CHAPTER V

EXISTENCE OF ROOM SQUARE OF SIDE $3n$; $n > 1$

5.1 Existence of Room Square of side $3n$; $n > 1$

Lemma 5.1.1 Given a Room Square $\mathcal{R}$ of side r ; where $r = 2s + 1$, there are s permutations $\phi_1, \phi_2, \ldots, \phi_s$ of $\{1, 2, \ldots, r\}$ with properties that

(I) $k\phi_i = k\phi_j$ never occurs unless $i = j$,

(II) $(k, k\phi_i)$ cell is empty for $1 \leq k \leq r$; $1 \leq i \leq s$.

Proof Let $M = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & ; \quad \text{if } (i,j) \text{ cell of } \mathcal{R} \text{ is empty} , \\ 0 & ; \quad \text{otherwise} . \end{cases}$$

M is a matrix of zeros and ones with every row and column sum equal to s .

So by theorem 5.1.9 of [1] M is a sum of s permutation matrices, say

$$M = P_1 + P_2 + \ldots + P_s .$$

For each $i = 1, 2, \ldots, s$, let ϕ_i be defined by putting $k\phi_i = l$ if and only if (k, l) entry of P_i is 1 . Then ϕ_i are permutation on the set $\{1, 2, \ldots, r\}$.

If $k\phi_i = k\phi_j$ for some i, j such that $i \neq j$, then P_i and P_j would both have 1 in position (k, l) ; where $l = k\phi_i = k\phi_j$, so M would have an entry equal to 2 or more.

Since for each k, $1 \leq k \leq r$, the $(k, k\phi_i)$ cell of P_i equal to 1, hence the $(k, k\phi_i)$ cell of M is equal to 1. Therefore $(k, k\phi_i)$ cell of $\mathcal{R}$ is empty for $1 \leq k \leq r$, $1 \leq i \leq s$.

Therefore the lemma follows.

Q.E.D.

<u>Theorem 5.1.2</u> If there exists a Room Square of side $n > 1$, then there is a Room Square of side 3n.

<u>Proof</u>. Let $\mathcal{R}$ be a standardized Room Square of side n based on $\{0, 1, 2, \ldots, n\}$.

For $i, j \in \{1, 2, 3\}$, we defined $\mathcal{R}_{ij}$ for the array formed from $\mathcal{R}$ in the following way :

(i) delete all diagonal entries ;

(ii) if $x < y$ replace the entry $\{x, y\}$ of $\mathcal{R}$ by $\{x_i, y_j\}$.

Let ϕ be a permutation that satisfies the condition (II) of Lemma 5.1.1. Let $\mathcal{R}_{ij}\phi$ denote the array obtained by carrying out the permutation ϕ on the column of $\mathcal{R}_{ij}$; i.e column k of $\mathcal{R}_{ij}$ becomes column $k\phi$ of $\mathcal{R}_{ij}\phi$.

Let $\mathcal{L}$ be the 3n x 3n array whose n x n subarrays are displayed as follow :

$$\mathcal{L} = \begin{array}{|c|c|c|} \hline \mathcal{R}_{11} & \mathcal{R}^{\phi}_{22} & \mathcal{R}^{\phi}_{33} \\ \hline \mathcal{R}^{\phi}_{23} & \mathcal{R}_{31} & \mathcal{R}_{12} \\ \hline \mathcal{R}^{\phi}_{32} & \mathcal{R}_{13} & \mathcal{R}_{21} \\ \hline \end{array} .$$

Figure 5.1

Let $S = \{0, 1_1, 2_1, \ldots, n_1, 1_2, \ldots n_2, 1_3, \ldots. n_3\}$. Since $\mathcal{R}_{11}$ is obtained by deleting all diagonal entries, hence 0, i_1 do not appear in the i^{th} row of $\mathcal{R}_{11}$. By the same argument, we see that 0, i_2 do not appear in the i^{th} row of $\mathcal{R}_{22}$. Since the i^{th} row of $\mathcal{R}_{22}$ and $\mathcal{R}^{\phi}_{22}$ consist of the same set of elements, hence 0, i_2 do not appear in the i^{th} row of $\mathcal{R}^{\phi}_{22}$.

