CHAPTER II

SOME SEMIGROUPS OF NUMBERS

The purpose of this chapter is to characterize all congruence-free congruences on the semigroup of integers under multiplication, the semigroup of nonnegative integers under addition and the semigroup of nonnegative real numbers r such that $r \leq 1$ under multiplication.

Let the notations $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of positive integers, the set of integers and the set of real numbers, respectively.

Let S be a semigroup with zero 0 and identity 1. Then the congruence ρ on S is a universal congruence on S if and only if $0\rho 1$.

Let ρ be a congruence on a semigroup S with zero 0. It is clear that 0ρ is an ideal of the semigroup S. However, it need not be completely prime eventhough S has an identity and is commutative. The identity congruence on a semigroup obtained by adjoining an identity to a nontrivial zero semigroup is an example.

2.1 Proposition. Let S be a commutative semigroup with zero 0 and identity 1. If ρ is a congruence on S such that S/ρ is 0-simple, then the ρ-class 0ρ is a completely prime ideal of the semigroup S.

Proof : Let a, b $\in$ S such that ab $\in 0\rho$. Assume a $\notin 0\rho$. Let $A = 0\rho \cup Sa$. Since 0ρ and Sa are ideals of S, A is an ideal of S. Let $\bar{A} = \{x\rho \mid x \in A\}$. Then $a\rho \neq 0\rho$ and $a\rho \in \bar{A}$. Thus $\bar{A}$ is a nonzero ideal of the semigroup S/ρ. Because S/ρ is 0-simple, $\bar{A} = S/\rho$. Then

$1\rho = x\rho$ for some $x \in A$. If $1\rho = 0\rho$, then $\rho = \omega$, the universal congruence on S and hence $a\rho = 0\rho$, a contradiction. Thus $1\rho = sa\rho$ for some $s \in S$. It then follows that $b\rho = 1b\rho = sab\rho = s\rho ab\rho = s\rho 0\rho = 0\rho$. Hence $b \in 0\rho$. #

2.2 Corollary. Let S be a commutative semigroup with zero 0 and identity 1. If ρ is a congruence-free congruence on S, then the ρ-class 0ρ is a completely prime ideal of S.

Proof : Since S/ρ is a semigroup with zero and it is congruence-free, S/ρ is either a 0-simple semigroup or a zero semigroup with $|S/\rho| \leq 2$. If $|S/\rho| = 2$, then $S/\rho = \{0\rho, 1\rho\}$ and $0\rho \neq 1\rho$ and therefore S/ρ is a 0-simple semigroup. Hence either S/ρ is 0-simple or $0\rho = S$. By Proposition 2.1, 0ρ is a completely prime ideal of S. #

2.3 Proposition. Let S be a commutative semigroup with zero 0 and identity 1, and ρ be a congruence on S. Then ρ is a congruence-free congruence on S if and only if $|S/\rho| \leq 2$; or equivalently, $S/\rho = \{0\rho, 1\rho\}$.

Proof : Assume that ρ is a nonuniversal congruence-free congruence on S. By Corollary 2.2, 0ρ is a completely prime ideal of S. Let $\bar{\delta} = \{(0\rho, 0\rho)\} \cup [(S/\rho \setminus \{0\rho\}) \times (S/\rho \setminus \{0\rho\})]$. Because 0ρ is completely prime, $\bar{\delta}$ is a congruence on the semigroup S/ρ. But $\bar{\delta}$ is not the universal congruence on the semigroup S/ρ which is congruence-free. Hence $\bar{\delta}$ is the identity congruence on S/ρ which implies that $|S/\rho \setminus \{0\rho\}| = 1$ and therefore $|S/\rho| = 2$.

The converse is trivial. #

Under usual multiplication, $\mathbb{Z}$ is a commutative semigroup having 0 and 1 as its zero and identity, respectively. For the remainder of this chapter, whenever we say the semigroup $\mathbb{Z}$, the usual multiplication is considered as its operation.

Let ρ be a congruence on the semigroup $\mathbb{Z}$, let $z \in \mathbb{Z}$. Then the following obviously follow :

(1) $z\rho 0$ if and only if $-z\rho 0$.

