CHAPTER IV

A RECURSIVE CONSTRUCTION

4.1 Construction of Room Square of side $v_1(v_2 - v_3) + v_3$ from a Room Square of side v_1 and a Room Square of side v_2 which contains a subsquare of side v_3 where $v_2 - v_3 \neq 2, 6$.

Definition 4.1.1 Let R be a finite set. An array L based on a set R is said to be a Latin Square if every member of R appears precisely once in each row and once in each column. The number of elements of R will be called the order of the Latin Square.

Suppose that $L = (l_{ij})$ and $M = (m_{ij})$ are two Latin Squares of same order. We write (L,M) to denote the array whose the (i,j) entry is (l_{ij}, m_{ij}). Two Latin Squares L, M are called orthogonal if all entries of (L, M) are distinct.

Theorem 4.1.2 Suppose there exists a Room Square $\mathcal{R}_1$ of side v_1, a Room Square $\mathcal{R}_2$ of side v_2 with a subsquare $\mathcal{R}_3$ of side v_3 such that $n = v_2 - v_3 \neq 2$ or 6. Then there is a Room Square $\mathcal{R}$ of side

$$v = v_1(v_2 - v_3) + v_3 ,$$

with subsquare isomorphic to $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$.

Proof. Assume that $\mathcal{R}_1$ is standardized based on $\{0, 1, 2, \ldots, v_1\}$. Let $\mathcal{R}_2$ be a standardized Room Square based on $\{0, 1, 2, \ldots, n, n+1, \ldots, v_2\}$. We relabel the objects so that $\mathcal{R}_3$ based on $\{0, n+1, n+2, \ldots, v_2\}$

and then reorder rows and columns so that $\mathcal{R}_3$ occupied the last v_3 rows and columns of $\mathcal{R}_2$ as follows

$$\mathcal{R}_2 = \begin{array}{|c|c|} \hline A^{(1)} & A^{(2)} \\ \hline A^{(3)} & \mathcal{R}_3 \\ \hline \end{array} .$$

Figure 4.1

Since $n \neq 2,6$, by Theorems 13.2.2 and 13.4.1 of [1], a pair of orthogonal Latin Squares exist.

Let L and M be a pair of orthogonal Latin Squares of order n based on $\{1, 2, \ldots, n\}$. We arrange the first column of L and M to the form $(1, 2, \ldots, n)$. For each $i, j = 1,2, \ldots, v_1$, where $i \neq j$, let L_i be the array obtained from L with each entry x replaced by x_i. Similarly, let M_j be the array obtained from M with each entry x replaced by x_j.

Let Z_{ij} be the array obtained from (L_i, M_j) by replacing each ordered pair (x_i, y_j) by the unordered pair $\{x_i, y_j\}$.

Now for each $j = 1, 2, 3$; $i = 1,2, \ldots, v_1$, let $A^{(j)}_{(i)}$ be obtained from $A^{(j)}$ by replacing the pair $\{x, y\}$ in $A^{(j)}$ by $\{g_i(x), g_i(y)\}$; where

$$g_i(x) = \begin{cases} x_i & ; \text{ if } 1 \leq x \leq n \\ x & ; \text{ otherwise.} \end{cases}$$

We shall construct a Room Square $\mathcal{R}$ of side v based on the set

$$S = \{ 0, 1_1, 1_2, \ldots, 1_{v_1}, 2_1, 2_2, \ldots, 2_{v_1}, \ldots\ldots, n_1, \ldots\ldots, n_{v_1}, n+1, n+2, \ldots, v_2 \},$$

which consists of $v + 1$ symbols.

We first convert $\mathcal{R}_1$ into $nv_1 \times nv_1$ array $\mathcal{S}$ by replacing each of its cells by an $n \times n$ array. An empty cell is replaced by an empty $n \times n$ array, the entry $\{0,i\}$ is replaced by $A_i^{(1)}$ and the entry $\{j,k\}$; $j \neq 0$; $k \neq 0$ is replaced by Z_{jk} .

