CHAPTER III

SYMMETRIC INVERSE SEMIGROUPS

In this chapter, Trotter's work of characterizing congruence-free inverse semigroups with zero is used to determine all congruence-free congruences on the symmetric inverse semigroups on any finite set and any denumerable set. It is proved that the symmetric inverse semigroup on any nonempty countable set has exactly one nonuniversal congruence-free congruence and its explicit form is also given.

Let S be a semigroup. A congruence ρ on S is said to be <u>idempotent-separating</u> if each ρ-class of S contains at most one idempotent.

It has been proved by Howie in [3] that the maximum idempotent-separating congruence μ on an inverse semigroup S always exists and

$$\mu = \{(a, b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\},$$

or equivalently,

$$\mu = \{(a, b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\},$$

and moreover, $\mu \subseteq \mathcal{H}$.

A semigroup S is said to be <u>fundamental</u> if the identity congruence is the only congruence contained in the Green's relation $\mathcal{H}$ of S.

Let S be a semigroup. Any $\mathcal{H}$-class of S contains at most one idempotent [1, Lemma 2.15]. Then any congruence on S contained in $\mathcal{H}$ is an idempotent-separating congruence. Thus, an inverse semigroup

S is fundamental if and only if the maximum idempotent-separating congruence μ of S is the identity congruence.

Let T be a subset of a semigroup S. The <u>centralizer of T in</u> S, $C_S(T)$, is the set $\{x \in S \mid xt = tx \text{ for all } t \in T\}$.

Let S be an inverse semigroup. Then $C_S(E(S)) = \{x \in S \mid xe = ex \text{ for all } e \in E(S)\}$. Because any two idempotents of S commute with each other, $E(S) \subseteq C_S(E(S))$. It has been proved by Howie in [3] that an inverse semigroup S is fundamental if and only if $C_S(E(S)) = E(S)$. Hence, an inverse semigroup S is fundamental if and only if for any nonidempotent a of S, there exists an idempotent e of S such that $ae \neq ea$.

A semilattice E with zero 0 is said to be <u>disjunctive</u> if for any e, $f \in E$ such that $e < f$, there exists $g \in E$ such that $g > 0$, $g \leq f$ and $eg = 0$.

The following fact is a very important motivation of our studying congruence-free congruences on symmetric inverse semigroups :

Congruence-free inverse semigroups with zero 0 have been characterized by P. G. Trotter in [4] as follows : An inverse semigroup S with zero is congruence-free if and only if (1) S is 0-simple, (2) S is fundamental and (3) E(S) is disjunctive.

Let X be a set. Then I_X, the symmetric inverse semigroup on X, is an inverse semigroup with zero. Hence, if ρ is a congruence on I_X, then ρ is congruence-free if and only if I_X/ρ is a congruence-free inverse semigroup with zero.

For any set X, the symmetric inverse semigroup, I_X, is congruence-free if and only if $|X| \leq 1$. Let X be a set such that I_X is congruence-free. Then I_X is a 0-simple semigroup or I_X is a zero semigroup of order less than 3. If I_X is a zero semigroup, then $X = \phi$, so $|X| = 0$. Next, assume I_X is 0-simple. Let $A = \{\alpha \in I_X \mid |\Delta\alpha| \leq 1\}$. It is clear that A is an ideal of the semigroup I_X. Therefore A is either $\{0\}$ or I_X. Hence $|X| \leq 1$. The converse is trivial.

Let X be a set and α, β be elements of the symmetric inverse semigroup on the set X. Then $\Delta\alpha\beta \subseteq \Delta\alpha$ and $\nabla\beta\alpha \subseteq \nabla\alpha$. Thus, $|\Delta\alpha\beta| \leq |\Delta\alpha|$ and $|\nabla\beta\alpha| \leq |\nabla\alpha|$. Because α and $\beta\alpha$ are one-to-one maps, $|\Delta\alpha| = |\nabla\alpha|$ and $|\Delta\beta\alpha| = |\nabla\beta\alpha|$. Hence $|\Delta\beta\alpha| = |\nabla\beta\alpha| \leq |\nabla\alpha| = |\Delta\alpha|$. It then follows that if ν is a cardinal such that $\nu > 0$, then the set $\{\alpha \in I_X \mid |\Delta\alpha| < \nu\}$ and $\{\alpha \in I_X \mid |\Delta\alpha| \leq \nu\}$ are ideals of I_X.

The following proposition shows that the symmetric inverse semigroup on a nonempty set has a maximum proper ideal :

3.1 <u>Proposition</u>. Let X be a nonempty set. Then the set $\{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$ is the maximum proper ideal of I_X.

<u>Proof</u> : As the above proof, $\{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$ is an ideal of I_X. Let A be a proper ideal of I_X. To show that $A \subseteq \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$, suppose not. Then there exists $\beta \in A$ such that $\beta \notin \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$. Hence $|\Delta\beta| = |X|$, and thus there exists $\gamma \in I_X$ with $\Delta\gamma = X$, $\nabla\gamma = \Delta\beta$. Hence $\gamma\beta \in A$ which implies that $\gamma\beta(\gamma\beta)^{-1} \in A$. But $\Delta\gamma\beta = X$ and $(\gamma\beta)(\gamma\beta)^{-1} \in E(I_X)$, so $(\gamma\beta)(\gamma\beta)^{-1}$ is

the identity of I_X and belongs to A. Therefore $A = I_X$ which contradicts the assumption. Then $A \subseteq \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$.

Therefore, $\{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$ is the maximum proper ideal of I_X. #

By Proposition 3.1, the next two corollaries are directly obtained.

3.2 <u>Corollary</u>. Let X be a finite set of cardinality $n > 0$. Then the set $\{\alpha \in I_X \mid |\Delta\alpha| \leq n - 1\}$ is the maximum proper ideal of I_X.