(2) $z\rho 1$ implies $z^n\rho 1$ for all $n \in \mathbb{N}$.

It follows from Proposition 2.3 that ρ is a congruence-free congruence on the semigroup $\mathbb{Z}$ if and only if $|\mathbb{Z}/\rho| \leq 2$.

Let ρ be a congruence-free congruence on the semigroup $\mathbb{Z}$. Then by Proposition 2.3, $\mathbb{Z}/\rho = \{0\rho, 1\rho\}$. If $-1 \in 0\rho$, then $1 \in 0\rho$ and therefore $-1\rho = 0\rho = 1\rho$. If $-1 \in 1\rho$, then $-1\rho = 1\rho$. Therefore, for any $x \in \mathbb{Z}$, $x\rho 1$ implies that $-x\rho -1$ and hence $-x \in -1\rho = 1\rho$. Thus if $x\rho 1$, then $-x\rho 1$. Therefore, for any $x \in \mathbb{Z}$, $-x\rho 1$ iff $x\rho 1$.

A characterization of all congruence-free congruences on the semigroup $\mathbb{Z}$ is given in the next theorem. To prove the theorom, the following lemmas are required :

2.4 <u>Lemma</u>. Let ρ be a congruence-free congruence on the semigroup $\mathbb{Z}$. Let $a, b \in \mathbb{Z}$. Then $ab\rho 1$ if and only if $a\rho 1$ and $b\rho 1$.

Proof : It is obvious for the case $\rho = \omega$, the universal congruence on S. Assume $\rho \neq \omega$. By Corollary 2.2 and Proposition 2.3, 0ρ is completely prime ideal of S and $S = (0\rho) \cup (1\rho)$ which is a disjoint union. Then $1\rho = S \setminus (0\rho)$ is a filter of S. Hence the lemma follows. #

Let P be the set of prime numbers. For each subset A of P, let

$$A^* = \{\pm p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \mid p_i \in A,\ r_i \in \mathbb{N} \cup \{0\},\ n \in \mathbb{N}\}.$$

Then $1 \in A^*$ for all nonempty subsets A of P and $A^* \subsetneq \mathbb{Z}$ for all $A \subseteq P$.

For each subset A of P, let ρ^A be the relation on the semigroup $\mathbb{Z}$ defined by

$$(x, y) \in \rho^A \text{ if and only if either } x, y \in A^* \text{ or } x, y \notin A^*,$$

that is, $\rho^A = (A^* \times A^*) \cup [(\mathbb{Z} \setminus A^*) \times (\mathbb{Z} \setminus A^*)]$.

Trivially, ρ^A is an equivalence relation on $\mathbb{Z}$ for every subset A of P. Note that $\rho^\phi = \omega$, the universal congruence on $\mathbb{Z}$.

Let $A \subseteq P$, $A \neq \phi$. To show A^* is a filter of the semigroup $\mathbb{Z}$, let $a, b \in \mathbb{Z}$ such that $ab \in A^*$. Then $ab = (-1)^m p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ for some primes $p_1, p_2, \ldots, p_k \in A$, $k \in \mathbb{N}$ and $r_1, r_2, \ldots, r_k \in \mathbb{N} \cup \{0\}$ and for some $m \in \{1, 2\}$. Hence $a = (-1)^{m_0} p_{i_1}^{s_1} p_{i_2}^{s_2} \cdots p_{i_j}^{s_j}$ for some $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, k\}$ and for some $s_1, s_2, \ldots, s_j \in \mathbb{N} \cup \{0\}$ and $m_0 \in \{1, 2\}$. By the definition of A^*, we have that $a \in A^*$. Similarly, $b \in A^*$. Hence A^* is a filter of the semigroup $\mathbb{Z}$. Thus, $\mathbb{Z} \setminus A^*$ is a completely prime ideal of the semigroup $\mathbb{Z}$.

2.5 <u>Lemma</u>. For each subset A of P, ρ^A is a congruence-free congruence on the semigroup $\mathbb{Z}$.