We now arrange arrays $\mathcal{S}$, $A_i^{(2)}$, $A_i^{(3)}$ and $\mathcal{R}_3$ in a master array of side $(nv_1 + v_3) \times (nv_1 + v_3)$ as follows :-

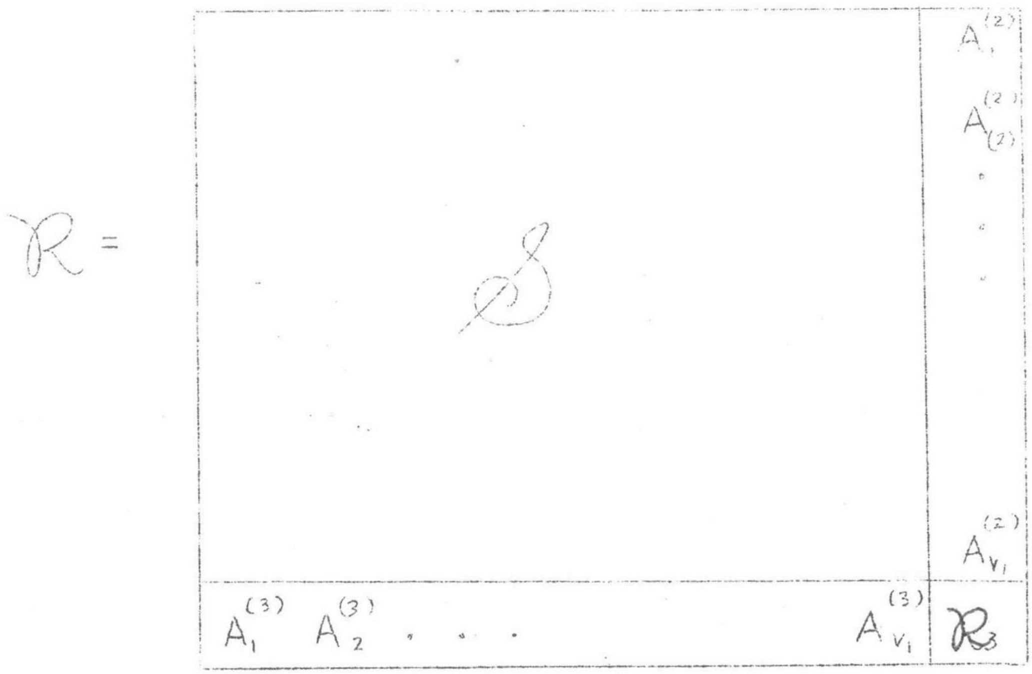

Figure 4.2

We shall show that $\mathcal{R}$ is a Room Square of side $nv_1 + v_3$ based on S. It is clear from the construction of $\mathcal{R}$ that each cell of $\mathcal{R}$ may contain an unordered pair of distinct elements of S or may be empty.
Now we shall show that each row of $\mathcal{R}$ contains all elements of S precisely once.

First consider the last v_3 row of $\mathcal{R}$. These v_3 rows comprise the following subarray of $\mathcal{R}$.

$$\boxed{A_1^{(3)} \; A_2^{(3)} \; \cdot \cdot \qquad A_{v_1}^{(3)} \; \mathcal{R}_3}\,.$$

Observe that each row of $A_i^{(3)}$ contains elements of $\{1_i, 2_i, \ldots, n_i\}$ precisely once and each row of $\mathcal{R}_3$ contains elements of $\{\infty, n+1, \ldots v_2\}$ precisely once. hence each row of these subarrays $\boxed{A_1^{(3)} \; A_2^{(3)} \ldots \; A_{v_1}^{(3)} \mathcal{R}_3}$ contains elements of S precisely once.

It remains tobe verified that for each $i = 1, 2, \ldots, nv_1$, the i^{th} row of $\mathcal{R}$ contains all elements of S precisely once.

Let i be such that $1 \leq i \leq nv_1$. Let s be an arbitrary element of S. We shall show that s appears in row i of $\mathcal{R}$.

Observe that we may write

$i = (i'-1)n + i''$ where $1 \leq i'' \leq n$ and $1 \leq i' \leq n$.

<u>case 1</u>. If $s = x_{i'}$, for some $x = 1,2,\dots, n$ and $i' = 1, 2,\dots, v_1$

Then s appears exactly once in every row of $\begin{array}{|c|c|}\hline A^{(1)}_{i'} & A^{(2)}_{i'} \\ \hline\end{array}$.

In particular, $s = x_i$ appears in the i''^{th} row of $\begin{array}{|c|c|}\hline A^{(1)}_{i'} & A^{(2)}_{i'} \\ \hline\end{array}$.