3.3 <u>Corollary</u>. If X is a denumerable set, then the set $\{\alpha \in I_X \mid \Delta\alpha \text{ is finite}\}$ is the maximum proper ideal of I_X.

Let X be a set. Then the permutation group (the symmetric group) on the set X, G_X, is the set $\{\alpha \in I_X \mid \Delta\alpha = \nabla\alpha = X\}$ and it is a group of units of I_X, the symmetric inverse semigroup on the set X.

Let X be a finite nonempty set. Then the set $I_X \setminus G_X = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$ and G_X is a filter of I_X. To show G_X is a filter of I_X, let $\alpha, \beta \in I_X$ such that $\alpha\beta \in G_X$. Then $\Delta\alpha\beta = X = \nabla\alpha\beta$. But generally, we have that $\Delta\alpha\beta \subseteq \Delta\alpha$ and $\nabla\alpha\beta \subseteq \nabla\beta$, so $\Delta\alpha = X = \nabla\beta$. Since X is finite, $\nabla\alpha = X = \Delta\beta$, thus $\alpha, \beta \in G_X$. Therefore G_X is a filter of I_X. Hence $I_X \setminus G_X$ is a completely prime ideal of I_X.

3.4 <u>Corollary</u>. For any finite nonempty set X, $I_X \setminus G_X$ is the maximum proper ideal of I_X and it is also completely prime.

We give a remark that for any infinite set X, $I_X \setminus G_X$ is not an ideal of I_X. A proof is given as follows : Let X be an infinite set. Let x be a fixed element in X. Since X is infinite, $|X| = |X \setminus \{x\}|$, so there exists a one-to-one map α from X onto $X \setminus \{x\}$. Then $\alpha \in I_X$ but $\alpha \notin G_X$, that is, $\alpha \in I_X \setminus G_X$. But $\alpha\alpha^{-1}$ is an idempotent of I_X with $\Delta\alpha\alpha^{-1} = \Delta\alpha = X$. Thus $\alpha\alpha^{-1}$ is the identity map on X, so $\alpha\alpha^{-1} \in G_X$. Therefore $\alpha \in I_X \setminus G_X$ but $\alpha\alpha^{-1} \notin I_X \setminus G_X$. Hence $I_X \setminus G_X$ is not an ideal of I_X.

Let X be a set. For each nonnegative integer n, let $A_n = \{\alpha \in I_X \mid |\Delta\alpha| \leq n\}$.

The following proposition characterizes all ideals of the symmetric inverse semigroup on any finite set X.

3.5 <u>Proposition</u>. Let X be a finite set and A be a nonempty subset of I_X. Then A is an ideal of I_X if and only if $A = A_n$ for some nonnegative integer n.

<u>Proof</u> : First, assume that A is an ideal of I_X. Let m be the maximum element of the set $\{|\Delta\alpha| \mid \alpha \in A\}$. Then there is an $\alpha_0 \in A$ such that $|\Delta\alpha_0| = m$. Since $A_m = \{\alpha \in I_X \mid |\Delta\alpha| \leq m\}$, $A \subseteq A_m$. Next, we are to show that $A_m \subseteq A$. Let $\beta \in A_m$. Then $|\Delta\beta| \leq m$.

<u>Case $|\Delta\beta| = m$.</u> Then $|\Delta\beta| = m = |\Delta\alpha_0|$, and so there exists $\gamma \in I_X$ such that $\Delta\gamma = \Delta\beta$, $\nabla\gamma = \Delta\alpha_0$. Let $\lambda = \alpha_0^{-1}\gamma^{-1}\beta$. Because $\Delta\gamma\alpha_0 = (\nabla\gamma \cap \Delta\alpha_0)\gamma^{-1} = (\Delta\alpha_0)\gamma^{-1} = \Delta\beta$, $(\gamma\alpha_0)(\alpha_0^{-1}\gamma^{-1})$ is the identity map on $\Delta\beta$. Hence $\gamma\alpha_0\lambda = (\gamma\alpha_0)(\alpha_0^{-1}\gamma^{-1})\beta = \beta$. Since $\alpha_0 \in A$, we have that $\beta \in A$.

Case $|\Delta\beta| < m$. Let $|\Delta\beta| = k$. Assume that $|X| = n$. Then $k < m \leq n$, hence $|X \setminus \Delta\beta| \geq m - k$. Let $a_1, a_2, \ldots, a_{m-k}$ be $m - k$ distinct elements in $X \setminus \Delta\beta$. Let γ be the identity map on the set $\Delta\beta \cup \{a_1, a_2, \ldots, a_{m-k}\}$. Then $\gamma \in I_X$ and $|\Delta\gamma| = m$, so $\gamma \in A_m$. By the first case, $\gamma \in A$. But $\gamma\beta = \beta$, thus $\beta \in A$.

Therefore, $A = A_m$.

The converse follows from the previous mention. #

Let X be a finite set of n elements. Then $A_0, A_1, \ldots, A_n$ are all the ideals of I_X and they are totally ordered by inclusion as follows : $\{0\} = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = I_X$.

Let ρ be a congruence on a semigroup S. Then for any ideal A of S, the set $\{a\rho \mid a \in A\}$ is then an ideal of the semigroup I_X/ρ, and for convenience, we will denote $\{a\rho \mid a \in A\}$ by $\bar{A}$.

3.6 Proposition. Let X be a nonempty set and ρ be a congruence on I_X. Then the semigroup I_X/ρ is 0-simple if and only if $0\rho = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$.

Proof : Assume that I_X/ρ is 0-simple. Then ρ is not a universal congruence on I_X and 0ρ is a proper ideal of I_X. By Proposition 3.1, $0\rho \subseteq \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$. Now, we will show for equality.

