CHAPTER II

PRELIMINARIES

In this chapter we will give some definitions and results from topology and group theory which will be a basic requirement for our investigation. The materials of this chapter were extracted from reference [2], [3], [4], [5], [7]. We shall assume that the reader is familiar with common terms used in set theory. Some details about ordinals and cardinals which will be used in the sequel can be found in the appendix

2.1 Algebraic Concepts

Following the usual custom, the value of the binary operation $\circ$ at (x, y) is written $x \circ y$ instead of the usual functional notation $\circ(x, y)$.

A <u>group</u> is a pair $(G, \circ)$, where G is a non - empty set and $\circ$ is a binary operation on G such that

a) the operation is associative, that is ,

$x \circ (y \circ z) = (x \circ y) \circ z$ for all elements x, y, z of G ;

b) there is an element e of G such that

$e \circ x = x \circ e = x$ for each x in G ;

c) for each x in G there is an element x^{-1} in G such that

$$x \circ x^{-1} = x^{-1} \circ x = e .$$

It can be shown that the element e in (b) is unique, it is known as the identity of G. For each $x \in G$, the element x^{-1} in (c) is also unique. It is known as the inverse of x. We shall sometimes write G for (G, $\circ$). For any element $x \in G$, the order of x is the least positive integer m such that $x^m = e$. If no such integer exists we say that x is of infinite order. If G is a group in which every element other than the identity is of infinite order, G is said to be torsion-free. If G is group in which every element is of finite order, G is said to be a a torsion group. A group G is abelian or commutative, if and only if $x \circ y = y \circ x$ for all members x and y of G. A group H is a subgroup of G if and only if $H \subseteq G$ and the group operation of H is the restriction of that of G. Let S be any subset of G. (The intersection of all subgroups of G that contains S form a subgroup of G. It is called the subgroup generated by S and will be denoted by $\langle S \rangle$). It can be shown that when $S \neq \emptyset$, $\langle S \rangle$ is the set of all finite product of elements of S or their inverses. When $S = \emptyset$, $\langle S \rangle$ consists of the identity alone.

A non - empty set F with two binary operations +, $\circ$, known as addition and multiplication respectively, is said to form a field if :

i) F forms a commutative group under addition.

ii) $F - \{0\}$, where 0 is the additive identity forms a commutative group under multiplication.

iii) For any a, b, $c \in F$, we have

$$a(b + c) = ab + ac .$$

Let $(F, +, \cdot)$ be a field and $(V, +)$ be a commutative group with a rule of multiplication which assigns to any $a \in F$, $u \in V$, a product $au \in V$. Then V is called a <u>vector space</u> over F if the following axioms hold :

1) For any $a \in F$ and any $u, v \in V$,

$$a(u + v) = au + av .$$

2) For any $a, b \in F$ and any $u \in V$,

$$(a + b) u = au + bu .$$

3) For any $a, b \in F$ and $u \in V$,

$$a(bu) = (ab)u .$$

4) For $v \in V$, $1v = v$ where 1 is the multiplicative identity of F .

The elements of F and V will be refered to as <u>scalars</u> and <u>vectors</u>, respectively. A subset E of a vector space V is said to to be a <u>subspace</u> if E is a vector space over the same field with addition and scalar multiplication induced from V, i.e. the operations of E are the restriction of that of V. If V is a vector space over the field F and $\{x_i\}$ $(1 \leq i \leq n)$ is a finite subset of V, then for $a_i \in F, 1 \leq i \leq n$, $\sum_{i=1}^{n} a_i x_i$ is called a <u>linear combination</u> of the x_i . The vectors $x_1, x_2, \ldots, x_n \in V$ are said to be <u>linearly dependent</u> over F, or simply <u>dependent</u>, if there exist scalars $a_1, \ldots, a_n \in F$, not all of them zero, such that $\sum_{i=1}^{n} a_i x_i = 0$. An arbitrary set **A** of vectors is said to be a <u>linearly dependent set</u> if some finite subset of **A** is linearly dependent. Otherwise, the set **A** is called a <u>linearly independent</u> or simply <u>independent</u>. If $\mathcal{B}$ is a linearly independent subset of V such that for every $v \in V$, v can be written as a linear combination of vectors in $\mathcal{B}$, we say that $\mathcal{B}$ is a <u>basis</u> of V. It can be shown that every vector in V has a unique representation as a linear combination of elements of $\mathcal{B}$ and that every basis of V has the same cardinal number. The cardinal number of a basis of a vector space is called its <u>dimension</u>. If the cardinal number of a basis of a vector space is finite, the vector space is called <u>finite dimensional</u>.