It can be seen from the construction of $\mathcal{R}$ that the i^{th} row of $\mathcal{R}$ contains the i''^{th} row of $\begin{array}{|c|c|}\hline A^{(1)}_{i'} & A^{(2)}_{i'} \\ \hline\end{array}$. Hence $s = x_i$ appears in the i^{th} row of $\mathcal{R}$. Note also that s appears exactly once in this row, since it appears exactly once in $\begin{array}{|c|c|}\hline A^{(1)}_{i'} & A^{(2)}_{i'} \\ \hline\end{array}$ and does not appear in any of Z_{jk}.

<u>case 2</u> $s = x_j$ where $j \neq i'$ and $j \in \{1,2,\dots, v_1\}$. Since j appears in the i'^{th} row of $\mathcal{R}_1$, hence there exists $k \neq j$ such that $\{j, k\}$ is an entry of the i'^{th} row of $\mathcal{R}_1$. Since x appears in the i''^{th} row of both L and M, hence there exist y, z such the (x, y) and (z, x) appears in the i''^{th} row of (L,M). Hence $\{x_j, y_k\}$ and $\{z_j, x_k\}$ appears in the i''^{th} row of Z_{jk}. From the construction of $\mathcal{R}$, we see that the i^{th} row of $\mathcal{R}$ contains the i''^{th} row of Z_{jk}. Hence x_j appears in the i^{th} row of $\mathcal{R}$. It may happen that (x, y) = (z, x)

or $(x, y) \neq (z, x)$. If $(x, y) = (z, x)$, then (x, x) appears only once in (L, M). Since $j \neq k$, hence x_j appears exactly once in i''^{th} row Z_{jk}.

If $(x, y) \neq (z, x)$, then the order pairs (x, y) and (z, x) give rise to the pairs $\{x_j , y_k\}$ and $\{z_j , x_k\}$ in the i''^{th} row of Z_{jk}. Since $j \neq k$, hence x_j appears only once in the i''^{th} row of Z_{jk}. From the construction of $\mathcal{R}$, x_j may not appears elsewhere in the i^{th} row of $\mathcal{R}$. Hence $s = x_j$ appears precisely once in the i^{th} row of $\mathcal{R}$.

<u>case 3</u> $n + 1 \leq s \leq v_2$ or $s = 0$.

Observe that $s = 0$, appears exactly once in the i''^{th} row of $A^{(1)}_{i'}$.

Therefore $s = 0$ appears precisely once in the i^{th} row of $\mathcal{R}$.
Let $s = n+1, n+2, \ldots, v_2$. Observe that each row of $\begin{array}{|c|c|}\hline A^{(1)} & A^{(2)} \\ \hline\end{array}$ contains all elements $\{0, 1, 2, \ldots, n, n+1, \ldots, v_2\}$ precisely once. Hence each row of $\begin{array}{|c|c|}\hline A_{i'}^{(1)} & A_{i'}^{(2)} \\ \hline\end{array}$ contains all elements of $\{0, 1_{i'}, 2_{i'}, \ldots, n_{i'}, n+1, \ldots, v_2\}$ precisely once. Therefore s appears exactly once in i''^{th} row of $\begin{array}{|cc|}\hline A_{i'}^{(1)} & A_{i'}^{(2)} \\ \hline\end{array}$. Hence s appears precisely once in the i^{th} row of $\mathcal{R}$.

Therefore all elements $s \in S$ appears precisely once in each row of $\mathcal{R}$.

A similar proof appied to column.
Next, we shall show that every unordered pair of elements of S appears precisely once in $\mathcal{R}$.

Consider the whole of $\mathcal{R}$. Since L, M are orthogonal Latin Square, hence every (x, y) appears in (L,M). Since $\mathcal{R}_1$ is a Room Square, hence every $\{j, k\}$, $j \neq k$; $1 \leq j, k \leq v$, appears in $\mathcal{R}_1$. Hence for each $x, y = 1, 2, \ldots, n$ and distinct $j, k = 1, \ldots, v_1$, the pair $\{x_j, y_k\}$ appears in Z_{jk}, which is a subarray of $\mathcal{R}$. Hence for each $x, y = 1, 2, \ldots n$ and all distinct $j, k = 1, 2, \ldots, v_1$, the pair $\{x_j, y_k\}$ appears in $\mathcal{R}$.

For distinct $x, y = 1, 2, \ldots, v_1$, we see that $\{x_i, y_i\}$ appears in $A_i^{(1)}$ or $A_i^{(2)}$ or $A_i^{(3)}$ which are subarray of $\mathcal{R}$.