Case X is finite. Then $|X| = n$ for some positive integer n and therefore $\{\alpha \in I_X \mid |\Delta\alpha| < |X|\} = A_{n-1}$. Since X is finite and 0ρ is an ideal of I_X, $0\rho = A_m$ for some nonnegative integer m (by Proposition 3.5). Then $m \leq n - 1$. We are to show $m = n - 1$. Because

$A_{m+1} = \{\alpha \in I_X \mid |\Delta\alpha| \leq m+1\}$ is an ideal of I_X, $\bar{A}_{m+1}$ is an ideal of I_X/ρ and $\{0\rho\} = \bar{A}_m \subsetneq \bar{A}_{m+1}$. Since I_X/ρ is 0-simple, $\bar{A}_{m+1} = I_X/\rho$. Hence $1\rho\beta$ for some $\beta \in A_{m+1}$. Then $|\Delta\beta| \leq m+1$. If $|\Delta\beta| \leq m$, then $\beta \in A_m = 0\rho$, so $1\rho 0$ which implies ρ is the universal congruence on I_X, a contradiction. Therefore $|\Delta\beta| = m+1$. Suppose $m+1 < n$. Then there exists $a \in X$ such that $a \notin \Delta\beta$. Let $b \in \Delta\beta$ and γ be the identity map on the set $(\Delta\beta \cup \{a\}) \setminus \{b\}$. Then $|\Delta\gamma| = m+1$ and $|\Delta\gamma\beta| = |\Delta\beta \setminus \{b\}| = m$ and therefore $\gamma\beta\rho 0$. Because $1\rho\beta$, $\gamma\rho\gamma\beta$. Hence $\gamma \in 0\rho = A_m$, it is a contradiction because $|\Delta\gamma| = m+1$. Therefore $m+1 = n$. Hence $0\rho = A_{n-1} = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$.

<u>Case X is infinite</u>. Let $A = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$. Since A is an ideal of I_X, $\bar{A}$ is an ideal of I_X/ρ, and hence $\bar{A} = \{0\rho\}$ or $\bar{A} = I_X/\rho$. Suppose $\bar{A} = I_X/\rho$. Then $1\rho\alpha$ for some $\alpha \in A$. Since $\alpha \in A$, $|\Delta\alpha| < |X|$ and therefore $|X \setminus \Delta\alpha| = |X|$. Let β be the identity map on $X \setminus \Delta\alpha$. Then $|\Delta\beta| = |X|$, so $\beta \notin 0\rho$. Since $1\rho\alpha$, $(\beta, \beta\alpha) = (\beta, 0) \in \rho$. This leads to a contradiction. Hence $\bar{A} = \{0\rho\}$ which implies that $A = 0\rho$. Therefore $0\rho = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$.

Conversely, assume that $0\rho = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$. Then $1\rho \neq 0\rho$ and therefore $(I_X/\rho)^2 \neq \{0\rho\}$. Let B be an ideal of I_X/ρ such that $B \neq \{0\rho\}$. Then there exists $\beta \in I_X$ such that $|\Delta\beta| = |X|$ and $\beta\rho \in B$. Let β' be a one-to-one, onto map such that $\Delta\beta' = X$ and $\nabla\beta' = \Delta\beta$. Thus $\beta'\beta\rho \in B$ and $\Delta\beta'\beta = X$. Since $(\beta'\beta)(\beta'\beta)^{-1}$ is the identity of I_X, the identity of I_X/ρ belongs to B. Hence $B = I_X/\rho$. Therefore I_X/ρ is 0-simple. #

The following corollaries are the immediate consequences of Proposition 3.6 :

3.7 <u>Corollary</u>. Let X be a finite nonempty set and ρ be a congruence on I_X. Then the semigroup I_X/ρ is 0-simple if and only if $0\rho = I_X \setminus G_X$.

3.8 <u>Corollary</u>. Let X be a denumerable set and ρ be a congruence on I_X. Then the semigroup I_X/ρ is 0-simple if and only if $0\rho = \{\alpha \in I_X \mid \Delta\alpha \text{ is finite}\}$.

Let X be a finite nonempty set. Because $I_X \setminus G_X$ is an ideal of I_X and G_X is a subgroup of I_X, we clearly have that $\delta = [(I_X \setminus G_X) \times (I_X \setminus G_X)] \cup (G_X \times G_X)$ is a nonuniversal congruence on I_X. The following theorem shows that the congruence δ is the only nonuniversal congruence-free congruence on I_X :

3.9 <u>Theorem</u>. Let X be a finite nonempty set. Then $\delta = [(I_X \setminus G_X) \times (I_X \setminus G_X)] \cup (G_X \times G_X)$ is the only nonuniversal congruence-free congruence on I_X.

<u>Proof</u> : As the above mention, δ is a nonuniversal congruence on S. Since $|I_X/\delta| = 2$, δ is congruence-free.

Let ρ be a nonuniversal congruence-free congruence on I_X. Then I_X/ρ is a 0-simple inverse semigroup. Hence by Corollary 3.7, $0\rho = I_X \setminus G_X$, this implies $\rho \subseteq \delta$. Therefore by Corollary 1.3, $\rho = \delta$. This shows that δ is the only nonuniversal congruence-free congruence on I_X. Hence the theorem is now completely proved. #

Let X be a set. For $\alpha, \beta \in I_X$, let the notation $D(\alpha, \beta)$ denote the set $\{x \in \Delta\alpha \cap \Delta\beta \mid x\alpha = x\beta\}$. Then the following clearly follow : For $\alpha, \beta, \gamma \in I_X$, $D(\alpha, \alpha) = \Delta\alpha$, $D(\alpha, \beta) = D(\beta, \alpha) \subseteq \Delta\alpha \cap \Delta\beta$ and $D(\alpha, \beta) \cap D(\beta, \gamma) \subseteq D(\alpha, \gamma)$.