2.2 <u>Topological Spaces</u>.

Let X be a set and $\mathcal{T}$ be a collection of subsets of X. The collection $\mathcal{T}$ is called a <u>topology</u> on X provided $\mathcal{T}$ satisfies the following conditions :

a) $\emptyset$ and X are elements of $\mathcal{T}$.

b) The intersection of any two members of $\mathcal{T}$ is in $\mathcal{T}$.

c) The arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$.

If $\mathcal{T}$ is a topology on a set X, then $(X, \mathcal{T})$ is said to be a topological space. Occasionally, we shall denote any topological space $(X, \mathcal{T})$ simply by X. The members of $\mathcal{T}$ are called $\mathcal{T}$ - open sets of X(or simply open sets of X). If a topological space X has the property that for any x, y in X there exist open sets O_1, O_2 such that $x \in O_1$, $y \in O_2$ and $O_1 \cap O_2 = \emptyset$, we say that X is a Hausdorff space. For any topological space $(X, \mathcal{T})$, it can be shown that if Y is any subset of X, then the family $\mathcal{T}_y = \{T \cap Y : T \in \mathcal{T}\}$ is a topology on Y ; it is called the relative topology of Y and the topological space $(Y, \mathcal{T}_y)$ is called a subspace of $(X, \mathcal{T})$.

By a neighborhood of a point x in a topological space X, we shall mean a set N for which there exists an open set T such that $x \in T \subset N$. For $x \in X$, the collection $\mathcal{N}_x$ of all neighborhoods of x is called the neighborhood system of x. A subset A of a topological space X is said to be closed if and only if its relative complement $X \setminus A$ is open. An element x in a topological space X is an accumulation point of a subset A of X if and only if every neighborhood of x contains points of A other than x. The closure of a subset A is the intersection of all the closed sets that contains A. The closure of A is denoted by $\overline{A}$. It can be shown that the accumulation points of A are contained in $\overline{A}$. A set A is dense in a topological space X if and only if $\overline{A} = X$.

A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is said to be a <u>base</u> of $\mathcal{T}$ provided the following condition holds : for each $T \in \mathcal{T}$ and $x \in T$, there is a $W_x \in \mathcal{B}$ such that $x \in W_x \subset T$, or equivalently, each T in $\mathcal{T}$ is a union of members of $\mathcal{B}$. It can be shown that if a family $\mathcal{B}$ of subsets of a set X has the properties,

i) the union of sets in $\mathcal{B}$ is X.

ii) for each B_1, $B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is the union of members of $\mathcal{B}$, then $\mathcal{B}$ is a base for some topology for X. This topology consists of all sets that can be written as unions of sets in $\mathcal{B}$. Observe that the family of all open intervals form a base for a topology on the set $\mathbb{R}$ of real numbers. This topology is called the <u>usual topology</u> on $\mathbb{R}$.

A subfamily $\mathcal{S}$ of $\mathcal{T}$ is a <u>subbase</u> of the topology $\mathcal{T}$ on X if and only if the set of all finite intersections of members of $\mathcal{S}$ form a base for $\mathcal{T}$

A <u>base for the neighborhood system</u> $\mathcal{N}_x$ of a point x is a subcollection $\mathcal{B}_x$ of $\mathcal{N}_x$ such that for each $N_x \in \mathcal{N}_x$, there is a $U_x \in \mathcal{B}_x$ such that $x \in U_x \in N_x$. If all sets in $\mathcal{B}_x$ are open, we say that $\mathcal{B}_x$ is an <u>open base for the neighborhood system</u> $\mathcal{N}_x$.

A function f of a topological space $(X, \mathcal{T})$ into a topological space $(Y, \mathcal{U})$ is <u>continuous</u> at a point x if and only if, given any neighborhood V_y of the point $y = f(x)$, there is a neighborhood U_x of the point x such that $f(U_x) \subset V_y$. The mapping f is said to be <u>continuous</u> on X if it is continuous at every point of X.

<u>Theorem</u> 2.2.1 If X and Y are topological space and f is a function on X to Y, then the following statements are equivalent.

a) The function f is continuous.

b) For each subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

c) For any open set $T \subset Y$, the preimage $f^{-1}(T)$ is open in X. For the proof of this theorem see [5].

