CHAPTER 2

MATHEMATICAL ANALYSIS

2.1 Introduction

For a certain class of nonlinear systems, a single loop servomechanism which has only a single nonlinearity is one of the most interesting type used in practical problems. The components of the loop may usually be seperated into two parts, as shown in Fig. 2.1 the linear portion has transfer function G(jw) It is a frequency-sensitive function and does not depend on amplitude. The nonlinear transfer function N, on the other hand depends only on amplitude and is frequency insensitive functions.

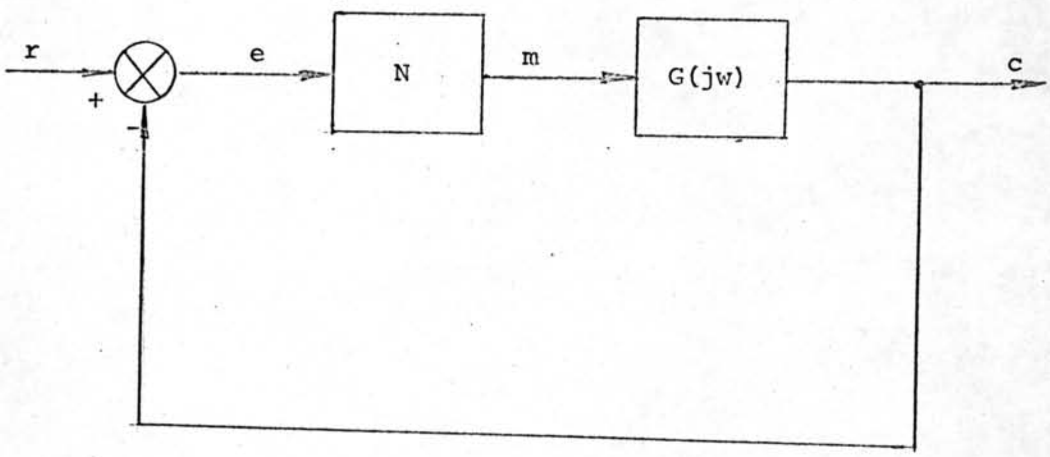

Fig. 2.1 A typical single loop nonlinear system.

In general the nonlinear transfer function N can be represented by an expression of the transfer functional relationship between e. and m. in the form [3]

$$m = F(e) \qquad (2.1a)$$

It is more convenient in this development to represent the relationship with an expression consisting of two parts : a quasilinear gain and a distortion term[3]

$$m = f(e) + f_d(e) \qquad (2.1b)$$

where e is a sinusoidal input signal[3]

$$e = E \sin(wt + \phi) = E \sin\Theta \qquad (2.2)$$

From Fig. 2.1, we have

$$m(\dot{j}W) = Keq(E) . e(\dot{j}W) \qquad (2.3)$$

In which Keq (E) is the equivalent gain and known as the describing function which is a function of the input signal amplitude or the fundamental component of the output m.

The describing function Keq (E) may be written in the complex form as[3]

$$Keq(E) = g(E) + jb(E) \qquad (2.4)$$

where

$$g(E) \triangleq \frac{1}{\pi E} \int_0^{2\pi} f(E\sin\Theta) \sin\Theta d\Theta$$

$$b(E) \triangleq \frac{1}{\pi E} \int_0^{2\pi} f(E\sin\Theta) \cos\Theta d\Theta \qquad (2.5)$$

The various types of the describing functions of common nonlinearities have been given by many authors.[1,2,3] However, some particular types have been listed in appendix A. for easy references.

2.2 Frequency Response

In 1953, Levinson suggested a graphical method for the determination of the frequency response of a certain class of nonlinear systems. This method can be applied to the nonlinear system shown in Fig. 2.1. However, his method has some practical limitations because of time consumption during the construction of the family of ellipses for various frequency w.[4,5]

A new graphical method is proposed in the thesis to overcome those limitations. In this method, the family of circles for various values of the amplitude of the error signal are obtained instead of ellipse.

2.3 Mathematical Analysis

It is assumed that the input signal r (t) is a sinusoidal and the linear transfer function portion G(jw) shown in Fig. 2.1 is a low-pass filter so that the output signal may be approximated as a sinusoidal and when the input signal is removed the nonlinear system remains stable.

