CHAPTER IV

THE FUNCTIONAL EQUATION $f(x * f(y)) = f(y * f(x))$.

The materials of this chapter are drawn from reference [3].

In this chapter we will prove some relations of a multiplicative symmetric function and a demi-multiplicative symmetric function. This proposition will be used to solve the functional equation characterizing the multiplicative-symmetric functions.

Before we state and prove our proposition, let us give some examples of multiplicative-symmetric functions.

<u>Example 4.1.</u> Consider the Dirichlet function defined by

$$f(x) = 0 \quad \text{if } x \text{ is rational numbers}$$
$$= 1 \quad \text{if } x \text{ is irrational numbers.}$$

Clearly f is everywhere discontinuous and is a measurable function. To show that f is multiplicative-symmetric function.

<u>Recall</u> : A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is <u>multiplicative</u> <u>symmetric</u> (MS), if

$$f(x\, f(y)) = f(y\, f(x)).$$

Suppose x and y are any rational numbers; then

$$f(x\, f(y)) = f(x.0) = f(0) = 0 = f(y.0) = f(y\, f(x)).$$

Hence in this case f is MS. Similarly, for the other cases, such as x and y are both irrational or x is rational, y is irrational, we can show that f is MS.

Thus there exists a measurable, everywhere discontinuous multiplicative symmetric function over $(\mathbb{R}, .)$.

<u>Example 4.2.</u> Let f be a function defined over $(\mathbb{R}, .)$ by

$$f(x) = 0 \quad , x \leq 0$$
$$= \lambda^{n+k} \quad , x \in [\lambda^n, \lambda^{n+1}[\; n \in \mathbb{Z}, \lambda > 0,$$
$$x > 0, \text{ and } k \in \mathbb{Z} (>0).$$

Clearly f is continuous for all $x \leq 0$ and for $x \in]\lambda^n, \lambda^{n+1}[$. But f is not continuous at the point of the denumerable set $\{\lambda^{n+1} \mid n \in \mathbb{Z}\}$.

It follows from the definition of MS and the definition of f that f is MS for all $x \leq 0$. Now consider the case when x and y are both in $[\lambda^n, \lambda^{n+1}[$. Then

$$f(x\, f(y)) = f(x \,.\, \lambda^{n+k}).$$

But $x \in [\lambda^n, \lambda^{n+1}[$, hence $x \,.\, \lambda^{n+k} \in [\lambda^{2n+k}, \lambda^{2n+k+1}[$.

Thus $f(x \,.\, f(y)) = \lambda^{(2n+k)+k}$.

But x and y are arbitrary in $[\lambda^n, \lambda^{n+1}[$, hence

$$f(x\, f(y)) = f(y\, f(x)),$$

i.e., f is MS for all $x \in [\lambda^n, \lambda^{n+1}[$.

Similarly, for the case $x \in [\lambda^{n+p}, \lambda^{n+p+1}[$ and $y \in [\lambda^{n+q}, \lambda^{n+q+1}[$, $(n, p, q \in \mathbb{Z})$ or the case $x \leq 0$ and $y \in [\lambda^{n+p}, \lambda^{n+p+1}[$, we can show that f is MS.

Therefore there exists a multiplicative symmetric function over $(\mathbb{R}, .)$ continuous except on a denumerable set.

<u>Example 4.3.</u> There exists a non-measurable bounded multiplicative symmetric functions in $(\mathbb{R}, +)$ [see Example 6.18 of Chapter **VI**].

<u>Lemma 4.4.</u> Let $(G, *)$ be a semi-group and $f : (G, *) \longrightarrow (G, *)$ a multiplicative symmetric function. Then

$$f(y * f(z) * f(x)) = f(z * f(y) * f(x)).$$

<u>Proof.</u> Since f is MS,

$$(4.1) \qquad f(x * f(y * f(z))) = f(y * f(z) * f(x)),$$

and

$$f(x * f(y * f(z))) = f(x * f(z * f(y)))$$

$$= f(z * f(y) * f(x)).$$

Therefore the last equation and Eq (4.1) imply that

$$f(y * f(z) * f(x)) = f(z * f(y) * f(x)).$$

Hence the lemma is proved /

<u>Theorme 4.5.</u> Let f be a non-constant everywhere continuous MS function on $(\mathbb{R}, .)$. Then there exists an $\alpha \neq 0$ such that f_α or $-f_\alpha$ is a DMS function.