Similarly, we see that 0, i_3 do not appear in the i^{th} row of $\mathcal{R}^{\phi}_{33}$. Therefore elements 0, i_1, i_2, amd i_3 do not appear in the i^{th} row of $\mathcal{L}$ where $1 \leq i \leq n$.

By the same arguments, it can be seen that 0, i_1, i_2, and i_3 do not appear in the $(n + i)^{th}$ row and $(2n + i)^{th}$ row of $\mathcal{L}$.

Let $\mathcal{L}^*$ be the array obtained from $\mathcal{L}$ by placing

$\{0, i_1\}$ in the (i, i) cell of $\mathcal{L}$,

$\{i_2, i_3\}$ in the $(i, i\phi)$ cell of $\mathcal{L}$,

$\{0, i_2\}$ in the $(i+n, i\phi+n)$ cell of $\mathcal{L}$,

$\{i_1, i_3\}$ in the (i+n, i+n) cell of $\mathcal{L}$,

$\{0, i_3\}$ in the $(i + 2n, i\phi + 2n)$ cell of $\mathcal{L}$,

$\{i_1, i_2\}$ in the $(i + 2n, i + 2n)$ cell of $\mathcal{L}$,

for all $i = 1, 2, \ldots, n$.

We shall show that $\mathcal{L}^*$ is a Room Square of side $3n$ based on S. It is clear from the construction that each cell of $\mathcal{L}^*$ is empty or contains two distinct elements from S.

Next, we shall show that each row i of $\mathcal{L}^*$ contains all elements of S precisely once. This will be done in three cases.

<u>case 1</u> $1 \leq i \leq n$

We shall show that for all $s \in S$, there exists a unique j such that s is in the (i, j) cell of $\mathcal{L}^*$.

<u>case 1.1</u> If $s = 0$ or i_1, then s appears in the (i, i) cell of $\mathcal{L}^*$.

<u>case 1.2</u> If $s = i_2$ or i_3 then s appears in the $(i, i\phi)$ cell of $\mathcal{L}^*$.

<u>case 1.3</u> If $s \neq 0, i_1, i_3$, then there exist $x \neq i, 0$ such that $s = x_1$ or $s = x_2$ or $s = x_3$. Since $\mathcal{R}$ is a Room Square, hence x appears at least once in the i^{th} row of $\mathcal{R}$. Since $x \neq i$, x is not a diagonal entry of $\mathcal{R}$. Hence if s is x_1 or x_2 or x_3, it must appear in the i^{th} row of $\mathcal{R}_{11}$ or $\mathcal{R}_{22}^{\phi}$ or $\mathcal{R}_{33}^{\phi}$ respectively. Therefore every element of S must be in the (i, j) cell of $\mathcal{L}^*$ for some j ; $1 \leq j \leq 3n$.

Next, we shall show that each element of S appears at most once in any row i of $\mathcal{L}^*$.

It can be seen that 0, i_1, i_2 and i_3 appear precisely once in each row i of $\mathcal{L}^*$. Therefore, we only need to prove the case $s = x_1$, $s = x_2$ and $s = x_3$ where $x \neq i$. Let $s \in S$ be such that $s \neq 0$, i_1, i_2 and i_3.

Suppose that s appears twice in row i of $\mathcal{L}^*$; therefore there exist y_k, $z_1 \in S$ and $1 \leq j, j' \leq 3n$ such that

$\{s, y_k\}$ and $\{s, z_1\}$ are in the (i, j) cell and the (i, j') cell of $\mathcal{L}^*$ respectively where $j \neq j'$.

If $s = x_1$, then $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_1, y_1\}$ and $\{x_1, z_1\}$. Therefore, they must appear in the i^{th} row of $\mathcal{R}_{11}$. Hence the i^{th} row of $\mathcal{R}_{11}$ contains x_1 twice. From the definition of $\mathcal{R}_{11}$ this can happen only if x appears twice in the i^{th} row of $\mathcal{R}$, which is not possible. Hence we have a contradiction.

