g}$ is an extension of g. We will prove that $\hat{g}$ is well - defined on $S[t] - S$. Let $y \in S[t] -$

Assume that

$$y = pt + s \qquad \text{and} \qquad y = p't + s'$$

where p and p' are positive integers and $s, s' \in S$.

Write $$p = qm_o + r \quad \text{and} \quad p' = q'm_o + r'$$

where p, q, r, p', q', r' are integers such that $0 \leq r, r' < m_o$.

Since $y \notin S$, hence $r, r' > 0$. Without loss of generality we may assume that $r \geq r'$.

Since

$$(qm_o + r)\,t + s = y = (q'm_o + r')t + s'.$$

Hence $(r - r')t = (q' - q)\, m_o t + s' - s$.

Since $(q' - q)m_o t + s' - s \in S$, hence $(r-r')t \in S$.

But $0 \leq r - r' < m_o$. Therefore, by the minimality of m_o we have $r - r' = 0$.

Hence $$qm_o t + s = q'm_o t + s'.$$

By (3.5.5) we have

$$\hat{g}((q'm_o + r)t + s') = ra + g(q'm_o t + s') - \sum_{j=1}^{r} f(t,(q'm_o+r-j)t+s')$$

$$\hat{g}((qm_o + r)t+s) = ra + g(qm_o t + s) - \sum_{j=1}^{r} f(t,(qm_o+r-j)t+s)$$

Since $q'm_o t + s' = qm_o t + s$, hence

$$g(q'm_o t + s') = g(qm_o t + s)$$

and

$$f(t, (q'm_o + r - j)t + s') = f(t,(qm_o + r-j)t+s)$$

for each j. Therefore

$$\hat{g}((qm_o' + r')t + s') = \hat{g}((qm_o + r)t + s).$$

Hence

$$\hat{g}(pt + s) = \hat{g}(p't + s').$$

Observe that the definition of $\hat{g}$ given in (3.5.4) and (3.5.5) are the same as those given by (3.5.1) and (3.5.3) in case I. The proof of case I shows that $\hat{g}$ and f satisfy (B$'$) . This completes the prove of lemma 3.5 .

Lemma 3.6 Let G be a group and γ be an ordinal. Let $\{S_\alpha\}(\alpha<\gamma)$ be a family of subgroups of G such that for each $\alpha < \beta < \gamma$, $S_\alpha \subset S_\beta$. For each $\alpha < \gamma$, let x_α be an element of $S_{\alpha+1}$ such that $x_\alpha \notin \bigcup_{\eta<\alpha} S_\eta$. If γ is a cardinal number, then $\gamma \leq \overline{\overline{\bigcup_{\alpha<\gamma} S_\alpha}}$, where $\overline{\overline{\bigcup_{\alpha<\gamma} S_\alpha}}$ denotes the cardinal number of $\bigcup_{\alpha<\gamma} S_\alpha$.

Proof. If γ is finite then

$$\{e, x_0, \ldots, x_{\gamma-2}\} \subseteq S_{\gamma-1} ,$$

$$= \bigcup_{\alpha<\gamma} S_\alpha .$$

Hence

$$\gamma \leq \overline{\overline{\bigcup_{\alpha<\gamma} S_\alpha}}$$

If γ is infinite cardinal then by lemma A-35, γ is a limit ordinal. Since

$$\{x_\alpha\} \subseteq S_{\alpha+1} ,$$

$$\bigcup_{\alpha<\gamma}\{x_\alpha\} \subseteq \bigcup_{\alpha<\gamma} S_{\alpha+1} .$$

Since γ is a limit ordinal, $\bigcup_{\alpha<\gamma} S_{\alpha+1} = \bigcup_{\alpha<\gamma} S_\alpha$.