<u>Proof</u> : Let A be a subset of P. Then $\mathbb{Z} \setminus A^*$ is a completely prime ideal of the semigroup $\mathbb{Z}$. Thus it follows that $\rho^A (= (A^* \times A^*) \cup [(\mathbb{Z}\setminus A^*) \times (\mathbb{Z}\setminus A^*)])$ is a congruence on $\mathbb{Z}$ and $|\mathbb{Z}/\rho^A| \leq 2$. Hence ρ^A is a congruence-free congruence on the semigroup $\mathbb{Z}$. #

2.6 <u>Theorem</u>. Let ρ be a congruence on the semigroup $\mathbb{Z}$. Then ρ is congruence-free if and only if $\rho = \rho^A$ for some subset A of P.

<u>Proof</u> : Assume that ρ is a congruence-free congruence on the semigroup $\mathbb{Z}$. Then by Proposition 2.3, $\mathbb{Z}/\rho = \{0\rho, 1\rho\}$. If $0\rho = 1\rho$, then $\rho = \omega = (\phi \times \phi) \cup [(\mathbb{Z}\setminus \phi) \times (\mathbb{Z}\setminus \phi)] = \rho^{\phi}$. Suppose $0\rho \neq 1\rho$. Let $A = P \cap (1\rho)$. Claim that $\rho = \rho^A$. Since $\mathbb{Z}/\rho = \{0\rho, 1\rho\}$ and $\rho^A = (A^* \times A^*) \cup [(\mathbb{Z}\setminus A^*) \times (\mathbb{Z}\setminus A^*)]$, it suffices to show that $1\rho = A^*$.

Let $x \in 1\rho$. Then $x \neq 0$. Assume without loss of generality that $x > 0$ (because $x\rho 1$ if and only if $-x\rho 1$). Then $x = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$ for some $n \in \mathbb{N}$, some primes $p_1, p_2, \dots, p_n$ and $r_1, r_2, \dots, r_n \in \mathbb{N} \cup \{0\}$. Since $1\rho\, p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$ and by Lemma 2.4, $1\rho p_i$ for all i, $1 \leq i \leq n$, so $p_i \in P \cap (1\rho)$ for all $i \in \{1, 2, \dots n\}$ which implies that for each $i \in \{1, 2, \dots, n\}$, $p_i \in A$, hence by the definition of A^*, $x = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n} \in A^*$.

Next, let $x \in A^*$. Then $|x| = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$ for some $n \in \mathbb{N}$, $p_1, p_2, \dots, p_n \in A$ and $r_1, r_2, \dots, r_n \in \mathbb{N} \cup \{0\}$. Because

$A = P \cap (1\rho)$, $p_i \in 1\rho$ for all i, hence $1\rho p_i$ for all i, thus, $1\rho p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$ and so 1ρ-$p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$. Therefore $x \in (1\rho)$.

The converse follows from Lemma 2.5. #

Let $\bar{\mathbb{N}}$ denote the set of all nonnegative integers, that is, $\bar{\mathbb{N}} = \{0, 1, 2, 3, \ldots\}$. Then, under the usual addition, $\bar{\mathbb{N}}$ is a semigroup without zero, but having 0 as its identity. In this chapter, by the semigroup $\bar{\mathbb{N}}$ we mean it is the semigroup under usual addition.

For each $n \in \bar{\mathbb{N}}$, let $I_n = \{n, n+1, n+2, \ldots\}$. Then for each $n \in \bar{\mathbb{N}}$, I_n is clearly an ideal of the semigroup $\bar{\mathbb{N}}$. Moreover, $\{I_n \mid n \in \bar{\mathbb{N}}\}$ is the set of all ideals of the semigroup $\bar{\mathbb{N}}$. To show this, let I be an ideal of $\bar{\mathbb{N}}$. Let m be the minimum element of I. Therefore $I \subseteq \{m, m+1, m+2, \ldots\} = I_m$. Because $m \in I$ and I is an ideal of the semigroup $\bar{\mathbb{N}}$, $m+k \in I$ for all $k \in \bar{\mathbb{N}}$. Thus $I_m \subseteq I$ and so $I = I_m$.