From Fig. 2.1 and the above assumption, we may write

$$r = R \sin wt \tag{2.6}$$

or $$r(jw) = R\, e^{jwt} \tag{2.6a}$$

$$e = E \sin (wt + \phi) \tag{2.7}$$

or $$e(jw) = E\, e^{j(wt + \phi)} \tag{2.7a}$$

$$c = C \sin (wt + \Theta) \tag{2.8}$$

or $$c(jw) = C\, e^{j(wt + \Theta)} \tag{2.8a}$$

which $$G(jw) = g_1(w) + jg_2(w) \tag{2.9}$$

$$e(jw) = r(jw) - c(jw) \tag{2.10}$$

and $$c(jw) = Keq(E).G(jw).e(jw) \tag{2.11}$$

From eqns. (2.10) and (2.11) we obtain:-

$$\frac{e(jw)}{r(jw)} = \frac{1}{1 + Keq(E)\ G(jw)} \tag{2.12}$$

Substituting eqn. (2.11) into eqn. (2.12) we have

$$c(jw) = \frac{Keq\ (E)\ G(jw)}{1 + Keq\ (E)\ G(jw)} \cdot r(jw) \tag{2.13}$$

Square of the magnitude of the ratio $\frac{e(jw)}{r(jw)}$ of eqn. (2.12)

$$\left|\frac{E}{R}\right|^2 = \left|\frac{1}{1 + \{g(E) + jb(E)\}\{g_1(w) + jg_2(w)\}}\right|^2$$

or $$\frac{R^2}{E^2} = \left[1 + g(E)\ g_1(w) - b(E)\ g_2(w)\right]^2 + \left[g(E)\ g_2(w) + b(E)\ g_1(w)\right]^2 \tag{2.14}$$

Dividing both sides by $\left[g^2(E) + b^2(E)\right]$, the eqn. (2.14) becomes

$$\frac{R^2}{E^2\left[g^2(E) + b^2(E)\right]} = g_1^2(w) + g_2^2(w) - \frac{2b(E)g_2(w)}{g^2(E)+b^2(E)} + \frac{2g(E)\ g_1(w)}{g^2(E) + b^2(E)} + \frac{1}{g^2(E) + b^2(E)} \tag{2.15}$$

Adding both sides of eqn. (2.15) by $\left[\frac{g(E)}{g^2(E) + b^2(E)}\right]^2$ and $\left[\frac{b(E)}{g^2(E) + b^2(E)}\right]^2$ and simplified to obtain

$$\left[g_1(w) + \frac{g(E)}{g^2(E) + b^2(E)}\right]^2 + \left[g_2(w) - \frac{b(E)}{g^2(w) + b^2(E)}\right]^2 = \frac{R^2}{E^2\left(g^2(E) + b^2(E)\right)} \tag{2.16}$$

It can be seen that the above equation is an equation of the circle which has the center at

$$\left[-\frac{g(E)}{g^2(E) + b^2(E)},\ \frac{b(E)}{g^2(E) + b^2(E)}\right]$$

with the radius $\frac{R}{E\sqrt{g^2(E) + b^2(E)}}$

This equation describes the relations between the linear transfer function $G(jw)$ and the various values of the amplitude of the error signal E and the input signal R.

A family of these circles can be constructed in the G(jw) plane by varying the values of the error signal amplitude E when the amplitude of the input signal is kept at any constant value R. Typical examples of family of normalized circles derived for various types of nonlinearities are presented in Appendix C.

2.4 Determination of Frequency Response

The frequency response of the nonlinear system given in Fig. 2.1 can be obtained by the following procedures:-

(a) Plotting the frequency response of the linear transfer function G(jw) into the same complex plane in which the family circle curves have already been constructed.

(b) The intersections between the linear frequency response curve and the family of circle curves will yield the values of the error amplitudes E corresponding to the input signal frequencies w.

(c) The magnitude of the frequency response of the nonlinear system is directly determined by

$$C = E \, K_{eq}(E) \, G(jw) \qquad (2.17)$$

In addition, the phase angle of the frequency response of the nonlinear system may be calculated by eqn. (2.13)

This method can be extended to cover various special parameters in the nonlinearity and a typical example is given in section 2.5

A useful tables for the construction of the family of circle curves for various types of nonlinearties has been given in Appendix C. And the complete procedures for the determination of the frequency response of difference types of nonlinear system have been demonstrated in the next chapter.

2.5 A Typical Example

To obtain the family of circle curves, the similar technique described in the previous sections can be used. But it requires some modifications.

(a) The circle curves are plotted in the com plane as the valuesof new variables (E/b) instead of E, where b is a constant.

(b) The new transfer function $\left[\frac{G(jw)}{1/n}\right]$ is also plotted in the same complex plane, where n is a constant.