If $s = x_2$, then $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_2, y_2\}$ and $\{x_2, z_2\}$. Therefore, they must appear in the i^{th} row of $\mathcal{R}_{22}^{\emptyset}$. Hence the i^{th} row of $\mathcal{R}_{22}^{\emptyset}$ contain x_2 twice. From the definition of $\mathcal{R}_{22}^{\emptyset}$, this can happen only if x appears twice in i^{th} row of $\mathcal{R}$, which is not possible. Hence we have a contradiction. By similar argument if $s = x_3$, we can show that s can not appear twice in row i of $\mathcal{L}^*$.

<u>case 2.</u> $n < i \leq 2n$.

Since $n < i \leq 2n$, hence we may write $i = n + i'$; where $1 \leq i' \leq n$. We shall show for all $s \in S$, there exists a unique j such that s is in the (i, j) cell of $\mathcal{L}^*$.

<u>case 2.1</u> If $s = 0$ or i'_2, then s appears in the $(n + i', n + i'\emptyset)$ cell of $\mathcal{L}^*$.

<u>case 2.2</u> If $s = i'_1$ or i'_3 , then s appears in the $(n + i', n + i')$ cell of $\mathcal{L}^*$.

<u>case 2.3</u> If $s \neq 0, i'_1, i'_2$, and i'_3 , then there exist $x \neq 0, i'$ such that $s = x_1$ or $s = x_2$ or $s = x_3$.

Assume that $s = x_1$.

Since $\mathcal{R}$ is a Room Square, hence x appears in the i'^{th} row of $\mathcal{R}$. Therefore there exists $a \neq x$ such that $\{a, x\}$ appears in the i'^{th} row or $\mathcal{R}$. Since $x \neq a$, then $x < a$ or $a < x$.
If $x < a$, then the pair $\{x_1, a_2\}$ appears in the i'^{th} row of $\mathcal{R}_{12}$.
If $a < x$, then the pair $\{a_3, x_1\}$ appears in the i'^{th} row of $\mathcal{R}_{31}$.
Hence, either $\mathcal{R}_{12}$ or $\mathcal{R}_{31}$ must contain x_1 **in the i'^{th} row,** that is $s = x_1$ appears in the $(n + i', j)$ cell of $\mathcal{L}^*$ for some j ; $1 \leq j \leq 3n$.

Suppose $s = x_2$. Since $\mathcal{R}$ is a Room Square, hence x appears in the i^{th} row of $\mathcal{R}$. Therefore there exists $b \neq x$ such that $\{x, b\}$ appears in the i'^{th} row of $\mathcal{R}$. Since $b \neq x$, hence $b < x$ or $x < b$.

If $x < b$, then $\{x_2, b_3\}$ appears in the i'^{th} row of $\mathcal{R}_{23}$.
Therefore $\{x_2, b_3\}$ appears in the i'^{th} row of $\mathcal{R}_{23}\phi$.

If $b < x$, then $\{b_1, x_2\}$ appears in the i'^{th} row of $\mathcal{R}_{12}$.
Hence, either $\mathcal{R}_{12}$ or $\mathcal{R}_{23}\phi$ must contain x_2 in the i'^{th} row.

By similar argument if $s = x_3$, we can show that s appears in the $(n + i', j)$ cell of $\mathcal{L}^*$ for some j, $1 \leq j \leq 3n$.

Next we shall show that each element s of S appears at most once in any row i of $\mathcal{L}^*$. It can be seen from the definition of $\mathcal{L}^*$ that 0, i', i'_2, and i'_3 appear precisely once in each row i of $\mathcal{L}^*$. So, we only need to prove the case $s \neq 0, i'_1, i'_2$ and i'_3.

Let $s \in S$ be such that $s \neq 0, i'_1, i'_2, i'_3$. Hence there exists $x \neq 0, i'$ such that $s = x_1$ or $s = x_2$ or $s = x_3$.