Let ρ be a congruence on the semigroup $\bar{\mathbb{N}}$. Then ρ is a Rees congruence if and only if there exists $m \in \bar{\mathbb{N}}$ such that $\rho = \{(k, k) \mid k \in \bar{\mathbb{N}}, k < m\} \cup \{(x, y) \mid x, y \in \bar{\mathbb{N}}, x, y \geq m\}$.

Let ρ be a congruence on the semigroup $\bar{\mathbb{N}}$. It is then easily seen that either $0\rho = \{0\}$ or 0ρ contains infinitely many elements of $\bar{\mathbb{N}}$.

For $n \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$, let ρ_n^k be the relation on the semigroup $\bar{\mathbb{N}}$ defined as follows : For $x, y \in \bar{\mathbb{N}}$,

$$(x, y) \in \rho_n^k \Longleftrightarrow \begin{cases} x = y \text{ if } x, y < k, \\ x \equiv y \pmod{n} \text{ if } x, y \geq k. \end{cases}$$

Hence ρ_n^k is a congruence on the semigroup $\bar{\mathbb{N}}$ for all $n \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$. Moreover, a congruence ρ on the semigroup $\bar{\mathbb{N}}$ is a Rees congruence if and only if $\rho = \rho_1^k$ for some $k \in \bar{\mathbb{N}}$. Note that ρ_1^0 is the universal congruence on the semigroup $\bar{\mathbb{N}}$. For $n \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$, ρ_n^k has $k + n$ classes.

The following proposition shows that the congruences ρ_n^k, $n \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$ and the identity congruence on $\bar{\mathbb{N}}$ are all the congruences on the semigroup .

2.7 <u>Proposition</u>. Let ρ be a relation on $\bar{\mathbb{N}}$. Then ρ is a congruence on the semigroup $\bar{\mathbb{N}}$ if and only if either ρ is the identity congruence on $\bar{\mathbb{N}}$ or $\rho = \rho_n^k$ for some $n \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$.

<u>Proof</u> : Let ρ be a congruence on the semigroup $\bar{\mathbb{N}}$ such that ρ is not the identity congruence on $\bar{\mathbb{N}}$. Then there exists $x \in \bar{\mathbb{N}}$ such that $|x\rho| > 1$. Let k be the minimum element of the set $\{x \in \bar{\mathbb{N}} \mid |x\rho| > 1\}$. Then $i\rho = \{i\}$ for all $i \in \bar{\mathbb{N}}$ such that $i < k$ and $|k\rho| > 1$. Let m be the smallest positive integer such that $k\rho(k+m)$. That is, m is the minimum element of the set $\{x-k \mid x \in k\rho, x \neq k\}$. Hence $k\rho(k+m)\rho(k+2m)\rho(k+3m)\rho \ldots$. Thus by the transitivity of ρ, $k\rho(k+jm)$ for all $j \in \bar{\mathbb{N}}$.

Claim that $\rho = \rho_m^k$. We already have that $i\rho = \{i\} = i\rho_m^k$ for all $i \in \bar{\mathbb{N}}$ such that $i < k$. Let $a, b \in \bar{\mathbb{N}}$ such that $a \geq b \geq k$ and

$a\rho_m^k b$. Then $a \equiv b \pmod m$, so $a - b = tm$ for some $t \in \bar{\mathbb{N}}$. Thus $k + a - b = k + tm$. But $(k + tm)\rho k$, so $(k + a - b)\rho k$. Since $b - k \in \bar{\mathbb{N}}$, $a\rho = (k + a - b + (b - k))\rho = (k + b - k)\rho = b\rho$, therefore $(a, b) \in \rho$. This proves that $\rho_m^k \subseteq \rho$. Hence we have the following :

$$\left.\begin{array}{l} i\rho = \{i\} = i\rho_m^k \\ \text{for all } i < k \text{ and} \\ k\rho \supseteq \{k + nm \mid n \in \bar{\mathbb{N}}\} = k\rho_m^k, \\ (k + 1)\rho \supseteq \{(k + 1) + nm \mid n \in \bar{\mathbb{N}}\} = (k + 1)\rho_m^k, \\ \vdots \\ \text{and } (k + (m - 1))\rho \supseteq \{(k + (m - 1)) + nm \mid n \in \bar{\mathbb{N}}\} = (k + (m - 1))\rho_m^k. \end{array}\right\} (*)$$