(c) The intersections between the new transfer function and the circle curves will yield the relations of new variables

(E/b) and the frequencies w. Therefore the output frequency response can be determined by eqns. (2.13) and(2.17). Now consider a typical nonlinearity as shown in Fig. 2.2. The describing function is written in the form[2]

$$Keq\ (E) = n \qquad \text{for } E \leq b$$

and

$$Keq\ (E) = \frac{2n}{\pi}\left[\sin^{-1}\left(\frac{1}{E/b}\right) + \frac{1}{E/b}\sqrt{1 - \left(\frac{1}{E/b}\right)^2}\right]$$

$$\text{for } E > b$$

In this case, the value of n is

$$n = M/b$$

where M is a constant value shown in Fig. 2.2

Therefore

$$\frac{g(E)}{n} = 1 \qquad \text{for } E/b \leq 1 \qquad (2.18a)$$

$$\frac{g(E)}{n} = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{E/b}\right) + \frac{1}{E/b}\sqrt{1 - \left(\frac{1}{E/b}\right)^2}\right] \qquad (2.18b)$$

$$\text{for } E/b > 1$$

and $b(E) = 0$ for all E/b (2.19)

It can be seen that a modified method for the construction of the family of circle curves is required in order to determine the frequency response.

Dividing both sides of eqn. (2.16) by $(1/n)^2$, we obtain.

$$\left[\frac{g_1(w)}{1/n} - \frac{\frac{g(E)}{n}}{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}\right]^2 + \left[\frac{g_2(w)}{1/n} + \frac{\frac{b(E)}{n}}{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}\right]^2 = \left[\frac{R/b}{E/b\sqrt{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}}\right]^2 \quad (2.20)$$

It can be seen that the above equation is a circle with the center at

$$\left[\frac{\frac{g(E)}{n}}{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}, -\frac{\frac{b(E)}{n}}{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}\right]$$

and the radius

$$\frac{R/b}{(E/b)\sqrt{\left(\frac{g(E)}{n}\right)^2 + \left(\frac{b(E)}{n}\right)^2}}$$

The graph of the family of circle curves of this example is given in Fig. 2.3. The frequency response of the nonlinearity Keq(E) followed by the linear transfer function $G(jw) = \frac{0.7(1+jw)}{(jw)^2}$ corresponding to the values of E/b for various frequency w is listed in Table 2.1.