Let X be an infinite set and let δ_X be the relation on I_X defined as follows : For $\alpha, \beta \in I_X$,

$$\alpha\delta_X\beta \text{ if and only if } |\Delta\alpha \setminus D(\alpha, \beta)| < |X| \text{ and } |\Delta\beta \setminus D(\alpha, \beta)| < |X|.$$

Note that for $\alpha, \beta \in I_X$, if $\alpha\delta_X\beta$, then $|\Delta\alpha \setminus \Delta\beta| < |X|$ and $|\Delta\beta \setminus \Delta\alpha| < |X|$.

It will be shown that for any infinite set X, δ_X, is a congruence on I_X. The following lemma is required.

3.10 <u>Lemma.</u> Let X be an infinite set. Then the following hold :

(1) If $\alpha, \beta, \gamma \in I_X$, then $(D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1} = D(\gamma\alpha, \gamma\beta)$.

(2) If $\alpha, \beta, \gamma \in I_X$, then $(D(\alpha, \beta)\alpha \cap \Delta\gamma)\alpha^{-1} \subseteq D(\alpha\gamma, \beta\gamma)$.

(3) For $\alpha, \beta, \gamma \in I_X$, if $\alpha\delta_X\beta$ and $\beta\delta_X\gamma$, then $|\Delta\alpha \setminus D(\beta, \gamma)| < |X|$.

<u>Proof</u> : (1) Let $x \in (D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1}$. Then $x\gamma \in D(\alpha, \beta) \cap \nabla\gamma$, so $x\gamma \in \Delta\alpha \cap \Delta\beta \cap \nabla\gamma$ and $(x\gamma)\alpha = (x\gamma)\beta$. Therefore $x \in \Delta\gamma\alpha \cap \Delta\gamma\beta$ and $x\gamma\alpha = x\gamma\beta$. Thus $x \in D(\gamma\alpha, \gamma\beta)$.

Conversely, let $x \in D(\gamma\alpha, \gamma\beta)$. Then $x \in \Delta\gamma\alpha \cap \Delta\gamma\beta$ and $x\gamma\alpha = x\gamma\beta$. But $\Delta\gamma\alpha \cap \Delta\gamma\beta = (\nabla\gamma \cap \Delta\alpha)\gamma^{-1} \cap (\nabla\gamma \cap \Delta\beta)\gamma^{-1} = ((\Delta\alpha \cap \Delta\beta) \cap \nabla\gamma)\gamma^{-1}$. Thus $x\gamma \in \Delta\alpha \cap \Delta\beta \cap \nabla\gamma$. Since $x\gamma \in \Delta\alpha \cap \Delta\beta$ and $(x\gamma)\alpha = (x\gamma)\beta$, $x\gamma \in D(\alpha, \beta)$. Hence $x\gamma$ belongs to $D(\alpha, \beta) \cap \nabla\gamma$ and thus $x \in (D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1}$.

Hence, $(D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1} = D(\gamma\alpha, \gamma\beta)$.

(2) Let $x \in (D(\alpha, \beta)\alpha \cap \Delta\gamma)\alpha^{-1}$. Then $x\alpha \in D(\alpha, \beta)\alpha \cap \Delta\gamma$, so $x \in D(\alpha, \beta)$ (since α is one-to-one) and $x\alpha \in \Delta\gamma$. Thus $x \in \Delta\alpha \cap \Delta\beta$, $x\alpha = x\beta$ and $x\alpha \in \Delta\gamma$. Since $x\beta = x\alpha \in \Delta\gamma$, $x \in \Delta\beta\gamma \cap \Delta\alpha\gamma$ and $x\beta\gamma = x\alpha\gamma$. Hence $x \in D(\alpha\gamma, \beta\gamma)$.

(3) Assume that $\alpha\delta_X\beta$ and $\beta\delta_X\gamma$. Then $|\Delta\alpha \setminus \Delta\beta| \leq |\Delta\alpha \setminus D(\alpha, \beta)| < |X|$ and $|\Delta\beta \setminus D(\beta, \gamma)| < |X|$. Since X is infinite $|(\Delta\beta \setminus D(\beta, \gamma)) \cup (\Delta\alpha \setminus \Delta\beta)| < |X|$. Because $\Delta\alpha \setminus D(\beta, \gamma) \subseteq (\Delta\beta \setminus D(\beta, \gamma)) \cup (\Delta\alpha \setminus \Delta\beta)$. Therefore $|\Delta\alpha \setminus D(\beta, \gamma)| < |X|$. #

3.11 <u>Proposition</u>. For any infinite set X, δ_X is a congruence on I_X.

<u>Proof</u> : The relation δ_X is clearly reflexive and symmetric on I_X. To show δ_X is transitive, let $\alpha, \beta, \gamma \in I_X$ such that $\alpha\delta_X\beta$ and $\beta\delta_X\gamma$. Then $|\Delta\alpha \setminus D(\alpha, \beta)| < |X|$ and $|\Delta\beta \setminus D(\alpha, \beta)| < |X|$, $|\Delta\beta \setminus D(\beta, \gamma)| < |X|$ and $|\Delta\gamma \setminus D(\beta, \gamma)| < |X|$. We are to show $\alpha\delta_X\gamma$, that is to show $|\Delta\alpha \setminus D(\alpha, \gamma)| < |X|$ and $|\Delta\gamma \setminus D(\alpha, \gamma)| < |X|$. Since $D(\alpha, \beta) \cap D(\beta, \gamma) \subseteq D(\alpha, \gamma)$, $\Delta\alpha \setminus D(\alpha, \gamma) \subseteq \Delta\alpha \setminus (D(\alpha, \beta) \cap D(\beta, \gamma))$. But $\Delta\alpha \setminus (D(\alpha, \beta) \cap D(\beta, \gamma)) = (\Delta\alpha \setminus D(\alpha, \beta)) \cup (\Delta\alpha \setminus D(\beta, \gamma))$, so $\Delta\alpha \setminus D(\alpha, \gamma) \subseteq (\Delta\alpha \setminus D(\alpha, \beta)) \cup (\Delta\alpha \setminus D(\beta, \gamma))$ which implies $|\Delta\alpha \setminus D(\alpha, \gamma)| \leq |\Delta\alpha \setminus D(\alpha, \beta)| + |\Delta\alpha \setminus D(\beta, \gamma)|$. By Lemma 3.10 (3), we have that $|\Delta\alpha \setminus D(\beta, \gamma)| < |X|$. Then $|\Delta\alpha \setminus D(\alpha, \gamma)| < |X|$. By the symmetry of δ_X, we can similarly prove that $|\Delta\gamma \setminus D(\alpha, \gamma)| < |X|$. Hence $\alpha\delta_X\gamma$. Therefore δ_X is an equivalence relation on I_X.