Let $f : D \longrightarrow Y$ be a map from a subset D of a topological space X into a topological space Y. Let x_o be an accumulation point of D. If for every neighborhood V of y_o, there is a neighborhood U of x_o such that $f(U - \{x_o\}) \subset V$, then y_o is a <u>limit</u> of f at x_o, It can be shown that when Y is a Hausforff space, f can have at most one limit at each point x_o. Hence, if a limit of f at x_o exists, it is unique. When this is the case, we shall use the notation $y_o = \lim\limits_{x \longrightarrow x_o} f(x)$ to indicate that y_o is the limit of f at x_o. When Y is $\mathbb{R}$, the extended real line, we then define functions on $\overline{D}$ to $\overline{\mathbb{R}}$, called the limit superior and the limit inferior of f, the value of these limits at x are written as $\limsup_{t \to x} f(t)$ and $\liminf_{t \to x} f(t)$ respectively. To define $\limsup_{t \to x} f(t)$ and $\liminf_{t \to x} f(t)$, let $\mathcal{N}$ be the class of all sets $N = D \cap U$ when U can be any neighborhood of x. We assume $x \in \overline{D}$; therefore $N \neq \emptyset$. Then, by definition

$$\limsup_{t \to x} f(t) = \inf_{N \in \mathcal{N}} \sup \{ f(t) : t \in N \}, \ x \in \overline{D},$$

$$\liminf_{t \to x} f(t) = \sup_{N \in \mathcal{N}} \inf \{ f(t) : t \in N \}, \ x \in \overline{D}.$$

It can be shown that if f is continuous at $x \in D$, then

$$\liminf_{t \to x} f(t) = f(x) = \limsup_{t \to x} f(t).$$

Let $\{ X_\alpha \mid \alpha \in A \}$ be a family of sets. $X = \prod_{\alpha \in A} X_\alpha$ denotes the set of all mappings $x : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for each $\alpha \in A$. X is called the Cartesian product or product of X_α's. For each $x \in X$ and each $\alpha \in A$, $x(\alpha)$ is called the projection of x on X_α. We shall denote $x(\alpha)$ by x_α. The mapping $P_\alpha : X \longrightarrow X_\alpha$ defined by $P_\alpha(x) = x_\alpha$, is called the α th projection mapping. It can be seen that P_α, is a mapping from X onto X_α . If $\{X_\alpha / \alpha \in A\}$ is a family of topological spaces, then the family of sets of the form $p_\alpha^{-1}(T_\alpha)$, where T_α is a $\mathcal{T}_\alpha$- open set, forms a subbase of a topology $\mathcal{T}$ for the product $\prod_{\alpha \in A} X_\alpha$. This topology is known as the product topology. The topological space $(\prod X_\alpha , \mathcal{T})$ will be called the product space of $\{X_\alpha / \alpha \in A\}$.

Let D be a set . A binary relation $\geq$ on D is said to direct D if the following hold :

a) for any m, n and p in D if $m \geq n$ and $n \geq p$, then $m \geq p$;

b) for any m in D, we have $m \geq m$;

c) for any m and n in D, there exists a p in D such that $p \geq m$ and $p \geq n$.

A directed set is a pair $(D, \geq)$, where $\geq$ directs D.

Let X be a set. A net y in X is a map $y : D \longrightarrow X$ where $(D, \geq)$ is a directed set. For $\alpha \in D$ we usually write y_α for $y(\alpha)$. The notation $\{y_\alpha\} (\alpha \in D)$ will be used to designate a net defined on the directed set $(D, \geq)$.

Remark One immediatly observes that the concept of a directed set is a generalization of the positive integers with their natural ordering, and that of a net is the generalization of a sequence.

Let Γ be any limit ordinal number. Hence $(\Gamma, \geqslant)$ is a directed set. Any net having $(\Gamma, \geqslant)$ as a directed set will be called Γ - net.

A net $\{y_\alpha\}_{(\alpha \in D)}$ in a topological space $(X, \mathcal{T})$ converges to $x \in X$ if for each neighborhood U of x there exists a $\beta \in D$ such that for all $\alpha \geqslant \beta$, $y_\alpha \in U$. When $\{y_\alpha\}_{(\alpha \in D)}$ converges to x, we say that x is a limit of $\{y_\alpha\}_{(\alpha \in D)}$.

Suppose $\{y_\alpha\}_{(\alpha \in D)}$ is a net in a set X. Let J be a directed set and k be a function from J to D such that

i) if $\alpha \geqslant \alpha_0$, then $k(\alpha) \geqslant k(\alpha_0)$;

ii) if $\gamma, \beta \in D$ then there is $j \in J$ such that $k(j) \geqslant \gamma$ and $k(j) \geqslant \beta$.