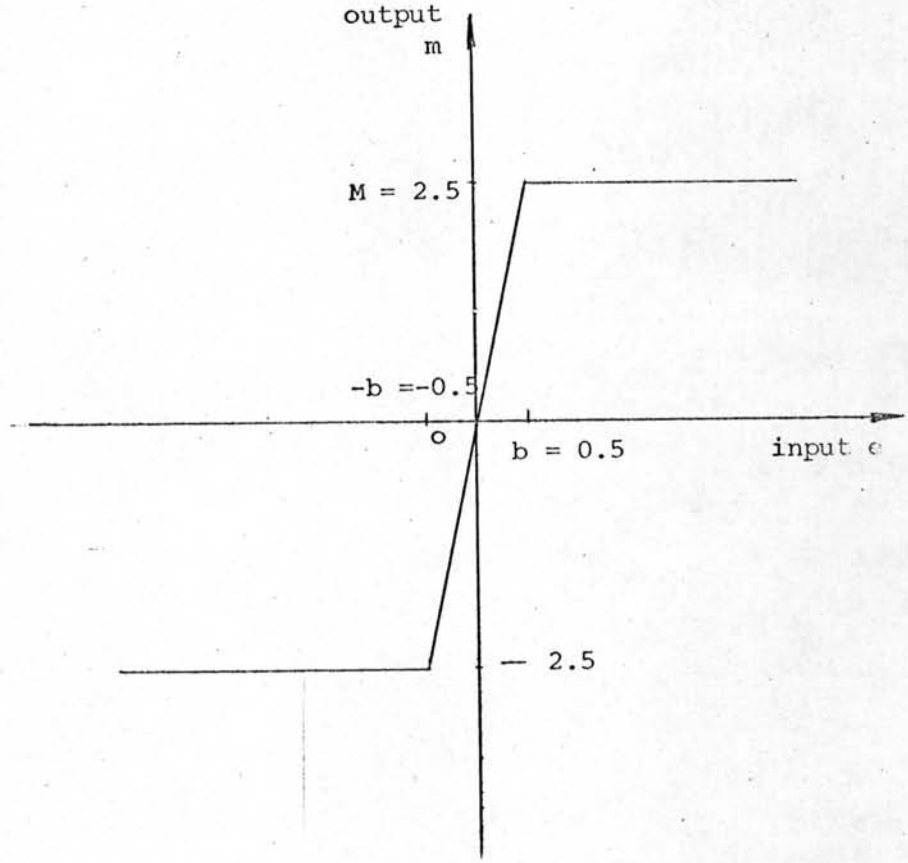

Fig. 2.2 A typical saturation nonlinearity

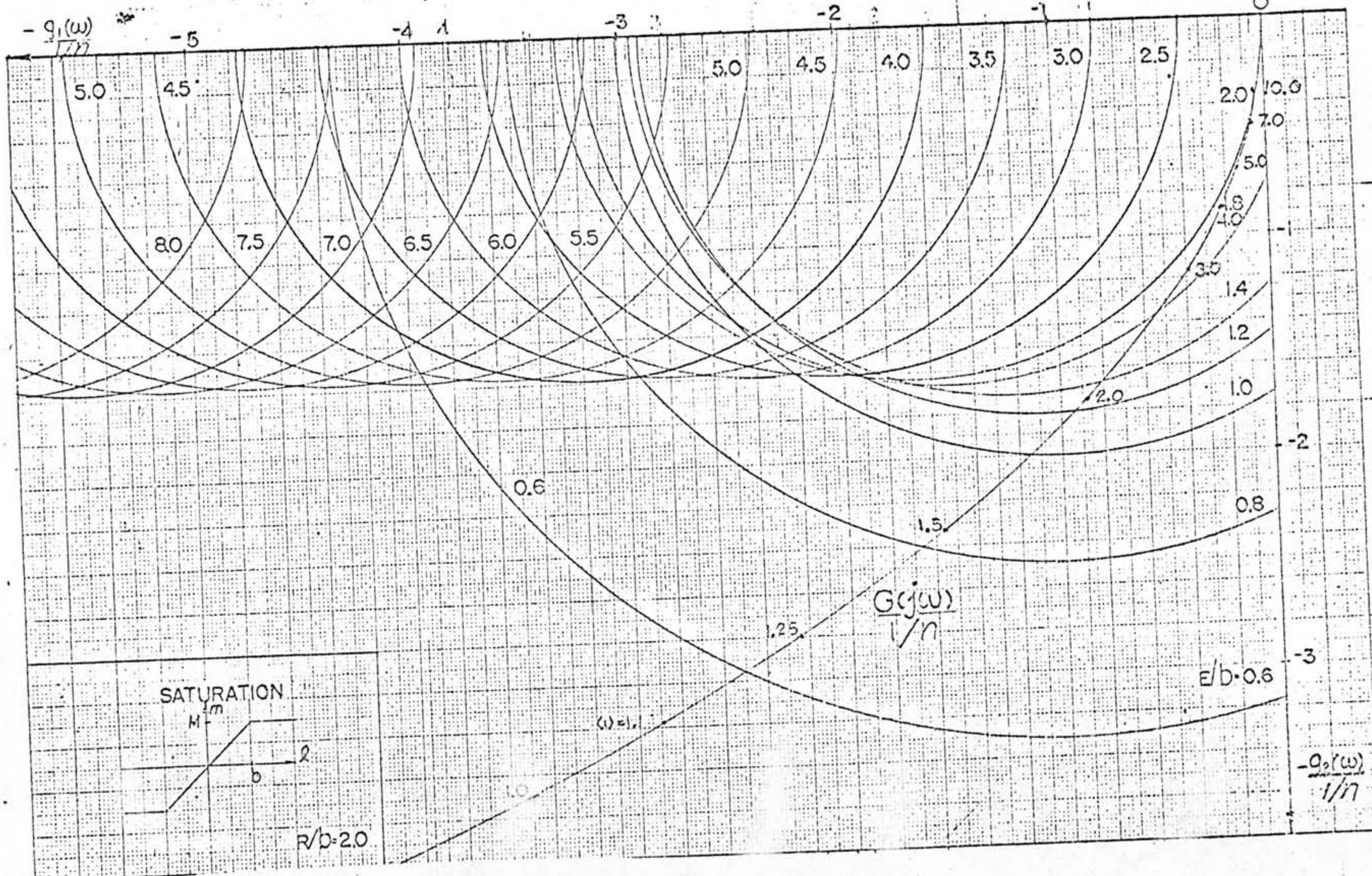

Fig. 2.3 Family of circle curves for saturation and the curve of $G(jw) = \frac{0.7\,(1+jw)}{(jw)^2}$

Table 2.1 Calculated results of example in section 2.5.