Next, to show δ_X is compatible, let $\alpha, \beta, \gamma \in I_X$ such that $\alpha\delta_X\beta$. Then $|\Delta\alpha \setminus D(\alpha, \beta)| < |X|$ and $|\Delta\beta \setminus D(\alpha, \beta)| < |X|$. We will show that $|\Delta\gamma\alpha \setminus D(\gamma\alpha, \gamma\beta)| < |X|$ and $|\Delta\gamma\beta \setminus D(\gamma\alpha, \gamma\beta)| < |X|$, $|\Delta\alpha\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$ and $|\Delta\beta\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$. Because γ is a one-to-one map and $(D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1} = D(\gamma\alpha, \gamma\beta)$ (Lemma 3.10 (1)), $|(\Delta\alpha \setminus D(\alpha, \beta)) \cap \nabla\gamma| = |((\Delta\alpha \setminus D(\alpha, \beta)) \cap \nabla\gamma)\gamma^{-1}| =$ $|((\Delta\alpha \cap \nabla\gamma) \setminus (D(\alpha, \beta) \cap \nabla\gamma))\gamma^{-1}| = |(\Delta\alpha \cap \nabla\gamma)\gamma^{-1} \setminus (D(\alpha, \beta) \cap \nabla\gamma)\gamma^{-1}|$ $= |\Delta\gamma\alpha \setminus D(\gamma\alpha, \gamma\beta)|$. Since $|\Delta\alpha \setminus D(\alpha, \beta)| < |X|$, $|\Delta\gamma\alpha \setminus D(\gamma\alpha, \gamma\beta)| =$ $|(\Delta\alpha \setminus D(\alpha, \beta)) \cap \nabla\gamma| < |X|$. Similarly, $|\Delta\gamma\beta \setminus D(\gamma\alpha, \gamma\beta)| < |X|$. Hence $\gamma\alpha\delta_X\gamma\beta$.

Next, we are to show $|\Delta\alpha\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$ and $|\Delta\beta\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$. Because $|\Delta\alpha \setminus D(\alpha, \beta)| < |X|$ and

$$
\begin{aligned}
|\Delta\alpha\gamma \setminus D(\alpha\gamma, \beta\gamma)| &\le |\Delta\alpha\gamma \setminus (D(\alpha, \beta)\alpha \cap \Delta\gamma)\alpha^{-1}| \quad \text{(Lemma 3.10 (2))} \\
&= |(\nabla\alpha \cap \Delta\gamma)\alpha^{-1} \setminus (D(\alpha, \beta)\alpha \cap \Delta\gamma)\alpha^{-1}| \\
&= |((\nabla\alpha \cap \Delta\gamma) \setminus (D(\alpha, \beta)\alpha \cap \Delta\gamma))\alpha^{-1}| \\
&= |(\nabla\alpha \cap \Delta\gamma) \setminus (D(\alpha, \beta)\alpha \cap \Delta\gamma)| \\
&\le |\nabla\alpha \setminus D(\alpha, \beta)\alpha| \\
&= |(\nabla\alpha \setminus D(\alpha, \beta)\alpha)\alpha^{-1}| \\
&= |\Delta\alpha \setminus D(\alpha, \beta)|,
\end{aligned}
$$

it follows that $|\Delta\alpha\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$. It can be proved similarly that $|\Delta\beta\gamma \setminus D(\alpha\gamma, \beta\gamma)| < |X|$. Thus $\alpha\gamma\delta_X\beta\gamma$.

Therefore, δ_X is a congruence on I_X, as desired. #

By the definition of δ_X, we have that $\alpha\delta_X 0$ if and only if $|\Delta\alpha| < |X|$. Then $0\delta_X = \{\alpha \in I_X \mid |\Delta\alpha| < |X|\}$. Hence, by Proposition 3.6, the following proposition is obtained :

3.12 <u>Proposition</u>. For any infinite set X, the semigroup I_X/δ_X is 0-simple.

Let X be an infinite set. Since I_X is an inverse semigroup and δ_X is a congruence on I_X, from Introduction page 4 , I_X/δ_X is an inverse semigroup and $E(I_X/\delta_X) = \{\alpha\delta_X \mid \alpha \in E(I_X)\}$.

Let $\alpha, \beta \in E(I_X)$. Then α and β are identity maps on $\Delta\alpha$ and $\Delta\beta$, respectively, so for $x \in \Delta\alpha \cap \Delta\beta$, $x\alpha = x = x\beta$. Hence $D(\alpha, \beta) = \Delta\alpha \cap \Delta\beta$. Therefore $\alpha\delta_X = \beta\delta_X$ if and only if $|\Delta\alpha \setminus (\Delta\alpha \cap \Delta\beta)| = |\Delta\alpha \setminus \Delta\beta| < |X|$ and $|\Delta\beta \setminus (\Delta\alpha \cap \Delta\beta)| = |\Delta\beta \setminus \Delta\alpha| < |X|$.

