CHAPTER I

INTERODUCTION

Suppose $\mathbb{K}$ is a set and "o", " $*$ " are binary operations on $\mathbb{K}$ with certain compatibility condition between "o" and " $*$ ". For example, ($\mathbb{K}$, $*$,+) (with + = o) might be a field or ($\mathbb{K}$, $*$) (with o = $*$) might be a group. We consider the Cauchy type function equation

$$(1.1) \qquad f(x \circ f(y)) = f(x) * f(y).$$

where f : $\mathbb{R} \rightarrow \mathbb{R}$. Our main concern is to find the algebraic solutions of Eq (1.1). As consequences, we also find continuous solutions f : $\mathbb{R} \rightarrow \mathbb{R}$ of Eq (1.1).

In particular in Chapter VI, we show that a non-zero function f mapping a field $\mathbb{K}$ of characteristic 0 into itself satisfies the equation

$$(1.2) \qquad f(x + f(y)) = f(x) \cdot f(y)$$

if and only if there exists a subgroup G of $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ satisfying the condition

$$(*) \qquad \sum a_i g_i = 0 \text{ implies } \prod g_i^{a_i} = 1$$

where a_i are integers and $g_i \in G$ and where the sum and product are finite, and there exists a function $\emptyset$ mapping the additive subgroup Λ of $\mathbb{K}$ generated by G into $\mathbb{K}$ such that $f(x + \lambda) = f(x) \emptyset(\lambda)$ for all $x \in \mathbb{K}$, $\lambda \in \Lambda$.

We also show in Chapter VI that a function f mapping G_o, a group G with O, into itself satisfies equation

$$(1.3) \qquad f(x * f(y)) = f(x) * f(y)$$

if and only if there is a subgroup M and G and a function $\tilde{f}$ from a set consisting of the elements of a set of representatives S of left cosets of $G/_M$ and the element O, onto $M \cup \{O\}$ such that $\tilde{f}(O) = O$ whenever $f \not\equiv$ constant and $f(x) = \tilde{f}(s_x) * m_x$; where each $x \in G$ has a unique representation $x = s_x * m_x$ ($s_x \in S$ and $m_x \in M$).

As application of the latter fact, we find in Chapter IV the continuous solutions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of equations

$$(1.4) \qquad f(x * f(y)) = f(y * f(x))$$

and in Chapter VI, those of the equation

$$(1.5) \qquad f(x * f(y)) = f(f(x) * f(y)),$$

where $*$ could be multiplication or addition in $\mathbb{R}$. These are done by reducing Eq (1.4) to Eq (1.5) by using the following theorem which is proved in Chapter IV: If $f : (\mathbb{R}, .) \longrightarrow (\mathbb{R}, .)$ is a non-constant continuous function satisfying Eq (1.4), then there exists an $\alpha \neq 0$ such that f_α or $-f_\alpha$ satisfies Eq (1.5), where $f_\alpha(x) = f(x.\alpha)$. Then we reduce Eq (1.5) to Eq (1.3) by using a theorem proved in Chapter V stating that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a non-constant continuous function satisfying Eq (1.5), then f or -f satisfies Eq (1.3).

Moreover, very general solutions $f : \mathbb{R} \to \mathbb{R}$ (for example, unbounded or non-measurable) of Eq (1.1) and Eq (1.2) are obtained in Chapter VI.

Of course, no monogram on functional equations is complete without making references to Cauchy's classical functional equations. In Chapter II of this thesis, these equations as well as some related generalizations are considered.

Finally, we have adopted the symbol "/" to mark the end of the proof.