Hence all pairs of distinct elements of the form $\{x_j, y_k\}$, $x, y = 1, 2, \ldots, n$, ; $j, k = 1, 2, \ldots, v_1$, appear in some cell of $\mathcal{R}$.

Note that any pair of the form $\{z, t\}$ where $z \neq t$; $n + 1 \leq z, t \leq v_2$, appears in $\mathcal{R}_3$, which is a subarray of $\mathcal{R}$. Hence all such pairs appear in $\mathcal{R}$. It remains to be verify that all pairs of the form $\{x_i, z\}$ where $i = 1, 2, \ldots, v_1$; $x = 1, 2, \ldots, n$ and $z = n+1, \ldots, v_2, 0$, appear in $\mathcal{R}$.

Since $\mathcal{R}_2$ is a Room Square based on $\{0, 1, 2, \ldots, n, n+1, \ldots v_2\}$ hence each pair $\{x, z\}$ where $1 \leq x \leq n$; $z = n+1, \ldots, v_2, 0$ appears in $\mathcal{R}_2$.

By definition of $A^{(j)}$, we see that such pair appears in some $A^{(j)}$, Hence $\{x_i, z\}$ appears in $A_i^{(j)}$ where $A_i^{(j)}$ is a subarray of $\mathcal{R}$. Hence all unordered pairs of elements of S appears in $\mathcal{R}$.

By counting the number of pairs that appear in each row of $A_i^{(j)}$ and Z_{jk} we find that ; each row of $\begin{array}{|c|c|} A_i^{(1)} & A_i^{(2)} \end{array}$ contains $\frac{1}{2}(v_2 + 1)$ pairs, each row of Z_{jk} contains n pairs, each row of $A_i^{(3)}$ contains $\frac{1}{2}n$ pairs and $\mathcal{R}_3$ has $\frac{1}{2}(v_3 + 1)$ pairs per row.

So the number of pairs in $\mathcal{R}$ is

$$v_1 n \left[\frac{1}{2}(v_1 - 1)n + \frac{1}{2}(v_2 + 1) \right] + v_3 \left[\frac{1}{2} v_1 n + \frac{1}{2}(v_3 + 1) \right]$$

$$= \frac{1}{2} \left[v_1 n (v_1 n - n + n + v_3 + 1) + v_3 (v_1 n + v_3 + 1) \right]$$

$$= \frac{1}{2}\left[(v_1 n + v_3)(v_1 n + v_3 + 1)\right]$$

$$= \frac{1}{2}\ v\,(v + 1)\ .$$

This is the number of unordered pairs which can be chosen from S, so each pair must appear precisely once in R. Therefore $\mathcal{R}$ is a Room Square based on S.

Note that

$A_1^{(1)}$	$A_1^{(2)}$
$A_1^{(3)}$	$\mathcal{R}_3$

Figure 4.3

is a subsquare of $\mathcal{R}$ which is isomorphic to $\mathcal{R}_2$. $\mathcal{R}_3$ is exhibited as a subsquare in the last v_3 rows and columns .

To show that $\mathcal{R}_1$ is a subsquare of $\mathcal{R}$, take the intersection of rows $1, n+1, 2n+1, \ldots, n(v_1 - 1) + 1$ and corresponding columns . The array formed has entry $\{0, 1_i\}$ where $\mathcal{R}_1$ has $\{0, i\}$ and entry $\{1_j, 1_k\}$ where $\mathcal{R}_1$ has $\{j, k\}$, so it is isomorphic to $\mathcal{R}_1$

Therefore the theorem follows .

Q.E.D.

Theorem 4.1.3 If there are Room Squares of sides v_1 and v_2, then there is a Room Square of side $v_1 \cdot v_2$ with subsquares of sides v_1 and v_2 which are isomorphic to the original squares.

Proof. The construction $\mathcal{S}$ in the proof of theorem 4.1.2 is carried out with $n = v_2$ and with $\mathcal{R}_2$ replacing $A^{(1)}$. By method described in that proof, it may be seen that $\mathcal{S}$ is a Room Square with the required properties.

Q.E.D.

Theorem 4.1.4 If there exist Room Squares of sides $v_1, v_2, \ldots, v_k$, then there is a Room Square of side $v = v_1 \cdot v_2 \cdot \ldots \cdot v_k$ with subsquares of sides $v_1, v_2, \ldots, v_k$.

Proof. This theorem follows from theorem 4.1.3 by induction on k.

Q.E.D.