Let X be an infinite set and $\alpha, \beta \in E(I_X)$. If $\alpha\delta_X \leq \beta\delta_X$, then $\alpha\delta_X = \alpha\delta_X\beta\delta_X = \alpha\beta\delta_X$ and so $|\Delta\alpha \setminus \Delta\alpha\beta| < |X|$. Because α and $\alpha\beta$ are idempotents of I_X, $x\alpha = x\alpha\beta$ for all $x \in \Delta\alpha \cap \Delta\alpha\beta = \Delta\alpha\beta$. Thus, if $|\Delta\alpha \setminus \Delta\alpha\beta| < |X|$, then $(\alpha, \alpha\beta) \in \delta_X$, so $\alpha\delta_X = \alpha\delta_X\beta\delta_X$ which implies $\alpha\delta_X \leq \beta\delta_X$.

Hence $\alpha\delta_X \leq \beta\delta_X$ if and only if $|\Delta\alpha \setminus \Delta\alpha\beta| < |X|$.

Assume that $\alpha\delta_X < \beta\delta_X$. Then $\alpha\delta_X \neq \beta\delta_X$, so we have that $|\Delta\alpha \setminus \Delta\beta| = |X|$ or $|\Delta\beta \setminus \Delta\alpha| = |X|$. But $\Delta\alpha\beta \subseteq \Delta\beta$, so $|\Delta\alpha \setminus \Delta\beta| \leq |\Delta\alpha \setminus \Delta\alpha\beta| < |X|$. Therefore $|\Delta\beta \setminus \Delta\alpha| = |X|$. This shows that for $\alpha, \beta \in E(I_X)$, if $\alpha\delta_X < \beta\delta_X$, then $|\Delta\beta \setminus \Delta\alpha| = |X|$.

3.13 <u>Proposition</u>. For any infinite set X, $E(I_X/\delta_X)$ is disjunctive.

<u>Proof</u> : Let $\alpha, \beta \in E(I_X)$ such that $\alpha\delta_X < \beta\delta_X$. Then $|\Delta\beta \setminus \Delta\alpha| = |X|$. Let γ be the identity map on the set $\Delta\beta \setminus \Delta\alpha$. Then $\gamma \in E(I_X)$ and $\Delta\gamma \subseteq \Delta\beta$. Since $|\Delta\gamma| = |\Delta\beta \setminus \Delta\alpha| = |X|$, $\gamma \notin 0\delta_X$ and

hence $0\delta_X < \gamma\delta_X$. Next, we will show that $\gamma\delta_X \leq \beta\delta_X$, that is to show $|\Delta\gamma \setminus \Delta\gamma\beta| < |X|$. Because $\Delta\gamma \setminus \Delta\gamma\beta = \Delta\gamma \setminus (\Delta\gamma \cap \Delta\beta) = \Delta\gamma \setminus \Delta\gamma = \phi$, $|\Delta\gamma \setminus \Delta\gamma\beta| = 0 < |X|$. Therefore $\gamma\delta_X \leq \beta\delta_X$.

Finally, we show that $\alpha\delta_X\gamma\delta_X = 0\delta_X$. Because $\Delta\alpha\gamma = \Delta\alpha \cap \Delta\gamma = \Delta\alpha \cap (\Delta\beta \setminus \Delta\alpha) = \phi$, $|\Delta\alpha\gamma| = 0 < |X|$ and thus $\alpha\gamma\delta_X 0$, this means $\alpha\delta_X\gamma\delta_X = 0\delta_X$.

Therefore, $E(I_X/\delta_X)$ is disjunctive. #

One of the main results of this research is to show that for any denumerable set X, δ_X is the only nonuniversal congruence-free congruence on the symmetric inverse semigroup on the set X.

3.14 Lemma. Let X be an infinite set and $\alpha \in I_X$. Then $\alpha\delta_X$ is an idempotent of the semigroup I_X/δ_X if and only if $|\{x \in \Delta\alpha \mid x\alpha \neq x\}| < |X|$.

Proof : Let $\alpha \in I_X$. Assume that $\alpha\delta_X$ is an idempotent of the semigroup I_X/δ_X. Since $E(I_X/\delta_X) = \{\beta\delta_X \mid \beta \in E(I_X)\}$, $\alpha\delta_X = \gamma\delta_X$ for some $\gamma \in E(I_X)$. Then $D(\alpha, \gamma) = \{x \in \Delta\alpha \cap \Delta\gamma \mid x\alpha = x\gamma\} = \{x \in \Delta\alpha \cap \Delta\gamma \mid x\alpha = x\}$ and $|\Delta\alpha \setminus D(\alpha, \gamma)| < |X|$. But $\Delta\alpha \setminus D(\alpha, \gamma) = \Delta\alpha \setminus \{x \in \Delta\alpha \cap \Delta\gamma \mid x\alpha = x\} \supseteq \Delta\alpha \setminus \{x \in \Delta\alpha \mid x\alpha = x\} = \{x \in \Delta\alpha \mid x\alpha \neq x\}$. Hence $|\{x \in \Delta\alpha \mid x\alpha \neq x\}| < |X|$.