Then the composition $y \circ k$ from J into X is said to be a subnet of the net $\{y_\alpha\}_{(\alpha \in D)}$. The subnet $y \circ k$ is usually written as $\{y_{k_j}\}_{(j \in J)}$.

It can be shown that every convergent net in a Hausdorff space has a unique limit. We shall write $\lim(y_\alpha, \alpha \in D) = p$ to mean that $\{y_\alpha\}_{(\alpha \in D)}$ converges to p.

Let S be a subset of a set X. A collection $\mathcal{A}$ of subsets of X is called a covering of S if and only if the union of the sets in $\mathcal{A}$ contains S. When $\mathcal{A}$ is a covering of S we also say that $\mathcal{A}$ covers S.

Let $\mathcal{A}$ be a covering of a subset S of a set X. A collection $\mathcal{H}$ is called a <u>subcovering</u> of $\mathcal{A}$ if and only if $\mathcal{H}$ covers S and every set in $\mathcal{H}$ is a set in $\mathcal{A}$.

A collection $\mathcal{A}$ of subsets of a topological space X is called an <u>open covering</u> of a subset A of a topological space X if and only if $\mathcal{A}$ is a covering of A and every set in $\mathcal{A}$ is open in X.

A subset A of a topological space X is <u>compact</u> if and only if every covering of A by sets which are open in X has a finite subcovering.

2.3 <u>Topological groups</u>

A triple $(G, \circ, \mathcal{T})$ is a <u>topological group</u> if and only if $(G, \circ)$ is a group, $(G, \mathcal{T})$ is a topological space, and the function whose value at a member (x, y) of $G \times G$ is $x \circ y^{-1}$ is continuous relative to the product topology for $G \times G$. We sometimes say " G is a topological group ".

<u>Example of topological groups</u>

a) The set $\mathbb{R}$ of real numbers with addition as the group operation and the usual topology form an additive abelian topological group.

b) The set $\mathbb{R}^{+}$ of positive real numbers with multiplication as the group operation and the relative topology of the usual topology of $\mathbb{R}$ form a topological group.

c) The set $\mathbb{R}^k$ of all real k - tuples with addition as the coordinate addition and the usual topology of $\mathbb{R}^k$ form a topological group.

d) A complex number can be considered as an ordered pair of real numbers. Hence the usual topology on $\mathbb{C}$, the set of complex numbers, shall mean the usual topology on $\mathbb{R}^2$. The set $\mathbb{C}$ of complex numbers with addition as its binary operation and the usual topology on $\mathbb{C}$ form a topological group.

All topological groups mentioned above are abelian and torsion free.

A topological group X will be said to be <u>σ^* - compact</u> if $X = \bigcup_{n=0}^{\infty} K_n$ for some sequence $\{K_n / n \in \omega\}$ of compact neighborhoods of e, the identity of X. Note that $\mathbb{R}^{(k)}$ is σ^* - compact.

2.4 <u>Uniform Spaces</u>

2.4.1 <u>Uniformities and the uniform topology</u>

By a relation on a set X we mean a subset of $X \times X$. If U is a relation on X, its <u>inverse</u>, denote by U^{-1}, is defined by

$$U^{-1} = \{(y, x) / (x, y) \in U\}$$

For any two relations U, V on X, their composition $U \circ V$ is defined by $U \circ V = \{(x, z) \,/\, x, z \in X$ and there exists $y \in X$ such that $(x, y) \in U$ and $(y, z) \in V\}$.

The relation

$$\Delta = \{(x, x) \,/\, x \in X\}$$

will be called the <u>diagonal</u>.

A <u>uniformity</u> for a set X is a non - empty family $\mathcal{U}$ of relations on X such that

a) each member of $\mathcal{U}$ contains the diagonal Δ,

b) if U is in $\mathcal{U}$, then U^{-1} is also in $\mathcal{U}$,

c) if U is in $\mathcal{U}$, then there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$,

d) if U and V are members of $\mathcal{U}$, then $U \cap V$ is in $\mathcal{U}$,

e) if U is in $\mathcal{U}$ and $U \subset V \subset X \times X$, then V is in $\mathcal{U}$.

A <u>uniform space</u> is a pair $(X, \mathcal{U})$, where X is a set and $\mathcal{U}$ is a uniformity on X. If A is a subset of X, the <u>relative uniformity</u> $\mathcal{U}'$ for A is the uniformity consisting of the sets $U \cap (A \times A)$, for $U \in \mathcal{U}$. $(A, \mathcal{U}')$ is called a (uniform) <u>subspace</u> of X.