Suppose $\rho \neq \rho_m^k$. From (*) and since $\bar{\mathbb{N}} = (\bigcup_{0 \leq i < k}\{i\}) \cup (\bigcup_{i=0}^{m-1}\{k + i + nm \mid n \in \bar{\mathbb{N}}\})$ and it is a disjoint union, there exist r, s belong to $\{0, 1, \ldots, m - 1\}$ such that $r > s$ and $(k + r)\rho(k + s)$. Because $m - s \in \bar{\mathbb{N}}$, $(k + r + m - s)\rho = (k + s + m - s)\rho = (k + m)\rho$. But $(k + m)\rho k$, so $k\rho = (k + r + m - s)\rho = (k + m + r - s)\rho = (k + (r - s))\rho$. Since $0 \leq s < r < m$, $0 < r - s < m$. Then the equality of $k\rho$ and $(k + r - s)\rho$ contradicts to the property of m.

Hence $\rho = \rho_m^k$, as required. #

Let n, k be positive integers. Then $0\rho_n^k = \{0\}$. Because the semigroup $\bar{\mathbb{N}}$ has no nontrivial unit, the relation

$$\delta = \{(0\rho_n^k, 0\rho_n^k)\} \cup [(\bar{\mathbb{N}}/\rho_n^k \setminus \{0\rho_n^k\}) \times (\bar{\mathbb{N}}/\rho_n^k \setminus \{0\rho_n^k\})]$$

is clearly a congruence on the semigroup $\bar{\mathbb{N}}/\rho_n^k$ and δ is not the universal congruence on $\bar{\mathbb{N}}/\rho_n^k$. Therefore if $\bar{\mathbb{N}}/\rho_n^k$ is congruence-free, then δ is the identity congruence on $\bar{\mathbb{N}}/\rho_n^k$ which implies that $k = n = 1$.

The congruences ρ_1^0 and ρ_1^1 are congruence-free congruence on $\bar{\mathbb{N}}$.

Let n be a composite number such that $n > 1$. Claim that ρ_n^0 is not congruence-free. Since n is a composite number and $n > 1$, $n = mj$ for some $m, j \in \mathbb{N}$ such that $1 < m, j < n$. Therefore for $a, b \in \bar{\mathbb{N}}$, if $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$, so $\rho_n^0 \subseteq \rho_m^0$. But $n > m$, thus $\rho_n^0 \subsetneq \rho_m^0$. Let $\bar{\rho}_m^0 = \{(a\rho_n^0, b\rho_n^0) \mid (a, b) \in \rho_m^0\}$. Then $\bar{\rho}_m^0$ is a congruence on the semigroup $\bar{\mathbb{N}}/\rho_n^0$. Because $\rho_n^0 \subsetneq \rho_m^0$, $\bar{\rho}_m^0$ is not the identity congruence on $\bar{\mathbb{N}}/\rho_n^0$. Since $m > 1$, ρ_m^0 is not the universal congruence on $\bar{\mathbb{N}}$ and hence $\bar{\rho}_m^0$ is not the universal congruence on $\bar{\mathbb{N}}/\rho_n^0$. This proves that ρ_n^0 is not a congruence-free congruence on the semigroup $\bar{\mathbb{N}}$ if n is a composite number and $n > 1$.