Hence $$\bigcup_{\alpha<\gamma}\{x_\alpha\} \subseteq \bigcup_{\alpha<\gamma} S_\alpha .$$

Therefore $$\overline{\overline{\bigcup_{\alpha<\gamma}\{x_\alpha\}}} \leq \overline{\overline{\bigcup_{\alpha<\gamma} S_\alpha}} .$$

Since $\{x_\alpha\}(\alpha<\gamma)$ is equipotent to γ , hence

$$\gamma = \overline{\overline{\bigcup_{\alpha<\gamma}\{x_\alpha\}}}$$

Hence $$\gamma \leq \overline{\overline{\bigcup_{\alpha<\gamma} S_\alpha}} .$$

Theorem 3.7 Given any group G, there exists an ordinal γ and a γ- sequence $\{x_\alpha\}(\alpha<\gamma)$ in G such that the γ - sequence $\{S_\alpha\}(\alpha<\gamma)$ of subgroup of G determined by $\{x_\alpha\}(\alpha<\gamma)$ has the property that $\bigcup_{\alpha<\gamma} S_\alpha = G$.

Proof. In the case that $G = \{e\}$, the ordinal $\gamma = 0$

$S_0 = \langle \phi \rangle = \{e\} = G.$

Assume that $G \neq \{e\}$. First we shall show that there exists an ordinal γ and a family of subgroups $\{S_\alpha\}(\alpha < \gamma)$ such that if $\alpha < \beta < \gamma$, then $S_\alpha \subset S_\beta$ and $\bigcup_{\alpha<\beta} S_\alpha = G$.

Let c be a choice function for G.

Let β be any nonzero ordinal such that the subgroups S_α have been defined for all $\alpha < \beta$ and $G - \bigcup_{\alpha<\beta} S_\alpha$ is not empty .

Case 1 $\quad \beta = \delta + 1$ for some ordinal δ .

Define $\quad y_\delta = c(G - \bigcup_{\alpha<\beta} S_\alpha)$

and $\quad S_\beta = \langle \{ y_\alpha / \alpha < \beta \} \rangle$.

Case 2 $\quad \beta$ is a limit ordinal.

Define $\quad S_\beta = \bigcup_{\alpha<\beta} S_\alpha$,

$$y_\beta = c\,(G - S_\beta)$$

We claim that there exists an ordinal number γ such that

$$G - \bigcup_{\alpha<\gamma} S_\alpha = \emptyset$$

Suppose the contrary, i.e, for all γ ,

$$\bigcup_{\alpha<\gamma} S_\alpha \subset G.$$

Take $\gamma' = \overline{\overline{\mathcal{P}G}}$ where $\overline{\overline{\mathcal{P}G}}$ is the power set of G.

Hence $\quad \bigcup_{\alpha<\gamma'} S_\alpha \subset G$.

Therefore $\quad \overline{\overline{\bigcup_{\alpha<\gamma'} S_\alpha}} \leq \overline{\overline{G}}$.

By lemma 3.6, we have

$$\overline{\overline{\bigcup_{\alpha < \gamma'} S_\alpha}} \geqslant \gamma' .$$

Hence $\overline{\overline{G}} \geqslant \overline{\overline{\mathcal{P}G}}$, which is a contradiction.

Therefore the assumption is false, hence there exists an ordinal γ and a family $\{S_\alpha\}(\alpha < \gamma)$ of subgroups of G such that $\bigcup_{\alpha<\gamma} S_\alpha = G$

Let

$$x_\beta = \begin{cases} c(G - \bigcup_{\alpha<\beta+1} S_\alpha) & \text{if } \beta \text{ is a non limit ordinal} \\ c(G - \bigcup_{\alpha<\beta} S_\alpha) & \text{if } \beta \text{ is a limit ordinal} . \end{cases}$$

Observe that $x_\beta = y_\beta$. By the above construction, we see that $\{x_\alpha\}(\alpha < \gamma)$ is a γ - sequence and $\{x_\alpha\}(\alpha < \gamma)$ generates the γ - sequence of subgroup $\{S_\alpha\}(\alpha < \gamma)$.

<u>Theorem 3.8</u> Let (G, +) and (G', +) be abelian groups. Let a symmetric function $f : G \times G \longrightarrow G'$ satisfy

(*) $f(e,e) = e'$,

(A) $f(x,y) + f(x + y, z) = f(y, z) + f(x,y + z)$

for all x, y, z, in G. Then there exists a function $g : G \longrightarrow G'$ such that

(B) $f(x,y) = g(x) + g(y) - g(x + y)$

for all x,y, in G.