<u>Proof.</u> By the hypothesis that f is non-constant, we have that $f \not\equiv 0$ so there exists z_o in $\mathbb{R}$ such that $f(z_o) \neq 0$ which by Lemma 3.16 implies that $z_o \neq 0$. Since we can write $f(z_o)$ as

$$f(z_o) = f(z_o) . z_o^{-1} . z_o,$$

there exists $\alpha = (f(z_o))^{-1} . z_o \neq 0$ such that

(4.2) $$f(z_o) = \frac{z_o}{\alpha}.$$

Since f is MS on $(\mathbb{R}, .)$, Lemma 4.4 implies that

$$f(y\, f(z)\, f(x)) = f(z\, f(y)\, f(x)).$$

Put $y = \alpha$ in the last equation to get

(4.3) $$f(\alpha\, f(z)\, f(x)) = f(z\, f(\alpha)\, f(x)).$$

Let

(4.4) $$Z = z_o\, f(x)$$

then from Eq (4.2),

(4.5) $$Z = \alpha\, f(z_o)\, f(x).$$

By applying f to Eq (4.5),

$$f(Z) = f(\alpha\, f(z_o)\, f(x))$$

which gives, by Eq (4.3) when $z = z_o$,

$$f(Z) = f(z_o\, f(\alpha)\, f(x))$$

$$= f(z_o\, f(x)\, f(\alpha)).$$

By Eq (4.4),

(4.6) $$f(Z) = f(Z\, f(\alpha)).$$

Note that, from Eq (4.4) and the fact that f is a MS function,

$$f(Z\, f(\alpha)\, f(y)) = f(z_o\, f(x)\, f(\alpha)\, f(y))$$

$$= f(y\, .\, f(z_o f(x)\, f(\alpha)))$$

$$= f(y\, .\, f(Z\, f(\alpha))).$$

Then by Eq (4.6),

$$f(Z\, f(\alpha)\, f(y)) = f(y.\, f(Z)) = f(Z\, f(y))$$

because f is MS. Then

(4.7) $$f(Z\, f(\alpha)\, f(y)) = f(Z\, f(y)).$$

We now claim that

(4.8) $$f(Z) = f(Z\,(f(\alpha))^n) \quad (n \in \mathbb{Z}\ (>0)).$$

Since Eq (4.8) is Eq (4.6) when $n = 1$, assume Eq (4.8) holds for $n = k$, i.e.,

$$f(Z) = f(Z(f(\alpha))^k).$$

Now
$$\begin{aligned} f(Z(f(\alpha))^{k+1}) &= f(Z\, f(\alpha)(f(\alpha))^k) \\ &= f(Z(f(\alpha))^k) \\ &= f(Z) \end{aligned}$$

by the induction hypothesis and Eq (4.7). Hence the case $n = k + 1$ holds. Therefore by induction on n, Eq (4.8) holds.

If $f(\alpha) = 0$, then from Eq (4.6) implies that $f(Z) = f(0)$ which implies, by Lemma 3.16, $f(Z) = 0$ so that by Eq (4.4) and f being MS,

$$0 = f(z_o f(x)) = f(x\, f(z_o))$$

for all x in $\mathbb{R}$. Hence for any y in $\mathbb{R}$

$$y = y \cdot (f(z_o))^{-1} f(z_o) = x\, f(z_o)$$

where $x = y\,(f(z_o))^{-1}$, so that

$$f(y) = f(x\, f(z_o)) = 0 \quad (y \in \mathbb{R}).$$

Therefore $f \equiv 0$, which we have excluded. Hence $f(\alpha) \neq 0$.

If $|f(\alpha)| < 1$, choose Z so that $f(Z) \neq 0$. Then by the continuity of f and Lemma 3.16,

$$f(Z) = \lim_{n\to\infty} f(Z(f(\alpha))^n) = f(Z. \lim_{n\to\infty} f(\alpha)^n) = f(0) = 0$$

which contradicts the choice of Z. Hence $|f(\alpha)| \geq 1$.

If $|f(\alpha)| > 1$, then $f(\alpha) > 1$ or $f(\alpha) < -1$. Since by Lemma 3.16, $f(0) = 0$ and by Intermediate Value Theorem, there exists a β between 0 and α such that $|f(\beta)| = 1$.