Hence, for any positive integer n, if ρ_n^0 is a congruence-free congruence on $\bar{\mathbb{N}}$, then n is either 1 or a prime number. The converse is true and a proof is given as follows : Let p be a prime. Let ρ be a congruence on $\bar{\mathbb{N}}$ such that $\rho \supseteq \rho_p^0$. By Proposition 2.7 we have that $\rho = \rho_m^k$ for some $m \in \mathbb{N}$, $k \in \bar{\mathbb{N}}$. If $k > 0$, then $0\rho_m^k = \{0\}$ which does not contain $0\rho_p^0 = \{0, p, 2p, 3p, \dots\}$. Therefore $k = 0$, so $\rho = \rho_m^0$. Because $\rho_p^0 \subseteq \rho = \rho_m^0$ and $(0, p) \in \rho_p^0$, $(0, p) \in \rho_m^0$. Then $p \equiv 0 \pmod{m}$ and so $m \mid p$. But p is a prime, then $m = 1$ or p. Hence $\rho = \rho_1^0$ which is the universal congruence on $\bar{\mathbb{N}}$ or $\rho = \rho_p^0$. Therefore, the congruence ρ_p^0 and the universal congruence on $\bar{\mathbb{N}}$ are the only congruences on $\bar{\mathbb{N}}$ which contain ρ_p^0. But there is an inclusion preserving one-to-one correspondence between the set of congruences on $\bar{\mathbb{N}}$ which contain ρ_p^0 and the set of congruences on $\bar{\mathbb{N}}/\rho_p^0$. Hence, the identity congruence and the universal congruence are the only congruences on the semigroup $\bar{\mathbb{N}}/\rho_p^0$. Therefore ρ_p^0 is a congruence-free congruence on the semigroup $\bar{\mathbb{N}}$.

Hence, the following theorem is obtained :

2.8 Theorem. The universal congruence, ρ_1^1 $(= \{(0, 0)\} \cup (\mathbb{N} \times \mathbb{N}))$ and ρ_p^0 $(= \{(a, b) \in \bar{\mathbb{N}} \times \bar{\mathbb{N}} \mid a \equiv b \pmod{p}\})$ for any prime p are all the congruence-free congruences on the semigroup $\bar{\mathbb{N}}$.

Under usual multiplication the set of real numbers x such that $0 \leq x \leq 1$ is a commutative semigroup with zero 0 and identity 1. Next, the characterization of congruence-free congruences on this semigroup is considered.

By the semigroup [0, 1] we mean the semigroup $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ under usual multiplication.

The next proposition shows that any congruence ρ on the semigroup [0, 1] is a Rees congruence if 0ρ contains more than one element. The characterization of ideals on [0, 1] is required to prove the proposition.

2.9 Lemma. Let A be a nonempty subset of [0, 1]. Then A is an ideal of the semigroup [0, 1] if and only if A is either [0, a) for some $a \in [0, 1]$ or [0, b] for some $b \in [0, 1]$.

Proof : Let A be an ideal of the semigroup [0, 1]. If $A = \{0\}$, then $A = [0, b]$ where $b = 0$. Assume that $A \neq \{0\}$. Then there exists $x \in A$ such that $x \neq 0$. Let b be the supremum of A. Thus $b \neq 0$ and $A \subseteq [0, b]$. We claim that if $b \in A$, then $A = [0, b]$ and if $b \notin A$, then $A = [0, b)$. Suppose first, $b \in A$. To show $A = [0, b]$, let $x \in [0, b]$. Then $0 \leq x \leq b$, and so $0 \leq \frac{x}{b} \leq 1$. Therefore $\frac{x}{b} \in [0, 1]$. Since $x = \frac{x}{b}b$ and b belongs to the ideal A of the semigroup [0, 1], $x \in A$. Thus $A = [0, b]$. Next, suppose that $b \notin A$. To show $[0, b) \subseteq A$, let $x \in [0, b)$. Then $x < b$. Since b is the supremum of A, there exists $y \in A$ and $x < y < b$, so $\frac{x}{y} \in [0, 1]$. Hence $x = \frac{x}{y} y \in A$. Therefore $A = [0, b)$. This proves the lemma, as desired. #

2.10 Proposition. Let ρ be a congruence on the semigroup [0, 1]. If $0\rho \neq \{0\}$, then ρ is a Rees congruence on the semigroup [0, 1].

Proof : Let ρ be a congruence on the semigroup [0, 1] such that $0\rho \neq \{0\}$. Then $0\rho = [0, a]$ for some $a \in (0, 1]$ or $0\rho = [0, b)$ for some $b \in (0, 1]$. To show that ρ is a Rees congruence on the semigroup [0, 1], it is enough to show that for all $c, d \notin 0\rho$ if $c\rho d$, then $c = d$. Let $c, d \in [0, 1]$ such that $c, d \notin 0\rho$ and $c\rho d$. Assume that $c \leq d$.