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a <u>base</u> for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. If a base $\mathcal{B}$ of a uniformity $\mathcal{U}$ is given, then each U is in $\mathcal{U}$

contains a member of $\mathcal{B}$. Hence any base for $\mathcal{U}$ determines $\mathcal{U}$ entirely.

It can be shown that a family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if

(a') each member of $\mathcal{B}$ contains the diagonal Δ ;

(b') if U is in $\mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$;

(c') if U is in $\mathcal{B}$, then $V \circ V \subset U$ for some V in $\mathcal{B}$; and

(d') The intersection of two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

When $X = \mathbb{R}$, the set of all real numbers, the collection $\mathcal{B}$ of all sets of the form $R_\delta = \{(x,y) / |x-y| < \delta\}$, where δ is any positive real number, satisfies (a'), (b'), (c'), (d'). Hence $\mathcal{B}$ is a base for some uniformity for $\mathbb{R}$. This uniformity is called the <u>usual uniformity</u> for $\mathbb{R}$.

If $(X, \mathcal{U})$ is a uniform space, C is any subset of X, and U is any element of $\mathcal{U}$, we denote by $U[C]$ the set of all points y in X such that (x, y) is in U for some point x in C. If can be verified that the family of all subsets A of X such that for each x in A there is U in $\mathcal{U}$ such that $U[x] \subset A$, is a topology on X. This topology is called <u>the uniform topology determined by $\mathcal{U}$</u>.

Let $\mathcal{N}$ be the neighborhood system of the identity e of a topological group $(G, o, \mathcal{T})$. For each N in $\mathcal{N}$, let

$$R(N) = \{(x, y) \in G \times G : yx^{-1} \in N\},$$

$$L(N) = \{(x, y) \in G \times G : xy^{-1} \in N\}.$$

It can be verified that $\mathcal{R}(\mathcal{N}) = \{R(N) / N \in \mathcal{N}\}$ and $\mathcal{L}(\mathcal{N}) = \{L(N) / N \in \mathcal{N}\}$ form bases for some uniformities. These uniformities are known as the right uniformity and the left uniformity on G, determined by $\mathcal{T}$ respectively. It can be shown that the uniformities with $\mathcal{L}(\mathcal{N}), \mathcal{R}(\mathcal{N})$ and $\mathcal{L}(\mathcal{N}) \cup \mathcal{R}(\mathcal{N})$ as bases determine the same uniform topology, which coincides with the original topology $\mathcal{T}$ of the topological group $(G, o, \mathcal{T})$. Hence the topology of any topological group is always a uniform topology.

Remark : Recall that the set of real numbers $\mathbb{R}$ with addition as the group operation and the usual topology, is a topological group. It can be seen that the usual uniformity of $\mathbb{R}$ can be considered as the uniformity determined by the usual topology of $\mathbb{R}$.

2.4.2 Uniform Continuity

A function f on a uniform space $(X, \mathcal{U})$ with values in a uniform space $(Y, \mathcal{V})$ is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ if and only if for each V in $\mathcal{V}$ the set $\{(x, y) : (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$. Let A be a subset of X with a relative uniformity $\mathcal{U}'$. A function f on $(A, \mathcal{U}')$ with values in a uniform space $(Y, \mathcal{V})$ is uniformly continuous relative to $\mathcal{U}'$ and $\mathcal{V}$

if and only if for each V in $\mathcal{V}$ the set $\{(x,y):(f(x), f(y))\in V\}$ is a member of $\mathcal{U}'$.

Remark 1 One can see that a function f on a subset A of a topological group $(G, o\, \mathcal{T})$ with values in a topological group $(G', *, \mathcal{T}')$ is uniform continuous on A if and only if for each $N'\in\mathcal{N}'$, where $\mathcal{N}'$ is an open base of neighborhoods of the identity e' of G', there exists a $N\in\mathcal{N}$, where $\mathcal{N}$ is an open base of the identity e of G, such that $f(y)\,[f(x)^{-1}]\in N'$ (or $[f(x)]^{-1}f(y)\in N'$) whenever $yx^{-1}\in N$ (or $x^{-1}y\in N$) for x, y in A .