If $\underline{f(\beta) = 1}$, define $g = f_\beta$ and due to Lemma 3.15, g is MS with $g(1) = f_\beta(1) = f(1.\beta) = f(\beta) = 1$. Since g is MS,

$$g(x\, g(y)) = g(y\, g(x)) \qquad (x, y \in \mathbb{R}).$$

Take $x = 1$ in the last equation, we have

$$g^{(2)}(y) = g(y\, g(1)).$$

But $g(1) = 1$; hence

$$g^{(2)}(y) = g(y) \quad .$$

Therefore

$$g(x\, g(y)) = g(y\, g(x)) = g(y\, g^{(2)}(x))$$

$$= g(g(x)\, g(y))$$

by the MS-property of g. Hence $g = f_\beta$ is DMS.

Assume now that $\underline{f(\beta) = -1}$. Define $g = f_\beta$, then

$$g(1) = f_\beta(1) = f(\beta) = -1.$$

Now $g^{(2)}(x) = f_\beta^{(2)}(x) = f(f(x.\beta)\beta) = f(\beta f(x.\beta))$

$= f(x\beta\, f(\beta))$, by the MS-property of f. But $f(\beta) = -1$, hence

$$g^{(2)}(x) = f(-x\beta) = f_\beta(-x)$$

by the definition of f_β. But $f_\beta = g$; then

$$g^{(2)}(x) = g(-x), \tag{4.9}$$

and

$$g(g(x)\, g(y)) = g(y\, g^{(2)}(x)) = g(y\, g(-x)).$$

By symmetry roles of x and y, we have

(4.10) $$g(y\, g(-x)) = g(x\, g(-y)).$$

Since g is MS, $g(y\, g(-x)) = g(-x\, g(y))$.

Now put $h = -g$; then $h(1) = -g(1) = -(-1) = 1$ and

$$g(y\, h(x)) = g(y.\, -g(x)) = g(-y\, g(x)) = g(x\, g(-y))$$

and by Eq (4.10), we get $g(y\, h(x)) = g(y\, g(-x))$

$$= g(-x\, g(y)) = g(x.\, -g(y)) = g(x\, h(y)).$$

So that $-g\,(y\, h(x)) = -g\,(x\, h(y))$; i.e.,

$$h(y\, h(x)) = h(x\, h(y)).$$

Hence h is MS and $h(1) = 1$ which implies that $h = -g = -f_\beta$ is DMS.

Therefore the theorem is now completely proved /

From now on assume that $f : \mathbb{R} \to \mathbb{R}$. We will solve the following functional equations :

(4.11) $$f(x\, f(y)) = f(y\, f(x))$$

(4.12) $$f(x + f(y)) = f(y + f(x))$$ and

(4.13) $$f(x + f(y)) = f(y\, .\, f(x)).$$

From these equations, it is clear that the constant functions immediately satisfy these equations. Therefore for the remainder of this chapter, assume f is non-constant.

In order to solve Eq (4.11), let us assume the validity of Theorem 5.2 of Chapter V which states that :

Theorem 5.2. If $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous DMS function, then f or -f is SMS.

<u>Theorem 4.6.</u> If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-constant continuous function satisfying :

(4.11) $$f(x\ f(y)) = f(y\ f(x)),$$

then $f(x) = cx$ or $f(x) = \text{Sup}\{ax, bx\}$, $a \leqslant 0$, $b \geqslant 0$, $b > a$ or $f(x) = -k \cdot |x|$, $(k \geqslant 0)$.

<u>Proof.</u> Since f is continuous on $\mathbb{R}$, it follows from Theorem 4.5 that there exists a real number $\alpha > 0$ such that f_α or $-f_\alpha$ is a DMS function. Since f is continuous, f_α or $-f_\alpha$ is SMS.

If f_α is SMS, then by assuming the validity of Theorem 6.5 of Chapter VI, f_α is of the form :

$$f_\alpha(x) = cx \quad \text{for some } c \in \mathbb{R}$$

$$\text{or} \quad f_\alpha(x) = \text{Sup}\{ax, bx\}, \ a \leqslant 0,\ b \geqslant 0,\ b > a.$$

Therefore by definition of $f_\alpha(x)$ we have

$$f_\alpha(x) = f(x\alpha) = cx$$

$$\text{or} \quad f_\alpha(x) = f(x.\alpha) = \text{Sup}\{ax, bx\},\ a \leqslant 0, b \geqslant 0, b > a.$$

Thus

(4.14) $$f(y) = \frac{c}{\alpha} \cdot y = ky \quad \text{for some } k = \frac{c}{\alpha} \text{ in } \mathbb{R}$$

or

(4.15) $$f(y) = \text{Sup}\{cy, dy\} \quad (y \in \mathbb{R})$$

where $c = \frac{a}{\alpha} \leqslant 0$, $d = \frac{b}{\alpha} \geqslant 0$ and $d > c$.