To prove the converse, assume that $|\{x \in \Delta\alpha \mid x\alpha \neq x\}| < |X|$. Let λ be the identity map on the set $\{x \in \Delta\alpha \mid x\alpha = x\}$. Then $\lambda \in E(I_X)$ and so $\lambda\delta_X \in E(I_X/\delta_X)$. Claim that $\alpha\delta_X\lambda$. By the definition of λ, $\Delta\alpha \setminus D(\alpha, \lambda) = \Delta\alpha \setminus \{x \in \Delta\alpha \cap \Delta\lambda \mid x\alpha = x\} = \Delta\alpha \setminus \{x \in \Delta\alpha \mid x\alpha = x\} = \{x \in \Delta\alpha \mid x\alpha \neq x\}$ and $\Delta\lambda \setminus D(\alpha, \lambda) = \Delta\lambda \setminus \{x \in \Delta\alpha \cap \Delta\lambda \mid x\alpha = x\} =$

$\Delta\lambda \setminus \Delta\lambda = \phi$. Thus $|\Delta\alpha \setminus D(\alpha, \lambda)| = |\{x \in \Delta\alpha \mid x\alpha \neq x\}| < |X|$ and $|\Delta\lambda \setminus D(\alpha, \lambda)| = 0 < |X|$. Therefore, $\alpha\delta_X\lambda$. #

3.15 <u>Lemma</u>. Let X be an infinite set and ρ be a congruence on I_X such that the semigroup I_X/ρ is 0-simple and $E(I_X/\rho)$ is disjunctive. Let $\alpha \in I_X$. If there exists a subset A of $\Delta\alpha$ such that $|\Delta\alpha \setminus A| < |X|$ and $a\alpha = a$ for all $a \in A$, then $\alpha\rho = \beta\rho \in E(I_X/\rho)$ where β is the identity map on the set A.

<u>Proof</u> : Let $\alpha \in I_X$. Let A be a subset of $\Delta\alpha$ such that $|\Delta\alpha \setminus A| < |X|$ and $a\alpha = a$ for all $a \in A$. Let β be the identity map on A. Then $\beta\rho \in E(I_X/\rho)$ and $\Delta\beta \subseteq \Delta\alpha$. Since I_X/ρ is 0-simple, by Proposition 3.6, $0\rho = \{\lambda \in I_X \mid |\Delta\lambda| < |X|\}$.

If $|\Delta\alpha| < |X|$, then $|\Delta\beta| \leq |\Delta\alpha| < |X|$ and hence $\alpha, \beta \in 0\rho$ which implies $\alpha\rho\beta$.

Assume that $|\Delta\alpha| = |X|$. Since $|\Delta\alpha \setminus \Delta\beta| < |X|$, $|\Delta\beta| = |X|$. Let γ be the identity map on $\Delta\alpha$. Then $\gamma\rho \in E(I_X/\rho)$. Because $\beta\gamma = \beta$, $\beta\rho = \beta\gamma\rho = \beta\rho\gamma\rho$ and hence $\beta\rho \leq \gamma\rho$. Suppose that $\beta\rho < \gamma\rho$. Since $E(I_X/\rho)$ is disjunctive, there exists $\lambda \in E(I_X)$ such that $0\rho < \lambda\rho \leq \gamma\rho$ and $\beta\rho\lambda\rho = 0\rho$. Thus $\beta\lambda\rho = 0\rho$ which implies $|\Delta\beta\lambda| = |\Delta\beta \cap \Delta\lambda| < |X|$. Because $|\Delta\alpha \setminus \Delta\beta| < |X|$ and $\Delta\gamma = \Delta\alpha$, $|\Delta\gamma \setminus \Delta\beta| < |X|$ and hence $|(\Delta\gamma \setminus \Delta\beta) \cap \Delta\lambda| < |X|$. Since $0\rho < \lambda\rho \leq \gamma\rho$, $0\rho < \lambda\rho = \lambda\gamma\rho$, and so $|\Delta\lambda \cap \Delta\gamma| = |\Delta\lambda\gamma| = |X|$. But $\Delta\lambda \cap \Delta\gamma \subseteq (\Delta\beta \cap \Delta\lambda) \cup (\Delta\gamma \setminus \Delta\beta)$ and $|\Delta\gamma \setminus \Delta\beta| < |X|$, then $|\Delta\beta \cap \Delta\lambda| = |X|$. This contradicts that we have obtained. Therefore $\beta\rho = \gamma\rho$, thus $\beta\alpha\rho\gamma\alpha$. But $\beta\alpha = \beta$ and $\gamma\alpha = \alpha$. Hence $\beta\rho\alpha$. #

3.16 <u>Theorem</u>. For any denumerable set X, δ_X is the only nonuniversal congruence-free congruence on the symmetric inverse semigroup on the set X.

<u>Proof</u> : We prove that δ_X is a congruence-free congruence on I_X first. From [4], it is equivalent to show that I_X/δ_X is 0-simple and fundamental and $E(I_X/\delta_X)$ is disjunctive. By Proposition 3.12, I_X/δ_X is 0-simple and by Proposition 3.13, $E(I_X/\delta_X)$ is disjunctive. To show that I_X/δ_X is fundamental, it is equivalent to show that $E(I_X/\delta_X)$ is the centralizer of $E(I_X/\delta_X)$ in I_X/δ_X or equivalently, for $\alpha \in I_X$ such that $\alpha\delta_X \notin E(I_X/\delta_X)$, there exists $\beta \in E(I_X)$ such that $(\alpha\beta, \beta\alpha) \notin \delta_X$.