Suppose that $s \in S$ appears twice in row i of $\mathcal{L}^*$, therefore there exist $y_k, z_l \in S$ and $1 \leq j, , j' \leq 3n$ such that

$\{s, y_k\}$ and $\{s, z_l\}$ are in the $(n + i', j)$ cell and the $(n + i', j')$ cell of $\mathcal{L}^*$ respectively where $j \neq j'$.

If $s = x_1$, it follows that s appears in $\mathcal{R}_{31}$ or $\mathcal{R}_{12}$. Hence there exists $t \neq x$ such that $\{x, t\}$ appears in the i'^{th} row of $\mathcal{R}$. Since $x \neq t$, then $x < t$ or $t < x$.

If $t < x$, the both pairs $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_1, t_3\}$ or $\{x_2, t_1\}$.

Since we only consider the case $s = x_1$, therefore $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_1, t_3\}$. Therefore they appear in the i'^{th} row of $\mathcal{R}_{31}$. Hence i'^{th} row of $\mathcal{R}_{31}$ contains x_1 twice. From the definition of $\mathcal{R}_{31}$, this can happen only if x appear twice in the i'^{th} row of $\mathcal{R}$, which is not possible. Hence we have a contradiction. By a similar argument the supposition $x < t$ also leads to a contradiction.

By similar arguments if $s = x_2$ or $s = x_3$ we can show that s can not appear twice in row i of $\mathcal{L}^*$.

<u>case 3</u> $2n < i \leq 3n$

By similar arguments we can show that each row i of $\mathcal{L}^*$ contains all element of S precisely once.

Therefore each row i of $\mathcal{L}^*$ contains all elements of S precisely once.

Next, we shall show that for each j, the j^{th} column of $\mathcal{L}^*$ contains all elements of S precisely once. This will be done in three cases.

<u>case 1</u> $1 \leq j \leq n$

We shall show that for all $s \in S$, there exists a unique i such that s is in the (i, j) cell of $\mathcal{L}^*$.

If $s = 0$, or j_1, then s appears precisely once in the (j, j) cell of $\mathcal{L}^*$.

If $s = (j\phi^{-1})_2$ or $(j\phi^{-1})_3$, then s appears precisely once in the $(j\phi^{-1}, j)$ cell of $\mathcal{L}^*$.

It remains to concider the cases where $s \neq 0$, j, $(j\phi^{-1})_2$, $(j\phi^{-1})_3$.

Therefore there exists $x \neq 0$, j, such that $s = x_1$ or there exists $x \neq 0$, $j\phi^{-1}$ such that $s = x_2$ or there exists $x \neq 0$, $j\phi^{-1}$ such that $s = x_3$.

We first show that s appears in the (i, j) cell of $\mathcal{L}^*$ for some i.

If $s = x_1$ where $x \neq 0$, j, then there exists $b \neq x$ such that $\{x, b\}$ is in the j^{th} column of $\mathcal{R}$. Hence $\{x_1, b_1\}$ appears in the j^{th} column of $\mathcal{R}_{11}$. Therefore $s = x_1$ appears in the (i, j) cell of $\mathcal{R}$ for some i; $1 \leq i \leq 3n$.

If $s = x_2$; where $x \neq 0$, $j\phi^{-1}$, then by a similar argument there exists $c \neq x$ such that $\{x, c\}$ is in the $j\phi^{-1}$ th column of $\mathcal{R}$. Since $x \neq c$, hence $x < c$ or $c < x$.

If $x < c$, then $\{x_2, c_3\}$ appears in the $j\phi^{-1}$ th column of $\mathcal{R}_{23}$.

Therefore $\{x_2, c_3\}$ appears in the j^{th} column of $\mathcal{R}_{23}\phi$.

If $c < x$, then $\{x_2, c_3\}$ appears in the $j\phi^{-1}$ th column of $\mathcal{R}_{32}$.

Therefore $\{x_2, c_3\}$ appears in the j^{th} column of $\mathcal{R}_{32}\phi$.