E/b	w	G(jw)Keq(E)	1+G(jw)Keq(E)	|C/R|	|C/R| db	Θ
0.6	1.18	3.87 ∠229.8°	3.32 ∠243.1°	1.17	1.36	346.7°
0.8	1.45	2.92 ∠235.3°	2.44 ∠254.6°	1.17	1.36	340.7°
1.0	1.74	2.32 ∠240.1°	2.00 ∠265.7°	1.16	1.29	334.4°
1.2	1.91	1.87 ∠242.6°	1.67 ∠-85.2°	1.12	0.98	327.8°
1.4	2.06	1.53 ∠244.1°	1.42 ∠-76.6°	1.08	0.67	320.7°
1.8	2.52	0.98 ∠248.4°	1.11 ∠-54.9°	0.88	-1.11	303.3°
2.0	8.75	0.21 ∠264.6°	1.00 ∠-12.1°	0.21	-13.56	276.7°

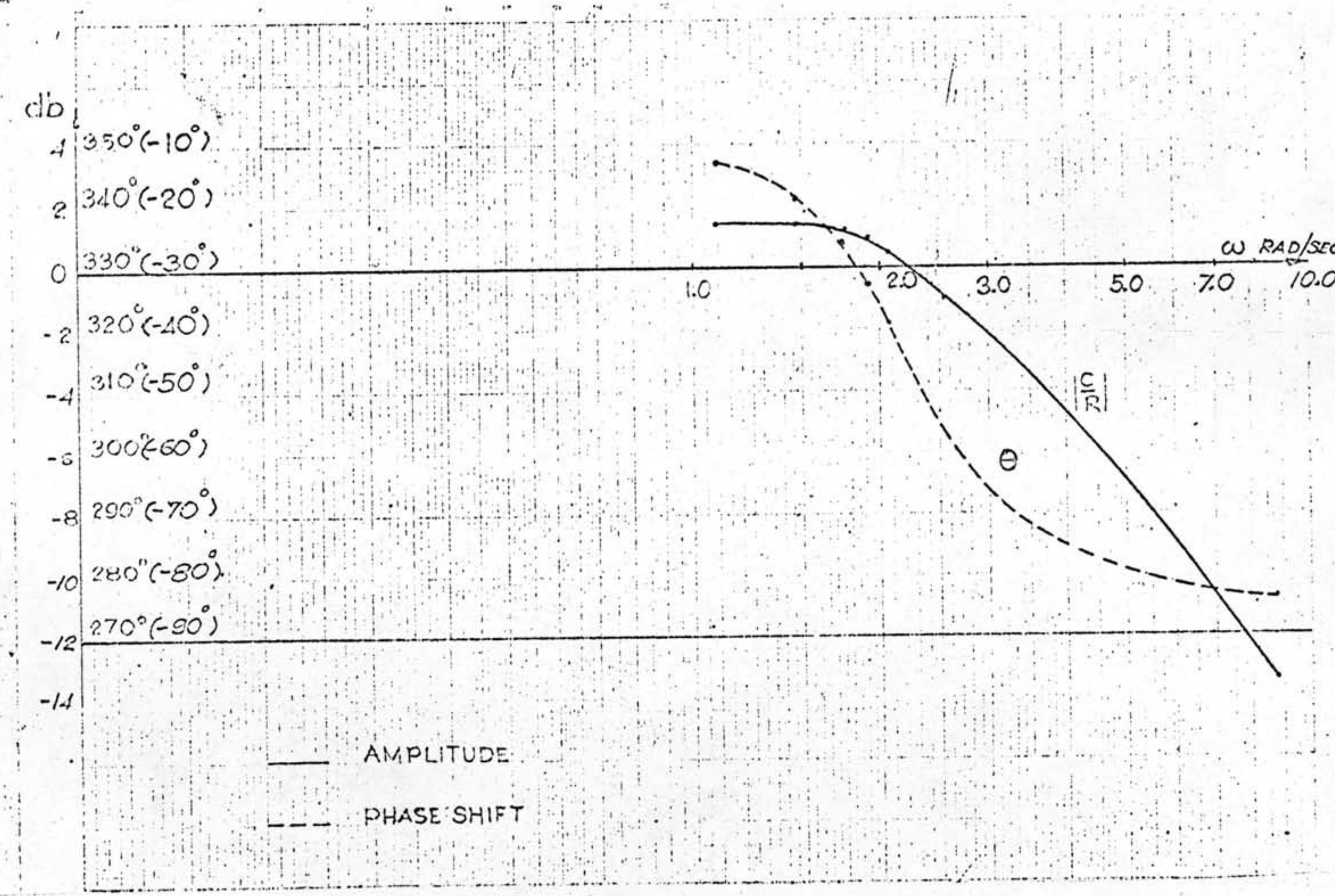

Fig. 2.4 Output frequency response of the typical example