Proof. From theorem 3.7 there exists an ordinal γ and a γ - sequence $\{x_\alpha\}(\alpha<\gamma)$ in G such that the γ - sequence $\{S_\alpha\}(\alpha<\gamma)$ of subgroups of G determined by $\{x_\alpha\}(\alpha<\gamma)$ has the property that

$$\bigcup_{\alpha<\gamma} S_\alpha = G .$$

For each $\alpha<\gamma$, we shall define g_α on S_α so that

(1) if $\alpha'<\alpha$, then $g_{\alpha'} \subseteq g_\alpha$,

(2) f and each g_α satisfy (B) on S_α.

This will be done by transfinite induction.

Define g_0 on $S_0 = \{e\}$ by putting

$$g_0(e) = e' .$$

Clearly f and g_0 satisfy (B) on S_0.

Let $\beta<\gamma$ be any ordinal number such that g_α have been defined so that f and g_α satisfy (B) on S_α for all $\alpha<\beta$.

Case 1 $\beta = \delta + 1$ for some ordinal δ.

Since g_δ has been defined on S_δ, hence by lemma 3.5, there exists an extension $\hat{g}_\delta$ on $S_\delta[x_\delta]$ such that f and $\hat{g}_\delta$ satisfy (B) on $S_\delta[x_\delta]$.

Put

$$g_\beta = \hat{g}_\delta .$$

Then g_β is defined on $S_\beta = S_\delta[x_\delta]$ and f and g_β satisfy (B) on S_β. It can be shown that (1) holds.

Case 2 β is a limit ordinal.

In this case, we put

$$g_\beta = \bigcup_{\alpha<\beta} g_\alpha .$$

Clearly (1) holds. From (1), it follows that g_β is well - defined on $S_\beta = \bigcup_{\alpha<\beta} S_\alpha$ and f, g_β satisfy (B) on S_β.

Hence, for each $\beta<\gamma$, if g_α has been defined on S_α, and f and g_α satisfy (B) on S_α for all $\alpha<\beta$, then g_β can be defined on S_β, and f and g_β satisfy (B) on S_β. Therefore, for all $\alpha<\gamma$, g_α can be defined on S_α and f and g_α satisfy (B) on S_α.

Define

$$g = \bigcup_{\alpha<\gamma} g_\alpha .$$

Hence, by(1), g is well - defined on $G = \bigcup_{\alpha<\gamma} S_\alpha$ and f and g satisfy (B) on G.

For certain group G, the symmetry of f can be derived from the functional equation(A). For such a group the symmetry of f needs not be assumed.

Theorem 3.9 Let the group G in theorem 3.8 be such that there exist a sequence of infinite cyclic subgroup $\{S_i\}$ with the following properties :

i) $G = \bigcup_{i=o}^{\infty} S_i \supset \ldots \supset S_i \supset \ldots \supset S_o$.

ii) For any $x \in S_i$, $2x \in S_{i-1}$.

iii) For all $x_i \in S_i$ and all $j > i$, there exists $x_j \in S_j$ such that

$$2^{j-i}(x_j) = x_i .$$

If a function $f : G \times G \longrightarrow G'$, where G' is an abelian group, satisfy

$$(*) \qquad f(e, e) = e' ,$$

and

$$(A) \qquad f(x,y) + f(x+y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z, in G, then there exists a function $g : G \rightarrow G'$ such that

$$(B) \qquad f(x,y) = g(x) + g(y) - g(x+y)$$

for all x, y, in G .

Proof From (*) and (A) it follows that

(A.1) $$f(x,e) = e'$$

for all x in G, and

(A.2) $$f(e,y) = e'$$

for all y in G.

Define

$$F(x, y) = f(x,y) - f(y,x) .$$

For all x in G and for each integers n, m, let C(m,n) be the statement

"$F(mx,nx) = e'$ for all $x \in G$."

For each nonnegative integer N, let P(N) be the proposition :

"C(m,n) holds for all integers m,n with $|m| \leq N$, $|n| \leq N$."

By (*) , P(0) holds.