Hence either $\mathcal{R}_{23}\phi$ or $\mathcal{R}_{32}\phi$ must contain x_2 in the j^{th} column, that is $s = x_2$ appears in the (i, j) cell of $\mathcal{L}^*$ for some i ; $1 \leq i \leq 3n$.

By similar arguments if $s = x_3$ where $x \neq 0$, $j\phi^{-1}$, we can show that $s = x_3$ appears in the j^{th} column of $\mathcal{L}^*$

Next, we shall show that $s \neq 0$, j_1, $(j\phi^{-1})_2$, $(j\phi^{-1})_3$ appear at most once in column j of $\mathcal{L}^*$. That is we only consider the cases $s = x_1$ where $x \neq 0, j$ and $s = x_2$ where $x \neq 0$, $j\phi^{-1}$ and $s = x_3$ where $x \neq 0$, $j\phi^{-1}$.

Suppose that s appears twice in the j^{th} column of $\mathcal{L}^*$, then there exist y_k, $z_l \in S$ and $1 \leq i, i' \leq 3n$ such that $i \neq i'$ and $\{s, y_k\}$ and $\{s, z_l\}$ appear in the (i, j) cell and (i', j) cell of $\mathcal{L}^*$ respectively.

If $s = x_1$, then the pairs $\{s, y_k\}$ and $\{s, z_l\}$ must be of the form $\{x_1, y_1\}$ and $\{x_1, z_1\}$ respectively. Therefore, they must appear in the j^{th} column of $\mathcal{R}_{11}$.

Hence the j^{th} column of $\mathcal{R}_{11}$ contains x_1 twice. From the definition of $\mathcal{R}_{11}$, this can happen only if x appears twice in the j^{th} column of $\mathcal{R}$, which is not possible. Hence we have a contradiction.

If $s = x_2$, it follows that s appears in $\mathcal{R}_{23}\phi$ or $\mathcal{R}_{32}\phi$. Hence there exists $r \neq x$ suct that $\{x, r\}$ is in the $j\phi^{-1}$th column of $\mathcal{R}$. Since $x \neq r$, hence $x < r$ or $r < x$.

If $x < r$, then the pairs $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_2, r_3\}$ or $\{x_3, r_2\}$. Since we only consider the case $s = x_2$, therefore both $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_2, r_3\}$. Therefore, they appears in the $j\phi^{-1}$ th column of $\mathcal{R}_{23}$, that is in the j^{th} column of $\mathcal{R}\phi_{23}$. Hence j^{th} column of $\mathcal{R}\phi_{23}$ contains x_2 twice. From the definition of $\mathcal{R}\phi_{23}$, this can happen only if x appears twice in the $j\phi^{-1}$ th column of $\mathcal{R}$ which is not possible. Hence we have a contradiction.

By a similar argument the supposition $r < x$ also leads to a contradiction. By similar argument if $s = x_3$, we can show that s can not appear twice in the j^{th} column of $\mathcal{L}^*$.

<u>case 2</u>. $n < j \leq 2n$

We shall that for all $s \in S$, there exists a unique i such that s is in the (i, j) cell of $\mathcal{L}^*$. Since $n < j \leq 2n$, we may write $j = n+j'$ where $1 \leq j' \leq n$.

If $s = 0$, or $(j'\phi^{-1})_2$, then s appears precisely once in the $(j'\phi^{-1}+n, j'+n)$ cell of $\mathcal{L}^*$.

If $s = j'$, or j'_3, then s appears precisely once in the $(j'+n, j'+n)$ cell of $\mathcal{L}^*$.

It remains to consider the case $s \neq 0, j'_1, j'_3, (j'\phi^{-1})_2$. Hence there exists $x \neq 0$, j' such that $s = x_1$ or there exists $x \neq 0$, j' such that $s = x_3$ or there exists $x \neq 0$, $j'\phi^{-1}$ such that $s = x_2$.

