CHAPTER VIII

EXISTENCE OF ROOM SQUARE OF SIDE $5p^n$; FOR ODD PRIME p.

8.1 Existence of Room Square of side $5p^n$; where p is an odd prime.

Lemma 8.1.1. If r is not divisible by any $F_k = 2^{2^k} + 1$; where k is a non - negative integer, then there exists a Room Square of side r.

Proof. Since r is not divisible by any $F_k = 2^{2^k} + 1$, hence by the fundamental theorem of arithmetic, r can be written in the form

$$r = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$; where p_i are primes and $p_i^{\alpha_i} \neq F_k$

and α_i are non - negative integers for all i.

Since , by theorem 3.1.5 Room Square of side $p_i^{\alpha_i}$ exists, therefore by theorem 4.1.4 Room Square of side r exists.

Q.E.D

Lemma 8.1.2   Room Square of side 15 exist.

Proof. We prove by displaying Room Square of side 15.

0,3	5,10				13,1			9,15	7,8		2,6	12,14		4,11
5,12	0,4	6,11				14,2			10,1	8,9		3,7	13,15	
	6,13	0,5	7,12				15,3			11,2	9,10		4,8	14,1
15,2		7,14	0,6	8,13				1,4			12,3	10,11		5,9
6,10	1,3		8,15	0,7	9,14				2,5			13,4	11,12	
	7,11	2,4		9,1	0,8	10,15				3,6			14,5	12,13
13,14		8,12	3,5		10,2	0,9	11,1				4,7			15,6
1,7	14,15		9,13	4,6		11,3	0,10	12,2				5,8		
	2,8	15,1		10,14	5,7		12,4	0,11	13,3				6,9	
		3,9	1,2		11,15	6,8		13,5	0,12	14,4				7,10
8,11			4,10	2,3		12,1	7,9		14,6	0,13	15,5			
	9,12			5,11	3,4		13,2	8,10		15,7	0,14	1,6		
		10,13			6,12	4,5		14,3	9,11		1,8	0,15	2,7	
			11,14			7,13	5,6		15,4	10,12		2,9	0,1	3,8
4,9				12,13			8,14	6,7		1,5	11,13		3,10	0,2

Figure 8.1

This Room Square is found by R.C. Mullin by using computer search [10].

<u>Lemma 8.1.3</u> Let $F_k = 2^{2^k} + 1$; where k is a non - negative integer then there is a Room Square of side $5F_k$ for all k = 1, 2, 3,...,

<u>Proof</u> By Lemma 8.1.2, Room Square of side 15 exists.

Since $25 = 2^3 . 3 + 1$, so by theorem 3.1.5 Room Square of side 25 exists.

We may assume that $k \geqslant 2$. Hence $5F_k = (5.2^{2^k} + 4) + 1$ $= 4(5.2^{2^k-2} + 1) + 1 = 4R + 1$; where $R = 5.2^{2^k-2} + 1$.

It can be shown by induction on $k \geqslant 2$, that $R \equiv 0 (\text{mod } 3)$.

Therefore $5F_k = 12s + 1$, for some positive integer s.

Observe that $F_k = F_{k-1} . F_{k-2} \cdots\cdots . F_1 F_o + 2$. Hence it follows from $5F_k = 12s + 1$ that $5F_1 . F_2 \cdots\cdots F_{k-1} = 4s - 3$.

From this relationship we see that (3, s) = 1 ; $(F_i, s) = 1$ for $i \geqslant 0$.

Therefore s is relatively prime to all F_i ; $i \geqslant 0$. Then there exists a Room Square of side s by lemma 8.1.1. Since Room Squares of sides s and 13 such that Room Square of side 1 is a subsquare of Room Square of side 13 exist, hence by theorem 4.1.2 Room Square of side $s(13 - 1) + 1 = 12s + 1$ exists, that is Room Square of side $5F_k$ exists.

Q.E.D.

Theorem 8.1.4. There is a Room Square of side 5p for all prime p.

Proof. Let p be any odd prime.

case 1 If p can be written in the form

$p = 2^k t + 1$; where k is a positive integer and t an odd integer greater than 1, then there exists a Room Square of side 5p From corollary 3.2.3.

case 2 If p is not of the form $p = 2^k t + 1$, then by theorem A1 p can be written in the form

$p = 2^{2^k} + 1$; where k is a non-negative integer.

Therefore by Lemma 8.1.3 there exists a Room Square of side 5p.

Q.E.D.

Theorem 8.1.5. For each positive integer $n \geq 1$ and each odd prime p, a Room Square of side $5p^n$ exists.

Proof. Let p be any odd prime. Hence by theorem 8.1.4 there exists a Room Square of side 5p.

case 1 Suppose that $p^{n-1} \neq 3, 5$. Then by theorem 7.1.5 a Room Square of side p^{n-1} exists. Therefore by theorem 4.1.3 a Room Square of side $(5p).(p^{n-1}) = 5p^n$ exists.

case 2 Suppose that $p^{n-1} = 3, 5$.

If $p^{n-1} = 3$, then $5p^n = (5.3).3 = 45 = 3.(15)$.
It is seen from Lemma 8.1.2 that Room Square of side 15 exists. Hence by theorem 5.1.2 Room Square of side $3.(15) = 45$ exists.

If $p^{n-1} = 5$, then $5p^n = (5.5).5 = 5.(25)$.
Since $25 = 2^3.3 + 1$, therefore by theorem 3.2.2, there exists a Room Square of side $5(25) = 125$.
Therefore the theorem follows.

Q.E.D.