Let k be any positive integer. Assume that P(k-1) holds. We shall show that P(k)holds. Let m,n be any integers such that $|m| \leq k$, $|n| \leq k$. By the assumption P(k-1), we have C(m,n) holding for all m,n such that $|m| < k$ and $|n| < k$. It remains to be verified that C(m,m) holds in the following cases :

Case 1. $|n| \leq k-1, \quad m = k$.

Case 2. $m = -k, \quad |n| \leq k-1$.

Case 3. $|m| \leq k-1, \quad n = k$.

Case 4. $|m| \leq k-1, \quad n = -k$.

Case 5. $m = k, \quad n = -k$.

Case 6. $m = k, \quad n = k$.

Case 7. $m = -k, \quad n = -k$.

Case 8. $m = -k, \quad n = k$.

Replacing $Z = x$ in (A), we have

$$f(x,y) + f(x+y,x) = f(y,x) + f(x,y+x)$$

This implies

$$f(x,y) - f(y,x) = f(x,y+x) - f(x+y,x).$$

Therefore

$$F(x,y) = F(x,x+y) . \tag{3.9.1}$$

Observe that

$$\begin{aligned} F(x,y) &= f(x,y) - f(y,x) , \\ &= -(f(y,x) - f(x,y)), \\ &= -F(y,x) . \end{aligned}$$

Therefore

(3.9.2) $$F(x,y) = -F(y,x) .$$

<u>Case</u> 1. $|n| \leq k-1 , \quad m = k .$

(1.1) Suppose that n is negative.

By (3.9.2), (3.9.1) and the inductive hypothesis, we have

$$F(kx, nx) = -F(nx, kx),$$

$$= -F(nx, (n+k)x) ,$$

$$= e' .$$

(1.2) Suppose that n is zero.

By (A.1) and (A.2) we have

$$F(kx, e) = f(kx, e) - f(e,kx),$$

$$= e' .$$

(1.3) Suppose that n is positive.

By (3.9.2), (3.9.1) and inductive hypothesis we have

$$F(kx, nx) = - F(nx, kx),$$

$$= -F(nx, (k-n)x),$$

$$= e' .$$

Case 2 $\quad m = -k, \quad |n| \leqslant k-1$.

If follows from (3.9 .2) that

$$F(kx,\ nx) \quad = \quad -\ F(nx,\ kx)\ .$$

Replacing n and x in the last equation by - n and - x respectively, we have

$$F(-kx,\ nx) \quad = \quad -F(nx,\ -kx)\ .$$

(2.1) Suppose that n is negative.

By (3.9 .1) and inductive hypothesis, we have

$$-F(nx,\ -kx) \ = \ -F(nx,\ -(k+n)x),$$

$$= \ e'\ .$$

Hence

$$F(-kx,\ nx) \quad = \quad e'\ .$$

(2.2) Suppose that n is zero .

By (A.1) and (A.2) we have

$$F(-kx,\ e) \quad = \quad f(-kx,\ e) - f(e,\ -kx),$$

$$= \ e'\ .$$

(2.3) Suppose that n is positive.

By (3.9 .1) and inductive hypothesis, we have

$$-F(nx,\ -kx) \ = \ -F(nx,\ (n-k)x),$$

$$= \ e'\ .$$

Hence

$$F(-kx,\ nx) \quad = \quad e' .$$

<u>Case</u> 3 $\quad |m| \leq k-1 ,\ n = k .$

(3.1) Suppose that m is negative.

By (3.9.1) and inductive hypothesis we have

$$F(mx,\ kx) \quad = \quad F(mx,\ (k+m)x),$$

$$= \quad e' .$$

(3.2) Suppose that m is zero .

By (A.1) and (A.2) , we have

$$F(e,\ kx) \quad = \quad f(e,\ kx) - f(kx,\ e),$$

$$= \quad e' .$$

(3.3) Suppose that m is positive.

By (3.9.1) and inductive hypothesis we have

$$F(mx,\ kx) \quad = \quad F(mx,(k-m)x),$$

$$= \quad e' .$$

<u>Case</u> 4. $\quad |m| \leq k-1 ,\ n = -k .$

(4.1) Suppose that m is negative.