Case $0\rho = [0, a]$ for some $a \in (0, 1]$. Then $0 < a < c$ and $0 < a < d$, so $0 < \frac{a}{c} < 1$ and $0 < \frac{a}{d} < 1$ and hence $\frac{a}{c}, \frac{a}{d}$ belong to [0, 1]. Now, suppose that $c \neq d$. Then $c < d$ and thus $\frac{a}{d} < \frac{a}{c}$, so there exists $h \in [0, 1]$ such that $\frac{a}{d} < h < \frac{a}{c}$. Claim that $ch \in 0\rho$ and $dh \notin 0\rho$.

Since $\frac{a}{d} < h < \frac{a}{c}$, $c\frac{a}{d} < ch < a$ and $a < dh < d\frac{a}{c}$. Thus $ch \in O\rho$ and $dh \notin O\rho$. Since $c\rho d$ and $h \in [0, 1]$, $ch\rho dh$. It contradicts that $ch \in O\rho$ but $dh \notin O\rho$. Therefore, $c = d$.

Case $O\rho = [0, b)$ for some $b \in (0, 1]$. Then $0 < b \leq c$ and $0 < b \leq d$, so $0 < \frac{b}{c} \leq 1$ and $0 < \frac{b}{d} \leq 1$ which imply that $\frac{b}{c}, \frac{b}{d} \in [0, 1]$. Suppose that $c < d$. Then $\frac{b}{d} < \frac{b}{c}$, so there exists h such that $\frac{b}{d} < h < \frac{b}{c}$ and hence $h \in [0, 1]$. Because $\frac{b}{d} < h < \frac{b}{c}$, $c\frac{b}{d} < ch < b$ and $b < dh < d\frac{b}{c}$, thus $ch \in O\rho$ and $dh \notin O\rho$, contradicting to that $c\rho d$. Hence $c = d$.

Therefore, ρ is a Rees congruence on the semigroup $[0, 1]$, so this proves the proposition. #

The relation $\{(0, 0)\} \cup ((0, 1] \times (0, 1])$ is clearly a congruence on the semigroup $[0, 1]$. A congruence on any semigroup S having exactly two classes is a nonuniversal congruence-free congruence on S. Then the congruences $\{(0, 0)\} \cup ((0, 1] \times (0, 1])$ and $\rho_{[0, 1)}$ (the Rees congruence on $[0, 1]$ induced by the ideal $[0, 1)$ of the semigroup $[0, 1]$) are nonuniversal congruence-free congruences on the semigroup $[0, 1]$. We show in the next theorem that these two congruences are the only nonuniversal congruence-free congruences on the semigroup $[0, 1]$.

Since the semigroup $[0, 1]$ is a commutative semigroup having 0 and 1 as its zero and identity, respectively, by Proposition 2.3, it follows that a congruence ρ on the semigroup $[0, 1]$ is congruence-free if and only if $|[0, 1]/\rho| \leq 2$.

2.11 Theorem. Let ρ be a nonuniversal congruence on the semigroup $[0, 1]$. Then ρ is congruence-free if and only if either $\rho = \{(0, 0)\} \cup ((0, 1] \times (0, 1])$ or $\rho = ([0, 1) \times [0, 1)) \cup \{(1, 1)\}$.

Proof : Let ρ be a nonuniversal congruence-free congruence on the semigroup $[0, 1]$. Then $|[0, 1]/\rho| = 2$. If $0\rho = \{0\}$, then $\rho = \{(0, 0)\} \cup ((0, 1] \times (0, 1])$ because $|[0, 1]/\rho| = 2$. Assume $0\rho \neq \{0\}$. Then by Proposition 2.10, ρ is a Rees congruence. But $|[0, 1]/\rho| = 2$, then $\rho = \rho_{[0,1)} = ([0, 1) \times [0, 1)) \cup \{(1, 1)\}$.

The converse is trivial. #