Remark 2 Let $\mathcal{N}$ be the set of all open neighborhoods of the identity e of the topological group G. It can be shown that the family $\{N'\times N' / N', N\in\mathcal{N}\}$ forms an open base of the neighborhoods of the identity (e,e) of the product space $G\times G$. Hence it follows from remark 1 that a function f from $G\times G$, into a topological group G' is uniformly continuous on $A\times A$, for a subset A of G, if and only if for each open neighborhood N' of e', the identity of G', there exists a neighborhood N of the identity e of G such that $f(x,y)\,[f(x', y')]^{-1}\in N'$ (or $[f(x', y')]^{-1} f(x,y)\in N'$) whenever $(x, y).\,(x', y')^{-1}\in N\times N$ (or $(x', y')^{-1}.\,(x, y)\in N\times N$) for all $(x, y), (x', y')\in A\times A$.

In [2] it has been proved that if f is a continuous function whose domain is a compact uniform space and whose range is a uniform space, then f is uniformly continuous.

2.4.3 Product Uniformities

For each a of an index set **A**, let $(X_a, \mathcal{U}_a)$ be a uniform space. The product uniformity for $X\{X_a : a \in \mathbf{A}\}$ is the smallest uniformity such that the projection into each coordinate space is uniformly continuous. It can be shown that the product uniformity describes precisely the product topology on $X\{X_a : a \in A\}$.

A net $\{y_\alpha\}_{(\alpha \in D)}$ in a uniform space **X** is a Cauchy net if for a given U in the uniformity $\mathcal{U}$ of **X** there exists $\alpha_0 \in D$ such that for all $\alpha, \beta \geq \alpha_0$, $(\alpha, \beta \in D)$, $(y_\alpha, y_\beta) \in U$.

Remark A net $\{y_\alpha\}_{(\alpha \in D)}$ is cauchy in a topological group $(G,o,\mathcal{T})$ if it is cauchy relative to the uniformity which determines the topology $\mathcal{T}$.

The following are well - known facts about nets in topological space and uniform space. We state these facts for later references. Their proofs are straight forward and will be omitted.

Theorem 2.5.1 If a net $\{y_\alpha\}_{(\alpha \in D)}$ in a topological space $(X, \mathcal{T})$ converges to x, then any subnet of $\{y_\alpha\}_{(\alpha \in D)}$ also converges to x.

Theorem 2.5.2 If $\mathcal{U}$ is a uniformity for a set X and U is any set in $\mathcal{U}$, then there exists a W of $\mathcal{U}$ such that $W o W^{-1} \subseteq U$. Furthermore, W can be chosen to be symmetric, i.e, $W^{-1} = W$.

<u>Theorem</u> 2.5.3 If a net $\{y_\alpha\}_{(\alpha \in D)}$ in a uniform space X converges to any point y_0 in X, then $\{y_\alpha\}_{(\alpha \in D)}$ is cauchy.

<u>Lemma</u> 2.5.4 Let Γ - nets $\{y_\alpha\}(\alpha<\Gamma)$ and $\{y'_\alpha\}(\alpha<\Gamma)$ in a uniform space $(X,\mathcal{U})$ converge to a point $x \in X$. Define

$$z_\alpha = \begin{cases} y_{\omega\xi+n} & \text{if } \alpha = \omega\xi + n2, \\ y'_{\omega\xi+n} & \text{if } \alpha = \omega\xi + n2 + 1, \end{cases}$$

where ξ is an ordinal and $n < \omega$. Then $\{z_\alpha\}(\alpha<\Gamma)$ is a cauchy net.

<u>Proof</u> First we shall verify that the net $\{z_\alpha\}(\alpha<\Gamma)$ are well - defined. For each $\alpha \in \Gamma$, α can be written in the form $\omega\xi + m$ where ξ and m are unique ordinal and m is finite. Since either $m = n2$ or $m = n2 + 1$ for some finite ordinal n. Hence for all $\alpha \in \Gamma$, either $\alpha = \omega\xi + n2$ or $\alpha = \omega\xi + n2 + 1$ for unique ξ and n.

Therefore net $\{z_\alpha\}$ is well - defined on Γ.

Next we will show that the net $\{z_\alpha\}(\alpha<\Gamma)$ is cauchy. Let V be in $\mathcal{U}$. By theorem 2.5.2, there exists a W of $\mathcal{U}$ such that

$$W \circ W^{-1} \subseteq V \quad \text{and} \quad W^{-1} = W.$$

Since the Γ - net $\{y_\alpha\}(\alpha<\Gamma)$ converges to a point $x \in X$, hence there exists $\gamma_0 < \Gamma$ such that $(y_\alpha, x) \in W$ for all $\alpha \geqslant \gamma_0$.