By (3.9.1) and inductive hypothesis, we have

$$F(mx, -kx) = F(mx, -(k+m)x),$$
$$= e' .$$

(4.2) Suppose that m is zero.

By (A.1) and (A.2) we have

$$F(e, -kx) = f(e, -kx) - f(-kx, e) ,$$
$$= e' .$$

(4.3) Suppose that m is positive.

By (3.9.1) and inductive hypothesis, we have

$$F(mx, -kx) = F(mx, (m-k)x),$$
$$= e' .$$

Case 5. $m = k, \quad n = -k$.

By (3.9.1), (A.1) and (A.2), we have

$$F(kx, -kx) = F(kx, e),$$
$$= f(kx, e) - f(e, kx),$$
$$= e' .$$

<u>Case</u> 6. $m = -k,\ n = k.$

By (3.9.1), (A.1) and (A.2), we have

$$F(-kx,\ kx) = F(-kx,\ e),$$

$$= f(-kx,\ e) - f(e, -kx),$$

$$= e' .$$

<u>Case</u> 7. $m = -k,\ n = -k$.

By (3.9.1), (A.1) and (A.2), we have

$$F(-kx,\ -kx) = F(-kx,\ e),$$

$$= f(-kx, e) - f(e, -kx) ,$$

$$= e' .$$

<u>Case</u> 8 $m = -k$, $n = k$.

By (3.9.1), (A.1) and (A.2), we have.

$$F(-kx,\ kx) = F(-kx,\ e) ,$$

$$= e' .$$

Therefore we have P(N) hold for all N. Thus

$$F(mx,\ nx) = e' .$$

for all x in G .

Let x, y be in G. Then there exist p and q such that $x \in S_p$ and $y \in S_q$.

Choose $r > p$ and $r > q$.

By (iii), there exist $z_1, z_2 \in S_r$ such that

$$x = 2^{r-p} . z_1$$

and

$$y = 2^{r-q} . z_2 .$$

Let z be a generator of S_r. Then

$$z_1 = az \quad \text{and} \quad z_2 = bz$$

for some integer a, b .

Therefore $x = a2^{r-p} . z$ and $y = b2^{r-q} . z$.

Set $a.2^{r-p} = m_1$ and $b.2^{r-q} = n_1$.

Therefore $F(x, y) = F(m_1 z, n_1 z)$.

We have proved that $F(m_1 z, n_1 z) = e'$ for all z in G .

Hence $F(x,y) = e'$ for all x,y in G.

Therefore

$$f(x,y) = f(y,x)$$

for all x,y, in G.

Hence, by theorem 3.8, the condusion of the theorem follows.

Theorem 3.10 Let $(G,+)$ be an abelian group and let $(G',+)$ be a 2 - divisible abelian groups. Let $F : G \times G \longrightarrow G'$ satisfy

$$\text{(A)} \qquad F(x,y) + F(x+y,z) = F(y,z) + F(x,y+z)$$

for all x, y, z in G. Then there exists a function $g : G \longrightarrow G'$ satisfying

$$\text{(3.10.1)} \qquad F(x,y) = B(x,y) + g(x) + g(y) - g(x+y),$$

where B is a skew-symmetric biadditive function; i.e, B satisfies

$$\text{(3.10.2)} \qquad B(x+y,z) = B(x,z) + B(y,z),$$

$$\text{(3.10.3)} \qquad B(x,y+z) = B(x,y) + B(x,z),$$

$$\text{(3.10.4)} \qquad B(x,y) + B(y,x) = 0$$

for all x, y, z, in G.

Proof. Let

$$B(x,y) = \frac{1}{2}\left[F(x,y) - F(y,x)\right].$$

We will show that B satisfies (3.10.2) (3.10.3) and (3.10.4)

By definition of B, we have

$$B(x,y) + B(y,x) = \frac{1}{2}\left[F(x,y) - F(y,x) + F(y,x) - F(x,y)\right].$$

$$= 0.$$

Hence (3.10.4) holds.