Similarly, the case in which $-f_\alpha$ is SMS will give :

$$f(y) = ky \quad \text{for some } k = -\frac{c}{\alpha}$$

or

(4.16) $$f(y) = \text{Inf}\{cy, dy\}$$

where $c = -\frac{a}{\alpha} \geq 0$, $d = -\frac{b}{\alpha} \leq 0$ and $c > d$.

Now Eq (4.14) and Eq (4.15) satisfy Eq (4.11). However Eq (4.16) satisfies Eq (4.11) only if $c = -k$. To prove this, it suffices to prove the case when $x > 0$ and $y < 0$. Eq (4.11) and Eq (4.16) give

$$f(x\,f(y)) = f(xcy) = c(xcy)$$
$$= f(y\,f(x)) = d(y\,dx).$$

Thus $c^2 = d^2$ which implies that either $c = d$ or $c = -d$. But $d < c$ from Eq (4.16); hence $d = -c$ and Eq (4.16) becomes

$$f(y) = \operatorname{Inf}\{cy, -cy\}$$
$$(4.17) \qquad = -c\,|y| \qquad (y \in \mathbb{R})$$

for some $c \geq 0$ in $\mathbb{R}$. Obviously Eq (4.17) satisfies Eq (4.11).

Hence the theorem is now completely proved /

Next we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$(4.12) \qquad f(x + f(y)) = f(y + f(x)).$$

The method of solution is to construct a new function which satisfies Eq (4.11).

Note. From Theorem 4.6, we have seen that if $f : \mathbb{R}(>0) \to \mathbb{R}(>0)$ satisfies Eq (4.11), then f is of the form

$$f(x) = cx \qquad (x \in \mathbb{R})$$

and some $c > 0$ in $\mathbb{R}$.

Theorem 4.7. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying :

$$(4.12) \qquad f(x + f(y)) = f(y + f(x)),$$

then $f(x) = x + k$ for some k in $\mathbb{R}$.

<u>Proof</u>. Consider the diagram :

$$\begin{array}{ccc} (\mathbb{R}, +) & \xrightarrow{\quad f \quad} & (\mathbb{R}, +) \\ \uparrow \ln & & \downarrow \exp \\ (\mathbb{R}(>0), .) & \xrightarrow{\quad g \quad} & (\mathbb{R}(>0), .) \end{array}$$

where

(4.18) $$g(x) = e^{f(\ln x)} \qquad (x \in \mathbb{R}(>0)).$$

Then g is continuous. It follows from Eq (4.18) and Eq (4.12) that

$$\begin{aligned} g(x \,.\, g(y)) &= e^{f(\ln(x\, e^{f(\ln y)}))} \\ &= e^{f(\ln x + f(\ln y))} \\ &= e^{f(\ln y + f(\ln x))} \\ &= g(y\, g(x)). \end{aligned}$$

Then from the note after Theorem 4.6,

$$g(x) = cx \quad (x \in \mathbb{R}(>0))$$

for some $c > 0$ in $\mathbb{R}$.

It follows from Eq (4.18) that

$$e^{f(\ln x)} = cx \quad (x \in \mathbb{R}(>0)).$$

Thus

$$f(\ln x) = \ln cx = \ln c + \ln x.$$

Therefore,

$$f(x) = k + x \quad (x \in \mathbb{R})$$

for some $k > 0$ in $\mathbb{R}$.

Moreover, this function satisfies Eq (4.12).

Hence the theorem is now completely proved //

Finally assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Eq (4.13) :

$$f(x + f(y)) = f(y.\, f(x)) \quad . \tag{4.13}$$

Since $f(x) = f(x - f(0) + f(0))$, we have

$$f(x) = f(0.\, f(x - f(0)))$$

$$= f(0) \quad (x \in \mathbb{R}).$$

Hence we get a theorem :

Theorem 4.8. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfies Eq (4.13), then f is identically constant.