We first show that s appears in the (i, j) cell of $\mathcal{L}^*$ for some i . If $s = x_2$ where $x \neq 0$, $j\phi^{-1}$, then there exists $k \neq x$ such that $\{k, x\}$ appears in the $j'\phi^{-1}$ th column of $\mathcal{R}$. Hence $\{k_2, x_2\}$ appears in the $j'\phi^{-1}$th column of $\mathcal{R}_{22}$, that is in the j' th column of $\mathcal{R}_{22}\phi$.

If $s = x_1$; where $x \neq 0, j'$, then there exists $e \neq x$ such that $\{x, e\}$ is in the j'th column of $\mathcal{R}$. Since $x \neq e$, hence $x < e$ or $e < x$.

If $x < e$, then $\{x_1, e_3\}$ appears in the j' th column of $\mathcal{R}_{13}$.

If $e < x$, then $\{e_3, x_1\}$ appears in the j' th column of $\mathcal{R}_{31}$.

Hence, either $\mathcal{R}_{13}$ or $\mathcal{R}_{31}$ must contain x_1 in the j' th column, that is $s = x_1$ appears in the $(i, j' + n)$ cell of $\mathcal{L}^*$ for some i ; $1 \leq i \leq 3n$.

By similar argument if $s = x_3$, we can show that $s = x_3$ appears in the j^{th} column of $\mathcal{L}^*$.

Next, we shall show that $s \neq 0$, j'_1, j'_3, $(j'\phi^{-1})_2$ appears at most once in column j of $\mathcal{L}^*$. That is we only consider the cases $s = x_1$ where $x \neq 0, j'$ and $s = x_3$ where $x \neq 0, j$ and $s = x_2$ where $x \neq 0$, $j\phi^{-1}$.

Suppose that s appears twice in the j^{th} column of $\mathcal{L}^*$, then there exist y_k, $z_l \in S$ and $1 \leq i, i' \leq 3n$ such that $i \neq i'$ and

$\{s, y_k\}$ and $\{s, z_1\}$ appear in the (i, j) cell and the (i', j) cell of $\mathcal{L}^*$ respectively.

If $s = x_2$ where $x \neq 0$, $j\phi^{-1}$, it follows that s appears in $\mathcal{R}_{22}\phi$. Hence there exists $p \neq x$ such that $\{p, x\}$ appears in the $j'\phi^{-1}$ th column of $\mathcal{R}$. Since $x \neq p$, hence $x < p$ or $p < x$.

If $x < p$, then the pairs $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_2, y_2\}$ and $\{x_2, z_2\}$ respectively. Therefore, they appear in the $j'\phi^{-1}$ th column of $\mathcal{R}_{22}$, that is in the j' th column of $\mathcal{R}_{22}\phi$.

Therefore $s = x_2$ appears twice in the j' th column of $\mathcal{R}_{22}\phi$, this can happen only if x appears twice in the $j'\phi^{-1}$ th column of $\mathcal{R}$, which is not possible. Hence we have a contradiction.

By a similar argument if $p < x$ also leads to a contradiction.

If $s = x_1$ where $x \neq 0$, j', it follows that s appears in $\mathcal{R}_{31}$ or $\mathcal{R}_{13}$. Hence there exists $d \neq x$ such that $\{x, d\}$ is in the j' th column of $\mathcal{R}$. Since $x \neq d$, hence $x < d$ or $d < x$.

If $x < d$, then the pairs $\{s, y_k\}$ and $\{s, z_1\}$ must be of the form $\{x_1, d_3\}$ or $\{x_3, d_1\}$. Since we only consider the case $s = x_1$, then $\{s, y_k\}$ and $\{s, z_1\}$ both must be of the form $\{x_1, d_3\}$.

Therefore they appear in the j' th column of $\mathcal{R}_{13}$. Hence j' th column of $\mathcal{R}_{13}$ contains x_1 twice. From the definition of $\mathcal{R}_{13}$, this can happen only if x appears twice in the j' th column of $\mathcal{R}$, which is not possible. Hence we have a contradiction.

By a similar if $d < x$, it also leads to a contradiction.

By similar arguments if $s = x_3$ where $x \neq 0, j'$ we can show that $s = x_3$ can not appear twice in the j^{th} column of $\mathcal{L}^*$.

case 3. $2n < j \leq 3n$.