Replacing x,y,z in (A) by z,x,y , respectively, we have

$$F(z,x) + F(z+x, y) = F(x,y) + F(z,x+y) .$$

This implies

$$(3.10.5) \qquad -F(z,x+y) = F(x,y) - F(z,x) - F(z+x,y).$$

Replacing y, z in (A) by z and y respectively we have

$$F(x,z) + F(x+z,y) = F(z,y) + F(x,y+z) .$$

This implies

$$(3.10.6) \qquad F(x,y+z) - F(x+z,y) = F(x,z) - F(z,y).$$

By the definition of B, we have

$$B(x+y,z) = \frac{1}{2}\left[F(x+y,z) - F(z,x+y)\right] .$$

By (A) and (3.10.5), we have

$$\begin{aligned}\frac{1}{2}\left[F(x+y,z) - F(z,x+y)\right] &= \frac{1}{2}\Big[F(x,y+z) + F(y,z) - F(x,y) + F(x,y) \\ &\qquad - F(z+x,y) - F(z,x)\Big] , \\ &= \frac{1}{2}\left[F(x,y+z) - F(z+x,y) + F(y,z) - F(z,x)\right] ; \\ &= \frac{1}{2}\left[F(x,z) - F(z,y) + F(y,z) - F(z,x)\right] , \\ &= \frac{1}{2}\left[F(x,z) - F(z,x)\right] + \frac{1}{2}\left[F(y,z) - F(z,y)\right] , \\ &= B(x,z) + B(y,z) ,\end{aligned}$$

where the third equality follows from (3.10.6) by replacing $F(x,y+z) - F(z+x,y)$ by $F(x,z) - F(z,y)$ and the last equality follows from the definition of B. Hence

$$B(x+y,z) = B(x,z) + B(y,z)$$

The verification of (3.10.3) is similar to that of (3.10.2) and will be omitted.

Thus B satisfies (3.10.2), (3.10.3) and (3.10.4).

Set $$f(x,y) = \frac{1}{2}\left[F(x,y) + F(y,x)\right] .$$

Since G' is abelian, hence

$$f(x,y) = f(y,x) .$$

Observe that

$$B(x,y) + f(x,y) = \frac{1}{2}\left[F(x,y) - F(y,x)\right] + \frac{1}{2}\left[F(x,y) + F(y,x)\right] ,$$

$$= F(x,y) .$$

Hence

(3.10.7) $$F(x,y) = B(x,y) + f(x,y) .$$

Next, we shall show that f satisfies (A).

Replacing $x = z$ and $z = x$ in (A), we have

(3.10.8) $$F(z,y) + F(z+y,x) = F(y,x) + F(z,x+y).$$

Observe that

$$\begin{aligned} f(x,y) + f(x+y,z) &= \frac{1}{2}\left[F(x,y)+ F(y,x)\right]+ \frac{1}{2}\left[F(x+y,z) + F(z,x+y)\right], \\ &= \frac{1}{2}\left[F(x,y)+ F(x+y,z) + F(y,x)+F(z,x+y)\right], \\ &= \frac{1}{2}\left[F(y,z) + F(x,y+z)+F(y+z,x) + F(z,y)\right], \\ &= \frac{1}{2}\left[F(y,z) + F(z,y)\right] + \frac{1}{2}\left[F(x,y+z)+F(y+z,x)\right] \\ &= f(y,z) + f(x,y+z), \end{aligned}$$

where the first and the last equalities follow from the definition of f, the third equality follows from (A) and (3.1o.8) by replacing $F(x,y) + F(x+y,z)$ by $F(y,z) + F(x,y+z)$ and replacing $F(y,x) + F(z,x+y)$ by $F(y+z,x) + F(z,y)$. Hence

$$(A') \qquad f(x,y) + f(x+y,z) = f(y,z) + f(x,y+z) .$$

Since f is symmetric and f satisfies (A'), hence by theorem 3.8, we can construct a function $g : G \rightarrow G'$ satisfying the identity

$$f(x,y) = g(x) + g(y) - g(x+y) .$$

This identity together with (3.1o.7) imply that

$$F(x,y) = B(x,y) + g(x) + g(y) - g(x+y)$$

This completes the proof of the theorem 3.1o .