CHAPTER II

PRELIMINARIES.

Let I, J be any two finite sets. Any set S of the form

$S = I \times J$ will be called a <u>rectangle</u>.

If $|I| = |J|$, then $S = I \times J$ will be called a <u>square</u>. The cardinality of I will be called the <u>side of the square</u>. Each element of the square will be called a <u>cell</u>. The element (i, j) will be called <u>the (i, j)- cell</u>.

The sets of the form $\{i\} \times I$ and $I \times \{j\}$ will be called the <u>row</u> and column, respectively, of the square $\{i\} \times I$ will be called the <u>i^{th} row</u>, and $I \times \{j\}$ will be called the <u>j^{ih} column of the square</u>.

Any function A on a square S into some set R will be called an <u>array</u>, it will be said to be <u>an array based on R</u>. The values of A will be called the <u>entries</u> of the array. A(i, j), the value of A at (i, j), will be called <u>the (i, j) - entry.</u> We shall write $A = (a_{ij})$ to indicate that A is an array whose (i, j) entry is a_{ij}.

If $I_1 \subseteq I$ and $J_1 \subseteq J$, then $S = I_1 \times J_1$ will be called a <u>subrectangle</u> $A\big|_{I_1 \times J_1}$ will be called a <u>sub array</u>.

Each square of side r can be represented physically by a square (geometric figure) which is subdivided by r - 1 vertical lines and r - 1 horizontal lines into r^2 small square.

Each row of the square can be represented by a rectangle determined by a pair of consecutive horizontal lines. Similarly a column of the square can be represented by a rectangle determined by a pair of consecutive vertical lines.

The (i, j) - cell can be represented by the small square which lies in the rectangle representing the i^{th} row and the j^{th} column.

An array based on R can be represented by a square whose small square representing (i, j) - cell is filled by the (i, j) - entry. For example if $I = \{1, 2\}$, $R = \{\alpha, \beta, \gamma\}$, $A(1, 1) = \beta$, $A(1, 2) = \gamma$, $A(2, 1) = \gamma$ and $A(2, 2) = \alpha$. Then the array can be represented as follows;

β	γ
γ	α

Figure 2.1

Definition 2.1 Let R be a finite set. An array $\mathcal{R}$ based on $\mathcal{P}(R)$, the set of all sub sets of R, will be said to be a Room Square if,

(1) each entry of $\mathcal{R}$ has cardinality 0 or 2,

(2) for every $a, b \in R$, if $a \neq b$, then $\{a, b\}$ appears precisely once in $\mathcal{R}$,

(3) every member of R appears precisely once in each row and precisely once in each column of $\mathcal{R}$.

Such a Room Square will be said to be a **Room Square based on R.** The number of elements of R will be called the order of $\mathcal{R}$.

For example, if R $= \{0, 1, 2, 3, 4, 5, 6, 7\}$, then Room Square can be represented as follows;

$\{0,1\}$	$\{3,7\}$	$\{5,6\}$	ϕ	$\{2,4\}$	ϕ	ϕ
ϕ	$\{0,2\}$	$\{4,1\}$	$\{6,7\}$	ϕ	$\{3,5\}$	ϕ
ϕ	ϕ	$\{0,3\}$	$\{5,2\}$	$\{7,1\}$	ϕ	$\{4,6\}$
$\{5,7\}$	ϕ	ϕ	$\{0,4\}$	$\{6,3\}$	$\{1,2\}$	ϕ
ϕ	$\{6,1\}$	ϕ	ϕ	$\{0,5\}$	$\{7,4\}$	$\{2,3\}$
$\{3,4\}$	ϕ	$\{7,2\}$	ϕ	ϕ	$\{0,6\}$	$\{1,5\}$
$\{2,6\}$	$\{4,5\}$	ϕ	$\{1,3\}$	ϕ	ϕ	$\{0,7\}$

Figure 2.2

To simplify the writing we shall omit the set notation in writing the unordered pairs and the empty set. Hence the above Room Square may be represented as follows :

0,1	3,7	5,6		2,4		
	0,2	4,1	6,7		3,5	
		0,3	5,2	7,1		4,6
5,7			0,4	6,3	1,2	
	6,1			0,5	7,4	2,3
3,4		7,2			0,6	1,5
2,6	4,5		1,3			0,7

Figure 2.3

Another example is the trivial Room Square of side 1.