Let $\alpha \in I_X$ such that $\alpha\delta_X \notin E(I_X/\delta_X)$. Then by Lemma 3.14, the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$ is denumerable. Let $A = \{x \in \Delta\alpha \mid x\alpha \neq x\}$. Then A is an infinite set. Let x_1 be an element of A. Then $x_1 \neq x_1\alpha$. Let $A_1 = A \setminus \{x_1, x_1\alpha, x_1\alpha^{-1}\}$ if $x_1 \in \nabla\alpha$ but if $x_1 \notin \nabla\alpha$, let $A_1 = A \setminus \{x_1, x_1\alpha\}$. Since A is infinite, $A_1 \neq \phi$. Let x_2 be an element of A_1. Then $x_2 \neq x_1$, $x_2 \neq x_2\alpha$ and $x_2\alpha \neq x_1$. Next, let $A_2 = A_1 \setminus \{x_2, x_2\alpha, x_2\alpha^{-1}\}$ if $x_2 \in \nabla\alpha$ and if $x_2 \notin \nabla\alpha$, let $A_2 = A_1 \setminus \{x_2, x_2\alpha\}$. Because A_1 is infinite, $A_2 \neq \phi$. Let x_3 be an element of A_2. Then $x_1 \neq x_3 \neq x_2$, $x_3 \neq x_3\alpha$ and $x_1 \neq x_3\alpha \neq x_2$. Assume that $x_1, x_2, \ldots, x_n$ are choosen from A as above. Then A_{n-1} is infinite. Let $A_n = A_{n-1} \setminus \{x_n, x_n\alpha, x_n\alpha^{-1}\}$ if $x_n \in \nabla\alpha$ and $A_n = A_{n-1} \setminus \{x_n, x_n\alpha\}$ if $x_n \notin \nabla\alpha$. Since A_{n-1} is an infinite set, $A_n \neq \phi$. Let $x_{n+1} \in A_n$. By induction process, we have the set $\{x_1, x_2, x_3, \ldots\}$

which is a denumerable subset of $\Delta\alpha$, $x_i \neq x_j$ if $i \neq j$, and for each i, $x_i \neq x_j\alpha$ for all j. Therefore the sets $\{x_1, x_2, x_3, \ldots\}$ and $\{x_1\alpha, x_2\alpha, x_3\alpha, \ldots\}$ are disjoint. Let β be the identity map on the set $\{x_1, x_2, x_3, \ldots\}$. Then $\beta \in E(I_X)$, so $\beta\delta_X \in E(I_X/\delta_X)$. Since $\Delta\beta = \nabla\beta = \{x_1, x_2, x_3, \ldots\} \subseteq \Delta\alpha$, $\Delta\beta\alpha = (\nabla\beta \cap \Delta\alpha)\beta^{-1} = \nabla\beta \cap \Delta\alpha = \nabla\beta = \{x_1, x_2, x_3, \ldots\}$. For each $i \in \{1, 2, 3, \ldots\}$, $x_i\alpha \notin \Delta\beta$, so $x_i \notin \Delta\alpha\beta$. Thus $\Delta\alpha\beta \cap \Delta\beta\alpha = \phi$, so $D(\alpha\beta, \beta\alpha) = \phi$. Therefore $|\Delta\beta\alpha \setminus D(\beta\alpha, \alpha\beta)| = |\Delta\beta\alpha| = |X|$ so $\beta\alpha\delta_X \neq \alpha\beta\delta_X$. Hence $\beta\delta_X\alpha\delta_X \neq \alpha\delta_X\beta\delta_X$. Thus $E(I_X/\delta_X)$ is the centralizer of $E(I_X/\delta_X)$ in I_X/δ_X. This proves that δ_X is a congruence-free congruence on I_X.

Finally, we prove that δ_X is the only nonuniversal congruence-free congruence on I_X. Let ρ be a nonuniversal congruence on I_X such that ρ is congruence-free. Then I_X/ρ is 0-simple and $E(I_X/\rho)$ is disjunctive. Let $\alpha, \beta \in I_X$ such that $\alpha\delta_X\beta$. Then $\beta^{-1}\alpha\delta_X\beta^{-1}\beta$ and $\alpha^{-1}\delta_X\beta^{-1}$. From $\alpha\delta_X\beta$ and $\alpha^{-1}\delta_X\beta^{-1}$, we get $\alpha\alpha^{-1}\delta_X\beta\beta^{-1}$. Hence $|\Delta\alpha\alpha^{-1} \setminus D(\alpha\alpha^{-1}, \beta\beta^{-1})| < |X|$ and $|\Delta\beta\beta^{-1} \setminus D(\alpha\alpha^{-1}, \beta\beta^{-1})| < |X|$. By definition of $D(\alpha\alpha^{-1}, \beta\beta^{-1})$ we have that, $x\alpha\alpha^{-1} = x = x\beta\beta^{-1}$ for all $x \in D(\alpha\alpha^{-1}, \beta\beta^{-1})$. Let λ be the identity map on the set $D(\alpha\alpha^{-1}, \beta\beta^{-1})$. Thus by Lemma 3.15, $\alpha\alpha^{-1}\rho\lambda$ and $\beta\beta^{-1}\rho\lambda$ and thus $\alpha\alpha^{-1}\rho\beta\beta^{-1}$. Because $\beta^{-1}\beta\delta_X\beta^{-1}\alpha$, $|\Delta\beta^{-1}\beta \setminus D(\beta^{-1}\beta, \beta^{-1}\alpha)| < |X|$ and $|\Delta\beta^{-1}\alpha \setminus D(\beta^{-1}\beta, \beta^{-1}\alpha)| < |X|$. Then $x\beta^{-1}\beta = x = x\beta^{-1}\alpha$ for all $x \in D(\beta^{-1}\beta, \beta^{-1}\alpha)$. Let γ be the identity map on the set $D(\beta^{-1}\beta, \beta^{-1}\alpha)$. Then by Lemma 3.15, $\beta^{-1}\beta\rho\gamma$ and $\gamma\rho\beta^{-1}\alpha$, thus $\beta^{-1}\beta\rho\beta^{-1}\alpha$. Hence $\alpha\rho = (\alpha\alpha^{-1}\alpha)\rho = (\alpha\alpha^{-1}\rho)\alpha\rho = (\beta\beta^{-1}\rho)\alpha\rho = \beta\rho(\beta^{-1}\alpha\rho) = \beta\rho\ (\beta^{-1}\beta\rho) = (\beta\beta^{-1}\beta)\rho = \beta\rho$, so $(\alpha, \beta) \in \rho$.

This proves that $\delta_X \subseteq \rho$. Therefore by Corollary 1.3, $\rho = \delta_X$.

Hence, the theorem is completely proved. #