By argument similar to those in case 2, we can show that for all $s \in S$ there exists a unique i such that $s \in (i, j)$ cell of $\mathcal{L}^*$.

It remains to be shown that every unordered pair of elements of S appears precisely once in $\mathcal{L}^*$.

Let s, t be any two distinct elements of S. We shall show that $\{s, t\}$ must appear in some cell of $\mathcal{L}^*$.

If s or $t = 0$, let us assume that $s = 0$. Since $t \neq s$ hence $t = x_i$ for some $i = 1, 2, 3$ and $1 \leq x \leq n$.

If $i = 1$, then $\{0, x_1\}$ appears in the (x, x) cell of $\mathcal{L}^*$.

If $i = 2$, then $\{0, x_2\}$ appears in the $(x + n, x\phi + n)$ cell of $\mathcal{L}^*$.

If $i = 3$, then $\{0, x_3\}$ appears in the $(x + 2n, x\phi + 2n)$ cell of $\mathcal{L}^*$,

If $s \neq 0$ and $t \neq 0$ then there exist x, y such that $x = x_i$ and $t = y_j$ where $1 \leq i, j \leq 3$ and $1 \leq x, y \leq n$.

<u>case 1</u> $x = y$.

Since $s \neq t$ hence $i \neq j$.

If $i = 1$; $j = 2$, then the pair $\{x_1, x_2\}$ appears in the $(x + 2n, x + 2n)$ cell of $\mathcal{L}^*$.

If $i = 1$; $j = 3$, then the pair $\{x_1, x_3\}$ appears in the $(x + n, x + n)$ cell of $\mathcal{L}^*$.

If $i = 2$; $j = 3$, then the pair $\{x_2, x_3\}$ appears in the $(x, x\emptyset)$ cell of $\mathcal{L}^*$.

<u>case 2</u> $x \neq y$.

Since $s \neq t$, hence $i \neq j$ or $i = j$.

<u>case 2.1</u> $i \neq j$

Since $x \neq y$, hence $\{x,y\}$ appears in some cell of $\mathcal{R}$. Hence $\{x_i, y_j\}$ appears in some cell of $\mathcal{R}_{ij}$ or $\mathcal{R}\emptyset_{ij}$.

<u>case 2.2</u> $i = j$

Since $x \neq y$, hence $\{x, y\}$ appears in some cell of $\mathcal{R}$. Hence $\{x_i, y_j\} = \{x_i, y_i\}$ appears in some cell of $\mathcal{R}_{ii}$ or $\mathcal{R}\emptyset_{ii}$ where $1 \leq i \leq 3$. Therefore every unordered pair of elements of S appears in $\mathcal{L}^*$.

Next we shall show that each unordered pair of elements of S appears at most once in $\mathcal{L}^*$.

Since each row of $\mathcal{R}$ contains $\frac{1}{2}(n+1)$ pairs, hence each row of $\mathcal{R}_{ij}$ or $\mathcal{R}\emptyset_{ij}$; where $1 \leq i, j \leq 3$, contains $\frac{1}{2}(n-1)$ pairs.

To obtain $\mathcal{L}^*$, we insert 2 new pairs of elements from S in some empty cell of each row of $\mathcal{L}$.

Therefore each row of $\mathcal{L}^*$ contains $3\left\{\frac{1}{2}(n-1)\right\}+2$ pairs .

Therefore the total number of pairs in $\mathcal{L}^*$ is

$$3n\left[3\left\{\frac{1}{2}(n-1)\right\}+2\right] = \frac{9n^2}{2} - \frac{9n}{2} + 6n \,,$$

$$= \frac{1}{2}(9n^2 - 9n + 12) \,,$$

$$= \frac{1}{2} \cdot 3n \; (3n+1) \,.$$

This is precisely the number of unordered pairs which can be formed from elements of S, so each unordered pairs of elements of S appears at most once in $\mathcal{L}^*$. Therefore $\mathcal{L}^*$ is a Room Square of side $3n$ based on S.

Q.E.D.