Similarly the Γ - net $\{y'_\alpha\}(\alpha<\Gamma)$ converges to the point $x \in X$, hence there exists $\gamma'_0 < \Gamma$ such that $(y'_{\alpha'}, x) \in W$ for all $\alpha' \geqslant \gamma'_0$.

Choose $\eta_1 < \Gamma$ such that $\eta_1 > \gamma_0$ and $\eta_1 > \gamma_0'$.

Therefore for all $\alpha, \alpha' \geq \eta_1$ we have

$$(y_\alpha, y'_{\alpha'}) \in W \circ W^{-1} \subseteq V .$$

Since $\{y_\alpha\}_{(\alpha < \Gamma)}$ and $\{y'_\alpha\}_{(\alpha < \Gamma)}$ are convergent, hence they are cauchy. Therefore

i) there exists $\eta_2 < \Gamma$ such that for all $\alpha_1, \alpha_2 < \Gamma$, if $\alpha_1, \alpha_2 \geq \eta_2$ then $(y_{\alpha_1}, y_{\alpha_2}) \in V$,

ii) there exists $\eta_3 < \Gamma$ such that for all $\alpha_1, \alpha_2 < \Gamma$, if $\alpha_1, \alpha_2 \geq \eta_3$ then $(y'_{\alpha_1}, y'_{\alpha_2}) \in V$.

Since η_1, η_2, η_3 are ordinals, they can be expressed in the form :

$$\eta_1 = \omega \xi_{\delta_1} + n_1 , \quad \eta_2 = \omega \xi_{\delta_2} + n_2 ; \quad \eta_3 = \omega \xi_{\delta_3} + n_3 ,$$

where ξ_1, ξ_2, ξ_3 are some ordinals and $n_1, n_2, n_3 < \omega$.

set
$$\eta_1' = \omega \xi_{\delta_1} + n_1 2 + 1 ,$$
$$\eta_2' = \omega \xi_{\delta_2} + n_2 2 ,$$
$$\eta_3' = \omega \xi_{\delta_3} + n_3 2 + 1 .$$

Take $\eta = \max (\eta_1', \eta_2', \eta_3')$.

We claim that for all $\alpha, \beta < \Gamma$, if $\alpha, \beta \geq \eta$ then $(z_\alpha, z_\beta) \in V$.

We shall prove this fact by cases :

i) both of α , β are in the form $\omega\xi_\gamma + n2$,

ii) both of α, β are in the form $\omega\xi_\gamma + n2 + 1$,

iii) only one of α , β is in the form $\omega\xi_\gamma + n2$.

Case 1 Suppose

$$\alpha = \omega\xi_\gamma + n2 \quad \text{and} \quad \beta = \omega\xi'_\gamma + n'2 \ ,$$

for some ordinals ξ_γ, ξ'_γ and $n, n' < \omega$. Then

$$z_\alpha = y_{\omega\xi_\gamma + n} \quad \text{and} \quad z_\beta = y_{\omega\xi'_\gamma + n'} .$$

Since

$$\omega\xi_\gamma + n2 \geq \eta \geq \eta'_2 = \omega\xi_{\gamma_2} + n_2 2 \ ,$$

hence, by theorem A - 30 , we have

$$\omega\xi_\gamma + n \geq \omega\xi_{\gamma_2} + n_2 = \eta_2$$

Similarly,

$\omega\xi'_\gamma + n'2 \geq \eta \geq \eta'_2 = \omega\xi_{\gamma_2} + n_2 2$, implies $\omega\xi'_\gamma + n' \geq \omega\xi_{\gamma_2} + n_2 = \eta_2$. Hence

$$(z_\alpha , z_\beta) = (y_{\omega\xi_\gamma + n}, y_{\omega\xi'_\gamma + n'}) \in V.$$

Case 2 Suppose

$$\alpha = \omega\xi_\gamma + n2 + 1 \quad \text{and} \quad \beta = \omega\xi'_\gamma + n'2 + 1$$

for some ordinals ξ_γ, ξ'_γ and $n, n' < \omega$. Then

$$z_\alpha = y'_{\omega\xi_\gamma + n} \quad \text{and} \quad z_\beta = y'_{\omega\xi'_\gamma + n'} .$$