0,1

Figure 2.4

<u>Theorem 2.2</u> If there exists a Room Square of side r based on R of order n, then r must be an odd integer and n must be even. Further more $r = n - 1$

<u>Proof</u> Let $\mathcal{R}$ be a Room Square of side r based on R

Since each entry of $\mathcal{R}$ has cardinality 0 or 2, and every member of R appears precisely once in each row, hence n must be even.

By counting the number of occurences of the n elements in two different ways, we have

$$nr = 2\binom{m}{2}.$$

The quantity nr on the left of this equation counts the occurences of the n elements by rows, while the quantity $2\binom{n}{2}$ on the right counts occurences of the n elements in pairs.

Hence we have

$$nr = 2\,\frac{n(n-1)}{2}$$

Therefore $r = n - 1$, which is an odd integer.

Q.E.D.

<u>Definition 2.3</u> Two Room Squares are called <u>isomorphic</u> if it is possible to obtain one from the other by a finite sequence of the following operations.

(i) permute the rows;

(ii) permute the columns;

(iii) relabel the elements ofthe set on which it is based.

A Room Square of sied r based on $R = \{0, 2, \ldots, r\}$ is called standardized if the pair $\{0, i\}$ appears in the (i, i) cell of Room square.

Definition 2.4 Let $\mathcal{R}$ be a Room Square. We say that $\mathcal{R}_1$ is a subsquare of $\mathcal{R}$ if $\mathcal{R}_1$ is a subarray of $\mathcal{R}$ which is itself a Room Square.

If r_1 and r_2 are sides of $\mathcal{R}_1$ and $\mathcal{R}$ respectively such that $r_1 < r_2$ then we say that $\mathcal{R}_1$ is a proper subsquare of $\mathcal{R}$.

From the definition 3.4 we see that every Room Square has itself as a subsquare and every Room Square has a subsquare of side 1 .

2.5 Non - Existence of Room Squares of sides 3 and 5 .

Suppose we has a Room Square of side $r = 3$ based on $\{0, 1, 2, 3\}$.

By reordering rows and columns we can ensure that $\{0, 1\}$ is in the $(1, 1)$ cell of the 3×3 array. In order to have a Room Square based on $\{0, 1, 2, 3\}$, the pair $\{2, 3\}$ must appear in row 1 as column 1, that is $\{2, 3\}$ must occur twice. This is a contradiction.

Hence no Room Square of side 3 exists .

In case $r = 5$, we can assume the first row is

0, 1	2, 3	4, 5	-	-

It does not matter whether the first column contains $\{2, 4\}$, $\{3, 5\}$ or $\{2, 5\}$, $\{3, 4\}$, as the first can be converted into the second case by the element permutation $4 \longleftrightarrow 5$, and this does not alter the pairs in the first row. So there is no loss generality in assuming a Room Square of side 5 has the form

0,1	2,3	4,5	–	–
2,4				
3,5				
–				
–				

Figure 2.5

where a dash indicates an empty cell.

The entries $\{2, 5\}$ and $\{3, 4\}$ must appear somewhere, but they can not be in row 1, row 2 or row 3 or column 1, column 2 or column 3. Moreover, they can not be in the same row or the same column, as this would entail $\{0, 1\}$ occuring again. Say $\{2, 5\}$ is in the (4, 4) cell and $\{3, 4\}$ is in the (5, 5) cell.

If there is an entry in the (4,5) cell it must be $\{0, 1\}$, which has already been used, so the (4, 5) cell is empty as in the (5, 4) cell.

The (4, 2) cell, the (4, 3) cell, the (5, 2) cell and the (5, 3) cell must be occupied in order that the fourth and fifth rows shall each contain three pairs. The (5, 2) cell can not contain 2, 3 or 4, so it may have $\{0,5\}$ or $\{1, 5\}$ in it; if it were $\{1, 5\}$ we could interchange 0 and 1 through out square, so assume the entry is $\{0, 5\}$

We can now fill in the (5, 3) cell, (4, 2) cell, and the (4, 3) cell, there being only one possibility in each case. So the square is a completion of

0,1	2,3	4,5	–	–
2,4	–	–		
3,5	–	–		
–	1,4	0,3	2,5	–
–	0,5	1,2	–	3,4

Figure 2.6

The pairs $\{0,2\}$ and $\{0,4\}$ remain to be placed. They can not both be in the third row, but neither one can be in row 2. So completion of the Square is impossible .