Since

$$\omega\xi_{\gamma} + n2 + 1 \geq \eta \geq \eta'_{3} = \omega\xi_{\gamma_3} + n_{3}2 + 1,$$

hence, by theorem A - 30 , we have

$$\omega\xi_{\gamma} + n \geq \omega\xi_{\gamma_3} + n_{3} = \eta_{3} .$$

Similarly

$$\omega\xi'_{\gamma} + n'2 + 1 \geq \eta \geq \eta'_{3} = \omega\xi_{\gamma_3} + n_{3}2 + 1 \quad \text{implies}$$

$$\omega\xi'_{\gamma} + n' \geq \omega\xi_{\gamma_3} + n_{3} = \eta_{3} .$$

Hence

$$(z_{\alpha} , z_{\beta}) = (y'_{\omega\xi_{\gamma} + n} , y'_{\omega\xi'_{\gamma} + n'}) \in V .$$

Case 3. Without loss of generality we may assume that

$$\alpha = \omega\xi_{\gamma} + n2 \quad \text{and} \quad \beta = \omega\xi'_{\gamma} + n'2 + 1$$

for some ordinals $\xi_{\gamma}, \xi'_{\gamma}$ and $n, n' < \omega$. Then

$$z_{\alpha} = y_{\omega\xi_{\gamma} + n} \quad \text{and} \quad z_{\beta} = y'_{\omega\xi'_{\gamma} + n'} .$$

Since

$$\omega\xi_{\gamma} + n2 \geq \eta \geq \eta'_{1} = \omega\xi_{\gamma_1} + n_{1}2 + 1,$$

hence, by theorem A - 30 , we have

$$\omega\xi_{\gamma} + n \geq \omega\xi_{\gamma_1} + n_{1} = \eta_{1} .$$

Similarly, since

$$\omega\xi'_{\gamma} + n'2 + 1 \geq \eta \geq \eta'_{1} = \omega\xi_{\gamma_1} + n_1 2 + 1 ,$$

it follows from theorem A - 30 that $\omega\xi'_{\gamma} + n' \geq \omega\xi_{\gamma_1} + n_1 = \eta_1$.

Hence $(z_{\alpha}, z_{\beta}) = (y_{\omega\xi_{\gamma} + n}, y'_{\omega\xi'_{\gamma} + n'}) \in V$.

Therefore $\{z_{\alpha}\}_{(\alpha < \Gamma)}$ is Cauchy .

2.6 Topological vector space

A topological vector space is the vector space E over the field k of real or complex number and a Hausdorff topology on E such that the function $f : E \times E \longrightarrow E$ and $g : K \times E \longrightarrow E$ defined by $f(x,y) = x + y$ and $g(\lambda x) = \lambda x$, are continuous, where the topology on K is the usual topology. However, in this thesis, by a topological vector space, we shall mean a real topological vector space.

2.7 Spaces with property (Γ N)

Let Γ be a limit ordinal. A topological space X will be said to have property (ΓN), if for each accumulation point x of any subset A of X, there exists a Γ - net in A which converges to x.

<u>Theorem</u> 2.7.1 Let $(X, \mathcal{T})$ be a topological space which has the property (ΓN). Let $g : X \longrightarrow Y$ be a function from space X into a topological Y. Then g is continuous if for each x in X and for each net $\{x_\alpha\}(\alpha < \Gamma)$ which converges to x, the net $\{g(x_\alpha)\}(\alpha < \Gamma)$ converges to $g(x)$.

<u>Proof</u>. Assume that for each x in X and for each Γ- net $\{x_\alpha\}(\alpha < \Gamma)$ which converges to x, the Γ- net $\{g(x_\alpha)\}(\alpha < \Gamma)$ converges to $g(x)$. Let A be a subset of X and x be an accumulation point of A. Since X has property (ΓN), hence there exists a Γ- net $\{x_\alpha\}(\alpha < \Gamma)$ in A which converges to x. Then $\{g(x_\alpha)\}(\alpha < \Gamma)$ converges to $g(x)$. We claim that $g(x) \in \overline{g(A)}$. Suppose $g(x) \notin \overline{g(A)}$. Then $g(x)$ must be in the complement of $\overline{g(A)}$ which is open. Hence there exists an open set O such that $g(x)$ is in O and $\overline{g(A)} \cap O$ is empty. But $g(x)$ is the limit of the Γ - net $\{g(x_\alpha)\}(\alpha < \Gamma)$ and $g(x)$ is in O, hence there exists an ordinal $\beta < \Gamma$ such that $g(x_\alpha) \in O$ for all $\alpha \geq \beta$. Therefore $g(x_\alpha) \notin g(A)$ for all $\alpha \geq \beta$, which is a contradiction. Hence $g(x) \in \overline{g(A)}$. Therefore $g(\bar{A}) \subseteq \overline{g(A)}$. Therefore, by theorem 2.2.1, g is continuous .